SEM-1 L.A.L.P

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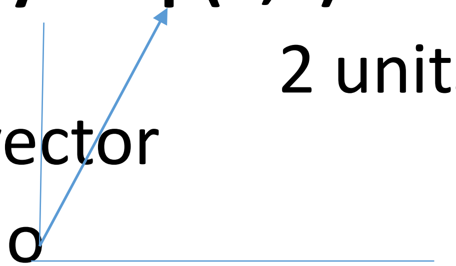
LALP.1

- Part 1 Quantities are of two types.
- 1 Scalar and 2 Vector
- 1 Scalar Quantities are the real values-- It has only magnitude. E.G. 4 , -8 , ½. Etc.
- 2 Vector Quantities are the entities with which two things are attached Magnitude and Direction.
- E.g. (2,3) , (1,0) , (0,1) they are in two dimensions; (x,y) If we express (2,3) by a single alphabet then we write it as bold value; say **a. a =** (2,3) or in practice we write as a **= (2,3)– a line segment on the top of letter a.**
- **y p(2,3)**
- 2 units on positive direction of x axis and 3 units on y axis. OP is a vector
- o x



LALP.2

- Point P is a vector point. Both the components (2,3) determine the position and the real value; its length from origin determines its magnitude.
- Components x =2 and y = 3 determine its direction and
- Real positive value = $\sqrt{2^2 + \ 3^2}$ = √(4 + 9) = √13 is its magnitude– this stands for its distance from origin.
- Vector OP = (2,3), x= 2 and y =3 determine the direction and its magnitude denoted as |op|= √13

- In the same way we can introduce vector in three dimension.
- $\overrightarrow{OP}$ = (-2, 4. 5), x = -2, y = 4 , and z = 5 determine the vector point in three dimension while its magnitude = $\sqrt{-2^2 + \ 4^2 + 5^2}$ = √45 = |OP| = magnitude OP = its distance from origin

LALP-3

- Part 2
- Unit vector: A vector whose lenth i.e. its distance from origin is one unit is a unit vector.
- i.e for **a =** (x,y), |**a| =1 ;** i.e $\sqrt{x^2 + y^2}$ =1 is a unit vector.
- E.g. **i = (1,0) =** $\sqrt{1^2 + 0^2}$ =1
- Also **j** = (0,1) = $\sqrt{0^2 + 1^2}$ = 1 We have I and j as unit vectors.
- Now vector OP = (2, 3) can also be witten as , **OP= 2i + 3j = 2 (1,0) + 3 (0,1)**
- **Also for** a = (x, y ,z) , then |a|=$\sqrt{x^2 + y^2 + z^2}$ and x, y and z determine the direction.
- a = (-1,2,3) then |a| = $\sqrt{14}$.

LALP-4

- (1, 0, 0) denoted as **i so |i| =1 unit vector in the direction of x- axis**
- (0, 1, 0) denoted as **j so |j| =1 unit vector in the direction of y- axis**
- (0, 0, 1) denoted as **k so |k| =1 unit vector in the direction of z- axis**
- **Vector p = (2, 4 , -2) = 2i + 4 j -2k . |p| =** $\sqrt{2^2+4^2+(-2)^2}$ = √24 = 2 √6

- PART -3 R = Set OF REAL numbers
- R^2 = { (x,y) | x and y are real numbers. } = two dimensional space
- R^3 = { (x,y, z) | x and y, and z are real numbers. } = three dimensional space

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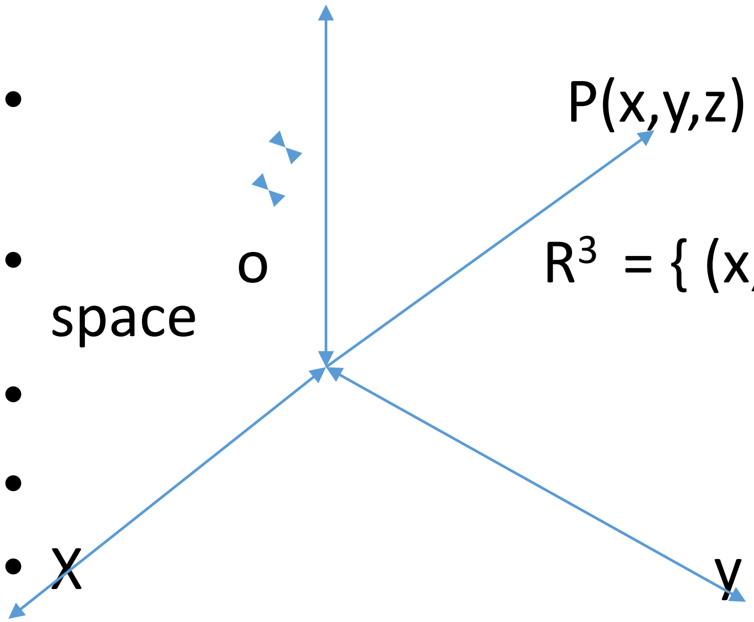
LALP-5

- R^2 = two dimensional space { (x,y) | x and y are real numbers. }
- y
- j =(0,1) a = (x,y)
- -x o i = (1,0) x |i|= |j| =1, I and j are unit vectors
-
- -y

LALP -6

- z
- P(x,y,z)
- o R^3 = { (x,y, z) | x and y, and z are real numbers. } = three dimensional space



- a =(x, y , z) vector OP,,
- magnitude of vector OP = |a|=$\sqrt{x^2+y^2+z^2}$
- X y
- (1, 0, 0) denoted as **i so |i| =1 unit vector in the direction of x- axis**
- (0, 1, 0) denoted as **j so |j| =1 unit vector in the direction of y- axis**
- (0, 0, 1) denoted as **k so |k| =1 unit vector in the direction of z- axis**

LALP-7

- Fundamentals:
- **1** Equality of two vectors of the same space, If P = (x, y, z) Q = (a, b, c)
- P = Q $\Longleftrightarrow$ corresponding components are equal.
- i.e. x = a, y = b , and z = c and vice- versa
- **2** Addition of two vectors. : Addition of two vectors of the same space is also a vector.
- P = (x, y, z), Q = (a, b, c) are the two ve ctors of R^3 then their addition is also a vector.
- **P + Q = R = (x +a, y +b, z +c)**
- **3** Multiplication by a scalar: Multilication of a vector y a scalar results in to a vector. For a eal value **c, and a vector p = (x,y, z) , it is denoted as**
- **Cp= c (x,y,z) = (cx, cy, cz) e.g 5(-1, 2, 14) = (-5, 10, 70)**

