Fuzzy Logic

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1 Fuzzy sets

The main idea is to model imprecise concepts, such as youngness or tallness. Although they are everyday concepts, there isn't a formal definition of their real meaning due to the fact that there is no clear border that delimits them.
Although at first glance it may not seem particularly useful, it can be an interesting tool for modeling imprecise dependencies (e.g. rules) for instance if Temperature is "low" and Oil is "cheap" then crank up the heating system.

1.1 Crisp Sets

Crisp sets A.K.A. Classical sets is one that can be described by the elements that belong it. It is possible to describe any set using a characteristic function:

$$m_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \qquad m_A(x) \in \{0, 1\}$$

Example Consider the following set $A = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, the corresponding characteristic function can be seen in Figure 1

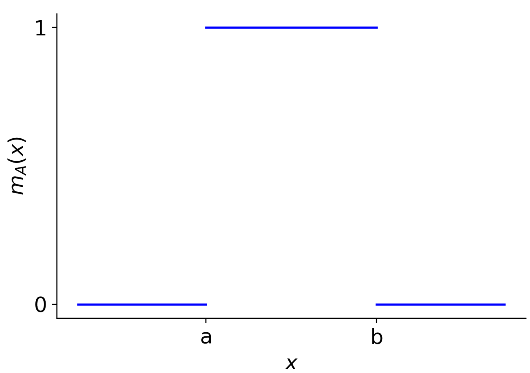


Figure 1: Plot of the characteristic function of a crisp set example

1.2 Fuzzy Sets (Zadeh, 1965)

A fuzzy set is a class of objects with a continuum of grades of membership, as stated above, it tries to model classes of concepts in which the notion of membership is ambiguous. It is possible to describe any fuzzy set using a so-called membership function, which is very similar to the characteristic function used to the crisp sets but now this function assigns values in the interval $[0, 1]$, so now it is possible to describe classes with intermediate degrees of membership

Example Consider the following fuzzy set $\hat{A} = x$ is roughly in $[a, b]$, the corresponding characteristic function can be seen in Figure 2

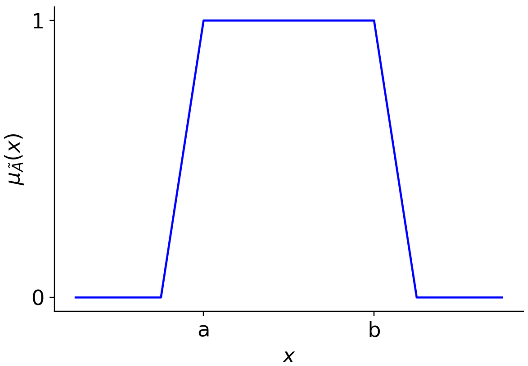


Figure 2: Plot of the membership function of a fuzzy set example

So basically, numbers very far from the interval simply do not belong to the set. Numbers in the interval strongly belong to the set. Numbers close to the extremes of the interval (i.e. roughly in $[a, b]$) belong to the set with a certain degree of membership between 0 and 1.

1.3 Linguistic Variables (Zadeh, 1975)

A linguistic variable is a variable whose values (interpretation) are natural language expressions. For example, *Age* is a linguistic variable if its values are *young, old, etc.*, rather than $20, 21, 22, \ldots$. Since words, in general, are less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means of approximate characterization of phenomena which are too complex or too ill-defined to be amenable to description in conventional quantitative terms.
More formally a linguistic variable can be characterized by the triplet $(\mathscr{X}, T(\mathscr{X}), U)$ in which $\mathscr{X}$ is the name of the variable i.e. *Age*; $T(\mathscr{X})$ is the *term-set* of $\mathscr{X}$, that is the collection of its linguistic values, i.e. *young, old, etc.*; U the universe of discourse, the range in which the variable would take numerical values i.e. As age can take values between 0 and 200, $U = \{x \in \mathbb{R} \mid 0 \leq x \leq 200\}$.
The meaning of the value X of a linguistic variable can be characterized by a fuzzy sets in which, its membership function $\mu : U \to [0, 1]$ tells how compatible are X to a certain numerical value in U. To characterize the meaning of a linguistic variable, the meaning of a representative subset of its values is typically used.

