Spectrum of Blackbody

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Overview

Part 1

Part 2

Intensity of a Black Body spectrum and the problem

Einstein's work on Black Bodies

Part 3

Part 1

An idealised object that can **absorb** and subsequently **emit**, all the radiation incident on it is called a **Blackbody**.

Rayleigh's law

- Consider gaseous medium, in which we have a charged oscilator.

- Due to random motion of the gaseous atoms, it often collides with the oscillator.

Rayleigh's law

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- Due to random motion of the gaseous atoms, it often collides with the oscillator.

- As a result the gaseous atoms imparts some of it's kinetic energy to the oscillator.

Rayleigh's law

- Mean kinetic energy acquired by the oscillator is, $\frac{1}{2} kT$.

- And the total kinetic energy will be kT.

Rayleigh's law

- Since the charge is radiating energy **the system cannot attain equilibrium**.

- How do we tackle this situation?

Rayleigh's law

- How do we tackle this situation?

- We confine the system.

Rayleigh's law

- Inner walls of the box is totally reflective.

- After some time the system will attains **thermal equilibrium**

Rayleigh's law

So, the energy radiated by the oscillator per second can be given by,

$$\frac{1}{Q} = \frac{\left(\frac{dW}{dt}\right)}{\omega_0 W}$$

Q is the radiation reaction.
ω_0 is the natural frequency of vibration.
W is the total energy content of the oscillator.

Rayleigh's law

Also, we have

$$\frac{1}{Q} = \frac{\gamma}{\omega_0}$$

Where γ is the dampping constant.

Remember the mean energy of the oscillator is kT? for three degrees of freedom, it is $3kT$.

Rayleigh's law

Average energy loss due to radiation is given by,

$$\left\langle \frac{dW}{dt} \right\rangle = \gamma kT$$

Rayleigh's law

- Let, $I(\omega)d\omega$ be the incident light energy with in a frequeny range $d\omega$.

- Assuming all the radiation which falls on a *cross section* 'σ_s' is absorbed completely,

- Scattered light must be the product of $I(\omega)d\omega$ and σ_s.

Rayleigh's law

Expression for the cross section is given by,

$$\sigma_s = \frac{8\pi r_0{}^2}{3} \left[\frac{\omega^4}{(\omega^2 - \omega_0{}^2)^2 + \gamma^2 \omega^2} \right]$$

When $\omega = \omega_0$,

$$\sigma_s = \frac{2\pi r_0{}^2 3\omega_0^2}{3(\omega - \omega_0)^2 + \dfrac{\gamma^2}{4}}$$

Rayleigh's law

$$\frac{dW_s}{dt} = \int_0^\infty \mathsf{I}(\omega)\, \sigma_s \, d\omega = \int_0^\infty \frac{2\pi {r_0}^2 \omega_0^2 \, \mathsf{I}(\omega)\, d\omega}{3\left[(\omega - \omega_0)^2 + \dfrac{\gamma^2}{4}\right]}$$

since we're considering the equilibrium condition ($\omega = \omega_0$), equation reduces to,

$$\frac{dW_s}{dt} = \frac{2}{3}\pi {r_0}^2 \omega_0^2 \, \mathsf{I}(\omega_0) \int_{-\infty}^\infty \frac{d\omega}{(\omega - \omega_0)^2 + \left(\dfrac{\gamma}{2}\right)^2}$$

Rayleigh's law

Since the integral reduces to arctan function we get, $\frac{2\pi}{\gamma}$. and as
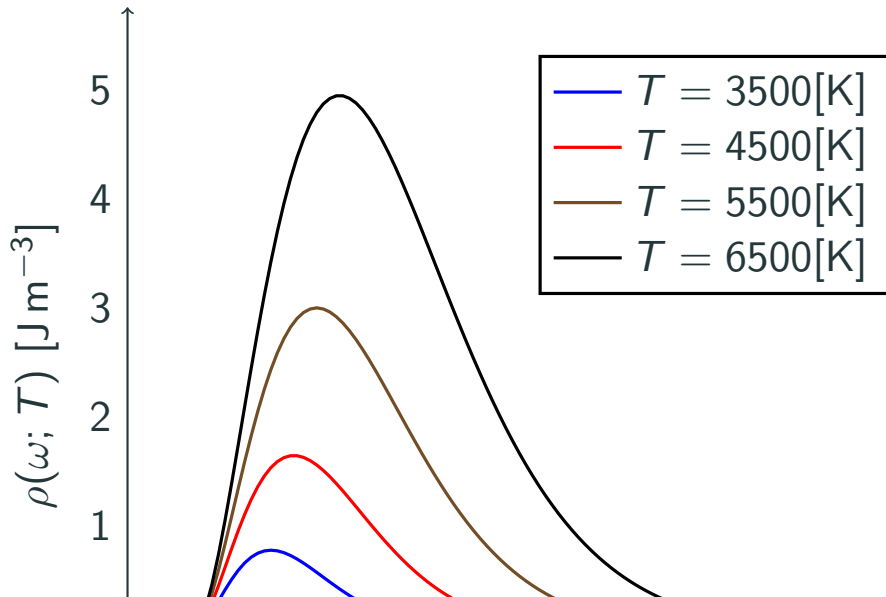
$$\frac{dW_s}{dt} = \frac{dW}{dt}$$

,

$$I(\omega) = \frac{9\gamma^2 \, kT}{4\pi^2 r_0^2 \omega_0^2}$$

The value of γ wa found to be equal to,

$$frac23\frac{r_0\omega_0^2}{c}$$

$$I(\omega) = \frac{\omega^2 \, kT}{\pi^2 c^2}$$

- Theoretical proof was needed for to explain the obtained experimental results.

