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Modeling infectious diseases in mixed household structures

Shrawan Parajuli

Mathematisches Institut
Universität Koblenz

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Contents

Background Study

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- Emergence of infectious diseases is concerning.
- spread of infectious disease gradually decreases due to immunity or become endemic due to lack of healthcare facilities.
- Study of infectious disease is an interdisciplinary fields is studied with different perspectives and purposes.
- Considering household structure has been an useful way of understanding spread of infectious diseases(for eg. Influenza) and it can be done with different epidemiological models such as SIS (Susceptible-Infectious-Susceptible).
- Also, the dynamics of infectious diseases is considered by studying equilibria.

Reference- [1],[3]

Objectives

- To analyze equilibria (Disease-Free Equilibrium(DFE) and Endemic Equilibrium(EE)).
- To observe the influence of infection and recovery parameters in Population dynamics.

Model and Method

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- Single Household (SHH) using SIS model.
- Master equation describes the time-evolution of P_k, P_k denotes the probability having k infected individuals and (N-k) denotes susceptible in a given household size N,
- External infection rate, $\epsilon = \tilde{\alpha}\frac{I}{T_p}$,
- Homogeneous household, Expectation: $\mathbb{E}I_H = \sum_{k=0}^{N} k * P_k$.
- $\frac{I}{T_p} \approx \frac{M_H * \mathbb{E}I_H}{M_H * N} = \frac{\mathbb{E}I_H}{N} = \frac{1}{N}\sum_{k=0}^{N} kP_k$
- In the SHH-N Model we have

$$\epsilon = \frac{\alpha}{N}\mathbb{E}I_H = \frac{\alpha}{N}\sum_{k=0}^{N} kP_k$$

Reference- [1]

Model and Method: State transition

- State transition in (SHH)

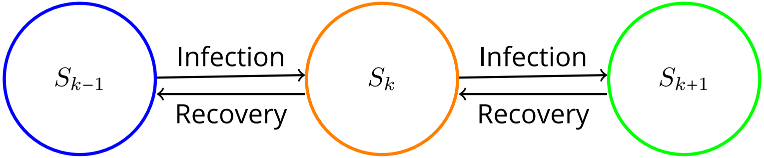

Figure 1: SHH diagram for state transition

Model and Method: SHH-N

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$$
\begin{aligned}
\frac{dP_k}{dt} &= (X_{\{\geq 1\}}(k))\ (\frac{\alpha}{\mathbf{N}}\sum_{k=1}^{\mathbf{N}} kP_k)\ (N-(k-1))\ P_{k-1} \\
&+ (X_{\{\geq 2\}}(k))\ \beta\ (N-(k-1))\ P_{k-1} \\
&+ (X_{\{\leq N-1\}}(k))\ \gamma\ (k+1)\ P_{k+1} \\
&- (X_{\{\leq N-1\}}(k))\ (\frac{\alpha}{\mathbf{N}}\sum_{k=1}^{\mathbf{N}} kP_k)\ (N-k)P_k \\
&- (X_{\{\geq 1\}}(k))(X_{\{\leq N-1\}}(k))\ \beta\ (N-k))\ P_k \\
&- (X_{\{\geq 1\}}(k))\ \gamma\ (k)P_k
\end{aligned}
$$

SHH-N

- Positive Contribution
  - (external infection) $S_{k-1} \to S_k$, $k \in \{1, 2, ..N\}$,
  - (internal infection) $S_{k-1} \to S_k$, $k \in \{2, ..N\}$
  - (recovery) $S_{k+1} \to S_k$, $k \in \{0, ..N-1\}$,
- Negative Contribution
  - (external infection) $S_k \to S_{k+1}$, $k \in \{0, ..N-1\}$,
  - (internal infection) $S_k \to S_{k+1}$, $k \in \{1, ..N-1\}$
  - (recovery) $S_k \to S_{k-1}$, $k \in \{1, ..N\}$,

