# STA 6866 Monte Carlo Statistical Methods
# Spring 2014

**Homework #2**
Assigned Friday January 24, 2014
Due Monday February 3, 2014

1 Suppose $X \sim \text{bin}(n, \theta)$ and suppose that $\theta$ has as prior the beta distribution with parameters $a$ and $b$, with density

$$f_{a,b}(\theta) = \frac{1}{B(a,b)} \theta^{a-1}(1-\theta)^{b-1}.$$

Here, $B(a,b)$ is the constant that makes $f_{a,b}$ integrate to $1$.

A Find the posterior distribution of $\theta$.

**Solution** The posterior is proportional to the likelihood times the prior. Therefore, if we let $\nu$ be the prior density and $\nu_x$ be the posterior density (given $X = x$), we have

$$\begin{aligned}
\nu_x(\theta) &= c\,\text{lik}(\theta)\,\nu(\theta) \\
&= c\binom{n}{x}\theta^x(1-\theta)^{n-x}\frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1} \\
&= c_1\theta^{a+x-1}(1-\theta)^{b+n-x-1}
\end{aligned}$$

where the $c$'s are constants. We recognize this as the density of the beta distribution with parameters $a+x$ and $b+n-x$. (And therefore the constant $c_1$ must be $B(a+x, b+n-x)$.)

B Find the mean and variance of this posterior distribution.

**Solution** Using standard formulas for the mean and variance of the beta, we have

$$E(\theta \mid X = x) = \frac{a+x}{a+b+n},$$

$$\text{Var}(\theta \mid X = x) = \frac{a+x}{a+b+n}\,\frac{b+n-x}{a+b+n}\,\frac{1}{a+b+n+1}.$$

C Compare the mean and variance of the posterior with the standard frequentist answers (maximum likelihood estimate and inverse of observed Fisher information) when $n$ is large.

**Solution** We now write $X_n$ instead of $X$ and $x_n$ instead of $x$, in order to be careful in the asymptotics. Suppose that as $n \to \infty$, $x_n/n \to c \in (0,1)$. We then have $E(\theta \mid X_n = x_n) \approx x_n/n$, which is the maximum likelihood estimate, and is the standard frequentist estimate for this situation. Also,

$$n\text{Var}(\theta \mid X_n = x_n) \approx \frac{x_n}{n}\,\frac{n-x_n}{n}, \qquad \text{(this is } \hat{p}(1-\hat{p}) \text{ in the usual notation)},$$

and this is $n$ times the inverse of the observed Fisher information. (The reciprocal of the observed Fisher information is the standard frequentist estimate for this situation.)

2 Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where $\sigma$ is known, and suppose that $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$. Find the posterior distribution of $\mu$ and its mean and variance.

**Solution** The joint pdf of $\boldsymbol{X} = (X_1, \cdots, X_n)$ given $\mu$ is given by

$$\begin{aligned} f_\mu(\boldsymbol{X}) &= c_1 \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2\right] \\ &= c_2 \exp\left[-\frac{n}{2\sigma^2} \left(\bar{X} - \mu\right)^2\right], \end{aligned}$$

where $c_1$ and $c_2$ are constants which do not depend on $\mu$. Let $\pi$ be the prior density of $\mu$. We have

$$\pi(\mu) \propto \exp\left[-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2\right].$$

Therefore, the posterior density of $\mu$ given $\boldsymbol{X}$ is

$$\begin{aligned} \pi_{\boldsymbol{X}}(\mu) &\propto \exp\left[-\frac{1}{2}\left\{\frac{n}{\sigma^2}(\bar{X} - \mu)^2 + \frac{1}{\sigma_0^2}(\mu - \mu_0)^2\right\}\right] \\ &\propto \exp\left[-\frac{1}{2}\left\{\mu^2\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right) - 2\mu\left(\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right)\right\}\right] \\ &\propto \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)\left(\mu - \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}\right)^2\right] \end{aligned}$$

Therefore, the posterior distribution of $\mu$ given $\boldsymbol{X}$ is the normal distribution $\mathcal{N}(\mu_{\boldsymbol{X}}, \sigma_{\boldsymbol{X}}^2)$, where

$$\mu_{\boldsymbol{X}} = \left(\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right) \bigg/ \left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right) \qquad \text{and} \qquad \sigma_{\boldsymbol{X}}^2 = \left[\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right]^{-1}.$$

3 Assume the random censorship model of survival analysis. That is, we have random variables $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F$; these are the random variables of interest. We have censoring variables $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} G$ (with density $g$), independent of the $X_i$'s, and we observe only the minima $Z_i = X_i \wedge Y_i$, and the indicators $\delta_i = I(X_i \leq Y_i)$.

Consider the Bayesian model in which we assume that the distribution of the $X_i$'s is exponential with parameter $\lambda$ (equal to the reciprocal of the mean) and the prior distribution of $\lambda$ is the Gamma distribution with parameters $a$ and $b$. (We also assume that the distribution $G$ does not depend on $\lambda$.) Recall that this Gamma distribution has density

$$f(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda) \qquad \text{for } \lambda > 0.$$

Find the posterior distribution of $\lambda$ given $(Z_1, \delta_1), \ldots, (Z_n, \delta_n)$.

**Solution** The posterior is proportional to the likelihood times the prior. Therefore, if we let $\nu$ be the prior density and $\nu_D$ be the posterior density (given the data $(Z_1, \delta_1), \ldots, (Z_n, \delta_n)$), we have

$$\begin{aligned}
\nu_D(\lambda) &= c_1 \,\mathrm{lik}(\lambda)\, \nu(\lambda) \\
&= c_1\, \lambda^{n_u} \, \exp\left(-\lambda \sum_{i=1}^{n} Z_i\right) c_2(G, (\boldsymbol{Z}, \boldsymbol{\delta}))\, \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda) \\
&= c_3\, \lambda^{a+n_u-1} \exp\left[-\left(b + \sum_{i=1}^{n} Z_i\right)\lambda\right],
\end{aligned}$$

where the $c$'s are constants. Here, we have used the derivation of the likelihood developed in class, and we are using the same notation (see around page 18 of the class notes). We recognize this as the gamma density with parameters $a + n_u$ and $b + \sum_{i=1}^{n} Z_i$.