# Staring log factors down the barrel of the QSVT gun

Words: 2649

# 1. Preliminaries

## 1.1. Markov chains and their quantization

A Markov chain is a graph with vertices in the finite set $X$ and a probabilistic transition matrix $P$. The matrix is row stochastic (rows sum to one) so that for any distribution $\vec{\rho}_t$ over $X$ at time $t$, $\vec{\rho}_{t+1}^\top = \vec{\rho}_t^\top P$. For the entirety of this paper we assume that the Markov chain is ergodic and reversible.

Ergodicity implies that the eigenvalues of $P$ may be written

$$1 = \lambda_1 > \lambda_2 \geq ... \geq \lambda_l > -1 \tag{1}$$

with possible repetition. The left and right eigenvectors associated with $\lambda_1 = 1$ are $\vec{\pi}$ and $\vec{1}$ respectively, where $\vec{\pi}$ is the so-called stationary probability distribution of $P$ whose components are all positive and $\vec{1}$ is the vector of all ones.

Reversibility implies that $P$ satisfies the equations

$$\vec{\pi}_x P_{x,y} = \vec{\pi}_y P_{y,x} \qquad \forall x, y \in X \tag{2}$$

where $\vec{\pi}_x$ is the $x$ component of $\vec{\pi}$ and $P_{x,y}$ is the $x$-$y$ component of $P$. While both ergodicity and reversibility have an intuitive, graph theoretic interpretation, we omit this perspective for brevity.

We can quantize the Markov chain as follows. Let $\{|x\rangle : x \in X\}$ be an orthonormal basis of the Hilbert space $\mathcal{H}_t$ and let $\mathcal{H}_a = \operatorname{span}\{|\overline{0}\rangle\} \cup \mathcal{H}_t$ be another Hilbert space with a special basis state $|\overline{0}\rangle$ used for control operations. For convenience we define $\mathcal{H}_{at} \coloneqq \mathcal{H}_a \otimes \mathcal{H}_t$ and $\mathcal{H}_0 \coloneqq \operatorname{span}\{|\overline{0}\rangle\} \otimes \mathcal{H}_t$. We assume access to the unitaries $U_L, U_R \in \operatorname{End}(\mathcal{H}_{at})$, whose action on states in $\mathcal{H}_0$ is

$$\begin{aligned} U_R|\overline{0}\rangle|x\rangle &= \sum_{y\in X} \sqrt{P_{x,y}}|x\rangle|y\rangle \qquad \forall x \in X \\ U_L|\overline{0}\rangle|y\rangle &= \sum_{x\in X} \sqrt{P_{y,x}}|x\rangle|y\rangle \qquad \forall y \in X \end{aligned} \tag{3}$$

The quantum walk operator is defined $W \coloneqq U_L^\dagger U_R$. We now list some useful properties of $W$, leaving their proofs to the appendix.

*Fact 1: Define the discriminant matrix $D = \operatorname{diag}(\vec{\pi})^{\frac{1}{2}} P \operatorname{diag}(\vec{\pi})^{-\frac{1}{2}}$, where $\operatorname{diag}(\vec{\pi})$ is a diagonal matrix of the coordinates of $\vec{\pi}$. $D_{x,y} = \sqrt{P_{x,y}P_{y,x}}$.*

Note that $D$ is just a similarity transform of $P$ and therefore shares the same eigenvalues with the same geometric multiplicity (see here for proof). Moreover, $D$ is symmetric since $D_{x,y} = D_{y,x}$.

*Fact 2: Let $|\pi_t\rangle$ denote the quantum state that is the component-wise square root of $\vec{\pi}$. The singular value decomposition of $D$ may be written*

$$D = |\pi_t\rangle\langle\pi_t| + \sum_{i=1}^{\dim(\mathcal{H}_t)-1} \sigma_i |\psi_i\rangle\langle\psi_i| \tag{4}$$

*where* $0 \leq \sigma_i < 1$ *and* $|\pi_t\rangle$ *and* $|\psi_i\rangle$ *are its eigenvectors.*
For later convenience, we define the singular value gap $\Delta := 1 - \max \sigma_i$.

*Fact 3:* $\left(|\bar{0}\rangle\langle\bar{0}| \otimes I_t\right) W \left(|\bar{0}\rangle\langle\bar{0}| \otimes I_t\right) = |\bar{0}\rangle\langle\bar{0}| \otimes D$.

*Fact 4:* $|\pi\rangle := |\bar{0}\rangle|\pi_t\rangle$ *is an eigenvector of* $W$.

## 1.2. QSVT

As a preliminary result, quantum signal processing shows that for and $d \geq 0$

$$QSP_\Phi(x) = e^{i\phi_0 Z} \prod_{j=1}^{d} R(x) e^{i\phi_j Z} = \begin{pmatrix} P(x) & i\sqrt{1-x^2}Q(x) \\ i\sqrt{1-x^2}Q^*(x) & P^*(x) \end{pmatrix} \tag{5}$$

where

$$R(x) = \begin{pmatrix} x & \sqrt{1-x^2} \\ \sqrt{1-x^2} & -x \end{pmatrix} \tag{6}$$

and $P, Q \in \mathbb{C}[x]$ are complex polynomials satisfying
(1) $\deg(P) \leq d$ and $\deg(Q) \leq k - 1$,
(2) $P$ has the same parity as $k$ and $Q$ has the same parity as $k - 1$
(3) $|P(x)|^2 + (1 - x^2)|Q(x)|^2 = 1$

Note that a polynomial has odd (even) parity if only the coefficients of odd (even) powers of $x$.
Now let $U$ be a unitary and $\Pi, \tilde{\Pi}$ be orthogonal projectors satisfying

$$A = \tilde{\Pi} U \Pi \tag{7}$$

where $A$ has the singular value decomposition $A = \sum_i \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$. Further define the unitary rotations

$$\begin{aligned} \Pi_\varphi &= e^{\frac{i\varphi}{2}(2\Pi - I)} = e^{\frac{i\varphi}{2}}\Pi + e^{\frac{-i\varphi}{2}}(I - \Pi) \\ \tilde{\Pi}_\phi &= e^{i\phi(2\tilde{\Pi} - I)} = e^{\frac{i\phi}{2}}\tilde{\Pi} + e^{\frac{-i\phi}{2}}\left(I - \tilde{\Pi}\right) \end{aligned} \tag{8}$$

and, for each $i$, the states (which are provably unit vectors)

$$\begin{aligned} |\psi_i^\perp\rangle &= \frac{(I - \Pi)U^\dagger|\tilde{\psi}\rangle}{\|(I - \Pi)U^\dagger|\tilde{\psi}\rangle\|}, \\ |\tilde{\psi}_i^\perp\rangle &= \frac{\left(I - \tilde{\Pi}\right)U|\psi\rangle}{\left\|\left(I - \tilde{\Pi}\right)U|\psi\rangle\right\|}. \end{aligned} \tag{9}$$

Let $d$ be even and $P, Q$ be QSP achievable polynomials with degree at most $d$. There exists phases $\varphi_j, \phi_j$ for $j \in [d/2]$, such that the action of the unitary

$$U_\Phi = \prod_{j=1}^{d/2} \Pi_{\varphi_j} U^\dagger \tilde{\Pi}_{\phi_j} U \tag{10}$$

on each of the subspaces $\operatorname{span}\left\{|\psi_i\rangle, |\psi_i^\perp\rangle\right\}$ is governed by the $SU(2)$ operator $U_\Phi = QSP(\sigma_i)$.
If we instead let $d$ be odd and $P, Q$ be QSP achievable polynomials of degree at most $d$ then.
There exists phases $\varphi_j, \phi_j$ for $j \in \left[\frac{d-1}{2}\right]$ such that the unitary

$$U_\Phi = \tilde{\Pi}_{\varphi_0}\left(\prod_{j=1}^{(d-1)/2} U\Pi_{\varphi_j}U^\dagger\tilde{\Pi}_{\phi_j}\right)U \tag{11}$$

on the input basis $\operatorname{span}\{|\psi_i\rangle, |\psi_i^\perp\rangle\}$ and output basis $\operatorname{span}\{|\tilde{\psi}_i\rangle, |\tilde{\psi}_i^\perp\rangle\}$ is given by $QSP_\Phi(\sigma_i)$.

