Problem 8:

1)

$Suppose\ that\ f(n)\ =\ n\ and\ g(n)\ =\ n^2$

$Then\ there\ does\ not\ exist\ a\ positive\ real\ number\ c\ and\ real\ number\ k\ such\ that$
$c \cdot n\ \geq n^2$ for all n ≥ k

$Hence\ g(n)\ \neq O(f(n))$

2)

$Suppose\ than\ f(n)\ =\ n\ and\ g(n)\ =\ n^2$

$Then\ max(f(n), g(n))\ = n^2$

$Then\ there\ exist\ a\ positive\ real\ number\ c1\ and\ real\ number\ k1\ such\ that$
$c1 \cdot (n + n^2) \leq n^2\ for\ n \geq k1$

$Then\ there\ exist\ a\ positive\ real\ number\ c2\ and\ real\ number\ k2\ such\ that$
$c2 \cdot (n + n^2) \geq n^2\ for\ n \geq k2$

$Hence\ f(n) + g(n)\ = \Theta(max(f(n), g(n)))$

3)

$f(n)\ =\ O(g(n))$

$f(n)\ \leq\ c\ \cdot g(n))$

$taking\ log\ at\ both\ sides$

$log(f(n))\ \leq\ log(c \cdot g(n))$

$log(f(n)))\ \leq\ log(c) + log(g(n))$

since $log(g(n))\ \geq 1$

$\log(f(n))\ \leq log(c) log(g(n))\ + log(g(n))$

$log(f(n))\ \leq\ log(g(n))[log(c) + 1]$

$let\ log(c)\ +\ 1\ =\ c1$

$then\ log(f(n))\ \leq c1 \cdot log(g(n))$

$Hence, log(f(n)) = O(log(g(n)))$

4)

$suppose\ f(n) = 2lg(n)\ and\ g(n) = lg(n)$

$there\ exists\ a\ positive\ real\ number\ c\ and\ real\ number\ k\ such\ that$

$2lg(n) \leq c \cdot lg(n)\ for\ all\ n \geq k$

$which\ implies\ that\ f(n) = O(g(n))$

$2^{f(n)} = 2^{2lg(n)} = 2^{lg(n^2)} = n^2$

$2^{g(n)} = 2^{lg(n)} = n$

$there\ does\ not\ exist\ a\ positive\ real\ number\ c1\ and\ real\ number\ k1\ such\ that$

$n^2 \leq c1 \cdot n$ for all n ≥ k1

$Hence\ 2^{f(n)} \neq O(2^{g(n)})$

5)

$suppose\ f(n) = n\ and\ (f(n))^2 = n^2$

$then\ there\ exist\ a\ positive\ real\ number\ c\ and\ real\ number\ k\ such\ that$

$n \leq c \cdot n^2$ for all n ≥ k

$hence\ f(n) = O(f(n))^2)$

6)

$f(n) = O(g(n))$

$then\ there\ exist\ a\ positive\ real\ number\ c\ and\ real\ number\ k\ such\ that$

$f(n) \leq c \cdot g(n)$ for all n ≥ k

$g(n) \geq \frac{1}{c} \cdot f(n)$

$let\ \frac{1}{c} = z$

$then\ there\ exist\ a\ positive\ real\ number\ z\ and\ real\ number\ k1\ such\ that$

$g(n) \geq z \cdot f(n)$ for all n ≥ k1

$implying\ that\ g(n) = \Omega(f(n))$

7)

$Suppose\ f(n) = 4^n\ then,$

$f(\frac{n}{2}) = 4^{\frac{n}{2}} = (4^{\frac{1}{2}})^n = 2^n$

$Since\ 4^n \neq \Theta(2^n), hence\ the\ conjecture\ is\ false$

8)

$Suppose f(n) = n^2,$

$then\ \Omega(f(n)) = n$

$Then\ there\ exist\ a\ positive\ real\ number\ c1\ and\ real\ number\ k1\ such\ that$

$c1 \cdot (n + n^2) \leq n^2$ for all n ≥ k1

$Then\ there\ exist\ a\ positive\ real\ number\ c2\ and\ real\ number\ k2\ such\ that$

$c2 \cdot (n + n^2) \geq n^2$ for all n ≥ k2

$implying\ that\ n^2 + n = \Theta(n^2)$

$hence, f(n) + \Omega(f(n)) = \Theta(n^2)$