Model and Method : SHH-N = 1,2

- General Master Equation for SHH in case of household size N=1,(SIS Model)

$$\frac{dP_0}{dt} = \gamma P_1 - \alpha P_1 P_0$$

$$\frac{dP_1}{dt} = \alpha P_1 P_0 - \gamma P_1$$

- General Master Equation for SHH in case of household size N=2,

$$\frac{dP_0}{dt} = \gamma P_1 - 2(\frac{\alpha}{2} P_1 + \alpha P_2) P_0$$

$$\frac{dP_1}{dt} = 2(\frac{\alpha}{2} P_1 + \alpha P_2) P_0 + 2\gamma P_2 - (\frac{\alpha}{2} P_1 + \alpha P_2) P_1 - \beta P_1 - \gamma P_1$$

$$\frac{dP_2}{dt} = (\frac{\alpha}{2} P_1 + \alpha P_2) P_1 + \beta P_1 - 2\gamma P_2$$

Reference- [1]

Model and Method : Analytical Approach

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- Non-linear differential equations is studied with stability analysis, phase portraits and Numerical simulation.
- Total infection rate :

$$\frac{d\mathbb{E}[I_H]}{dt} = \frac{dP_1}{dt} + 2\frac{dP_2}{dt}$$

$$I_{Total} = \int_0^T \mathbb{E}[I_H(t)]\,dt$$

- Equilibrium
  - DFE, $lim_{t\to\infty}(P_0(t), P_1(t), P_2(t)) = (1,0,0)$.
  - In EE, $lim_{t\to\infty}(P_0(t), P_1(t), P_2(t)) \neq (1,0,0)$ (or (1,0,0) does not exist).
- In equilibrium :

$$dP_k/dt = f_k(P_k) = 0$$

Model and Method : Linearization, DFE

- Linearization in SHH-2
- DFE, Jacobian Matrix (J) at $(1,0,0)$

$$\begin{pmatrix} 0 & -\alpha+\gamma & -2\,\alpha \\ 0 & \alpha-\beta-\gamma & 2\,\alpha+2\,\gamma \\ 0 & \beta & -2\,\gamma \end{pmatrix}$$

- characteristic equation of the form is $|J-\lambda E| = 0$,

$$\begin{pmatrix} \alpha-\beta-\gamma & 2\,\alpha+2\,\gamma \\ \beta & -2\,\gamma \end{pmatrix}$$

$$\lambda_i = \frac{Tr - \sqrt{Tr^2-4D}}{2}$$

$$\lambda_{ii} = \frac{Tr + \sqrt{Tr^2-4D}}{2}$$

Reference- [2]

Model and Method : Trace and Determinant

- Let A be 2 X 2 square matrix of form written in

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- $Trace(Tr) = a_{11} + a_{22} = \alpha - \beta - 3\gamma$
- $Determinant(D) = a_{11}a_{22} - a_{12}a_{21} = 2(\gamma^2 - \alpha(\beta + \gamma))$

Model and Method : EE

- EE, endemic equilibrium point (P_0, P_1, P_2) or $(P_0, P_1, 1 - P_0 - P_1)$
- P_1,

$$P_1 = \frac{2\alpha P_0(1 - P_0)}{(\gamma + \alpha P_0)} \tag{1}$$

- positive root P_0,

$$P_0 = \frac{\gamma}{2\alpha\beta}(-(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4\beta\gamma}) \tag{2}$$

- jacobian at (P_0,P_1,1-P_0-P_1)

$$\begin{pmatrix} -\alpha(2 - 2P_0 - P_1) & -\alpha P_0 + \gamma & -2\,\alpha P_0 \\ \alpha(2 - 2P_0 - P_1) & P_0(1+\alpha) - 1 - \beta - \gamma & 2\,\alpha P_0 - \alpha P_1 + 2\,\gamma \\ 0 & \alpha(1 - P_0) + \beta & \alpha P_1 - 2\,\gamma \end{pmatrix} \tag{3}$$