- The harmonic oscillator can take up energies only in the multiples of $\hbar\omega$.

- The probability of occupying an energy level **E** is $\mathbf{P(E)} = \alpha e^{\frac{-\hbar\omega}{kT}}$.

- The number of oscillators in ground state is N_0 .

- The number of oscillator in first state is $N_1 = N_0 e^{-\hbar\omega/kT}$.

- Then $N_n = N_0 x^n$, where $x = e^{-\hbar\omega/kT}$

- The energy in the ground state is zero.

- The energy in the first state is $N_1 \hbar \omega$ or $N_0 \hbar \omega x$

- Total energy is given by
$$E_{(total)} = N_0 \hbar \omega (0 + x + 2x^2 + \cdots)$$

Part 2

- The intensity of radiation for frequency ω is given by

$$I(\omega)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2 \left(e^{\frac{\hbar\omega}{kT}} - 1\right)}$$

- Planck considered that the matter was quantized and not light.

- Einstein came up with an alternate idea to solve Black body problem.

- Einstein considered that even the light was quantized, by taking that light is actually photons with energy $\hbar\omega$

- a pic is needed.

- Einstein assumed that Planck's formula was right.

- Consider two energy levels , say , the nth level and m^{th} level.

- A light of certain frequency is incident on atom and by absorbing a photon, transition from state n to state m occurs.

- The probability of the transition is proportional to intensity of the light.

- The proportionality constant is B_{nm} and mathematically can be written as,

$$R_{n \to m} = N_n B_{nm} I(\omega) \tag{1}$$

- Einstein considers that there are two types of emissions that take place in an atom.

- Spontaneous emission.

- Stimulated emission.

- The combined mathematical expression is,

$$R_{m \to n} = N_m[A_{mn} + B_{mn}I(\omega)] \tag{2}$$

- At thermal equilibrium, the number of atoms going to higher energy level ,ust be equal to the number of atoms coming to lower energy state.

- The ratio of N_m to N_n is given by $e^{\frac{-\hbar\omega}{kT}}$

- At thermal equilibrium both $R_{n \to m}$ and $R_{m \to n}$ are equal.

- $N_n B_{nm} I(\omega) = N_m [A_{mn} + B_{mn} I(\omega)]$

- On dividing the previous equation by N_m we get,

- $B_{nm}I(\omega)e^{\frac{\hbar\omega}{kT}} = A_{mn} + B_{mn}I(\omega)$

- So the equation that Einsteim got for intensity of radiation for frequency is

$$I(\omega) = \frac{A_{mn}}{B_{nm}e^{\frac{\hbar\omega}{kT}} - B_{mn}} \tag{3}$$

- Since Einstein considered that the formula given by Planck was correct, we should get ,

- $B_{nm} = B_{mn}$ and also

$$\frac{A_{mn}}{B_{nm}} = \frac{\hbar \omega^3 d\omega}{\pi^2 c^2}$$

This is how Einstein rederived Planck's formula from a quantum mechanical viewpoint.

Part 3

- **P** = probability

- ϕ = Probability amplitude

- **P** = $\left| \phi \right|^2$

- $\phi = \phi_1 + \phi_2$

- $\mathbf{P} = \left| \phi_1 + \phi_2 \right|^2$

Case A

- Detector 1 is set to detect only α particles and Dectector 2 is set to detect only oxygen atoms.

- Probability amplitude of the scattering is given as $f(\theta)$ when they are at an angle θ.

- The probability of this event $= \left| f(\theta) \right|^2$

Case B

- Set up the dectectors such that the detectors would dectect either α particle or oxygen atom.

- We will not distinguish which particle is which entering the detector.

- This means that if oxygen atom in position θ , then α particle on the opposite side is at an angle $\pi - \theta$.

- Probability amplitude of oxygen atom $= f(\pi - \theta)$

- Probability amplitude of α particle $= f(\theta)$

- The probability of a particle being detected at detector $1 = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2$

- Consider if both are α particles,

- Then we would not know which particle entered the detector, so the total probability changes to,

- The probability of a α particle being detected at detector $1 = \left| f(\theta) + f(\pi - \theta) \right|^2$

- If $\theta = \dfrac{\pi}{2}$, then applying this to the expression $\left| f(\theta) + f(\pi - \theta) \right|^2$ we get,

- Probability $= 4 \left| f\left(\dfrac{\pi}{2}\right) \right|^2$, if the particles are indistinguishible.

