Homework 5

1. Suppose

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

Then $X_i|Y_{i-1} \sim \mathcal{N}(\rho Y_{i-1}, (1-\rho^2))$ and $Y_i|X_i \sim \mathcal{N}(\rho X_i, (1-\rho^2))$. Marginally, $X_i \sim \mathcal{N}(0,1)$ and $Y_i \sim \mathcal{N}(0,1)$. We can use the following estimators for $E[X]$:

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$\hat{I}_{RB} = \frac{1}{n}\sum_{i=1}^{n} E[X_i|Y_{i-1}] = \frac{1}{n}\sum_{i=1}^{n} \rho Y_{i-1}$$

Hence

$$\begin{aligned} Var[\hat{I}] &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j] \\ &= \frac{1}{n^2}\sum_{i}^{n} Var[X_i] + 2\sum_{i<j} Cov[X_i, X_j] \\ &= \frac{1}{n^2}\big(\sum_{j=1}^{n}(1-\rho^{4j}) + 2\sum_{j=1}^{n-1}(n-j)\rho^{2n}x_0(1-\rho^{2j}x_0)\big) \end{aligned}$$

On the other hand,

$$\begin{aligned}
Var[\hat{I}_{RB}] &= \frac{1}{n^2} Var\big[\sum_{i=1}^{n} E[X_i|Y_{i-1}]\big] \\
&= \frac{1}{n^2} \sum_{i=1}^{n} Var\big[E[X_i|Y_{i-1}]\big] + 2\sum_{i<j} Cov\big[E[X_i|Y_{i-1}], E[X_j|Y_{j-1}]\big] \\
&= \frac{1}{n^2} \sum_{i=1}^{n} Var\big[\rho Y_{i-1}\big] + 2\sum_{i<j} Cov\big[\rho Y_{i-1}, \rho Y_{j-1}\big] \\
&= \frac{1}{n^2} \sum_{j=1}^{n} \rho^2 Var\big[Y_{i-1}\big] + 2\sum_{i<j} \rho^2 Cov\big[Y_{i-1}, Y_{j-1}\big] \\
&\leq \frac{\rho^2}{n^2}\big(\sum_{j=1}^{n}(1-\rho^{4j}) + 2\sum_{j=1}^{n-1}(n-j)\rho^{2n} x_0(1-\rho^{2j}x_0)\big) \\
&= \rho^2 Var[\hat{I}]
\end{aligned}$$

as $\rho << 1$. Hence, for small ρ, $Var[\hat{I}_{RB}]$ is much smaller than $Var[\hat{I}]$.