LALP-8

- Case: For x = (a,b) in R^2 or Y =(a,b,c) in R^3 if the scalar; say k = 0 then
- Kx = k(a,b) = 0(a,b) = (0,0) this is called a null vector denoted as **O or Ɵ.**
- **Ɵ = (0,0)----- a null vector with magnitude = 0**
- Ky= k(a,b, c) = 0(a,b,c) = (0,0, 0) this is called a null vector denoted as **O or Ɵ.**
- **Ɵ = (0, 0, 0)----- a null vector in** R^3 **with magnitude = 0**
-
-

1. LALP-9

- Assignment -1A
- Ex-1 for x = (2,3,5) and y = (-1,2,4) find x+ y = (1,5, 9),$\sqrt{1} + 25 + 81 = \sqrt{107}$ $2x + 3y = 2(2,3,5) - 3(-1,2,4) = (4,6,10) + (3,-6,-12)$
- = (7, 0,-2) = 2x -3y
- 1 x + y, 2 |x + y| 3 2x – 3y 4 |2x -3y|
- Ex-2 If x = (2,4, t) and y = (h, f, 5) then x – 2y to be a null vector find the values of h, f, and t. (x-2y)=(2-2h, 4-2f, t-10) = (0,0,0)
- Ex-3 If x = (2,4, t) is a unit vector then find the value of ‘ t’.
- Ex-4 For x = (2,3,5) and y = (-1,2,4) find 2x – 3y and 3x-2y.
- Ex-5 for x = (4, -2), y = (-4,2) is |x| + |y| = |x + y|? X+y = (0,0)

LALP-10

- Part -4
- Binary operation and notion of Group
- Let G be a non empty Set. G ≠ϕ {ϕ is a null set in which there is no member.
- Let us define + an addition--- an operation (process) on the members of G.
- For any two members' say   a and b of G, th result of addition process
- a + b is also a member of the set G; i.e a + b $\in G$ *the operation is a binary operation on the set* G.

LALP-11

- The set R of real numbers and the continuing on the set R^2, R^3 also follow some important properties that gives a structure of , what we call
- **‘ vector space’**
- **1 Non empty set. 2 Two binary operations defined– generally + and .**
- **3 + is commutative 4 + is associative, 5 existence of additive identity**
- **7 existence of additive inverse**
- **8 ‘.’ Is associative**
- **9 existence of multiplicative identity**
- **10 doubly distributive laws are satisfied a.(b+c) =a.b + a.c**
- **(b +c).a =b.a +c.a**
- **These are some important properties of vector space R,** R^2, and R^3
-

LALP-12

	R	R^2	R^3
•	R	R^2	R^3
• Identity	0	(0,0)	(0,0,0)
• + Inverse of x,(a,b), (a,b,c)	-x	(-a, -b)	(-a, -b, -c)
binary for +, . ,	yes	yes	yes
commutative	+ and .	Yes	yes
	not for – and ÷	do	do

distributive law a.(b + c) = a.b + a.c , left distributive,

(b + c).a = b.a + c.a right distributi0 ve law

If p and q ≠ 0 then inverse of p/q is q/p

LALP-13

- Some examples:
- **1** consider the set N. ----- set of positive integers.
- The operations + and ‘ .’ Multiplication are binary operations.
- The operation $\div$ *is not a binary operation*.
- **2** Consider the set Z of negative integers, zero and positive integers is
- a binary operation for +, -, ‘.’ and division (except division by zero.)
- **3** The same logic continues for the set **Q of rational numbers and the set R of real numbers follow** binary process for all such standard operations too.
- **4 Associative property basically involves three elements of the same set.**
- E.g for three elements of the set N ; we have a +(b +c) = (a +b) +c which is true. E.g. 2 + (9 + 7) = (2+9) +7 ; both sides gives same result.
- Also 2.(9 .7) = (2.9) .7 it is for multiplication that the result holds true

LALP 14

- It hold for subtraction also.
- Facts:
- 1 Technically a vector is written as a column vector.
- E.g. x = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{3} \\ \mathbf{14} \end{pmatrix}$ **Just in order to save space we write it in a row form.**
- **2 i = (1,0) we** normally write it but it actually mean that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- **j = (0,1) we** normally write it but it actually mean that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

LALP 15

- **i = (1,0,0) we** normally write it but it actually mean that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- **j = (0,1,0) we** normally write it but it actually mean that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- **K= (0,0,1) we** normally write it but it actually mean that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- **these are unit vectors; |i|= |j| = |k| =1**

LALP 16

- **Fact: 1**
- Any vector of R^2 R^3 an be written as linear combination of unit vectors i and j (i ,j, and k).
- E,g. x = $\begin{pmatrix} 2 \\ -3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$= 2i -3j = $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ + $\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$
- y = $\begin{pmatrix} 5 \\ 7 \\ 1/2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ +1/2 $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ = 5i + 7j +(1/2) k
- **Fact 2:** If x and y are vectors of the same space then for some real values c1 and c2, c1 x + c2 y is called a linear combination of vectors x and y. The result is also a vector of the same space.
- This can be extended to many vectors of the same space.

LALP-17

- C1 x + c2 y = 5 $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ -4 $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$ = $\begin{pmatrix} 10 \\ 15 \end{pmatrix} + \begin{pmatrix} 4 \\ -24 \end{pmatrix} = \begin{pmatrix} 14 \\ -11 \end{pmatrix}$
- The vector $\begin{pmatrix} 14 \\ -11 \end{pmatrix}$ is a linear combination of x =$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and y = $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$
- Say the vectors are x1= $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, x2 = $\begin{pmatrix} 7 \\ 1/2 \end{pmatrix}$, and x3= $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ then for c1=5,
- C2 = 2, and c3 = -4 their linear combination is 5x1 + 2x2 – 4x3 =
- $\begin{pmatrix} 10 \\ 15 \end{pmatrix} + \begin{pmatrix} 14 \\ 1 \end{pmatrix}$ + $\begin{pmatrix} -24 \\ 12 \end{pmatrix}$ = $\begin{pmatrix} 0 \\ 28 \end{pmatrix}$ this is a linear combination of vectors x1 , x2, and x3,

LALP-18

- Assignment -1b
- 1 For the vectors x1 $=\begin{pmatrix}2\\3\end{pmatrix}$ $and\ x2 = \begin{pmatrix}12\\18\end{pmatrix}$, find 6x1 –x2
- 2 If x1 $=\begin{pmatrix}-2\\k\end{pmatrix}$ and x2 = $\begin{pmatrix}p\\4\end{pmatrix}$ $and\ 4x1 + 2x2 = \begin{pmatrix}2\\3\end{pmatrix}$ then find the values of p and k.

$$\begin{pmatrix}-8\\4k\end{pmatrix}+\begin{pmatrix}2p\\8\end{pmatrix} = \begin{pmatrix}-8+2p\\4k+8\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

- -8+2p = 2, 4k + 8 = 3
- 3 If x1= $\begin{pmatrix}5\\1\\8\end{pmatrix}$ and x2 = $\begin{pmatrix}-5\\7\\5\end{pmatrix}$ then find 3x1 -5x2
- 4 If x1= $\begin{pmatrix}4\\-k\\8\end{pmatrix}$ and x2 = $\begin{pmatrix}-5\\7\\p\end{pmatrix}$ then find 3x1 -5x2 = $\begin{pmatrix}37\\1\\8\end{pmatrix}$ then find the values of k and p.

