Project Proposal

Let $f \in \mathbb{R}^N$ denote an image with N pixels and F be the 2-D Discrete Fourier Transform restricted to a partial set of frequencies. The partial Fourier data p containing $n < N$ measurements, is obtained as $p = Ff$. Let D_x and D_y denote horizontal and vertical differential operators. For the image f, the partial derivatives are obtained by $f_x = D_x f$ and $f_y = D_y f$. The total variation-baed CS reconstruction is the result of the following optimization problem:

$$\hat{f} = \arg\min_{u \in \mathbb{R}^N} \lambda \|u\|_{BV} + \|Fu - p\|_2^2$$

A similar problem is to solve the following problem:

$$[\hat{f}_x, \hat{f}_y]^T = \arg\min_{vin\mathbb{R}^{2N}} \lambda \|v\|_1 + \|Gv - p'\|_2^2$$

where $p_x = FD_x f$, $p_y = FD_y f$ and

$$G = \begin{pmatrix} F & 0 \\ 0 & F \\ \gamma D_y & \gamma D_x \end{pmatrix}, p' = \begin{pmatrix} p_x \\ p_y \\ 0 \end{pmatrix}$$

What we propose to do in this project is add a group lasso penalty to the last objective function. Basically divide v into groups, say $v_g, g \in G$ and add a penalty term to solve:

$$[\hat{f}_x, \hat{f}_y]^T = \arg\min_{v \in \mathbb{R}^{2N}} \lambda_1 \|v\|_1 + \lambda_2 \sum_{g \in G} \|v_g\|_2 + \|Gv - p'\|_2^2$$

Some ideas for the groups are:

1. $g_i = \{v_i, v_{N+i}\}$
2. $g_i = \{v_{i-1}, v_i, v_{i+1}, v_{N+i-1}, v_{N+i}, v_{N+i-1},\}$

and so on. For any group lasso problem, it is common to use the ADMM. Let $P(v) = \lambda_1 \|v\|_1 + \lambda_2 \sum_{g \in G} \|v_g\|_2$. Then for ADMM the objective is defined as follows:

$$\arg\min_{u,v \in \mathbb{R}^{2N}} \lambda_1 \|u\|_1 + \lambda_2 \sum_{g \in G} \|u_g\|_2 + \|Gv - p'\|_2^2 + \frac{\rho}{2} \|u - v\|_2^2 \text{ s.t. } u = v$$

which becomes the augmented lagrangian

$$L(u, v, \lambda_3) = \lambda_1 \|u\|_1 + \lambda_2 \sum_{g \in G} \|u_g\|_2 + \|Gv - p'\|_2^2 + \frac{\rho}{2}\|u - v\|_2^2 + \lambda_3^t(u - v)$$

The ADMM algorithm in this case is:

1. $u^{k+1} = \arg\min_u L(u, v^k, \lambda_3^k)$
2. $v^{k+1} = \arg\min_v L(u^k, v, \lambda_3^k)$
3. $\lambda_3^{k+1} = \lambda_3^k + \rho(u - v)$

In the above, step 3 is trivial. In step 2 we have to minimize

$$\|Gv - p'\|_2^2 + \frac{\rho}{2}\|u - v\|_2^2 + \lambda_3^t(u - v)$$

Now,

$$\begin{aligned}&\frac{d}{dv}\|Gv - p'\|_2^2 + \frac{\rho}{2}\|u - v\|_2^2 + \lambda_3^t(u - v)\\ &= 2G^t(Gv - p') + \rho(v - u) + \lambda_3\end{aligned}$$

Finally, to solve for step 1 we need to minimize w.r.t. u:

$$\lambda_1 \|u\|_1 + \lambda_2 \sum_{g \in G} \|u_g\|_2 + \frac{\rho}{2}\|u - v\|_2^2 + \lambda_3^t(u - v)$$

Let $k \in G$. Then,

$$\begin{aligned}&\partial_{u_k}(\lambda_1 \|u\|_1 + \lambda_2 \sum_{g \in G} \|u_g\|_2 + \frac{\rho}{2}\|u - v\|_2^2 + \lambda_3^t(u - v))\\ &= (\lambda_3)_k + \rho(u_k - v_k) + \lambda_1 t_k + \lambda_2 s_k\end{aligned}$$

where

$$(t_k)_j = \begin{cases} \text{sign}((u_k)_j) \text{ if } (u_k)_j \neq 0 \\ \in [-1, 1] \text{ if } (u_k)_j = 0 \end{cases}$$

and

$$s_k = \begin{cases} \frac{u_k}{\|u_k\|_2} \text{ if } u_k \neq 0 \\ \in \{u_k : \|u_k\|_2 \leq 1\} \text{ if } u_k = 0 \end{cases}$$

First consider

$$
\begin{aligned}
&\min_u \frac{\rho}{2}\|u-v\|_2^2 + \lambda_3^t(u-v)) + \lambda_1\|u\|_1 \\
\iff &\min_u \frac{\rho}{2}(u^t u - 2v^t u) + \lambda_3^t u + \lambda_1\|u\|_1 \\
\iff &\min_u \sum_{i=1}^n \big(\frac{\rho}{2}(u_i^2 - 2v_i u_i) + (\lambda_3)_i u_i + \lambda_1|u_i|\big) \\
\iff &\sum_{i=1}^n \min_{u_i} \big(\frac{\rho}{2}(u_i^2 - 2v_i u_i) + (\lambda_3)_i u_i + \lambda_1|u_i|\big)
\end{aligned}
$$

Now consider

$$
\min_{u_i} \big(\frac{\rho}{2}(u_i^2 - 2v_i u_i) + (\lambda_3)_i u_i + \lambda_1|u_i|\big)
$$

and suppose $u_i > \lambda_1/\rho$. Then

$$
\begin{aligned}
&\min_{u_i} \big(\frac{\rho}{2}(u_i^2 - 2v_i u_i) + (\lambda_3)_i u_i + \lambda_1 u_i\big) \\
\implies &\rho(u_i - v_i) + (\lambda_3)_i + \lambda_1 \overset{set}{=} 0 \\
\iff &u_i = \big(v_i - \frac{(\lambda_3)_i}{\rho}\big) - \frac{\lambda_1}{\rho}
\end{aligned}
$$

Similarly, if $u_i < -\lambda_1/\rho$, then $u_i\big(v_i - \frac{(\lambda_3)_i}{\rho}\big) + \frac{\lambda_1}{\rho}$. In short, $u = S(v - \frac{1}{\rho}\lambda_3, \frac{\lambda_1}{\rho})$. Then

$$
\begin{aligned}
&(\lambda_3)_k + \rho(u_k - v_k) + \lambda_1 t_k + \lambda_2 s_k = 0 \\
&u_k = 0 \\
&\quad \iff \|S(v_k - \frac{1}{\rho}(\lambda_3)_k, \frac{\lambda_1}{\rho})\|_2 \le \lambda_2
\end{aligned}
$$

and

$$
\begin{aligned}
&u_k \ne 0 \\
&\quad \iff ((\lambda_3)_k)_j + \rho((u_k)_j - (v_k)_j) + \lambda_1 \operatorname{sign}((u_k)_j) + \lambda_2 \frac{u_k}{\|u_k\|_2}
\end{aligned}
$$

What happens when $(u_k)_j = 0$ vs $(u_k)_j \ne 0$? When $(u_k)_j = 0$, we have that $|(v_k)_j - \frac{1}{\rho}((\lambda_3)_k)_j| \le \frac{\lambda_1}{\rho}$. When $(u_k)_j \ne 0$, ...