Homework 5

1. Suppose

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

Then $X_i|Y_i \sim \mathcal{N}(\rho Y, (1-\rho^2))$ and $Y_i|X_i \sim \mathcal{N}(\rho X, (1-\rho^2))$. Marginally, $X_i \sim \mathcal{N}(0,1)$ and $Y_i \sim \mathcal{N}(0,1)$. We can use the following estimators for $E[X]$:

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$\hat{I}_{RB} = \frac{1}{n}\sum_{i=1}^{n} E[X_i|Y_i]$$

Then

$$Var[\hat{I}] = 1/n$$

and,

$$Var[\hat{I}_{RB}] = \rho^2/n$$

As $\rho \leq 1$, $Var[\hat{I}_{RB}] \leq Var[\hat{I}]$. In particular, if $\rho << 1$, then $Var[\hat{I}_{RB}] << Var[\hat{I}]$.