LALP-19

- Fact: Any vector of $R^{2,}$ R^3 is a linear combination of basic vectors i , j, and I,j k
- E.g. $\binom{5}{8} = 5\binom{1}{0}$ + 8 $\binom{0}{1}$ = 5 i + 8 j where |i | = |j| = 1 , I and j are unit vectors in x and y directions. These vectors are linearly independent.
- **These are the basic vectors and they form standard Basis of R^2**
- c1 i + c2 j = $c1\binom{1}{0}$ + c2$\binom{0}{1}$ = $\binom{c1}{0}$ + $\binom{0}{c2}$ = $\binom{c1}{c2}$ thi s is a linear combination of the vectors I and j.
- If we wish $\binom{c1}{c2}$ = $\binom{0}{0}$ then we have to take c1 = c2 = 0 or else it is not possible.
- In the same way standard basis of R^3 is the set of vectors I , j , and k.
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LALP-20

Important concept:

For the vectors x1, x2, x3,…… of the same space, R^2 , $R^{3,}$ R^4 …and the real constant c1,c2,c3 ….

X = c1x1+ c2x2 + c3x3+……….is a linear combination of x1, x2, x3,…..

Note that it is one vector X of the same space from which vectors are taken.

If we wish X = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or X = =$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e a null vector for which all **c1, c2, c3**

(1) are zero then x1, x2, x3,……..are linearly independent.

(2) some of c1, c2 , c3 ae not zero then the vectors are linearly Dependent.

- Some examples:
- **1** x1 = $\begin{pmatrix}2\\3\end{pmatrix}$ $x2$ = $\begin{pmatrix}5\\4\end{pmatrix}$ then we want c1x1 + c2x2= c1$\begin{pmatrix}2\\3\end{pmatrix}$ +c2 $\begin{pmatrix}5\\4\end{pmatrix}$ =$\begin{pmatrix}2c1+5c2\\3c1+4\ c2\end{pmatrix}$is a linear combination of x1 and x2. If we want this combination = $\begin{pmatrix}0\\0\end{pmatrix}$ then we must have 2c1 +5c2 = 0 and 3c1 + 4c2 =0 so we have, on solving these, c1 = -5/2 c2 and c1 = -4/3 c2,
- the vectors x1 and x2 are linealy independent.[c1 = -5/2 c2 and c1 = -4/3 c2, not possible]
- **2** x1 = $\begin{pmatrix}2\\3\end{pmatrix}$ and x2 = $\begin{pmatrix}6\\9\end{pmatrix}$ then c1 x1 + c2 x2 = $\begin{pmatrix}2c1+6c2\\3c1+9c2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$will give
- 2c1 + 6c2 =0 and 3c1 + 9c2 = 0 gives c1 = -3c2 from both; it means that there are many values of c1 and c2 for which these equation will be satisfied. These vectors are linearly dependent.

LALP -22

- Example-2 x1 = $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, x2 = $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and x3 = $\begin{pmatrix} 8 \\ 10 \end{pmatrix}$ are they lin. Dep.?
- Linear combination = c1x1+ c2x2 + c3x3 = $\begin{pmatrix} c1 \\ 4c1 \end{pmatrix}$ + $\begin{pmatrix} -2c2 \\ 5c2 \end{pmatrix}$ + $\begin{pmatrix} 8c3 \\ 10c3 \end{pmatrix}$ = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- It means c1-2c2 +8c3 =0 and 4c1+5c2 + 10c3 =0 ,
- From these two equations find c1.
- c1 = 2c2 - 8c3 = - 5/4 c2 -10/4 c3, so we have c1 = 8c2-32c3 = -5c2 -10c3
- So it gives 13c2 =22c3 . They are equal if c2 = 22 and c3 = 13
- c1 =-240 . means all constants c1, c2, and c3 are non zero . So they are L.D
- [**more than two different vectors of R^2 are dependent]**

- Example-4
- Let x1 = $\begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix}$, x2 = $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$, x3 = $\begin{pmatrix} 7 \\ -2 \\ 10 \end{pmatrix}$
- Linear combination = c1x1 + c2x2 + c3x3 = $\begin{pmatrix} 4c1 + 3c2 + 7c3 \\ -7c1 + 5c2 - 2c3 \\ 3c1 + 7c2 + 10c3 \end{pmatrix}$ = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- 4c1 = -3c2 -7c3 from first,
- -7c1 = -5c2 +2c3 from second, and
- 3c1 = -7c2 -10c3 from third one if we add all then we get 0 = -15c2 - 15c3
- It implies 15c2 = -15 c3 means c2 = - c3, we can take say c2 = 1 and c3 =-1 then we get c1= 1.
- i.e c1, c2, and c3 they are not all zero. Hence they are L.D

- EXAMPLE-5 WE ARE GIVEN x1 = $\begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix}$, x1 = $\begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$, AND x1 = $\begin{pmatrix} 4 \\ -7 \\ 0 \end{pmatrix}$,
- LINEAR COMBINATION IS C1X1+C2X2 = X3
- C1x1 +C2X2 = $\begin{pmatrix} 4C1 \\ -7C1 \\ 3C1 \end{pmatrix} + \begin{pmatrix} 2C2 \\ -7C2 \\ 1C2 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 0 \end{pmatrix}$, We can solve first two equations and get the values of c1 and c2.
- Do these values satisfy the equation 3c1 +c2 =0 ? If yes then such combination is possible.