Simulation and Results

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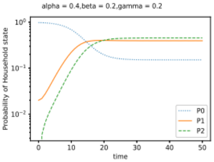
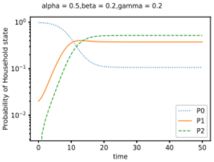
- Numerical method: Runge-Kutta Method, $(RK45)$
- Initial conditions, P_0 is $P_0(0) = 0.98$, P_1 is $P_1(0) = 0.02$ and P_2 is $P_2(0) = 0$,
- timestep : [0,50]
- parameters
  - α : between-household infection rate.
  - β : within-household infection rate.
  - γ : household recovery rate
- vary only one parameter and keep the other two parameter fixed
- regular spaced (0.2,0.3,0.4,0.5,0.6,0.7,0.8) rates
- initial rates , α = 0.4, β = 0.2, γ = 0.2
- for example : Influenza recovered rate 0.2 (takes 5 days to recover)

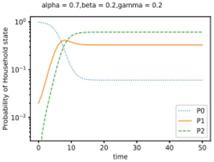
Probability of household state vs time (α)



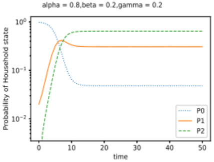
(a) $\alpha = 0.4, \beta = 0.2, \gamma = 0.2$



(b) $\alpha = 0.5, \beta = 0.2, \gamma = 0.2$



(c) $\alpha = 0.7, \beta = 0.2, \gamma = 0.2$



(d) $\alpha = 0.8, \beta = 0.2, \gamma = 0.2$

Probability of household state vs time (α):cont.

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- Probability of household states(P_k)

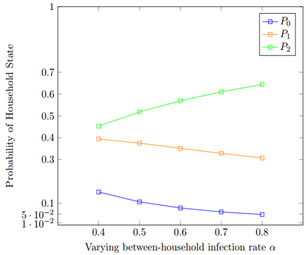
$$|P_{k_{t_{i+1}}} - P_{k_{t_i}}| < 10^{-3}$$

- Total infection rate(I_{Total})

$$|I_{Total_{t_{i+1}}} - I_{Total_{t_i}}| < 10^{-3}$$

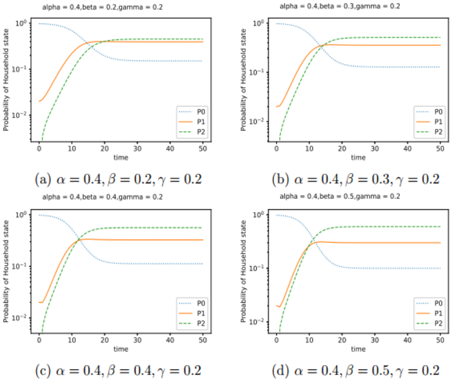
Probabilities of household states	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
P_0	0.151	0.106	0.078	0.060	0.048
P_1	0.394	0.375	0.351	0.328	0.307
P_2	0.453	0.518	0.569	0.610	0.644

Name	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
Total infection rate (I_{Total})	(1.290)	(1.402)	(1.482)	(1.543)	(1.592)
Total recovered rate (R_{Total})	(0.258)	(0.280)	(0.296)	(0.308)	(0.318)
Total Timestep	(35)	(28)	(26)	(23)	(19)



P_0,P_1,P_2 on y-axis and α on x-axis

Probability of household state vs time (β)



(a) $\alpha = 0.4, \beta = 0.2, \gamma = 0.2$ (b) $\alpha = 0.4, \beta = 0.3, \gamma = 0.2$

(c) $\alpha = 0.4, \beta = 0.4, \gamma = 0.2$ (d) $\alpha = 0.4, \beta = 0.5, \gamma = 0.2$

Probability of household state vs time (β):cont.