Example

- Consider the linguistic variable *Age*, in Figure 3 we can see the graph of the fuzzy sets that represents *young* and *old*
- Consider the linguistic variable *Size*, in Figure 4 we can see the graph of the fuzzy sets that represents *small, medium* and *tall*

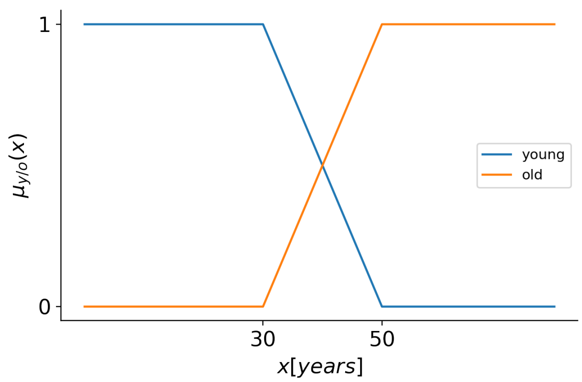


Figure 3: Plot of meaning of young and old

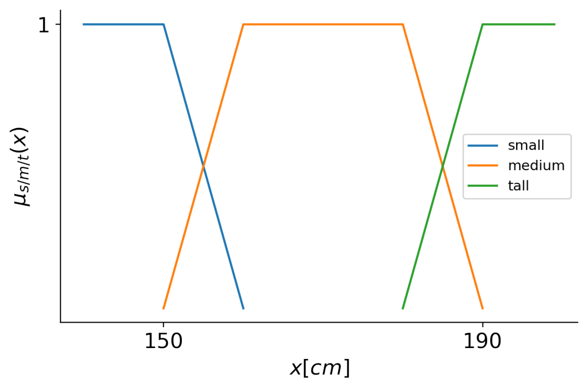


Figure 4: Plot of meaning of small, medium, tall

Context is critical

Ambiguity is part of human language and it is this that allows to express incredibly complex ideas with a relatively small number of bits, linguists agree that the context in which a word appears is essential to resolve the possible ambiguities associated with the meaning of a word (Bransford and Johnson, 1972) On a similar way, the meaning of a linguistic variable should depends on the context in which it is being used.

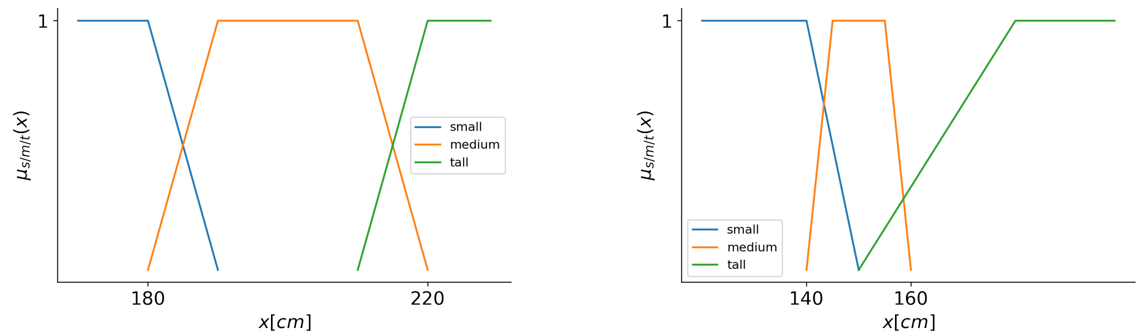
Example The notion of tallness strongly depends on the context in which is being used, a small person in the Netherlands may not be that small in Yemen, even more, a small basketball player may be a huge jockeys rider, in Figure 5 the comparison of the fuzzy sets that represents this last scenario is shown