## 1.3. Amplitude amplification

Amplitude amplification is a solution to the unordered search problem where we begin in state $|s\rangle$ and the goal is to measure a element in set of orthogonal states $\{|\mu_i\rangle\}$; the so-called 'marked' states. Let $\Pi = |s\rangle\langle s|$ and $\tilde{\Pi} = \sum_i |\mu_i\rangle\langle\mu_i|$. In addition to the state $|s\rangle$, we also assume access to the unitary operators $\Pi_\varphi$, $\tilde{\Pi}_\phi$ and the state preparation unitary $U$ satisfying $\tilde{\Pi}U\Pi = x|t\rangle\langle s|$ for $|t\rangle = \frac{\tilde{\Pi}U|s\rangle}{\|\tilde{\Pi}U|s\rangle\|}$. In amplitude amplification we choose the phases $\varphi = \phi = \pi$ such that,

$$\Pi_\pi = -i(I - 2\Pi) \text{ and } \tilde{\Pi}_\pi = -i\big(I - 2\tilde{\Pi}\big). \tag{12}$$

We then implement the following iteration

$$\begin{aligned} U_\Phi|s\rangle &= \left(\prod_{i=1}^{(n-1)/2} U\Pi_\pi U^\dagger \tilde{\Pi}_\pi\right)U|s\rangle \\ &= \left(\prod_{i=1}^{(n-1)/2} -\big(I - 2U|s\rangle\langle s|U^\dagger\big)(I - 2|t\rangle\langle t|)\right)U|s\rangle \end{aligned} \tag{13}$$

for odd $n$. Clearly this is just a trivial instance of QSVT. Indeed, it can be shown that this induces the QSP achievable polynomials $P(x) = (-1)^{\frac{n-1}{2}}T_n(x)$ and $Q(x) = (-1)^{\frac{n+1}{2}}iU_{n-1}(x)$, where $T_n$ and $U_{n-1}$ are the first and second order Chebyshev polynomials of degree $n$ and $n-1$ respectively. Note that for $x = \sin(\theta)$, $(-1)^{\frac{n-1}{2}}T_n(x) = \sin(n\theta)$ so long as $n$ is odd, which is the more familiar characterisation of the induced polynomial. Moreover since $\sin(n\theta) \approx \sin(nx)$ for small $x$, by choosing $n$ close to $\frac{\pi}{2x}$, the probability of measuring $|t\rangle$ is close to one. However, a significant limitation of this algorithm is that we require prior knowledge of $x$ to choose an appropriate $n$ and too low or too high an estimate can lead to the souffle problem, where the probability of measuring a marked state is still far from one.

## 1.4. Optimal fixed point amplitude amplification.

To address this limitation, Yoder, Low and Chuang devise the oblivious fixed point amplitude amplification algorithm whereby, for any error tolerance $\varepsilon > 0$, the probability of measuring $|t\rangle$ gets boosted and remains above the fixed point $1 - \varepsilon^2$ after any number of iterations $n \geq \frac{\log(\frac{2}{\varepsilon})}{x}$, which is optimal. Thus, one can measure $|t\rangle$ with high probability, knowing only a lower bound on $x$

For integer $n > 0$ and $\varepsilon \in (0,1)$ we find it convenient to define the polynomial

$$f_n(x,\varepsilon) = \varepsilon T_n\left(T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)x\right). \tag{14}$$

Note that Chebyshev polynomials of the first kind are generalised to non-integer degrees through the identity

$$T_y(x) \equiv \cosh(y\cosh^{-1}(x)) \equiv \cos(y\cos^{-1}(x)). \tag{15}$$

The results of optimal fixed point amplitude amplification is formalised through the following lemma.

**Lemma 1** *(Due to Low, Yoder and Chuang). Let $|s\rangle$ be an initial state, $\Pi = |s\rangle\langle s|$, $\tilde{\Pi} = \sum_i |\mu_i\rangle\langle\mu_i|$ be a projection onto a set of orthogonal marked states, $|t\rangle = \frac{\tilde{\Pi}U|s\rangle}{\|\tilde{\Pi}U|s\rangle\|}$ and $U$ be a unitary satisfying*

$$\tilde{\Pi}U\Pi = x|s\rangle\langle t|. \tag{16}$$

*Further, assume access to $U$, $U^\dagger$, $\Pi_\varphi$ and $\tilde{\Pi}_\phi$. The circuit*

$$U_\Phi = \left(\prod_{j=1}^{(n-1)/2} U\Pi_{\varphi_j}U^\dagger\tilde{\Pi}_{\phi_j}\right)U \tag{17}$$

*where $n$ is odd and*

$$\varphi_j = \phi_{(\frac{n-1}{2})-j+1} = -2\cot^{-1}\left(\tan\left(\frac{2\pi j}{n}\right)\sqrt{1 - T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)^{-2}}\right), \tag{18}$$

*$\varepsilon \in (0,1)$, induces the QSP achievable polynomials $P, Q$ such that $\sqrt{1-x^2}|Q(x)| = f_n\left(\sqrt{1-x^2}, \varepsilon\right)$ and $\langle t|U_\Phi|s\rangle = P(x)$.*

We want $|P(x)|^2 = 1 - (1-x^2)|Q(x)|^2 \geq 1 - \varepsilon^2$ which requires $f_n\left(\sqrt{1-x^2}, \varepsilon\right) \leq \varepsilon$. Since $|T_n(a)| \leq 1$ if and only if $|a| \leq 1$, $f_n\left(\sqrt{1-x^2}, \varepsilon\right) \leq \varepsilon$ if and only if

$$\left|T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)\right|\sqrt{1-x^2} \leq 1 \tag{19}$$

$$\Rightarrow x^2 \geq 1 - T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)^{-2} \geq \left(\frac{\log(\frac{2}{\varepsilon})}{n}\right)^2 \tag{20}$$

$$\Rightarrow n \geq \frac{\log(\frac{2}{\varepsilon})}{x} \in O\left(\frac{\log(\frac{1}{\varepsilon})}{x}\right). \tag{21}$$

which is optimal. Note that the second inequality in Eq. 20 is due to Yoder, Low and Chuang. While Lemma 1 shows how we can induce a polynomial on a singular value $x > 0$ to amplify it to one with precision $\varepsilon$, a very similar set of phases can be used to induce a polynomial on a singular value $x < 1$ to suppress it to zero with precision $\varepsilon$. Notably, the polynomial is real, unlike those in Lemma 1.