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- Probability of household states(P_k)

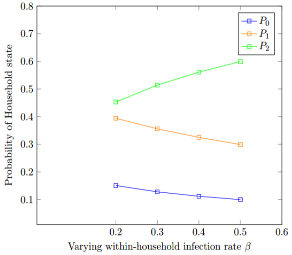
$$|P_{k_{t_{i+1}}} - P_{k_{t_i}}| < 10^{-3}$$

- Total infection rate(I_{Total})

$$|I_{Total_{t_{i+1}}} - I_{Total_{t_i}}| < 10^{-3}$$

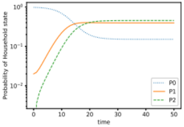
Probability of HouseHold state	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
P_0	0.151	0.128	0.112	0.100
P_1	0.394	0.356	0.325	0.299
P_2	0.453	0.514	0.561	0.599

Name	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
Total infection rate (I_{Total})	(1.290)	(1.375)	(1.441)	(1.493)
Total recovered rate (R_{Total})	(0.258)	(0.275)	(0.288)	(0.298)
Timestep	(35)	(31)	(30)	(27)

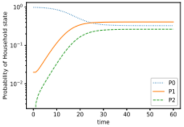


P_0,P_1,P_2 on y-axis and β on x-axis

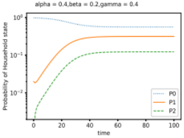
Probability of household state vs time (γ)



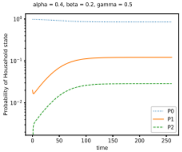
(a) $\alpha = 0.4, \beta = 0.2, \gamma = 0.2$



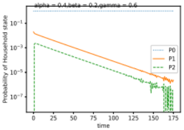
(b) $\alpha = 0.4, \beta = 0.2, \gamma = 0.3$



(c) $\alpha = 0.4, \beta = 0.2, \gamma = 0.4$



(d) $\alpha = 0.4, \beta = 0.2, \gamma = 0.5$



(e) $\alpha = 0.4, \beta = 0.2, \gamma = 0.6$

Probability of household state vs time (γ):cont.

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- Probability of household states(P_k)

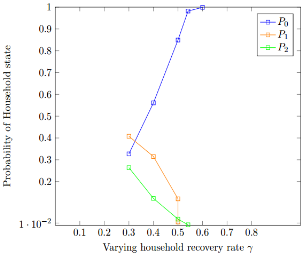
$$|P_{k_{t_{i+1}}} - P_{k_{t_i}}| < 10^{-3}$$

- Total infection rate(I_{Total})

$$|I_{Total_{t_{i+1}}} - I_{Total_{t_i}}| < 10^{-3}$$

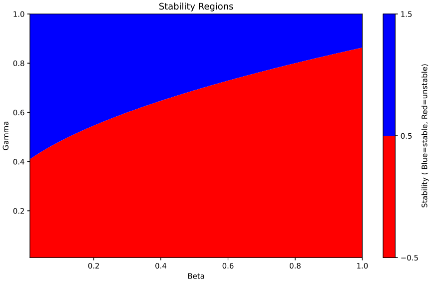
Probability of Household state	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$
P_0	0.327	0.561	0.849	0.9999
P_1	0.408	0.315	0.121	-
P_2	0.264	0.123	0.028	-

Name	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$
Total infection rate (I_{Total})	(0.922)	(0.545)	(0.161)	-
Total recovered rate (R_{Total})	(0.276)	(0.218)	(0.080)	-
Timestep	(54)	(74)	(185)	-



P_0,P_1,P_2 on y-axis and γ on x-axis

Disease-Free equilibrium



γ range on $y - axis$ and β range on $x - axis$ with constant α at 0.4.