LALP-25

- Assignment -3
- (a) Explain linear dependent and independent vectors and check the following vectors for dependency
- (a.1) x1 = (2, 3), and x2 = (4,6) (a.2) x1 = (-1, 5) and x2 = (7, 4)
- (a.3) y1 = (1,0) and y2 =(0,1) (a.4) x1 = (4,2,2) and x2 = (4, 2,1)
- { Note: If the vectors are linearly dependent the any one of them is a linear combination of the remaining vecors; if not possible then they are linearly independent}
- (b) check for dependency
- (b-1) x1 = (2,3,4), x2 = (3,4,5) and x3 = (5, 7, 9)
- (b.2) x1 = (7, 1, 0) x2 = (4, 6, 8) and x3 = (10, 7, 7)

LALP-26

- Transformations: We have the vector space R, R^2, and R^3. Our idea is to move from one space to another space or mutual travel within the space. This journey is called ‘Transformation’. We, can, using proper tool of function, move from R to R^2, and R^3. Also we can, using proper tool , can go come back from one space other higher spaces also.
- E.g. $T : R \to R$, $T : R \to R^2$, $T : R \to R^3$,
- $T : R^2 \to R^3$, $T : R^3 \to R^3$, $T : R^3 \to R^2$ etc.
- But moving must conform from the laws of one space to where we want to go.
- E.g. $T : R^2 \to R^3$. WE know that in r^2 there are ordered pairs like (x,y) and in R^3 there are ordered triplets like (x,y,z). Also do not forget that members of R^2, and R^3 are column vectors but we write it as rowwise.
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LALP-27

- $T : R^2 \rightarrow R^3$, For (a,b) in R^2 thee result of application of T, written as T(,b)
- We design a rule, (GHAR KA RULE), say T (a,b) = (a+b, a-b, a+2b). Let us understand it by an example.
- Say (a,b) = (2, 3) then T(2,3) = (2+3, 2-3, 2+ 2.(3)) = (5,-1,8)
- So under the said transformation, the vector (2,3) of R^2 is transferred to the vector (5,-1,8)
- Some more illustrations; 1 T: $R^2 \rightarrow R^2$, for (a,b) of R^2 T (a,b) = (a-b, a)
- Then t(3, -5) = (3-(-5), 3) = (8,3)
- 2 T: $R^3 \rightarrow R^2$ for (a,b,c) of R^3 T(a,b,c)= (a+c, b-c) ,T(2,-4, 3)= (2+3, 4-3)= (5,1)
- 3 T: R^3 R^2, for (a,b,c) of R^3 T(a,b,c)= (a^2 , c+ b^2) ,T(2,-4, 3)= (4, 3+16)=
- (4,19)

LALP-28

- 5 T: R to R^2, for p $\in R, T(p) = (\,2p, 3p)$
- e.g. say we take p = 5 and so T(5) = (10, 15)
- 6 T : R^2 to R, such that for T(x,y) = $\sqrt{x^2 + y^2}$, T(3,4) = $\sqrt{3^2 + 4^2}$ =5

- For X1, and x2 of R^2, say x1 = (2,3) and x2 =(-1,5) , x1+ x2 = (2,3) + (-1,5)
- = (1, 8). Let T(a,b) = (a+2b, b+2a)
- Then T(2,3) = T (x1) = (8, 7),
- T (-1,5) = T(x2) =(9,3),
- and T(1, 8) = T (x1 + x2) = (17, 10)
- SO WE HAVE T (x1 + x2) = (17, 10) = T (x1) + T(x2) = (8,7) + (9,3)

LALP PART -2

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LALP PART 2.1

- Subspace:
- Let V be space of a vector space. Let + and . Be the two binary operations defined on it. Let V be defined on the set R of real numbers. Let S be a subset of V. If the set S itself is a vector space itself. In this case the set 'S' is called a subspace of V.
- A non-empty subset S of the vector space V. is a subspace of V if and only if the following conditions are satisfied. (1) for x, and y of S , x + y is also in S. and (2) for x of S, and $\propto$ *any real number* , *we have* $\propto$x also in S.
- The set S = {ϴ } and S = V, both are also subspace of V. These are called improper subspace of V.
- Illustation-1 Let U1= { (a1,a2,a3)| a1-2a2+a3 = 0 }, is it a subpaceof R^3 ?
- a1, a2, and a3 are real numbers.

LALP PART 2.3

Facts:

1 That two vectors of R^2 can be linearly independent or dependent. This depends on condition of independency or dependency. S1= { (1,2), (3,4) }, S2 ={(1,0), (0,1)}, S3 = { (3,-4), (2,6), S4 ={ (5, 15), (1,3)}, which are L.D ?

2 More than two diferent vectors of R^2 are linearly dependent if any two of them are linearly independent. E.g. { (1,2), (3,4), (1,7)} are linearly dependent as any two of them are linearly independent. S2 = { (3,4) ,(2,4) , (5,-7)} are linearly dependent[(3,4) = c1(2,4) + c2 (5,-7) ; we can find c1 and c2 which are other than zero.

3 In the same way we can state for the vectors of R^3.

4 A Basis is a set of basic vectors. Standard basic vectors are { (1,0), (0,1)}

5 (6,8) = 6(1,0) + 8 (0,1)= 6i + 8j, |i| = |j| =1

LALP PART 2.2

- For x1, and x2 of u1 and $\propto$ and β numbers linear combination
- $\propto$x1 + βx2 = {$\propto$ (a1,a2,a3) + β(y1, y2,y3)} has property that
- $\propto$ (a1-2a2+a3) =0 and β (y1- 2y2+y3) = 0 given property
- {$\propto$ (a1,a2,a3) + β(y1, y2,y3)} = { ($\propto$a1 ,$\propto$ a2, $\propto$a3) + (βy1 , βy2, βy3)}
- { ($\propto$a1 + βy1, $\propto$ a2+ βy2 , $\propto$a3 + βy3)} has the same poperty that
- $\propto$a1 + βy1 – 2 ($\propto$ a2+ βy2) + $\propto$a3 + βy3 =0 whch is always true as it can be written as $\propto$ (a1-2a2+a3) + β (y1- 2y2+y3) = $\propto$(0) + β(0) = 0 +0 = 0
- Hence the set u1 with the given poperty is a subspace of R^3 = 0

LALP PART 2.4

- Facts
- **4:** Any two **L. independent** vectors of R^2 form a **basis** of R^2 Any other vector of
- R^2 can be represented by linear combination of these two vectors.
- S1 = {(2,5), (1,4)} ; these are linearly indep. Say there is a vector (-2,7) of R^2
- Now, (-2,7) = c1(2,5) + c2(1,4) and so -2 = 2c1 + 1c2, 7 = 5c1 + 4c2, We can solve these two equations for non zero c1 and c2.
- It means that any vector of R^2 can be constructed with these two vectors of S1.
- S1 is a basis. The set , denoted as **B1=** { (1,0) , (0,1)} is a basis– standard basis of $R^{2.}$
- **5 : Linear Span: The set of all possible combinations like** $\propto$x1 + βx2 or
- $\propto$x + βy for $\propto$, and β *as real numbers is called a* ***linear span*****.**
- **This is denoted as [s].**
-

