CS-210 Homework 5

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November 15, 2020

1. Lemma 1: The product of two positive or two negative numbers is always positive.

   Lemma 2: The sum of positive integers is greater than any of the individual integers being added.

   ## Proof:

   If n is an integer then, it can be safely assumed that n can only exist in three states. Either $n = 0$ or $n > 0$ or $n < 0$.

   - case 1: $n = 0$, then $n^2 = (0)^2 = 0$. Hence, $n = n^2$ satisfies the theorem.

   - case 2: $n > 0$
     We can also write $n^2 = (n \times n)$ as $n^2 = n + n + n + n...$(where number of n's $= n$. Now since $n > 0$ then, by lemma 2, $n^2 \geq n$. Hence, theorem is satisfied.

   - case 2: $n < 0$
     We can also write $(-n)^2 = (-1)^2(n \times n)$ as,
     $(-n)^2 = (-1 \times -1)(n + n + n + n...)$(where number of n's $= n$.
     By Lemma 1, $(-n)^2 = (1)(n + n + n + n...)^2 = n^2$. Now since $n < 0$ and $(n)^2 > 0$ therefore, $(-n)^2 > -n$. Hence, theorem is satisfied.

   All cases are satisfied therefore, the theorem is proved. QED

4. Lemma 1: If n is even then, n^3 is even.
   Lemma 2: If n is odd then, n^3 is odd.

Proof:
Let's assume that $\sqrt[3]{2}$ is rational.
This implies that $\sqrt[3]{2} = \frac{p}{q}$ p,q $\in \mathbb{Z}$, q $\neq 0$
Suppose p and q have no common factors.
After taking cube on both sides, we get $2 = \frac{p^3}{q^3} \rightarrow p^3 = 2q^3$.
Now by the definition of even numbers $2q^3$ is even.
As a result, by equality p^3 is even and therefore, by lemma 1, p is even .
By definition of even numbers p= 2k then, $p^3 = 8k^3$.
Since, $p^3 = 2q^3$ hence, $8k^3 = 2q^3 \rightarrow 4k^3 = q^3$
$2(2k^3) = q^3$ now by the definition of even numbers $2(2k^3)$ is even so by equality q^3 is also even and then, finally by lemma 1 q is also even.

So if p and q both are even then this indicates that they have 2 as a common factor. This is a contradiction to our assumption that p and q have no common factors. Due to this contradiction it is proved that $\sqrt[3]{2}$ is not rational. QED

7. $n^2 = n \times n$
   Theorem: if $n \times n$ is even then n is even.

   Proof:
   Assume that for an even n^2, n is odd
   Then by the definition of odd numbers $n = 2k + 1$
   $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1$
   Let $2k^2 + 2 = a$ then, $n^2 = 2a + 1$
   So, by the definition of odd numbers n^2 is an odd number whereas according to our assumption it should be even. Hence, there is a contradiction which proves that if n^2 is even then n cannot be odd and should be even. QED

11. Lemma 1: If n is an integer then, $n^2 \geq n$
    Lemma 2: Sum of two or more positive integers is always greater than 1.

    Proof: Assume that there are positive integer solutions to the equation.

Let $x = a$ and $y = b$ $\quad a, b \in \mathbb{Z}^+$
Now by Lemma 1, we get, $x^2 \geq a$ and $y^2 \geq b$ therefore,
$x^2 + y^2 \geq a + b$
By Lemma 2: $a + b > 1$ which implies that $x^2 + y^2 > 1$.
As a result $x^2 + y^2 \neq 1$
This is a contradiction to our assumption that there are some positive integers for which $x^2 + y^2 = 1$
Hence, proved that no positive integers solution exist for this equation QED.

12. Lemma 1: If x is rational and y is irrational then, xy must by irrational.
Lemma 2: If x and y are rational numbers then x+y must be also a rational.

Proof:
If a is a rational number then by the definition of rational numbers, $a = \frac{p}{q}$ $p, q \in \mathbb{Z}, \quad q \neq 0$. Given that p and q do not have a common factor.
Now $a + b = \frac{p}{q} + b = \frac{p+bq}{q}$.
By Lemma 1, bq is an irrational number therefore, Lemma 2 is false so, $p + bq$ is also an irrational number.
Now $p + bq$ (numerator) is not an integer hence, the definition of rational numbers is not satisfied making $\frac{p+bq}{q}$ an irrational number. Since, $a + b = \frac{p+bq}{q}$, so by equality, $a + b$ must be an irrational number as well.
QED