Problem 1

Algorithm

Note: A hospital will be matched iff all its positions are filled

GALE–SHAPLEY (list of hospitals with their preference lists and number of positions, list of students with their preference list)

INITIALIZE M to empty matching.

WHILE (some hospital h is unmatched and has not proposed to every student)

    FOR (the number of empty positions that hospital h has)

        s ← first student on h's list to whom h has not yet proposed.

        IF (s is unmatched)

            Add h–s to matching M.

            Update empty positions of h

        ELSE IF (s prefers h to current partner h')

            Replace h'–s with h–s in matching M.

            Update empty positions of h and h's

        ELSE

            s rejects h.

RETURN stable matching M.

Proof that all hospitals are matched:

- For the sake of contradiction, suppose that after termination of Gale-Shapley algorithm a hospital is h still unmatched.
- After termination of algorithm, there are still some unmatched students left since there are more students than the number of positions
- Any student that remains unmatched is because no hospital including h has proposed to it.
- But algorithm makes sure that hospital h proposes to every student if is not matched.

- Hence, a contradiction.

Proof that all the matches are stable:

Suppose a pair h-s that didn't make it to the matching M returned by algorithm

Case 1:

- h never proposed to s
- h preferred all of its gale shapely partners to s
- Hence, no chance of side deal between h-s so h-s is stable

Case 2:

- h proposed to s
- s rejected h right away or later by accepting another hospital h' that was higher in his preference list than h
- Hence, s prefers h' over h
- Again, no chance of side deal between h-s so h-s is stable

In either case, h-s is stable which means that all the pairs in M are stable because no student and hospital can make a side deal and hence, create instability.