1.4 Membership functions

The membership function is a key concept in fuzzy logic as it represents the degree of truth of a fuzzy set. Beside in theory any mapping between U and $[0, 1]$ can be used as membership function, the fuzzy logic community has adopted a relatively low number of parameterizable membership functions with which it is possible to model a large number of scenarios, in Figure 6

There are several properties associated with a membership function, some of the most important (in fuzzy logic context) are:

- **Support** elements having non-zero degree of membership



(a) Size of basketball player

(b) Size of jockeys riders

Figure 5: Comparison of the meaning of Size on different sports

- **Core** Set with elements having degree of 1
- **α-cut** set of elements with degree $\geq \alpha$
- **Height** Maximum degree of membership

In Figure 7 these concepts are exemplified.

1.5 Operators on Fuzzy Sets

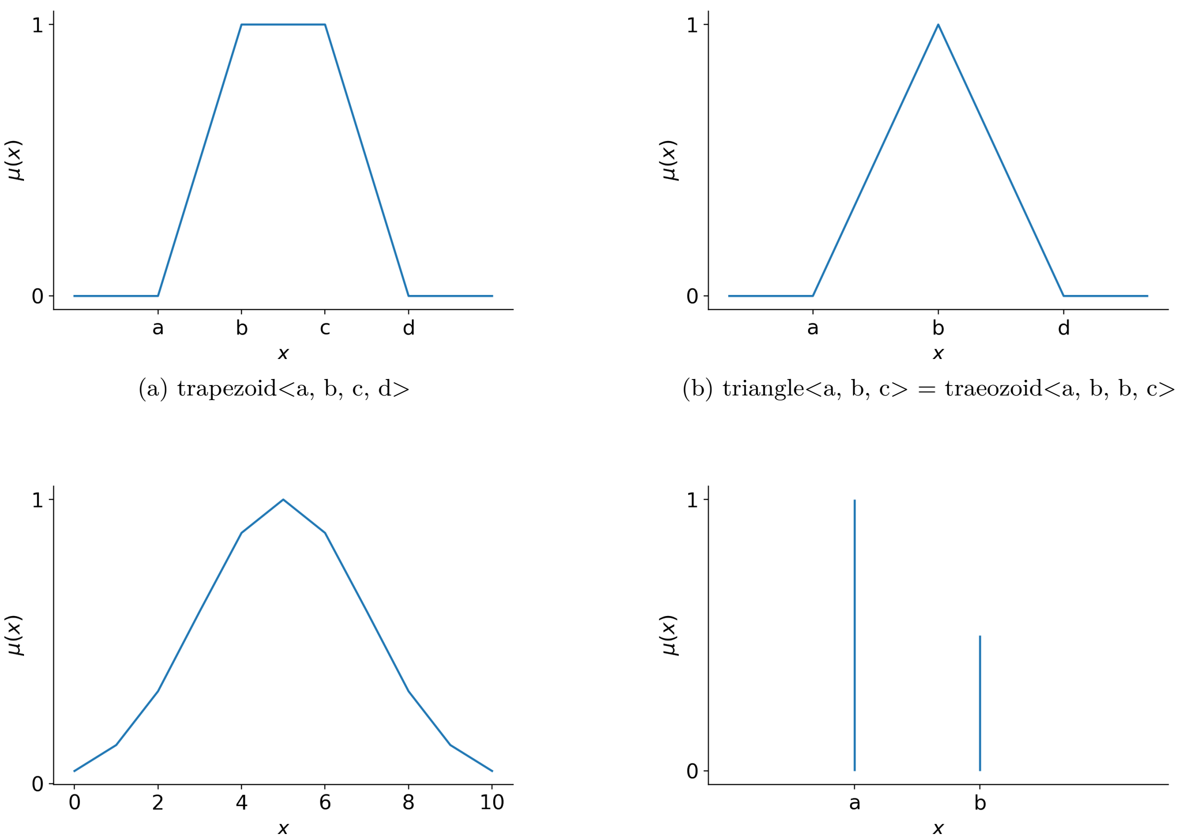
As in classic sets, the fuzzy sets can be operated between them, unfortunately, unlike the first, there is no a convincing arguments for something like the "right choice" of our operations (Zadeh, 1965). Of course, the problem in the pas has been discussed from different points of view e.g. Bellman and Giertz (1973), Yager (1979), Giles (1979). Beside those motivational discussions for the choice of "right" operators for fuzzy sets different families of operations have been discussed by e.g.Yager (1980), Dombi (1982), Weber (1983), all of which proved to be special cases of t-norms (Ben Schweizer, 1961), this is mainly due the t-norm generalize the usual conjunction operator and hence could be used to define intersection operation for fuzzy sets, correspondingly his dual, the t-conorms (AKA s-norm) generalize the usual disjunction operator and thus could be used to define union operation for fuzzy sets.

