CS-210 Homework 4

M.Shozab Hussain Student ID: 23100174

November 6, 2020

2. (a) $\sum_{j=0}^{8}(2^{j+1} - 2^j)$

$(2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + (2^4 - 2^3) + (2^5 - 2^4) + (2^6 - 2^5) + (2^7 - 2^6) + (2^8 - 2^7)$
$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$
255

(b) $\sum\limits_{i=1}^{5} i^2 + \sum\limits_{i=1}^{5} 7$

$\sum\limits_{i=1}^{5} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{5(5+1)(10+1)}{6} = \frac{5(6)(11)}{6} = \frac{330}{6} = 55$

$\sum\limits_{i=1}^{5} 7 = 7n = 7(5) = 35$

$\sum\limits_{i=1}^{5} i^2 + \sum\limits_{i=1}^{5} 7 = 55 + 35 = 90$

(c) $\sum\limits_{i=1}^{4}(i^2 + i)$

$\sum\limits_{i=1}^{4} i^2 + \sum\limits_{i=1}^{4} i$

$\sum\limits_{i=1}^{4} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{4(4+1)(8+1)}{6} = \frac{4(5)(9)}{6} = \frac{180}{6} = 30$

$\sum\limits_{i=1}^{4} i = \frac{n(n+1)}{2} = \frac{4(4+1)}{2} = \frac{4(5)}{2} = \frac{20}{2} = 10$

$\sum\limits_{i=1}^{4}(i^2 + i) = 30 + 10 = 40$

(d) $\sum_{k=1}^{4} k^2 + \sum_{k=1}^{4} k$

$\sum_{k=1}^{4} (k^2 + k)$

Hence, same as the previous part so;

$\sum_{k=1}^{4} k^2 + \sum_{k=1}^{4} k = 42$

(e) $\sum_{i=0}^{4} (3i^2 + 2i)$

$\sum_{i=0}^{4} 3i^2 + \sum_{i=0}^{4} 2i$

$3\sum_{i=0}^{4} i^2 + 2\sum_{i=0}^{4} i$

$\sum_{i=0}^{4} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{4(4+1)(8+1)}{6} = \frac{4(5)(9)}{6} = \frac{180}{6} = 30$

$\sum_{i=0}^{4} i = \frac{n(n+1)}{2} = \frac{4(4+1)}{2} = \frac{4(5)}{2} = \frac{20}{2} = 10$

$3\sum_{i=0}^{4} i^2 + 2\sum_{i=0}^{4} i = 3(30) + 2(10) = 90 + 20 = 110$

$\sum_{i=0}^{4} (3i^2 + 2i) = 3\sum_{i=0}^{4} i^2 + 2\sum_{i=0}^{4} i = 110$

(f) $3\sum_{k=0}^{4} k^2 + 2\sum_{k=0}^{4} k$

$\sum_{k=0}^{4} 3k^2 + \sum_{k=0}^{4} 2k$

$\sum_{k=0}^{4} (3k^2 + 2k)$

Hence, same as the previous part so;

$\sum_{k=0}^{4} (3k^2 + 2k) = 110$

(g) $\sum_{k=111}^{3000} k$

$\sum_{k=1}^{3000} k - \sum_{k=1}^{110} k$

$\sum_{k=1}^{3000} i = \frac{n(n+1)}{2} = \frac{3000(3000+1)}{2} = \frac{3000(3001)}{2} = \frac{9003000}{2} = 4501500$

$\sum_{k=1}^{110} k = \frac{n(n+1)}{2} = \frac{110(110+1)}{2} = \frac{110(111)}{2} = \frac{12210}{2} = 6105$

$\sum_{k=1}^{3000} k - \sum_{k=1}^{110} k = 4501500 - 6105 = 4495395$

$\sum_{k=111}^{3000} k = 4495395$

(h) $\sum_{k=-n}^{n} k$

$\sum_{k=-n}^{0} k + \sum_{k=0}^{n} k$

$\sum_{k=0}^{n} -k + \sum_{k=0}^{n} k$

$-\sum_{k=0}^{n} k + \sum_{k=0}^{n} k$

0

3. (a) $\sum_{i=0}^{n} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 ...$

$\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 ...$

(b) $\sum_{i=1}^{n} 2^{i-1} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 ...$

$\sum_{i=0}^{n} 2^{i-1} = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 ...$

(c) $\sum_{j=0}^{n}(2^{j+1} - 2^{j})$

$(2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + (2^4 - 2^3) + (2^5 - 2^4) + (2^6 - 2^5) + (2^7 - 2^6) + (2^8 - 2^7)...$
$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128...$

(d) $\sum\limits_{i=1}^{n} \frac{2^i}{2} = \frac{2^1}{2} + \frac{2^2}{2} + \frac{2^3}{2} + \frac{2^4}{2} + \frac{2^5}{2} + \frac{2^6}{2} + \frac{2^7}{2} + \frac{2^8}{2}...$

$= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8...$
$\sum\limits_{i=1}^{n} \frac{2^i}{2} = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256...$