**Lemma 2** *(Due to Yoder, Low and Chuang). Let $U$ be a unitary and $\Pi$, $\tilde{\Pi}$ be projectors such that*

$$A = \tilde{\Pi}U\Pi \tag{22}$$

*where $A$ is a matrix with singular value decomposition $A = \sum_{i=1}^{l} \sigma_i |w_i\rangle\langle v_i|$. Then the circuit*

$$U_\Phi = \left(\prod_{j=1}^{(n-1)/2} U\Pi_{\varphi_j}U^\dagger\tilde{\Pi}_{\phi_j}\right)U \tag{23}$$

*where*

$$\varphi_j = -2\cot^{-1}\left(\tan\left(\frac{(2j-1)\pi}{n}\right)\sqrt{1 - T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)^{-2}}\right),$$

$$\phi_j = -2\cot^{-1}\left(\tan\left(\frac{2j\pi}{n}\right)\sqrt{1 - T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)^{-2}}\right), \tag{24}$$

*for* $j \in \left[\frac{n-1}{2}\right]$ *and* $\varepsilon \in (0,1)$*, induces the QSP achievable polynomials,* $P, Q$ *such that* $P(x) = f_n(\sigma_i, \varepsilon)$ *for all* $i \in [l]$*.*

Note that for $\sigma_i = 1$

$$P(\sigma_i) = f_n(1, \varepsilon) = \varepsilon T_n\left(T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)\right) = 1 \tag{25}$$

where the last equality comes from the semi-group property of Chebyshev polynomials $T_n(T_m(x)) = T_{nm}(x)$ and the fact that $T_1(x) = x$. Thus $P$ does not effect unit singular values.

# 2. Quantum Search on a Markov chain

In Markov chain based search, we are promised the existence of a non-empty subset of vertices $M \subseteq X$ and are given access to an oracle $C_{\tilde{\Pi}} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathcal{H}_{at})$ that costs $\boldsymbol{C}$ to implement and satisfies

$$C_{\tilde{\Pi}} = X \otimes \tilde{\Pi} + I \otimes \left(I - \tilde{\Pi}\right) \tag{26}$$

where $\tilde{\Pi} = |\overline{0}\rangle\langle\overline{0}| \otimes \sum_{x \in M} |x\rangle\langle x|$. That is, $C_{\tilde{\Pi}}$ flips an ancilla qubit when $\mathcal{H}_{at}$ is a target state. We also assume access to the walk operator $W = U_L^\dagger U_R$, defined in Eq. 3, at cost $\boldsymbol{U}$ and the ability to prepare the state $|\pi\rangle$ at cost $\boldsymbol{S}$. Importantly, we assume $\boldsymbol{S}$ is large, so we should avoid preparing the initial state frequently. The goal is to find any state $|x\rangle$ where $x \in M$.

One can see by the results of Section 1.3 that with access to operators

$$R_\varphi = e^{i\frac{\varphi}{2}} |\pi\rangle\langle\pi| + e^{-i\frac{\varphi}{2}} (I - |\pi\rangle\langle\pi|),$$
$$\tilde{R}_\phi = e^{i\frac{\phi}{2}} \tilde{\Pi} + e^{-i\frac{\phi}{2}} \left(I - \tilde{\Pi}\right), \tag{27}$$

where we've replaced the '$\Pi$' for '$R$' to avoid overloading the symbol, we can implement any rendition of amplitude amplification, fixed point or not. We first show that building $\tilde{R}_\phi$ is computationally cheap with respect to $\boldsymbol{C}$.

**Lemma 3** (Exact rotations about the target state). *Given access to* $C_{\tilde{\Pi}} = X \otimes \tilde{\Pi} + I \otimes \left(I - \tilde{\Pi}\right)$*, where* $\tilde{\Pi} = |\overline{0}\rangle\langle\overline{0}| \otimes \sum_{x \in M} |x\rangle\langle x|$ *is an orthogonal projector onto the marked states, there is a unitary circuit with time complexity* $2\boldsymbol{C}$*, that requires 1 ancilla qubit and whose action on* $\mathrm{span}\{|0\rangle\} \otimes \mathcal{H}_{at}$ *is given by*

$$|0\rangle\langle 0| \otimes \tilde{R}_\phi \tag{28}$$

*where*

$$\tilde{R}_\phi = e^{i\frac{\phi}{2}} \tilde{\Pi} + e^{-i\frac{\phi}{2}} \left(I - \tilde{\Pi}\right). \tag{29}$$

*Proof.*

$$
\begin{aligned}
C_{\tilde{\Pi}}\left(e^{-i\frac{\phi}{2}Z}\otimes I_{at}\right)C_{\tilde{\Pi}}(|0\rangle\langle 0|\otimes I_{at}) &= \left(e^{i\frac{\phi}{2}Z}\otimes\tilde{\Pi}+e^{-i\frac{\phi}{2}Z}\otimes\left(I-\tilde{\Pi}\right)\right)(|0\rangle\langle 0|\otimes I_{at})\\
&= |0\rangle\langle 0|\otimes e^{i\frac{\phi}{2}}\tilde{\Pi}+|0\rangle\langle 0|\otimes e^{-i\frac{\phi}{2}}\left(I-\tilde{\Pi}\right)\\
&= |0\rangle\langle 0|\otimes\tilde{R}_{\phi}
\end{aligned}
\tag{30}
$$

■

Given our assumption on $\boldsymbol{S}$, building exact rotations around $|\pi\rangle$ would prove costly since it entails state preparation. Nonetheless, wielding QSVT in one hand and our quantum walk operator in the other, we build an approximation to $R_{\varphi}$ in Lemma 4 whose complexity is polynomial in $\boldsymbol{U}$ rather than $\boldsymbol{S}$ and show that it satisfies some convenient properties. A less general and more space inefficient construction was given in Theorem 6 of the original MNRS paper.

**Lemma 4** (Approximate rotations about the stationary state). *Let $\Pi=\left|\bar{0}\right\rangle\left\langle\bar{0}\right|\otimes I_{at}$, $W$ be the quantum walk operator, $\Delta$ be the singular value gap of discriminant matrix $\left|\bar{0}\right\rangle\left\langle\bar{0}\right|\otimes D=\Pi W\Pi$ and $R_{\varphi}=e^{\frac{i\varphi}{2}}|\pi\rangle\langle\pi|+e^{\frac{-i\varphi}{2}}(I-|\pi\rangle\langle\pi|)$. There exists a quantum circuit $R(\beta)$ with time complexity $O\left(\frac{\log(1/\beta)}{\sqrt{\Delta}}\boldsymbol{U}\right)$ and requires one ancilla qubit to build satisfying*
*(1) $R_{\varphi}(\beta)|\pi\rangle=e^{\frac{i\varphi}{2}}|\pi\rangle$, and*
*(2) $\left\|\left(R_{\varphi}(\beta)-R_{\varphi}\right)|v\rangle\right\|\leq\beta$ for $|v\rangle\perp|\pi\rangle$ and $|v\rangle\in\mathcal{H}_0$*

*Proof.* We have that $\left|\bar{0}\right\rangle\left\langle\bar{0}\right|\otimes D=\Pi W\Pi$ and

$$
\left|\bar{0}\right\rangle\left\langle\bar{0}\right|\otimes D=|\pi\rangle\langle\pi|+\sum_{i}^{\dim(\mathcal{H}_t)-1}\sigma_i|\psi_i\rangle\langle\psi_i|
\tag{31}
$$

where $\sigma_i$, $|\psi_i\rangle$ are singular value, singular vector pairs. We can immediately invoke Lemma 2 to say that there exists a unitary circuit $W_{\Phi}$ where $\Phi\in\mathbb{R}^n$ is a sequence of $n\in O\left(\frac{\log(2/\beta)}{\sqrt{\Delta}}\right)$ phases, such that

$$
D(\beta/2):=\Pi W_{\Phi}\Pi=|\pi\rangle\langle\pi|+\sum_{i}^{\dim(\mathcal{H}_t)-1}f_n\left(\sigma_i,\frac{\beta}{2}\right)|\psi_i\rangle\langle\psi_i|
\tag{32}
$$

and $f_n\left(\sigma_i,\frac{\beta}{2}\right)\leq\frac{\beta}{2}$. We let $R_{\varphi}(\beta)=W_{\Phi}\Pi_{\varphi}W_{\Phi}^{\dagger}$ so that its total time complexity is $O\left(\frac{\log(1/\beta)}{\sqrt{\Delta}}\boldsymbol{U}\right)$ and claim it satisfies both properties.