- **eigenvalues :** $\lambda_i < 0$, $\lambda_{ii} < 0$
- **case :** $Tr < 0$ and $Tr > \sqrt{Tr^2 - 4D}$, where $D > 0$ and $Tr^2 > 4D$
- **Parameter :** $Tr < 0$, means $\alpha < \beta + 3\gamma$ and $D > 0$ means $\alpha < \frac{\gamma^2}{\beta+\gamma}$

Endemic equilibrium: varying α

Probabilities of household states	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
P_0	0.151	0.106	0.078	0.060	0.048
P_1	0.394	0.375	0.351	0.328	0.307
P_2	0.453	0.518	0.569	0.610	0.644

Runge-Kutta Method: α

HouseHold states	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
P_0	0.151	0.106	0.078	0.060	0.048
P_1	0.394	0.375	0.351	0.328	0.307
P_2	0.454	0.518	0.569	0.610	0.644

Equilibrium: α

- **Numerical values:** RK method and endemic equilibrium values are similar.
- $P_0 = \frac{\gamma}{2\alpha\beta}\left(-(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4\beta\gamma}\right)$
- $P_1 = \frac{2\alpha P_0(1-P_0)}{(\gamma+\alpha P_0)}$
- $P_2 = 1 - P_0 - P_1$
- value of P_0 affects value in P_1 and summation of both P_0 and P_1 results change in P_2.
- **linearization :** Two negative eigenvalues(stable).

Endemic equilibrium: varying β

Probability of HouseHold state	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
P_0	0.151	0.128	0.112	0.100
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Runge-Kutta Method : β

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P_1	0.394	0.356	0.325	0.300
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Equilibrium: β

- **Numerical values:** RK method and endemic equilibrium values are similar.
- $P_0 = \frac{\gamma}{2\alpha\beta}\left(-(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4\beta\gamma}\right)$
- $P_1 = \frac{2\alpha P_0(1-P_0)}{(\gamma+\alpha P_0)}$
- $P_2 = 1 - P_0 - P_1$
- value of P_0 affects value in P_1 and summation of both P_0 and P_1 results change in P_2.
- **linearization :** Two negative eigenvalues(stable).

Endemic equilibrium: varying γ

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Probability of Household state	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$
P_0	0.327	0.561	0.849	0.9999
P_1	0.408	0.315	0.121	-
P_2	0.264	0.123	0.028	-

Runge-Kutta Method : γ

HouseHold states	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$
P_0	0.327	0.561	0.849	-
P_1	0.408	0.315	0.121	-
P_2	0.263	0.123	0.028	-

Equilibrium: γ

- **Numerical values:** RK method and endemic equilibrium values are similar.
- $P_0 = \frac{\gamma}{2\alpha\beta}(-(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4\beta\gamma})$
- $P_1 = \frac{2\alpha P_0(1-P_0)}{(\gamma+\alpha P_0)}$
- $P_2 = 1 - P_0 - P_1$
- value of P_0 affects value in P_1 and summation of both P_0 and P_1 results change in P_2.
- **linearization :** Two negative eigenvalues(stable).

Phase portraits

$$\begin{aligned} f_0(P_0) &= -\alpha(P_1 + 2(1 - P_0 - P_1))P_0 + \gamma P_1 \\ f_1(P_1) &= \alpha(P_1 + 2(1 - P_0 - P_1))P_0 \\ &\quad - ((\alpha/2)(P_1 + 2(1 - P_0 - P_1)) + \beta + \gamma)P_1 \\ &\quad + 2\gamma(1 - P_0 - P_1) \end{aligned} \tag{4}$$

vector plot and stream plot: α

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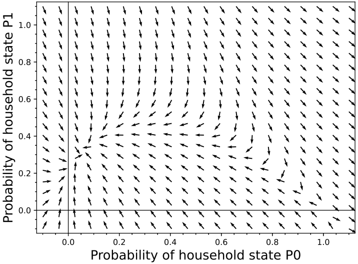
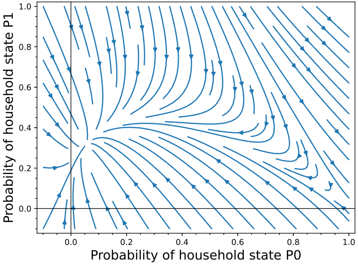
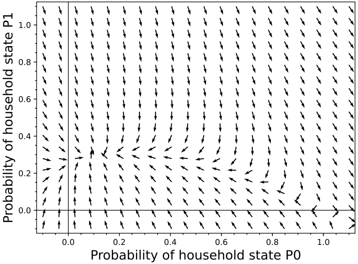