6. Suppose that$a_n = k$ then,

$k = \lfloor \frac{1}{2} + \sqrt{2n + \frac{1}{4}} \rfloor$

Hence, this means that $\sum_{i=1}^{k-1} i \leq n < \sum_{i=1}^{k} i$

Now if we remove the floor function then,

$k \leq \frac{1}{2} + \sqrt{2n + \frac{1}{4}} < k + 1$

$k - \frac{1}{2} \leq \sqrt{2n + \frac{1}{4}} < k + 1 - \frac{1}{2}$

$(k - \frac{1}{2})^2 - \frac{1}{4} \leq 2n < (k + \frac{1}{2})^2 - \frac{1}{4}$

$k^2 - k + \frac{1}{4} - \frac{1}{4} \leq 2n < k^2 + k + \frac{1}{4} - \frac{1}{4}$

$k^2 - k \leq 2n < k^2 + k$

$\frac{k^2-k}{2} \leq n < \frac{k^2+k}{2}$

$\frac{k(k-1)}{2} \leq n < \frac{k(k+1)}{2}$

Since, $\frac{k(k-1)}{2} = \sum_{i=1}^{k-1} i \; and \; \frac{k(k+1)}{2} = \sum_{i=1}^{k} i$ therefore,

$\sum_{i=1}^{k-1} i \le n < \sum_{i=1}^{k} i$

$\sum_{i=1}^{k} i - \sum_{i=1}^{k-1} i = k$

This shows that for every k, there are k number of terms, hence, proved.

7. $\sum\limits_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

   Let's differentiate both sides with respect to r

   $\sum\limits_{i=0}^{n} ir^{i-1} = \frac{-(n+1)r^n(1-r)-(-1)(1-r^{n+1})}{(1-r)^2}$

   $\sum\limits_{i=0}^{n} ir^{i-1} = \frac{-(n+1)r^n(1-r)+(1-r^{n+1})}{(1-r)^2}$

   $\sum\limits_{i=0}^{n} ir^{i-1} = \frac{(1-r^{n+1})-(n+1)r^n(1-r)}{(1-r)^2}$

   $\sum\limits_{i=0}^{n} ir^{i-1} = \frac{(1-r^{n+1})}{(1-r)^2} - \frac{(n+1)r^n(1-r)}{(1-r)^2}$

   $\sum\limits_{i=0}^{n} ir^{i-1} = \frac{(1-r^{n+1})}{(1-r)^2} - \frac{(n+1)r^n}{(1-r)}$

   Multiply both sides with r

   $\sum\limits_{i=0}^{n} ir^{i-1}r = \frac{r(1-r^{n+1})}{(1-r)^2} - \frac{(n+1)r^n r}{(1-r)}$

   $\sum\limits_{i=0}^{n} ir^{i} = \frac{r(1-r^{n+1})}{(1-r)^2} - \frac{(n+1)r^{n+1}}{(1-r)}$

   For n= ∞, we will have to take limits n $\rightarrow \infty$

   $\lim_{x\rightarrow\infty} \sum\limits_{i=0}^{n} ir^{i} = \lim_{x\rightarrow\infty} \frac{r(1-r^{n+1})}{(1-r)^2} - \lim_{x\rightarrow\infty} \frac{(n+1)r^{n+1}}{(1-r)}$

   Since $|r| < 1$ therefore, $r^{n+1} \rightarrow 0$ when $n \rightarrow \infty$ so this leaves as with,

   $\sum\limits_{i=0}^{\infty} ir^{i} = \frac{r}{(1-r)^2}$

8. If we suppose $n^2 \leq k < (n+1)^2$ then we get

$\lfloor \sqrt{k} \rfloor = n.$

Hence, let's suppose n as a positive integer and $n = \lfloor \sqrt{m} \rfloor$ so,

$n^2 \leq m < (n+1)^2$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = 0 + (2^2 - (2-1)^2).(2-1) + ... + (n^2 - (n-1)^2).(n-1) + (m - n^2 + 1).n$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \sum_{k=1}^{n-1} ((k+1)^2 - k^2))k + (m - n^2 + 1)n$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \sum_{k=1}^{n-1} (2k^2 + k) + (m - n^2 + 1)n$

Since, $\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$, and $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{2n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2} + mn - n^3 + n$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{2n(n-1)(2n-1)}{6} + \frac{3n(n-1)}{6} + mn - n^3 + n$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{(2n^2-2n)(2n-1)}{6} + \frac{3n^2-3n}{6} + mn - n^3 + n$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{4n^3-2n^2-4n^2+2n}{6} + \frac{3n^2-3n}{6} + mn - n^3 + n$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{4n^3-2n^2-4n^2+2n+3n^2-3n+6mn-6n^3+6n}{6}$

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{-2n^3-3n^2+5n+6mn}{6}$

Since, $n = \lfloor \sqrt{m} \rfloor$ so replace n

$\sum\limits_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{-2(\lfloor \sqrt{m} \rfloor)^3 - 3(\lfloor \sqrt{m} \rfloor)^2 + 5(\lfloor \sqrt{m} \rfloor) + 6m(\lfloor \sqrt{m} \rfloor)}{6}$