(1) First notice that $W_{\Phi}^{\dagger}|\pi\rangle=|\pi\rangle$ since

$$
\langle\pi|W_{\Phi}^{\dagger}|\pi\rangle=\langle\pi|\Pi W_{\Phi}\Pi|\pi\rangle=\langle\pi|D(\beta/2)|\pi\rangle=1
\tag{33}
$$

Furthermore, since $|\pi\rangle\in\mathcal{H}_0$, $\Pi_{\varphi}|\pi\rangle=e^{i\frac{\varphi}{2}}|\pi\rangle$ and statement 1 follows.

(2) It's not difficult to show that

$$
\left(R_{\varphi}(\beta)-R_{\varphi}\right)=\left(e^{i\frac{\varphi}{2}}-e^{-i\frac{\varphi}{2}}\right)\left(W_{\Phi}\Pi W_{\Phi}^{\dagger}-|\pi\rangle\langle\pi|\right).
\tag{34}
$$

Thus for $|v\rangle\in\mathcal{H}_0$ satisfying $|v\rangle\perp|\pi\rangle$ we have

$$\begin{aligned}\left\|\Big(R_\varphi(\beta)-R_\varphi\Big)|v\rangle\right\| &= \left\|\Big(e^{\frac{i\varphi}{2}}-e^{\frac{-i\varphi}{2}}\Big)W_\Phi\Pi W_\Phi^\dagger|v\rangle\right\| \\ &= \left|e^{\frac{i\varphi}{2}}-e^{\frac{-i\varphi}{2}}\right|\left\|D(\beta/2)^\dagger|v\rangle\right\| \\ &\le \left|\sin\frac{\varphi}{2}\right|\beta\le\beta. \end{aligned}\tag{35}$$

■

Based on Section 1.3, it seems reasonable to think for small enough $\beta$, $O\Big(\frac{1}{\sqrt{p_M}}\Big)$ iterations of $R_\pi(\beta)\tilde{R}_\pi$ transforms the state $|\pi\rangle$ to $|t\rangle$ with high probability. But how small should $\beta$ be? We provide one sufficing answer with Lemma 5, proved in the appendix.

**Lemma 5** (Naive error bound). *Let $\tilde{\Pi}$ be a projector on to the marked states and $|\pi\rangle$ be the initial state. Further, let $\tilde{R}_\phi=e^{\frac{i\phi}{2}}\tilde{\Pi}+e^{\frac{-i\phi}{2}}\big(I-\tilde{\Pi}\big)$, $R_\varphi=e^{\frac{i\varphi}{2}}|\pi\rangle\langle\pi|+e^{\frac{-i\varphi}{2}}(I-|\pi\rangle\langle\pi|)$ and $R_\varphi(\beta)$ satisfy $\max_{|v\rangle\in\mathcal{H}_0}\left\|\Big(R_\varphi(\beta)-R_\varphi\Big)|v\rangle\right\|\le\beta$. Then,*

$$\left\|\left(\prod_{i=1}^{n}R_{\varphi_i}(\beta)\tilde{R}_{\phi_i}-\prod_{i=1}^{n}R_{\varphi_i}\tilde{R}_{\phi_i}\right)|\pi\rangle\right\|\le n\beta \tag{36}$$

*Proof.* See Section 6. ■

Thus $\beta\in O\big(\sqrt{p_M}\big)$ suffices. But this implies the total time complexity of the algorithm is

$$O\left(\boldsymbol{S}+\frac{1}{\sqrt{p_M}}\left(\boldsymbol{C}+\frac{\log\Big(\frac{1}{\sqrt{p_M}}\Big)}{\sqrt{\Delta}}\boldsymbol{U}\right)\right). \tag{37}$$

MNRS unveil a clever way to remove this extra log factor which requires introducing the recursive amplitude amplification algorithm.

# 3. Recursive amplitude amplification

In this section we examine a recursive implementation of amplitude amplification with phases left unspecified for generality.

$$\begin{aligned} A_0 &= I, \\ A_i &= A_{i-1}R_{\varphi_i}A_{i-1}^\dagger\tilde{R}_{\phi_i}A_{i-1}. \end{aligned}\tag{38}$$

We see that $A_i$ simply enacts one iteration of amplitude amplification with the state preparation unitary equal to $A_{i-1}$. We find this form convenient to work with when comparing it to the error sensitive rendition

$$\begin{aligned} B_0 &= I, \\ B_i &= B_{i-1}R_{\varphi_i}(\beta_i)B_{i-1}^\dagger\tilde{R}_{\phi_i}B_{i-1}. \end{aligned}\tag{39}$$

Suppose we wish to apply a total of $t$ recursive calls. We are ultimately interested in finding a sequence of $\beta_i$ for $i\in[t]$ so that, for any error tolerance $\gamma$, we can upper bound some reasonable error function $e(t)\le\gamma$ and so that $R_{\varphi_i}(\beta_i)$ has complexity $O\Big(\frac{\log(1/\gamma)}{\sqrt{\Delta}}\boldsymbol{U}\Big)$, in which case, the factor $\log(1/\gamma)$ can be treated as a constant and dropped. Although a tall order, MNRS manage to prove this result for the non fixed point rendition of amplitude amplification, where $\varphi=\phi=\pi$ and the error function measures the absolute difference in marked state amplitudes. That is

$$e(t) = \|\tilde{\Pi}(B_t - A_t)|\pi\rangle\|. \tag{40}$$

However, it would also be interesting to examine if a similar bound can be derived for the amplitude difference of non-marked states.

$$e(t) = \|(I - \tilde{\Pi})(B_t - A_t)|\pi\rangle\|. \tag{41}$$

The remainder of this section offers some preliminary results which aid us in our eventual examination of the MNRS algorithm and discussions on how the analysis may be extended to the fixed point setting. We start with a simple result that can be viewed as a generalisation of Lemma 4 and that will come in handy in Lemma 7. Define $|t_i\rangle$, $|t_i^{\perp}\rangle$, $|b_i\rangle$ and $|b_i^{\perp}\rangle$ such that

$$|t_i\rangle \in \operatorname{img}(\tilde{\Pi}), |t_i^{\perp}\rangle \in \operatorname{img}(I - \tilde{\Pi}) \tag{42}$$

$$|b_i\rangle := B_i|\pi\rangle = x_i|t_i\rangle + \sqrt{1 - x_i^2}|t_i^{\perp}\rangle, \tag{43}$$

$$|b_i^{\perp}\rangle = \sqrt{1 - x_i^2}|t_i\rangle - x_i|t_i^{\perp}\rangle, \text{ and} \tag{44}$$

**Lemma 6** (Generalisation of Lemma 4). Let $E_{i+1} = B_i\left(R_{\varphi_{i+1}}(\beta_{i+1}) - R_{\varphi_{i+1}}\right)B_i^{\dagger}$.
(1) $E_{i+1}|b_i\rangle = 0$, and
(2) $\|E_{i+1}|b_i^{\perp}\rangle\| \leq \beta_{i+1}$.

*Proof.* (1)

$$\begin{aligned} E_{i+1}|b_i\rangle &= B_i\left(R_{\varphi_{i+1}}(\beta_{i+1}) - R_{\varphi_{i+1}}\right)B_i^{\dagger}|b_i\rangle \\ &= B_i\left(R_{\varphi_{i+1}}(\beta_{i+1}) - R_{\varphi_{i+1}}\right)|\pi\rangle \\ &= B_i 0 = 0 \end{aligned} \tag{45}$$

where we invoke Lemma 4 in the last equality.