- Suppose the particles were distinguishible, then the probability for $\theta = \dfrac{\pi}{2}$ when applied for $\left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2$ is given by,

- Probability $= 2 \left| f\left(\dfrac{\pi}{2} \right) \right|^2$

- This shows that the probability gets doubled for indistinguishible particles.

- *Can we apply the same logicto the electron-electron scattering ?*

- *OBSERVATION* : " When we have situation in which the identity of the electron which is arriving at a point is echanged with anothe one , the new amplitude interfere with old one with an opposite phase."

- In electrons case , the interfering amplitude for exchange interfere with a negative sign.
  Probability for electron $= \left| f(\theta) - f(\pi - \theta) \right|^2$

SPIN PROBABILITY TABLE

Identical particles, also called **indistinguishable particle**, are particles that cannot be distinguished from one another.

Identical - Indistinguishable Particles

- Consider particle 'a' and particle 'b'.

- Let the two particle collide and get sactttered in two different directions say '1' and '2' over a surface element ds_1 and ds_2 of the detector respectively.

- If the particles are indistinguishable then the amplitudes of these process will add up.

- Probability that the two particles arrive at ds_1 and ds_2 is
$$\Big| <1|a><2|b> + <2|a><1|b> \Big|^2 ds_1 ds_2$$

- Integrating over the area of the detector , if we let ds_1 and ds_2 rangr over the whole area $(\triangle S)$, we could count each part of the area twice since the expression $\left| <1|a><2|b> + <2|a><1|b> \right|^2 ds_1 ds_2$ contains everything that can happen with any pair of surface elements ds_1 and ds_2.

- $Probability_{BOSE} = \dfrac{\left(4 \left| a \right|^2 \left| b \right|^2 \right)}{2} (\triangle S)$

- This is just twice what we got the probability for distinguished particles.

State with n Bose particle

- Consider n particles say a, b, c... scattered in n direction say 1, 2, 3 ...

- Probability that each particle acting alone would go into an element of the surface ds of the detector is $\left| <> \right|^2 ds$.

- **Assumption** : All particles are distinguishable.

- Probability that n particles will be counted together in n different surface elements
  $$= \left| a_1 b_2 c_3 ... \right|^2 ds_1 ds_2 ...$$

- If the amplitude does not depend on where ds is located in the detector , then the
  $$Probability = \left(\left| a \right|^2 \left| b \right|^2 ... \right) (ds_1 ds_2 ...)$$

- Integrating each dS over the surface $\triangle S$ of the dectector

$$(P_n)_{different} = \left(\left| a \right|^2 \left| b \right|^2 ... \right) (\triangle S)^n$$

- Now suppose that all the particle are Bose particles.

- For n particles, there are n! different , but indistinguishable possibilities for which we must add the amplitudes.

- Probability that n particles will be counted on the n surface elements is given by

$$Probability = \left(\left| a_1 b_2 c_3 ... + a_1 b_3 c_2 ... \right|^2 \right) (ds_1 ds_2 ...)$$

- $Probability = \left(\left| n! abc... \right|^2 \right) (ds_1 ds_2 ...)$

- Integrate each ds over the area $\triangle S$ of the detector

$$(P_n)_{BOSE} = n! \left(\left| abc... \right|^2 \right) (\triangle S)^n$$

- Compareing the probability when the particles are distinguishable and indistinguishable

$$(P_n)_{BOSE} = n! \left(\left| abc... \right|^2 \right) (\triangle S)^n$$

$$(P_n)_{different} = \left(\left| a \right|^2 \left| b \right|^2 ... \right) (\triangle S)^n$$

- $(P_n)_{BOSE} = n!(P_n)_{different}$

What is the probability that a Bose particle will go into particular state when there are already n particles present ?

Emission and Absorption of photons

- When the light is emitted, a photon is "created".

- Consider that there are some atom emitting n photons.

- *OBSERVATION* : The probability that an atom will wmit a photon into a particular final state is increased by the factor $(n+1)$ if there are already n photons in that state.

The Blackbody Spectrum

- For each light frequency ω, there are certain N number of atoms which have two energy states separated, given by the equation $E = \omega\hbar$.

- Let N_e and N_g be the average numbers of atoms that are in excited state and ground state.

- In thermal equilibrium at temperature T, from statistical mechanics

- NOTE: Each atom in the ground stae can absorb a photon and go into the excited state and each atom in the excited state can emit a photon and go to the ground state.

- At equilibrium, the rate of these two process must be equal.

- Rate is proportional to the probability of the event and the number of atoms present.

- $\overline{n}$ is the average number of photons present in a given state with the frequency ω

- The absorption rate from the state is $N_g \bar{n} \left| a \right|^2$, and the emission rate into that state is $N_e(\bar{n} + 1) \left| a \right|^2$.
- At equilibrium $N_g \bar{n} \left| a \right|^2 = N_e(\bar{n} + 1) \left| a \right|^2$