1.5.1 t-norm (Ben Schweizer, 1961)

A t-norm (triangular norm) is a function $\underline{t}_\wedge : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

1. **Commutativity**: $\underline{t}_\wedge(a,b) = \underline{t}_\wedge(b,a)$
2. **Monotonicity**: $\underline{t}_\wedge(a,b) \leq \underline{t}_\wedge(c,d)$ if $a \leq c$ & $b \leq d$
3. **Associativity**: $\underline{t}_\wedge(a, \underline{t}_\wedge(b,c)) = \underline{t}_\wedge(\underline{t}_\wedge(a,b), c)$
4. The number 1 acts as neural element: $\underline{t}_\wedge(a,1) = a$

The boolean conjunction is both commutative and associative, its generalization must have these operational properties. The monotonicity property ensures that the degree of truth of conjuction does not decrease if the thuth values of conjucts increase. The requirement of 1 as neutral element corresponds to the interpretation of 1 as true.



(a) trapezoid<a, b, c, d>

(b) triangle<a, b, c> = traeozoid<a, b, b, c>

(c) gaussian<mean, std>

(d) singleton: (a, 1) and (b, 0.5)

Figure 6: some common membership functions with their respective parameterization

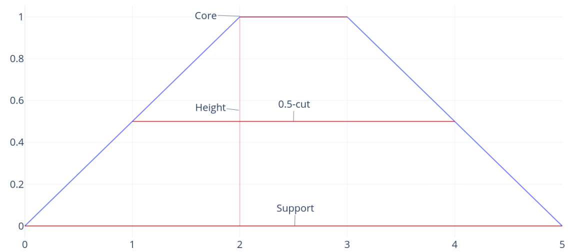


Figure 7: Diagram with annotated properties

1.5.2 Negation

The truth function of negation has to be non-incresing (and assing 0 to 1 and vice versa); the function $1 - x$ is the best known candidate.

1.5.3 t-conorm

Using the De morgan duality is possible to define a t-corom $\underline{s}_\vee$ with respect with respect to a negation operation. This new $\underline{s}_\vee$ would generalize the usual disjunction operation. More specific

$$\underline{s}_\vee(u, v) = 1 - \underline{t}_\wedge(1 - u, 1 - v)$$
$$\underline{t}_\wedge(u, v) = 1 - \underline{s}_\vee(1 - u, 1 - v)$$

Gödel norm It occurs in most t-norm based fuzzy logics as the standard semantics for conjunction

$$\underline{t}_\wedge(x, y) = \min(x, y)$$

$$\underline{s}_\vee(x, y) = \max(x, y)$$

This "and" must be interpreted in a "hard" sense, that is, we do not allow any tradeoff between x and y so long as $x > y$ or vice-versa. For instance, if $x = 0.8$ and $x = 0.5$, then $\underline{t}_\wedge(x, y) = 0.5$ so long as $x > 0.5$.

Product norm

$$\underline{t}_\wedge(x, y) = x \cdot y$$
$$\underline{s}_\vee(x, y) = x + y - x \cdot y$$

Lukasiewicz norm

$$\underline{t}_\wedge(x, y) = \max(0, x + y - 1)$$
$$\underline{s}_\vee(x, y) = \min(1, a + b)$$

1.5.4 Classic vs Fuzzy logic

2 Fuzzy Rule System

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