(2) First notice that $B_i^{\dagger}|b_i^{\perp}\rangle \in \mathcal{H}_0$ since $B_i^{\dagger}|b_i\rangle = |\pi\rangle \in \mathcal{H}_0$ implies that $B_i^{\dagger}|t_i^{\perp}\rangle, B_i^{\dagger}|t_i\rangle \in \mathcal{H}_0$. Further, $\langle\pi|B_i^{\dagger}|b_i^{\perp}\rangle = \langle b_i|b_i^{\perp}\rangle = 0$. We can therefore invoke Lemma 4 again to say

$$\begin{aligned} \|E_{i+1}|b_i^{\perp}\rangle\| &= \|B_i\left(R_{\varphi_{i+1}}(\beta_{i+1}) - R_{\varphi_{i+1}}\right)B^{\dagger}|b_i^{\perp}\rangle\| \\ &= \|\left(R_{\varphi_{i+1}}(\beta_{i+1}) - R_{\varphi_{i+1}}\right)B^{\dagger}|b_i^{\perp}\rangle\| \leq \beta_{i+1}. \end{aligned} \tag{46}$$

■

Now suppose we have access to the unitaries $B_i$, $B_i^{\dagger}$, $R_{\varphi_{i+1}}$ and $\tilde{R}_{\phi_{i+1}}$. From the results of QSVT discussed in section Section 1.2 we know that the sequence $B_i R_{\varphi_{i+1}} B_i^{\dagger} \tilde{R}_{\phi_i} B_i$ induces the polynomial transformation $x_i \to P(x_i)$ and $\sqrt{1 - x_i^2} \to i\sqrt{1 - x_i^2}Q(x_i)$, where $P$ and $Q$ are at most degree 3 QSP achievable polynomials. In Lemma 7, we bound how this algorithm differs from $B_{i+1}$ in terms of absolute amplitude differences.

**Lemma 7** (Single level error bound). *Let* $x_{i+1} = \langle t_{i+1}|b_{i+1}\rangle$, $\sqrt{1 - x_{i+1}^2} = \langle t_{i+1}^{\perp}|b_{i+1}\rangle$, $P(x_i) = \langle t_i|B_i R_{\varphi_{i+1}} B_i^{\dagger} \tilde{R}_{\phi_{i+1}}|b_i\rangle$ *and* $i\sqrt{1 - x_i^2}Q(x_i) = \langle t_i^{\perp}|B_i R_{\varphi_{i+1}} B_i^{\dagger} \tilde{R}_{\phi_{i+1}}|b_i\rangle$ *where* $P : \mathbb{C} \to \mathbb{C}$ *and* $Q : \mathbb{C} \to \mathbb{C}$ *are QSP achievable polynomials.*
*(1)* $\left|x_{i+1} - P(x_i)\right| \leq 2|x_i\sqrt{1 - x_i^2}|\ \beta_{i+1}$.
*(2)* $\left|\sqrt{1 - x_{i+1}^2} - i\sqrt{1 - x_i^2}Q(x_i)\right| \leq 2|x_i\sqrt{1 - x_i^2}|\ \beta_{i+1}$

*Proof.* First note that

$$
\begin{aligned}
E_{i+1}\tilde{R}_{\phi_{i+1}}|b_i\rangle &= E_{i+1}\Big(e^{\frac{i\phi_{i+1}}{2}}x_i|t_i\rangle + e^{\frac{-i\phi_{i+1}}{2}}\sqrt{1-x_i^2}\big|t_i^{\perp}\rangle\Big) \\
&= E_{i+1}\Big[\Big(e^{\frac{i\phi_{i+1}}{2}}x_i^2 + e^{\frac{-i\phi_{i+1}}{2}}(1-x_i^2)\Big)|b_i\rangle + \Big(e^{\frac{i\phi_{i+1}}{2}} - e^{\frac{-i\phi_{i+1}}{2}}\Big)x_i\sqrt{1-x_i^2}\big|b_i^{\perp}\rangle\Big] \\
&= \Big(e^{\frac{i\phi_{i+1}}{2}} - e^{\frac{-i\phi_{i+1}}{2}}\Big)x_i\sqrt{1-x_i^2}E_{i+1}\big|b_i^{\perp}\rangle
\end{aligned} \tag{47}
$$

where we use Lemma 6 statement 1 for the third equality. Then using statement 2 we get

$$
\begin{aligned}
\|E_{i+1}\tilde{R}_{\phi_{i+1}}|b_i\rangle\| &\leq \sqrt{2-2\cos(\phi_{i+1})}\Big|x_i\sqrt{1-x_i^2}\Big|\beta_{i+1} \\
&\leq 2\Big|x_i\sqrt{1-x_i^2}\Big|\beta_{i+1}.
\end{aligned} \tag{48}
$$

and

$$
\begin{aligned}
\big|x_{i+1} - P(x_i)\big| &= \sqrt{x_{i+1}^2 - 2x_{i+1}\operatorname{Re}(P(x_i)) + |P(x_i)|^2} \\
&\leq \sqrt{x_{i+1}^2 - 2x_{i+1}\operatorname{Re}\big(P(x_i)\langle t_i|t_{i+1}\rangle\big) + |P(x_i)|^2} \\
&= \big\|x_{i+1}\big|t_{i+1}\rangle - P(x_i)|t_i\rangle\big\| \\
&= \|\tilde{\Pi}E_{i+1}\tilde{R}_{\phi_{i+1}}|b_i\rangle\| \\
&\leq \|E_{i+1}\tilde{R}_{\phi_{i+1}}|b_i\rangle\| \\
&\leq 2\Big|x_i\sqrt{1-x_i^2}\Big|\beta_{i+1}
\end{aligned} \tag{49}
$$

A near identical argument proves (2).

# 4. Proof of the MNRS algorithms

First, we should check that the recursive algorithm introduced in Section 3 supports standard amplitude amplification. Let $\varphi_i = \phi_i = \pi$ for all $i$. Its clear from the results introduced in Section 1.3 that if $A_{i-1}$ is the state preparation unitary satisfying $x_{i-1} = \|\tilde{\Pi}A_{i-1}|\pi\rangle\|$ then algorithm $A_i$ satisfies $\|\tilde{\Pi}A_i|\pi\rangle\| = |T_3(x_i)| = \sin(3\arcsin x_i)$ which we can then show by induction is equal to $\sin\big(3^i\arcsin\sqrt{p_M}\big)$. Thus, choosing $t \approx \log_3\Big(\frac{\pi}{2\sqrt{p_M}}\Big)$ gives $\|\tilde{\Pi}A_t|\pi\rangle\| = \sin\big(3^t\arcsin\sqrt{p_M}\big) \approx 1$ and its not hard to show that this is optimal in the number of uses of $R_\pi, \widetilde{R_\pi}$.

Now consider the corresponding error sensitive case, $B_t$, with phases $\varphi = \phi = \pi$. The feat of MNRS is to show that for any $\gamma \in (0,1)$ there exists values $\beta_i$ for $i \in [t]$ such that $\|\tilde{\Pi}(B_t - A_t)|\pi\rangle\| \leq \gamma$ and the total time complexity of $B_t$ is

$$
O\Bigg(\boldsymbol{S} + \frac{1}{\sqrt{p_M}}\Bigg(\boldsymbol{C} + \frac{\log\big(\frac{1}{\gamma}\big)}{\sqrt{\Delta}}\boldsymbol{U}\Bigg)\Bigg). \tag{50}
$$

We start by proving the time complexity result in Lemma 8, which is the simpler of the two to prove. Then in Lemma 9 we prove the correctness of the algorithm.