LALP 2.5

- Vectors of the basis are called **basic vectors. One** can obtain any vector of R^2 or R^3
- **B2=** { (1,0,0) , (0,1,0), (0,0,1)} is a standard basis of R^3.
- (p,q,r) = p (1,0,0) + q (0,1,0) + r(0,0,1) for real values p,q and r.
- Remember that vectors of basis are
- (1) linearly independent
- (2) They span the given vector space.
- Example: Show that the vector (1,2,3)belongs to [(2,1,0),(3,0,1), (-2,5,0)]
- But does not belong to {(2,1,0),(-2,-1,0),(-2,5,0)}
- (1,2,3) ∈ $\alpha(2,1,0)$ +$\beta(3,0,1)$ +$\gamma(-2,5,0)$; find α, β, and γ [α =9/2, $\beta = -3$, and γ =-1/2]

LALP 2.6

- Prove that (1,2,3) $\notin \propto(2,1,0) + \beta(-2, -1,0) + \gamma(2,5,0)$
- [D = $\begin{vmatrix} 2 & -2 & 2 \\ 1 & -1 & 5 \\ 0 & 0 & 0 \end{vmatrix}$ = 0 implies that the vectors are L.D. They cannot span the space.]
- A non-empty subset B of a vector space V is a basis of V if each vector of V can be uniquely expressed as a linear combination of vectors of V in a unique way.
- A subset B1 of vector space V, which properly contains B, then vectors of B1 are linearly dependent.
- Say B = { (1,4), (-2,5)} ---- Is it a basis? Linearly ind. And span R^2.
- B1 = { (1,4), (2,5), (5,-2)} i.e.
- B $\sqsubset B1$ $It\ means\ that\ vectors\ of\ B1\ are\ l.d.$

LALP 2.7

- Dimension:
- Dimension of a vector space is the number that shows number of vectors in basis. Basis of R^2 has two vectors in the basis and hence dimension of R^2 is 2. In the same way dimension of R^3 is = 3.
- These are finite dimensional vector spaces.
- Is the set { (1,4), (2,5), (1,7),(2,8)} of vectors linearly dependent? Explain.
- The set of all 2x2 matrices has the rank 4.

- Basis of M2, the set of all 2x2 matrices has the basis = { $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$}
- Any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a linear combination of vectors in the basis.

- a $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ + b $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ + c $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ + d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

(,

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LALP 2.8

- In the same way the vector space M_3 of all 3x3 matrices have 9 elements(matrices) in its **basis**. E.g. the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- Out of 9 entries one is 1 and others are zero. Any matrix of the space M_3 is a linear combination of all this nine matrices of the basis.
- We can continue for higher dimensional spaces.

LALP 2.9

- Back to L.T.
- A transformation from one vector space U to the another vector space V, denoted as T: U V is called linear if for
- X1 and x2 as members of U and c1 and c2 as constants, we have either
- 1 T(x1 + x2) = T(x1) + T (x2)
- 2 T (c1x1) = c1 T(x1)
- or
- T(c1x1 + c2x2) = c1T(x1) + c2 T(x2)
- Illustration: Let T: R^2 R^2 so that for (a,b) of R^2 T(a,b) = (2a +b, a-b)

LALP 2.10

- Let a = (x1,x2) = (2,5) and so T(a) = T(x1,x2) = T(2.2 +5, 2-5) = **(9,-3)**
- Let b = (y1,y2) = (-2, 6) and so T(b) = T(y1,y2) =T(2. -2 +6 , -2-6) = **(2,-8**)
- **T(a) + T(b) = (9,-3)+(2,-8) = (11, -11)**
- T(a +b) = T((x1,x2) + (y1, y2)) = T ((2,5) + (-2,6)) = T(0,11) = (2.0+11, 0-11)
- **T(a + b) = (11,-11)**
- T(a+b) = T(a) + T(b) ………………..(1)
- For some real value c let us find T(ca) = T(c(x1, x2)) = T(c(2,5)) = T(2c, 5c)
- = (2. (2c) + 5c, 2c- 5c) = (9c, -3c) = c(9, -3) = c T(a)
- ∴ T(ca) = c T(a) ……………………..(2)
- ∴ It is a linear transformation.

LALP 2.11

- 1 Consider $T : R^2 \rightarrow R^2$ defined as $T(x, y) = (y, x)$
- a= (x1,y1) and b = (x2,y2), ∴ T(a) = T(x1,y1) = (y1,x1),
- T(b) = T (x2,y2) = (y2,x2)
- **∴ T(a) + T(b) = (y1,x1) + (y2 , x2) = (y1+y2, x1+x2) …………… (1)**
- We find **T(a+b) = T((x1,y1)+ (x2,y2)) = T(x1+x2, y1+y2) = (y1+y2, x1+x2)…..(2)**
- **For some c, T(ca) = T(c(x1,y1)0 = T (cx1, cy1) = (cy1. cx1) = c(y1,x1) = cT(a)**
- **∴ T(ca) = c(T(a) ..(3)**
- **∴ It is a linear transformation. It is reflection about the line y = x.**
- **In the same way T(x,y) = (y, -x) is a reflexion about y = -x**

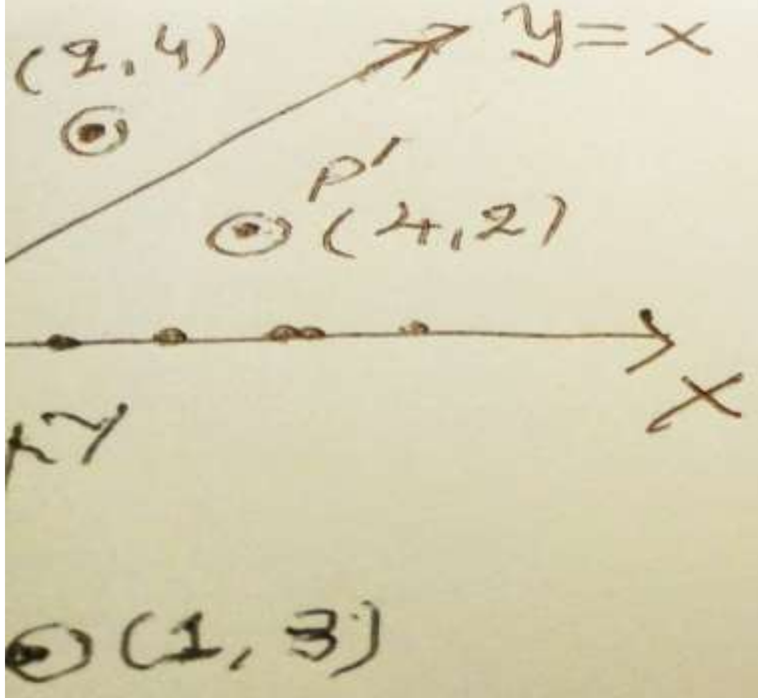
Lalp2.12

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$; $T: R^2 \to R^2$.

Prove that it is a L. T.