$\alpha = 0.8, \beta = 0.2, \gamma = 0.2$

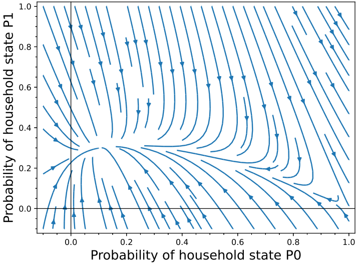


$\alpha = 0.8, \beta = 0.2, \gamma = 0.2$

vector plot and stream plot: β

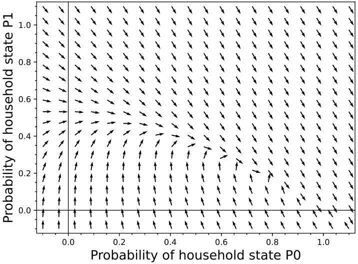


$\alpha = 0.4, \beta = 0.5, \gamma = 0.2$

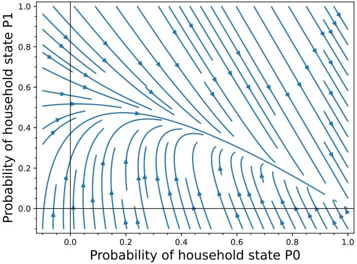


$\alpha = 0.4, \beta = 0.5, \gamma = 0.2$

vector plot and stream plot: γ



$$\alpha = 0.4, \beta = 0.2, \gamma = 0.6$$



$$\alpha = 0.4, \beta = 0.2, \gamma = 0.6$$

Numerical simulation results discussion

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- Total Timestep:
  - Gradual **increase** in α and β, **decreases** the **timestep** to reach steady values.
  - In **comparison**, there is **decrease** in the timestep for α but an **increase** in the total timestep β.
  - An **increase** in γ **increases** timestep.
- P_0,P_1,P_2 and I_{Total}:
  - An **increase** in α and β leads to a **decrease** in P_0 and P_1 but **increase** in P_2 and I_{Total}.
  - when **comparing increase** in α and β: P_0 **decreases more** in increase in α and **decreases less** in increase in β while P_1 **decreases less** in increase in α and **decrease more** in β whereas P_2 remains similar for both in α and β.
  - while **comparing** for I_{Total}: increase in α **increases more** in I_{Total} but **increases less** in gradual increase in β.
  - increase in γ **increases** P_0 while (P_1,P_2, I_{Total}) **decreases**.

Further discussion

- Stability in gradual increase in α,β and γ(till 0.5), EE.
- Stability seen at γ = 0.6, at (P_0 , P_1, P_2) = (P_0,0,0), DFE.
- At Phase portraits,
  - where EE stable (sink), DFE is seen unstable (saddle).
  - where DFE is seen stable (sink), EE is seen unstable (saddle).

Conclusion

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- In summary, a transition from EE to DFE is observed, at γ = 0.6, α = 0.4 and β = 0.2.
- Increase in α and β leads to EE and increase in γ leads to DFE.

Limitations and future research

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- Limitation :
  - Mixed household size rather than same household size.
  - Larger household size rather than only household of size two.
  - Strategies or policies.
  - Household structure (space) and behavioural patterns(movement).
- Further research :
  - Epidemiological models.
  - Higher dimension.
  - Larger population.
  - lower rates in parameters.
  - Intervention strategies.

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Thank You