**Lemma 8** (Time complexity of MNRS). *Let $A_0 = B_0 = I$, $A_i = A_{i-1}R_\pi A_{i-1}^\dagger\tilde{R}_\pi$, and $B_i = B_{i-1}R_\pi(\beta_i)B_{i-1}^\dagger\tilde{R}_\pi$ for*

$$
\beta_i = \frac{18\gamma}{4\pi^3 i^2}. \tag{51}
$$

*where $\gamma \in (0,1)$ is an error tolerance. Further let $t = \left\lfloor \log_3\left(\frac{\pi}{2\sqrt{p_M}}\right)\right\rfloor$. The total time complexity of computing $B_t|\pi\rangle$ is*

$$O\left(\boldsymbol{S} + \frac{1}{\sqrt{p_M}}\left(\boldsymbol{C} + \frac{\log\left(\frac{1}{\gamma}\right)}{\sqrt{\Delta}}\boldsymbol{U}\right)\right). \tag{52}$$

*Proof.* Let $c_1 = \boldsymbol{C}$ and $c_2 = \frac{\boldsymbol{U}}{\sqrt{\Delta}}$. The cost of implementing $B_i$ be given by the recurrence relation

$$\begin{aligned}\mathrm{Cost}(0) &= 0\\ \mathrm{Cost}(i) &= 3\mathrm{Cost}(i-1) + \left(c_1 + \log\left(\frac{1}{\beta_i}\right)c_2\right).\end{aligned} \tag{53}$$

Unravelling we get that

$$\begin{aligned}\mathrm{Cost}(t) &= \sum_{j=1}^{t} 3^{t-j}\left(c_1 + \log\left(\frac{4\pi^3 j^2}{18\gamma}\right)c_2\right)\\ &= 3^t\sum_{j=1}^{t}\frac{1}{3^j}\left(c_2\left(2\log(j) + \log\left(\frac{1}{\gamma}\right) + O(1)\right) + c_1\right)\\ &= 3^t\left(\left[c_2\left(\log\left(\frac{1}{\gamma}\right) + O(1)\right) + c_1\right]\sum_{j=1}^{t}\frac{1}{3^j} + 2c_2\sum_{j=1}^{t}\frac{\log(j)}{3^j}\right)\end{aligned} \tag{54}$$

Since $\sum_{j=1}^{t}\frac{1}{3^j}, \sum_{j=1}^{t}\frac{\log(j)}{3^j} \in O(1)$ the time complexity becomes

$$O\left(3^t\left[c_2\log\left(\frac{1}{\gamma}\right) + c_1\right] + 3^t c_2 O(1)\right) = O\left(\frac{1}{\sqrt{p_M}}\left(\boldsymbol{C} + \frac{\log\left(\frac{1}{\gamma}\right)}{\sqrt{\Delta}}\boldsymbol{U}\right)\right). \tag{55}$$

Preparing the initial state $|\pi\rangle$ at cost $\boldsymbol{S}$ completes the analysis.

■

Before moving onto the error analysis. Note that by Lipschitz continuity, for any complex polynomial $P$ and reals $x, y \in [-1,1]$, $|P(x) - P(y)| \leq \sup_{z\in[-1,1]}|P'(z)|\ |x-y|$.

**Lemma 9** (Correctness of MNRS). *Let $A_i$, $B_i$ $t$ and $\gamma$ be defined the same as in Lemma 8. $e(t) = \|\tilde{\Pi}(B_t - A_t)|\pi\rangle\| \leq \gamma$.*

*Proof.* Let $x_i = \|\tilde{\Pi}B_i|\pi\rangle\|$ and $y_i = \|\tilde{\Pi}A_i|\pi\rangle\|$. $A_i$ and $B_i$ simply apply a single iteration of amplitude amplification with state preparation operators $A_{i-1}, B_{i-1}$ respectively. Thus, from Section 1.3, its clear that $x_{i+1} = |T_3(x_i)|$ and $y_{i+1} = |T_3(y_i)|$ and indeed it can further be shown that $x_{i+1} = P(x_i) = 3x_i - 4x_i^3$ and $y_{i+1} = P(y_i) = 3y_i - 4y_i^3$. So

$$
\begin{aligned}
e(i+1) = \|x_{i+1} - y_{i+1}\| &= \|x_{i+1} - y_{i+1} + P(x_i) - P(x_i)\| \\
&\leq \|x_{i+1} - P(x_i)\| + \|P(x_i) - P(y_i)\| \\
&\leq 2\left|x_i\sqrt{1-x_i^2}\right|\beta_{i+1} + \sup_{z\in[-1,1]}|P'(z)|\ |x_i - y_i| \\
&= 2\left|x_i\sqrt{1-x_i^2}\right|\beta_{i+1} + 3e(i) \\
&\leq 2|x_i + y_i - y_i|\beta_{i+1} + 3e(i) \\
&\leq 2(|y_i| + e(i))\beta_{i+1} + 3e(i) \\
&\leq 2(\arcsin|y_i| + e(i))\beta_{i+1} + 3e(i)
\end{aligned} \tag{56}
$$

where we've used the fact that $\arcsin|x| \geq |x|$ for $\arcsin|x| \in [0, \pi]$. The use of arcsin in this context is natural since we know that $3\arcsin|y_i| = \arcsin|y_{i-1}|$, a fact that can be used to cancel the factor in front of $e(i)$. However, in other contexts, like fixed point search, it does not appear to naturally fall out of the computation. Define

$$
\begin{aligned}
\tilde{e}(0) &= 0 \\
\tilde{e}(i) &= 4\arcsin|y_{i-1}|\beta_i + 3e(i-1).
\end{aligned} \tag{57}
$$

Its not hard to prove by induction that $\tilde{e}(i) \geq e(i)$ so long as $\tilde{e}(i) \leq \arcsin|y_i|$. We can proceed to unravel $\tilde{e}(i)$ as follows

$$
\tilde{e}(i) = 4\sum_{j=1}^{i} 3^{i-j}\arcsin|y_{j-1}|\ \beta_j \tag{58}
$$

but $\arcsin|y_{j-1}| = 3^{j-i-1}\arcsin|y_i|$ so the above simplifies to

$$
\tilde{e}(i) = \frac{4}{3}\arcsin|y_i|\sum_{j=1}^{i}\beta_j = \frac{6\gamma}{\pi^3}\arcsin|y_i|\sum_{j=1}^{i}\frac{1}{j^2} \tag{59}
$$

and from the identity $\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{\pi^2}{6}$ we get $\tilde{e}(i) \leq \frac{\gamma}{\pi}\arcsin|y_i|$. This implies both that $\tilde{e}(i) \leq \arcsin|y_i|$ so $e_i \leq \tilde{e}(i)$ and since $\arcsin|y_t| \leq \frac{\pi}{2}$, $\tilde{e}(t) \leq \gamma$.

■

Much of the simplicity of the above proof is owed to some convenient properties of standard amplitude amplification. Importantly, the same phases $\varphi = \phi = \pi$ are used in every recursive level so the induced polynomial is not dependent on $i$, unlike the fixed point case, allowing for the simple Lipschitz constant of 3 to arise. Moreover, the induced polynomial has a convenient geometric interpretation which makes the unraveling of Eq. 56 an easy procedure. Surprisingly however, the procedure does not readily admit a bound for the amplitude of the unmarked states. Recall that the induced polynomial in this case is $\sqrt{1-x^2} \to \sqrt{1-x^2}|U_2(x)| = \sqrt{1-x^2}(4x^2-1) = 4(1-x^2)^{\frac{3}{2}} - 3\sqrt{1-x^2}$ whose Lipschitz constant with respect to $\sqrt{1-x^2}$ is 9. We can confirm this by checking that for any $\theta, \overline{\theta}$, $|\cos(3\theta) - \cos(3\overline{\theta})| \leq 9|\cos(\theta) - \cos(\overline{\theta})|$ with equality as $\theta \to \overline{\theta}$. The convenient cancellation we had earlier by introducing $\arcsin|y_i|$ is no longer sufficient and an open problem is whether a similar bound can be recovered.

