Homework 3

1 Let $I = \int_{-\infty}^{\infty} e^{-x^4} dx$. Then

$$I = \int_{-\infty}^{\infty} e^{-x^4} dx = \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-x^4+x^2/2} \phi(x) dx$$

where $\phi(x)$ denotes the density of the standard normal. Let $X_1, ..., X_{1000000} \overset{iid}{\sim} N(0,1)$. Thus using simple importance sampling we can let

$$\hat{I} = \frac{1}{n} \sum_{i}^{n} h(X_i) = 1.811491$$

$$SE(\hat{I}) = \frac{1}{n} \sum_{i=1}^{n} (h(X_i) - \hat{I})^2 = 0.002581333$$

where $h(x) = \sqrt{2\pi} e^{-x^4+x^2/2}$

2 To calculate the posterior expectation of θ given $X = 1$ we use importance sampling without a normalizing constant based of Hastings(1970). Let $\hat{I}$ denote our estimate. Also let $\theta_1, ..., \theta_{1000000} \overset{iid}{\sim} N(1,1)$. Then

$$\hat{I} = \sum_{i=1}^{n} w_i \theta_i = 0.5537485$$

where $w_i = \frac{1}{1+\theta_i^2} / (\sum_{j=1}^{n} \frac{1}{1+\theta_j^2})$ and

$$SE(\hat{I}) = \sqrt{\frac{(\hat{\sigma}_{11}^2/(\hat{v}^2) - 2\hat{\sigma}_{12}^2(\hat{u}/(\hat{v}^3)) + \hat{\sigma}_{22}^2(\hat{u}^2)/(\hat{v}^4))}{n}} = 0.0007334674$$

Via the central limit theorem and the delta method, we derive that

$$\sqrt{n}(\frac{\sum_{j=1}^{n} h(\theta_i) l(\theta_i)}{\sum_{j=1}^{n} l(\theta_i)} - \frac{u}{v}) \xrightarrow{d} N(0, \frac{\sigma_{11}^2}{v^2} - 2\frac{\sigma_{12}^2 u}{v^3} + \frac{\sigma_{22}^2 u^2}{v^4})$$

where $E[h(\theta)l(\theta)] = u$, $E[l(\theta)] = v$ and the variance-covariance matrix for $h(\theta)l(\theta)$ and $l(\theta)$ is denoted by

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix}$$

As in the previous problem we use the usual estimates for all of these. For example,

$$\hat{u} = \frac{\sum_{j=1}^{n} h(\theta_i) l(\theta_i)}{n},$$

$$\hat{\sigma}_{11}^2 = \frac{\sum_{j=1}^{n} (h(\theta_i) l(\theta_i) - \hat{u})^2}{n}$$

and

$$\hat{\sigma}_{12}^2 = \frac{\sum_{j=1}^{n} (h(\theta_i) l(\theta_i) - \hat{u})(l(\theta_i) - \hat{v})}{n}$$

Appendix 1: R code for Problem 1

```
set.seed(10)
n = 1000000
X=rnorm(n)
h = function(x){exp(-x^4+(x^2)/2)}
Ihat = sqrt(2*pi)*mean(h(X))
SE = (2*pi)*sqrt(var(h(X)))/sqrt(n)
```

Appendix 2: R code for Problem 2

```
set.seed(10)
n = 1000000
th = rnorm(n,1,1)
h1 = function (x) {x/(pi*(1+x^2))}
h2 = function (x) {1/(pi*(1+x^2))}
u = mean(h1(th))
v = mean(h2(th))
s1 = var(h1(th))
s2 = var(h2(th))
s12 = cov(h1(th),h2(th))
Ihat = u/v
SE = sqrt(s1/(v^2) - 2*s12*(u/(v^3)) + s2*(u^2)/(v^4))/sqrt(n)
```