Homework 5

1. Suppose

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

Then $X_i|Y_{i-1} \sim \mathcal{N}(\rho Y_{i-1}, (1-\rho^2))$ and $Y_i|X_i \sim \mathcal{N}(\rho X_i, (1-\rho^2))$. We can use the following estimators for $E[X]$:

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$\hat{I}_{RB} = \frac{1}{n}\sum_{i=1}^{n} E[X_i|Y_{i-1}] = \frac{1}{n}\sum_{i=1}^{n} \rho Y_{i-1}$$

Let $S_x = \sum_{i=1}^n X_i$ and $S_y = \sum_{i=1}^n Y_{i-1}$ where we start the Markov chain at $X_0 = x_0$ and generate $Y_0 \sim \mathcal{N}(\rho x_0, 1-\rho^2)$. Since the chain is generated using $X_i|Y_{i-1} \sim \mathcal{N}(\rho Y_{i-1}, (1-\rho^2))$ and $Y_i|X_i \sim \mathcal{N}(\rho X_i, (1-\rho^2))$, we have that $X_n|X_{n-1} \sim \mathcal{N}(\rho^2 X_{n-1}, 1-\rho^4)$, which implies $X_n|X_1 \sim \mathcal{N}(\rho^{2(n-1)}X_1, 1-\rho^{4(n-1)})$. Therefore,

$$Var(X_n|X_1) = 1 - \rho^{4(n-1)} \tag{1}$$

and

$$\begin{aligned}
Cov(X_i, X_j|X_1) &= E(X_i, X_j|X_1) - E(X_i|X_1)E(X_j|X_1) \\
&= E(E(X_i, X_j|X_1, X_i)) - E(X_i|X1)E(X_j|X_1) \\
&= E(\rho^{2(j-i)}X_i^2|X_1, X_i) - E(X_i|X_1)E(X_j|X_1) \\
&= \rho^{2(j-i)}E(X_i^2|X_1) - E(X_i|X_1)E(X_j|X_1) \\
&= \rho^{2(j-i)}(Var(X_i|X_1) + E(X_i|X_1)^2) - E(X_i|X_1)E(X_j|X_1) \\
&= \rho^{2(j-i)}\rho^{2(i-1)}X_1^2 + \rho^{2(j-i)}(1-\rho^{4(i-1)}) - \rho^{2(i-1)}\rho^{2(j-1)}X_1^2 \\
&= \rho^{2(j-i)}(1-\rho^{4(i-1)}) \qquad (2)
\end{aligned}$$

From equations 1 and 2

$$\begin{aligned}
Var(S_x|X_1) &= Var(\sum_{j=2}^{n} X_j|X_1) \\
&= \sum_{j=2}^{n} Var(X_j|X_1) + 2\sum\sum_{2\le i<j\le n} Cov(X_i, X_j|X_1) \\
&= \sum_{j=2}^{n} 1-\rho^{4(j-1)} + 2\sum\sum_{2\le i<j\le n} \rho^{2(j-i)}(1-\rho^{4(i-1)}) \overset{def}{=} V
\end{aligned}$$

As this doesn't depend on X_1 we can conclude that $Var(S_y|Y_0) = Var(S_x|X_1) = V$ using the fact that $X_n|X_1 \sim \mathcal{N}(\rho^{2(n-1)}X_1, 1-\rho^{4(n-1)})$ and $Y_{n-1}|Y_0 \sim \mathcal{N}(\rho^{2(n-1)}Y_0, 1-\rho^{4(n-1)})$ for all $n > 1$, i.e. given X_1 and Y_0 these are both identically distributed.

Note that V doesn't depend on X_1, Y_0 or x_0. Recall that $X_1 \sim \mathcal{N}(\rho^2 x_0, 1-\rho^4)$ and $Y_0 \sim \mathcal{N}(\rho x_0, 1-\rho^2)$. Then

$$\begin{aligned}
Var(S_x|X_0 = x_0) &= E(Var(S_x|X_1, X_0 = x_0)) + Var(E(S_x|X_1, X_0 = x_0)) \\
&= E(Var(S_x|X_1)) + Var(E(S_x|X_1, X_0 = x_0)) \\
&= E(V) + Var(\sum_{j=1}^{n} \rho^{2(j-1)} X_1) \\
&= V + (\sum_{j=1}^{n} \rho^{2(j-1)})^2 (1-\rho^4) \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
Var(S_y|X_0 = x_0) &= E(Var(S_y|Y_0, X_0 = x_0)) + Var(E(S_x|Y_0, X_0 = x_0)) \\
&= E(Var(S_y|Y_0)) + Var(E(S_x|Y_0, X_0 = x_0)) \\
&= E(V) + Var(\sum_{j=1}^{n} \rho^{2(j-1)} Y_0) \\
&= V + (\sum_{j=1}^{n} \rho^{2(j-1)})^2 (1-\rho^2) \\
&\le Var(S_x|X_0 = x_0) \qquad (4)
\end{aligned}$$

Hence, for $\rho \leq 1$

$$\begin{aligned}
Var(\hat{I}) &= Var(\frac{1}{n}(S_x|X_0 = x_0)) \\
&= \frac{1}{n^2}Var(S_x|X_0 = x_0) \\
&\geq \frac{\rho^2}{n^2}Var(S_x|X_0 = x_0) \\
&\geq \frac{\rho^2}{n^2}Var(S_y|X_0 = x_0) \\
&= Var(\hat{I}_{RB}) \qquad (5)
\end{aligned}$$