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ show that (1) $T(a+b) = T(a) + T(b)$

(2) $T(Ka) = K\,T(a)$

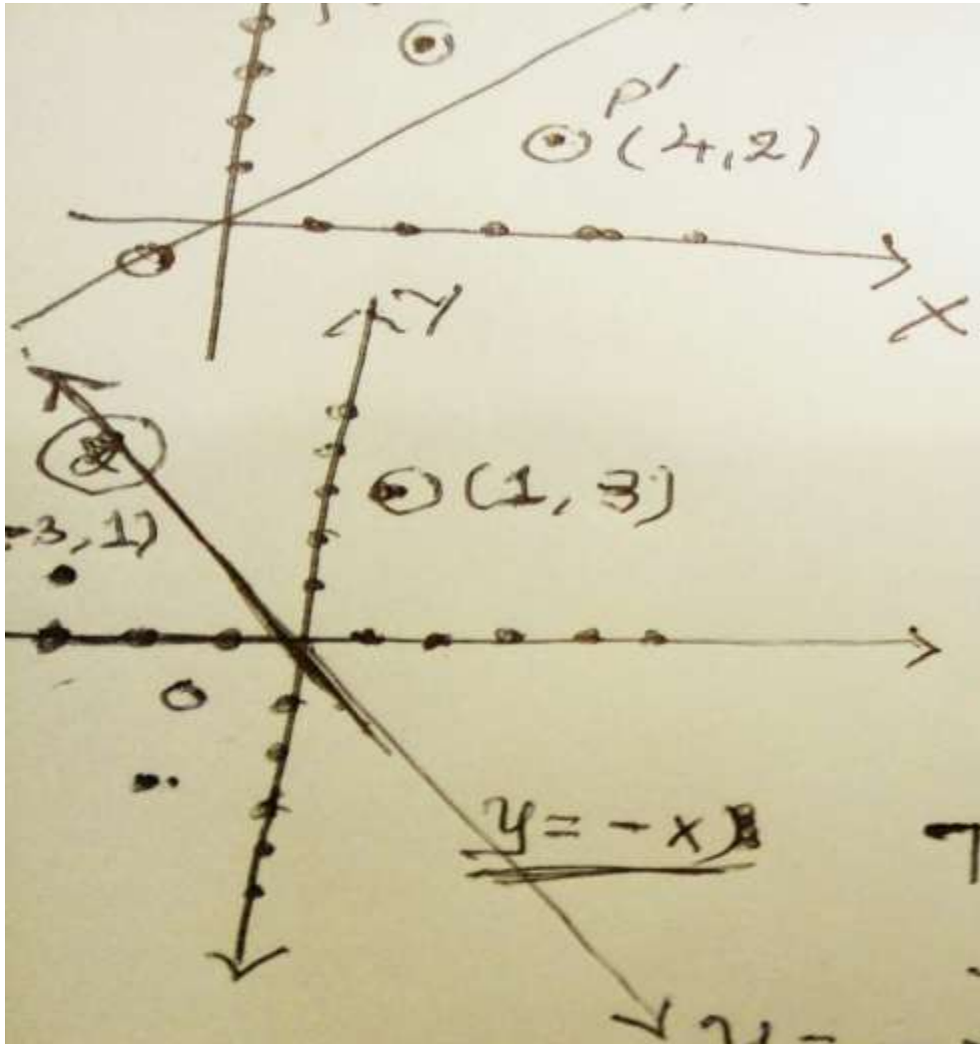


$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ is Reflexion about $y = x$.

$\boxed{T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}}$ $T: R^2 \to$

$\bar{a} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$(x_1 + y_1$



$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ is Reflexion about $y = x$

$\boxed{T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}}$ $T: R^2 \to$

$\bar{a} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \bar{b} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

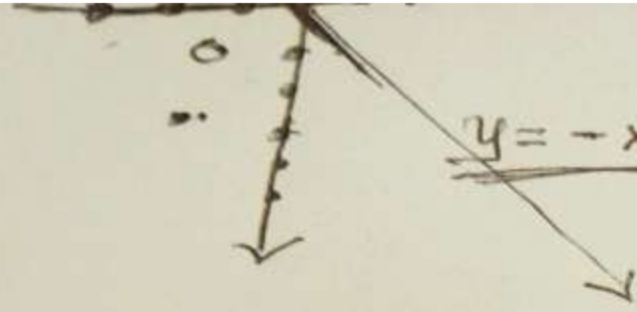
$\bar{a} + \bar{b} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$T(\bar{a}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ $T(\bar{b}) = (-$

$T(a+b) = \begin{pmatrix} -(x_2 + y_2) \\ x_1 + y_1 \end{pmatrix}$

$\therefore T(\bar{a} + \bar{b}) = T(\bar{a}) + T(\bar{b}) \Big\} \Rightarrow$ L.T

Also $T(ka) = k\,T(a)$



$a+b = \begin{pmatrix} \cdots \\ x_2+y_2 \end{pmatrix}$

$y=-x$ $\quad T(\bar{a}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad T(\bar{b}) = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$

$y=-x$ $\quad T(a+b) = \begin{pmatrix} -(x_2+y_2) \\ x_1+y_1 \end{pmatrix}$

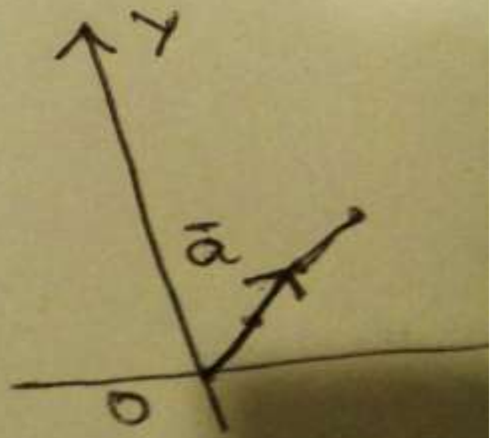
$$\left.\begin{aligned} &\therefore\ T(\bar{a}+\bar{b}) = T(\bar{a})+T(\bar{b}) \\ &\text{Also}\ \ T(ka) = K\,T(a) \end{aligned}\right\} \Rightarrow \text{L.T.}$$

(2) $T(\bar{a}) = T(x,y) = (\alpha x, \alpha y)$ $\quad \alpha \in R$

$\alpha = 0, \Rightarrow$ Null vector.

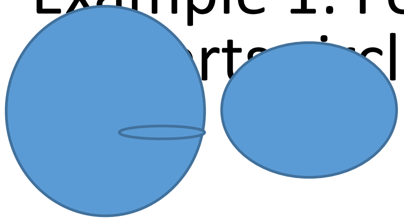
$|\alpha| < 1 \Rightarrow$ Contraction.