# 5. A fixed point rendition

As a sanity check, we should verify that a fixed point rendition of recursive amplitude amplification exists. For this, we present Lemma 8.

**Lemma 10** (Optimality of recursive fixed point search). *For any error tolerance* $\varepsilon \in (0,1)$ *and* $t \geq \log_3\left(\frac{\log\left(\frac{2}{\varepsilon}\right)}{\sqrt{p_M}}\right)$, *define the phases*

$$\varphi_i = \phi_i = -2\cot^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\sqrt{1 - T_{\frac{1}{3}}\left(\frac{1}{\varepsilon_i}\right)^{-2}}\right) \tag{60}$$

*where* $\varepsilon_i$ *is given by the recurrence relation*

$$\varepsilon_i = \begin{cases} T_{\frac{1}{3}}\left(\frac{1}{\varepsilon_{i+1}}\right)^{-1} & \text{if } i < t \\ \varepsilon & i = t \end{cases} \tag{61}$$

*for* $i \in [t]$. *Let* $A_0 = I$ *and* $A_i = A_{i-1}R_{\varphi_i}A_{i-1}^{\dagger}\tilde{R}_{\phi_i}$. $\|\tilde{\Pi}A_t|\pi\rangle\| \geq \sqrt{1-\varepsilon^2}$.

*Proof.* We first prove that $\|(I-\tilde{\Pi})A_i|\pi\rangle\| = f_{3^i}\left(\sqrt{1-p_M}, \varepsilon_i\right)$. Note that the base case, where $i = 1$, follows from Lemma 1. Assume for induction that $(I-\tilde{\Pi})A_{i-1}|\pi\rangle = f_{3^{i-1}}\left(\sqrt{1-p_M}, \varepsilon_{i-1}\right)|t_{i-1}\rangle$, where $|t_{i-1}\rangle$ absorbs any phase imparted.

$$\|(I-\tilde{\Pi})A_i|\pi\rangle\| = \|(I-\tilde{\Pi})A_{i-1}R_{\varphi_i}A_{i-1}^{\dagger}\tilde{R}_{\phi_i}A_{i-1}|\pi\rangle\| \tag{62}$$

$$= f_3\left(f_{3^{i-1}}\left(\sqrt{1-p_M}, \varepsilon_{i-1}\right), \varepsilon_i\right) \text{ by Lemma 1} \tag{63}$$

$$\begin{aligned} &= \varepsilon_i T_3\left[T_{\frac{1}{3}}\left(\frac{1}{\varepsilon_i}\right)\varepsilon_{i-1}T_{3^{i-1}}\left(T_{\frac{1}{3^{i-1}}}\left(\frac{1}{\varepsilon_{i-1}}\right)\sqrt{1-p_M}\right)\right] \\ &= \varepsilon_i T_3\left[T_{3^{i-1}}\left(T_{\frac{1}{3^{i-1}}}\left(T_{\frac{1}{3}}\left(\frac{1}{\varepsilon_i}\right)\right)\sqrt{1-p_M}\right)\right] \text{ from } \varepsilon_{i-1} = T_{\frac{1}{3}}\left(\frac{1}{\varepsilon_i}\right)^{-1} \\ &= \varepsilon_i T_{3^i}\left(T_{\frac{1}{3^i}}\left(\frac{1}{\varepsilon_i}\right)\sqrt{1-p_M}\right) \text{ by the semi-goup property of } T_n \\ &= f_{3^i}\left(\sqrt{1-p_M}, \varepsilon_i\right). \end{aligned} \tag{64}$$

Thus

$$\begin{aligned} \|\tilde{\Pi}A_t|\pi\rangle\| &= \sqrt{1 - f_{3^t}\left(\sqrt{1-p_M}, \varepsilon\right)^2} \\ &= \sqrt{1 - f_n\left(\sqrt{1-p_M}, \varepsilon\right)^2} \end{aligned} \tag{65}$$

for $n \geq \frac{\log\left(\frac{2}{\varepsilon}\right)}{\sqrt{p_M}} \Rightarrow p_M \geq 1 - T_{\frac{1}{n}}\left(\frac{1}{\varepsilon}\right)^{-2}$ which is sufficient for $\sqrt{1 - f_n\left(\sqrt{p_M}, \varepsilon\right)^2} \geq \sqrt{1-\varepsilon^2}$.

■

As a fun little fact, which we avoid calling a Lemma, it's easy to show that for $\sqrt{1-y_i^2} = \|(I-\tilde{\Pi})A_i|\pi\rangle\|$, $\sqrt{1-y_i^2} \leq \varepsilon_i$. In the base case $i = t$ so this is trivially satisfied (otherwise we've chosen $t$ too small!) Now suppose $\sqrt{1-y_{i+1}^2} \leq \varepsilon_{i+1}$, then

$$\begin{aligned} \sqrt{1-y_{i+1}^2} &= \varepsilon_{i+1}T_3\left(T_{\frac{1}{3}}\left(\frac{1}{\varepsilon_{i+1}}\right)\sqrt{1-y_i^2}\right) \leq \varepsilon_{i+1} \\ &\Rightarrow T_3\left(\frac{\sqrt{1-y_i^2}}{\varepsilon_i}\right) \leq 1 \Rightarrow \sqrt{1-y_i^2} \leq \varepsilon_i \end{aligned} \tag{66}$$

where the last implication follows from $|T_n(x)| \leq 1 \iff |x| \leq 1$. This actually presents a novel approach to the problem of bounding the error between $B_i$ and the ideal case since in fixed point search, we are ultimately interested in suppressing $\sqrt{1-p_M}$ to below $\varepsilon$. Thus the approach would be to bound the distance between $\sqrt{1-x_i^2} = \|(I-\tilde{\Pi})B_i|\pi\rangle\|$ and $\varepsilon_i$. Nonetheless, we proceed with the usual, MNRS-based, approach.

If we want to be true to MNRS by mirroring the proof of Lemma 9, then we would make use of the Lipschitz constant of the polynomial $P$ induced on the marked component of $A_i|\pi\rangle$ to bound the error $e(i) = \|\tilde{\Pi}(B_i - A_i)|\pi\rangle\|$ . Unfortunately, Lemma 1 and Lemma 8 only focus on the polynomial transformation, $Q$, applied to the unmarked component. However, we know that $P$ and $Q$ are QSP achievable functions meaning they satisfy $|P(x)|^2 + (1-x^2)|Q(x)|^2 = 1$, a fact we can use to deduce the Lipschitz constant of $P$.

