Sufficiency.

Q1 Let $x_1, x_2, \ldots x_n$ be a random sample from $N(\mu, \sigma^2)$ population. Find the sufficient estimators for μ and σ^2.

Solution: Here $\theta = (\mu, \sigma^2)$; $-\infty < \mu < \infty$
$0 < \sigma^2 < \infty$

We have $L = f(x_1, \theta_1) f(x_2, \theta_2) \ldots f(x_n, \theta_n)$

$$= \prod_{i=1}^{n} f(x_i, \theta_i) = \prod_{i=1}^{n} f_\theta(x_i)$$

(Since $x_1, x_2, \ldots x_n \sim N(\mu, \sigma^2)$)

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \cdots \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_n-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \cdot e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \cdots e^{\frac{-(x_n-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(x_i-\mu)^2\right\}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{\frac{-1}{2\sigma^2}\left\{\sum_i (x_i^2 - 2\mu \cdot x_i + \mu^2)\right\}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{\frac{-1}{2\sigma^2}\left\{\sum_i x_i^2 - 2\mu \cdot \sum_i x_i + n\mu^2\right\}}$$

$$= g_\theta[t(x)] \cdot h(x)$$

where $g_\theta[t(x)] = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\{t_2(x) - 2\mu t_1(x) + n\mu^2\}}$

$$t(x) = [t_1(x), t_2(x)] = [\Sigma x_i, \Sigma x_i^2]$$

and $h(x) = 1$.

Thus Σx_i is sufficient for μ and Σx_i^2 is sufficient for σ^2.

Q2 Let $x_1, x_2, \ldots x_n$ be a random sample from a population with p.d.f

$$f(x;\theta) = \theta x^{\theta-1} \quad ; \quad 0 < x < 1, \quad \theta > 0.$$

Show that $t_1 = \prod_{i=1}^{n} X_i$ is sufficient for θ.

Solution: We have $L = f(x_1,\theta).f(x_2,\theta) \cdots f(x_n,\theta)$.

$$L = \theta x_1^{\theta-1} \cdot \theta x_2^{\theta-1} \cdot \ldots \theta x_n^{\theta-1}$$

$$= \theta^n \cdot x_1^{\theta-1} \cdot x_2^{\theta-1} \cdot \ldots x_n^{\theta-1}$$

$$= \theta^n \cdot \prod_{i=1}^{n} x_i^{\theta-1}$$

$$= \theta^n \cdot \frac{\prod_{i=1}^{n} x_i^{\theta}}{\prod_{i=1}^{n} x_i}$$

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$$L = \theta^n \cdot \prod_{i=1}^{n} x_i^{\theta} \cdot \frac{1}{\prod_{i=1}^{n} x_i}$$

$$= g(t_1, \theta) \cdot h(x_1, x_2, \ldots x_n).$$

Hence by Factorization Theorem,

$t_1 = \prod_{i=1}^{n} x_i$ is sufficient for θ.

Q3. Let $x_1, x_2, \ldots x_n$ be a random sample of 'n' observations from Poisson population having parameter λ. Find a sufficient statistic for λ?

Solution: $P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$ $\quad x = 0, 1, 2, \ldots$
$\lambda > 0$.

$$L = P(x_1, \lambda) \cdot P(x_2, \lambda) \cdots P(x_n, \lambda).$$

$$= \frac{e^{-\lambda} \cdot \lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \cdots \frac{e^{-\lambda} \cdot \lambda^{x_n}}{x_n!}$$

$$= e^{-n\lambda} \cdot \frac{\lambda^{x_1 + x_2 + \cdots + x_n}}{x_1! \, x_2! \cdots x_n!}$$

$$= e^{-n\lambda} \cdot \frac{\lambda^{\sum_i x_i}}{\prod_{i=1}^{n} x_i!} = g(t, \lambda) \cdot h(x_1, x_2, \ldots x_n)$$

where $g(t, \lambda) = e^{-n\lambda} \cdot \lambda^{\sum x_i}$ and $h(x_1, x_2, \ldots x_n) = \frac{1}{\prod_{i=1}^{n} x_i!}$

Thus $t_1 = \Sigma x_i = n\bar{x}$ is a sufficient statistic for λ.

Q4. Let $x_1, x_2, \ldots x_n$ be a random sample of n observations from a distribution with p.d.f

$$f(x;\theta) = \frac{1}{\Gamma\theta} e^{-x} x^{\theta-1}, \quad 0 < x < \infty$$

Find a sufficient statistic for θ?

Solution: $L = f(x_1,\theta) \cdot f(x_2,\theta) \ldots f(x_n,\theta)$

$$= \frac{1}{\Gamma\theta} \cdot e^{-x_1} x_1^{\theta-1} \cdot \frac{1}{\Gamma\theta} e^{-x_2} \cdot x_2^{\theta-1} \ldots \frac{1}{\Gamma\theta} e^{-x_n} \cdot x_n^{\theta-1}$$

$$= \left(\frac{1}{\Gamma\theta}\right)^n \cdot e^{-\sum_i x_i} \prod_{i=1}^{n} x_i^{\theta-1}$$

$$= \frac{1}{(\Gamma\theta)^n} \cdot e^{-\sum_i x_i} \frac{\prod_{i=1}^{n} x_i^{\theta}}{\prod_{i=1}^{n} x_i}$$

$$= g(t_1,\theta) \cdot h(x_1, x_2, \ldots x_n)$$

where $g(t_1,\theta) = \frac{1}{(\Gamma\theta)^n} \cdot \prod_{i=1}^{n} x_i^{\theta}$

and $h(x_1, x_2, \ldots x_n) = \frac{e^{\sum_i x_i}}{\prod_{i=1}^{n} x_i}$

$\therefore \prod_{i=1}^{n} x_i$ is sufficient for θ.