$|\alpha| > 1 \Rightarrow$ Magnification



LALP 2.15

- of L.T
- Properties
- Example 1: For 0 < b< a, T(a1,a2) = (a1, (b/a)a2) is a linear transformation which converts circle $x^2 + y^2 = a^2$ ontothe ellipse $(x/a)^2 + (y/b)^2 = 1$



- 2 : For a L.T. T : U $\to$ V, for Ɵ ∈ U T(Ɵ) = Ɵ
- 3 : For a L.T, T : U $\to$ V, for any x ∈ U , T(-x) = - T(x)
- 4 : For a L.T, T : U $\to$ V, for any x1, x2 ∈ U, T (x1-x2) = t(x1) – T(x2)
- 5 For a L.T, T : U V, The set N(T) ={ x| T(x) = Ɵ} is called a null space, Ɵ ∈ V
- N(T) is called ‘ kernel’ of L.T
- 6 For a L.T. T:U V , the set R(T) = {T(x) = y , for each x ∈ U} is called the range of L.T

LALP 2.16

- If U = {x1,x2, … xn} then R(T) = { T(x1), T (x2),…… T(xn)}
- The null space N(T) is a subspace of U and the Range is a subspace of V.

- **Linear Transformaation on basis:**
- 1 Standard Basis of R^2 = {(1,0), (0,1)} = {I ,j } = standard basis of R^2
- T : $R^2 \to R^2$, so that for any (x,y) in R^2, T(x,y) = (2x+ y , x +2y)
- Is T linear?, In fact, T$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}$
- What is T$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.1 \ + 0 \\ 1+2.0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- In the same way, T$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.0 \ + 1 \\ 0+2.1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

LALP 2.17

- $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ and
- $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$
- The L.T. converts vectors of standard basis B1 = { i, j} in to {T(i), T(j)}
- In our case, T(i) = $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$, T(j) = $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$
- T ($\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ = { $T\begin{pmatrix}1\\0\end{pmatrix}, T\begin{pmatrix}0\\1\end{pmatrix}\} = \{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\} = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}$
- So the $\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}$ is the result of L.T. on the standard basis.
- The given L.T. converts/ transforms the basic vectors column vectors of domain set into

LALP 2.18

- $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. THIS IS CALLED A MATRIX OF L.T.
- EXAMPLE 2:
- Let T: $R^2 \rightarrow R^3$ defined as , in the form of row vectors, T(a,b) = (b, a, a+b)
- Standard basis of R^2 are {(1,0), (0,1)}. They will be converted using this L.T.
- T(a,b) = (b, a, a+b), T (1,0) = (0,1, 1+0) = **(0,1,1)**
- T(0,1) = (1,0, 0+1) = **(1,0,1)**
- **these are to be written in to the form of column form;** $\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$ **and** $\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$
- So we have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ as result. It is a matrix of the order 3x2.

LALP 2.19

- EXAMPLE 3:
- Let T: R^3 R^2 defined as , in the form of row vectors, T(a,b,c) = (a+b, c+b)
- Standard basis of R^2 are {(1,0,0), (0,1,0), (0,0,1)}. They will be converted using this L.T.
- T(a,b,c) = (a+b, c+b), T (1,0, 0) = (1+0, 0+0) = **(1,0)**
- T(0,1,0) = (0+1, 0+1) = **(1,1)**
- **T(0,0,1) = (0+0,1+0) = (0,1)**
- **these are to be written in to the form of column form;** $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$
- So we have as result $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$. It is a matrix of the order 2x3.

LALP 2.20

- Matrix is a result of linear transformation on the basis of the domain set.
- It is written as (T: B1, B2) where B1 and B2 are standard basis of the domain and codomain.
- There may be different basis of domain and codomain also.
- Assignment 1:
- Ex-1 Determine the matrix associated with T(a,b) = (a a+b)
- Ex-2 Determine the matrix associated with T(a,b) = (a a+b, a-b)
- Ex-3 Determine the matrix associated with T(a,b,c) = (2a a+b, a-b)
- Ex-4 Determine the matrix associated with T(a,b,c) = (2a+c, c+b)
- Ex-5 Determine the matrix associated with T(a,b, c,d) = (a+d, b+c, d+c+a)

Matrix Algebra

Dr. Pradeep .J.Jha

LALP. 3.1

- We know that a matrix is a result of Linear Transformation on basis of the first space onto the(basis) of the second space.
- Ex-1 Let T : $R^2 \rightarrow R^3$ defined as T(x,y) = (x-y, 2x, 3y). We find [t,b1,b2] where b1 and b2 are the standard basis of R^2 and R^3.
- We know that b1 = { (1,0), (0, 1)} and b2 = { (1, 0, 0) , (0 ,1, 0), (0, 0 ,1)}
- These are the standard basis of the two spaces. As T(x,y) = (x-y, 2x, 3y).
- We have T(1,0) = (1-0, 2 .1, 3.0) = (1,2,0); actually to be written as $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$
- Also T(0,1) = (0-1 , 2.0 , 3.1) = (-1, 0,3) actually to be written as $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

LALP. 3.2

- So we have T(1,0) = $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and T(0,1) = $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$
- The basis b1 = { (1,0), (0, 1)} by the action of L.T. is transformed in $\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}$; this is a matrix which has 3 rows and 2 columns. (T: R^2 to R^3)
- 1 Matrix is denoted by capital letter. Say A = $\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}$
- 2 Matrix is an arrangement of mn numbers in the form of m rows and n columns. A = $\left(a_{ij}\right)$or$\left(a_{i,j}\right)$is the general notation. i varies from 1 to m and j varies from 1 to n.[I for rows and j for columns]
- 3 It does not stand for some real value. Each entry is meaningful.

- 4 We A = $\left(a_{ij}\right)_{mxn}$ is general notation. m x n is called the **order** of the matrix.
- **2x3 shows that the matrix has 2 rows and 3 columns etc.**
- **If** m = n then it is a **square** matrix. If m ≠ n then is a **rectangular** matrix.
- **5 Transpose** of a matrix: Let A = $\left(a_{ij}\right)_{mxn}$ be a matrix.If we interchange rows in to columns then the new matrix is called transpose of the given matrix a ; it is denoted as A' or A^t. A' = $(a_{ji})_{nxm}$. This process changes the order of the matrix.
- For A = $\left(a_{ij}\right)_{mxn}$ = $\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}$, A' = $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ $observe\ the\ order$.
- Order of the square matrices do not change.