**Lemma 11** (Lipschitz constant for the marked state polynomial). *Let $P$ and $Q$ be QSP achievable polynomials of degree at most 3. Specifically, suppose*
*(1)* $\deg(P) = 3$, $\deg(Q) = 2$,
*(2) $P$ is odd and $Q$ is even,*
*(3)* $|P(x)|^2 + (1-x^2)|Q(x)|^2 = 1$.
*Further, let* $\sqrt{1-x^2}|Q(x)| = \varepsilon' T_3\left(\frac{\sqrt{1-x^2}}{\varepsilon}\right)$ *where* $\varepsilon = T_{\frac{1}{3}}\left(\frac{1}{\varepsilon'}\right)^{-1}$.

$$\sup_{z\in[-1,1]} |P'(x)| = \frac{\sqrt{48+24\varepsilon^2+9\varepsilon^4}}{4-3\varepsilon^2}. \tag{67}$$

*Proof.* Rearranging we get $\varepsilon' = T_3\left(\frac{1}{\varepsilon}\right)^{-1}$ so that $\sqrt{1-x^2}|Q(x)|$ expands to

$$\sqrt{1-x^2}|Q(x)| = \sqrt{1-x^2}\frac{4-3\varepsilon^2-4x^2}{4-3\varepsilon^2}. \tag{68}$$

Let $D = 4-3\varepsilon^2$.

$$1-(1-x^2)|Q(x)|^2 = \frac{1}{D^2}\left((8D+D^2)x^2 - 8(2+D)x^4 + 16x^4\right). \tag{69}$$

From conditions (1) and (2) we can write $P(x) = \frac{1}{D}(ax^3+bx)$ for $a, b \in \mathbb{C}$. Then from condition (3)

$$\begin{aligned} |P(x)|^2 &= \frac{1}{D^2}\left(|a|^2x^6 + 2\operatorname{Re}(a^*b)x^4 + |b|^2x^2\right) = 1-(1-x^2)|Q(x)|^2 \\ &\Rightarrow |a|^2 = 16, |b|^2 = 8D+D^2 \text{ and } \operatorname{Re}(a^*b) = -4(2+D). \end{aligned} \tag{70}$$

The $z \in [-1,1]$ that maximises $|P'(x)|$ must also maximise $|P'(x)|^2$.

$$|P'(x)|^2 = \frac{1}{D^2}(3a^*x^2+b^*)(3ax^2+b) = \frac{1}{D^2}(9|a|^2\ x^4 + 6\operatorname{Re}(a^*b)x^2 + |b|^2). \tag{71}$$

Since $|P'(x)|^2$ is an even polynomial whose leading coefficient is positive, its turning point must be a global minima, ensuring its maximum in the domain $z \in [-1,1]$ is at $z = \pm 1$. So

$$\sup_{z\in[-1,1]} |P'(x)| = \frac{1}{D}\sqrt{96-16D+D^2} = \frac{\sqrt{48+24\varepsilon^2+9\varepsilon^4}}{4-3\varepsilon^2}. \tag{72}$$

■

# 6. Appendix

**Proof of Fact 1.**

$$D_{x,y} = \langle x|D|y\rangle = \sqrt{\frac{\vec{\pi}_x}{\vec{\pi}_y}}\langle x|P|y\rangle = \sqrt{\frac{P_{y,x}}{P_{x,y}}}P_{x,y} = \sqrt{P_{x,y}P_{y,x}} \tag{73}$$

where the second last equality follows from the reversibility assumption.

**Proof of Fact 2.** First note that $|\pi\rangle$ is the eigenvector of $D$ with eigenvalue 1.

$$D|\pi\rangle = \mathrm{diag}(\vec{\pi})^{\frac{1}{2}}P\vec{1} = \mathrm{diag}(\vec{\pi})^{\frac{1}{2}}\vec{1} = |\pi\rangle \tag{74}$$

Because $D$ is symmetric and similar to $P$ it may be given the spectral decomposition

$$D = |\pi\rangle\langle\pi| + \sum_{i=1}^{|X|-1}\lambda_i|\psi_i\rangle\langle\psi_i| \tag{75}$$

where $|\lambda_i| < 1$. The right singular vectors of $D$ are the eigenvectors of $D^\dagger D$ and its left singular vectors are the eigenvectors of $DD^\dagger$ (see here for proof). Since $D^\dagger D = DD^\dagger = D^2$ and since the eigenvectors of $D^2$ are identical to $D$, the singular vectors and eigenvectors of $D$ coincide. The corresponding singular values are the eigenvalues of $\sqrt{D^\dagger D} = |D|$ and thus equal to $|\lambda_i|$.

**Proof of Fact 3.**

$$\begin{aligned}\left(|\bar{0}\rangle|x\rangle\langle\bar{0}|\langle x|\right)W\left(|\bar{0}\rangle|y\rangle\langle\bar{0}|\langle y|\right) &= \left(|\bar{0}\rangle|x\rangle\sum_{y'\in Y}\sqrt{P_{x,y'}}\langle y'|\langle x|\right)\left(\sum_{x'\in X}\sqrt{P_{y,x'}}|y\rangle|x'\rangle\langle\bar{0}|\langle y|\right)\\ &= |\bar{0}\rangle\langle\bar{0}|\otimes\sqrt{P_{x,y}P_{y,x}}|x\rangle\langle y| = |\bar{0}\rangle\langle\bar{0}|\otimes D_{x,y}|x\rangle\langle y|\end{aligned} \tag{76}$$

**Proof of Fact 4.**

$$\begin{aligned}W|\pi\rangle &= \sum_{x\in X}\sqrt{\vec{\pi}_x}W|\bar{0}\rangle|x\rangle = \sum_{x\in X}\sum_{y\in X}\sqrt{\vec{\pi}_xP_{x,y}}U_L^\dagger|x\rangle|y\rangle\\ &= \sum_{y\in X}\sqrt{\vec{\pi}_y}U_L^\dagger\sum_{x\in X}\sqrt{P_{y,x}}|x\rangle|y\rangle = \sum_{y\in X}\sqrt{\vec{\pi}_y}|\bar{0}\rangle|y\rangle = |\pi\rangle\end{aligned} \tag{77}$$

**Proof of Lemma 5** We'll use the convention $\prod_{i=1}^n A_i = A_1...A_n$, but it's easy to adapt the proof to the alternative. Let $U_i = R_{\varphi_i}(\beta)\tilde{R}_{\phi_i}$, $V_i = R_{\varphi_i}\tilde{R}_{\phi_i}$ and $|v\rangle \in \mathrm{span}\{|t\rangle, |t^\perp\rangle\}$. Note that $V_i|v\rangle \in \mathrm{span}\{|t\rangle, |t\rangle^\perp\}$ and, by the inductive hypothesis, $\left\|\left(\prod_{i=1}^{n-1}U_i - \prod_{i=1}^{n-1}V_i\right)|v\rangle\right\| \leq (n-1)\beta$.

$$\begin{aligned}\left\|\left(\prod_{i=1}^n U_i - \prod_{i=1}^n V_i\right)|v\rangle\right\| &= \left\|\left(\prod_{i=1}^n U_i - \prod_{i=1}^n V_i + \left(\prod_{i=1}^{n-1}U_i\right)V_n - \left(\prod_{i=1}^{n-1}U_i\right)V_n\right)|v\rangle\right\|\\ &\leq \left\|\left(\prod_{i=1}^n U_i - \left(\prod_{i=1}^{n-1}U_i\right)V_n\right)|v\rangle\right\| + \left\|\left(\left(\prod_{i=1}^{n-1}U_i\right)V_n - \prod_{i=1}^n V_i\right)|v\rangle\right\|\\ &= \left\|\left(\prod_{i=1}^{n-1}U_i\right)(U_n - V_n)|v\rangle\right\| + \left\|\left(\prod_{i=1}^{n-1}U_i - \prod_{i=1}^{n-1}V_i\right)V_n|v\rangle\right\|\\ &\leq n\beta\end{aligned} \tag{78}$$

**Proof of $\cos\left(2\cot^{-1}(x)\right) = \frac{x^2-1}{x^2+1}$.**

$$
\begin{aligned}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= \frac{\frac{\cos^2(\theta)-\sin^2(\theta)}{\sin^2(\theta)}}{\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2(\theta)}} \\
&= \frac{\cot^2(\theta)-1}{\cot^2(\theta)+1}.
\end{aligned} \tag{79}
$$

Setting $\theta = \cot^{-1}(x)$ completes the proof.