Homework 1

2. Newton's method to minimize

$$f(X) = \|X\|^3, X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathbb{R}^2.$$

using a constant step size.

Thus,

$$\nabla f = 3\|X\|X.$$

and

$$\nabla^2 f = 3 \begin{pmatrix} \frac{X_1^2}{\|X\|} + \|X\| & \frac{X_1 X_2}{\|X\|} \\ \frac{X_1 X_2}{\|X\|} & \frac{X_2^2}{\|X\|} + \|X\| \end{pmatrix}.$$

As a result,

$$(\nabla^2 f)^{-1} = \frac{1}{3(\frac{X_1 X_2}{\|X\|}^2 + X_1^2 + X_2^2 + \|X\|^2) - \frac{X_1 X_2}{\|X\|}^2} \begin{pmatrix} \frac{X_2^2}{\|X\|} + \|X\| & -\frac{X_1 X_2}{\|X\|} \\ -\frac{X_1 X_2}{\|X\|} & \frac{X_1^2}{\|X\|} + \|X\| \end{pmatrix}.$$

which implies,

$$\begin{aligned} (\nabla^2 f)^{-1}(\nabla f) =& \frac{1}{6\|X\|^2} \begin{pmatrix} \frac{X_2^2}{\|X\|} + \|X\| & -\frac{X_1 X_2}{\|X\|} \\ -\frac{X_1 X_2}{\|X\|} & \frac{X_1^2}{\|X\|} + \|X\| \end{pmatrix} \times 3\|X\| \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ =& \frac{1}{2\|X\|} \begin{pmatrix} \frac{X_1 X_2^2}{\|X\|} + \|X\|X_1 - \frac{X_1 X_2^2}{\|X\|} \\ -\frac{X_1^2 X_2}{\|X\|} + \frac{X_1^2 X_2}{\|X\|} + \|X\|X_2 \end{pmatrix} \\ =& \frac{1}{2} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{X}{2}. \end{aligned}$$

Thus, Newton's method for this problem implies

$$\begin{aligned} X^{k+1} =& X^k - \alpha(\nabla^2 f)^{-1}(\nabla f) \\ =& X^k(1 - \frac{\alpha}{2}) \\ =& X^{k-1}(1 - \frac{\alpha}{2})^2 \\ & \vdots \\ =& X^0(1 - \frac{\alpha}{2})^k. \end{aligned}$$

As $k \to \infty, X^k \to 0$ for $0 < |(1 - \frac{\alpha}{2})| < 1 \iff 0 < \alpha < 4.$