LALP 3.4

- Example: Let T: $R^3 \leftarrow R^2$, T(x, y, z) = (x+y –z, x-y+z) Find the corresponding matrix f linear transformation.
- **a = (x1,y1,z1) ,T(a) = (x1+y1-z1, x1-y1+z1)**
- **b = (x2,y2,z2) , T(b) =(x2+y2-z2, x2-y2+z2)**
- **To prove L.T. we prove that T(a+b) = T(a) + T(b), and T (ca) = c. T(a)**
- **a + b = (x1+x2, y1+y2, z1 +z2) , T(a+b) = T(x1+x2, y1+y2, z1 +z2)**
- **= (x1+x2+y1+y2-z1-z2, x1+x2-y1-y2+z1+z2) Is it equa to T(a) + T(b) ?so T(a+b) = T(a) + T(b), check T (ca) is it same as c. T(a)?**
- **T(cx1, cy1, cz1) apply and heck.**
- **Both rules are satisfied and so the transformation is linear., find**
- **T(1,0,0), T(0,1,0) and T (0,0,1)**

LALP 3.5

- Three Fundamentals: As we have three working laws in vector spaces, the same way the set of 2x2 , 3x3 (in general nxn) matrices is a vector space over the field of real numbers , set R,(the set of complex number also) with two standard operations + and x (.).
- **1** Equality of two matrices: Two matrices of the **same order** can be equated and their equality is that of corresponding elements.
- E.g. Let A =$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $and\ B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. A = B $\longleftrightarrow$ a =p, b = q, c = r, and d =s.
- **2** Addition of two matrices: Two matrices of the same order can be added and the resultant is also a matrix of the same order. Also their addition is of the corresponding elements.
- e.g. Let A =$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $and\ B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then their addition denoted as A+B

- Is also a matrix of the same order.
- A + B = C (say), and C = $\begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$
- **3** : Multiplication of a matrix by a scalar: Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and p be any number; then multiplication of the matrix by the number ‘ p’; denoted as
- pA is a matrix of the order obtained by multiplying the all the elements of the matrix by the number.
- For the matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $and\ the\ number'p';pA = \begin{pmatrix} pa & pb \\ pc & pd \end{pmatrix}$

LALP 3.7

- If the scalar 'p' = 0, then pA = 0A = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called a null matrix; denoted as **0.** It can be of any order
- $0_{2x2} = O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $0_{3x3} = O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ This is also known as an additive identity. For any matrix A , A + **0 = 0 + A = A.**
- **If the scalar 'p' = -1** then for any matrix A, -1A is obtained by multiplying all eements of the matrix by -1.
- For A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, -1A = -A = $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$
- For the matrix A, the matrix –A is called additive inverse.
- A + (-A) = (-A) + A = **0 = null matrix.**

LALP 3.8

- Illustrations:
- **1** Let A =$\begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$ $and\ B = \begin{pmatrix} 8 & 9 \\ 4 & -7 \end{pmatrix}$ then A + B =$\begin{pmatrix} 10 & 14 \\ 3 & -7 \end{pmatrix}$
- [Note, preservation of the order and addition of corresponding members]
- **2** Let A =$\begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$ $and\ B = \begin{pmatrix} 8 & 9 \\ 4 & -7 \end{pmatrix}$ then find 2A + 5B
- 2A=2$\begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -2 & 0 \end{pmatrix}$ $and\ \ 5B = 5\ \begin{pmatrix} 8 & 9 \\ 4 & -7 \end{pmatrix}$ = $\begin{pmatrix} 40 & 45 \\ 20 & -35 \end{pmatrix}$
- Now 2A + 5B = $\begin{pmatrix} 48 & 55 \\ 18 & -35 \end{pmatrix}$
- [Note, preservation of the order, multiplication by a scalar, and addition of corresponding members]

LALP 3.9

- **3 Let** A = $\begin{pmatrix} 5 & -2 & 4 \\ 1 & 8 & 0 \\ 5 & 7 & 1 \end{pmatrix}$ **,** B = $\begin{pmatrix} 4 & 2 & 5 \\ 7 & 2 & 4 \\ 8 & 10 & 5 \end{pmatrix}$ **and**

- **2A – 3B = 2**$\begin{pmatrix} 5 & -2 & 4 \\ 1 & 8 & 0 \\ 5 & 7 & 1 \end{pmatrix}$ **- 3** $\begin{pmatrix} 4 & 2 & 5 \\ 7 & 2 & 4 \\ 8 & 10 & 5 \end{pmatrix}$ **=**

- **=** $\begin{pmatrix} 10 & -4 & 8 \\ 2 & 16 & 0 \\ 10 & 14 & 2 \end{pmatrix}$ **-** $\begin{pmatrix} 12 & 6 & 15 \\ 21 & 6 & 12 \\ 24 & 30 & 15 \end{pmatrix}$ **=** $\begin{pmatrix} 22 & 2 & 23 \\ 23 & 24 & 12 \\ 34 & 44 & 17 \end{pmatrix}$

- Example (Assignment part 1):
- 1 For the matrices A = $\begin{pmatrix} 5 & -2 & 4 \\ 1 & 8 & 0 \\ 5 & 7 & 1 \end{pmatrix}$ **,** B = $\begin{pmatrix} 4 & 2 & 5 \\ 7 & 2 & 4 \\ 8 & 10 & 5 \end{pmatrix}$
- **find 2 A′ + 3B.**
- **2 Let A =** $\begin{pmatrix} \boldsymbol{p} & \mathbf{1} \\ \mathbf{2}\boldsymbol{p} & \mathbf{4} \end{pmatrix}$, B = $\begin{pmatrix} \mathbf{7} & \boldsymbol{q} \\ \mathbf{-4} & \boldsymbol{k} \end{pmatrix}$ and 2A’ + 3B’ = **0 then find the values of** P + q.
- **3 A =** $\begin{pmatrix} \mathbf{5} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$, B = $\begin{pmatrix} \mathbf{7} & \mathbf{6} \\ \mathbf{-4} & \mathbf{1} \end{pmatrix}$ then check whether (A + B) ‘ = A’ + B′ ?3
- 1 find A +B 2 find (A + B) ‘ = lhs then 3 find A’ 4 find B’ 5 find A ‘ + B’ = rhs

LALP 3.11

- As we discussed earlier, there are two fundamental operations 1 addition of matrices and 2 Multiplication of matrices.
- We have seen '+' of two matrices or more matrices.
- Let us study Multiplication of matrices/ Product of matrices.
- **Multiplication of Matrices.:**
- Let A = (a_{ij}) and B = (b_{jk}) be the two matrices of the given order.
- Product of two matrices denoted as AB. i changes from 1 to m, j varies from 1 to n, and k varies from 1 to p where m,n , and p are positive integers.
- Product is also a matrix; say AB = C; where we write C = (c_{ik}).
- C_{ik} = $\sum c_{ik}$ where i = 1 to m, k = 1 to p.
- c_{ik} = $\sum a_{ij}\ b_{jk}$ where the summation runs over j.

LALP 3.12

- As j is common in both; it means that the basic condition for matrix multiplication to exist is
- **Number of columns of the first matrix must be equal to the number of rows of the second matrix.**
- A = ($a_{ij})_{mxn}$ and B = $(b_{jk})_{nxp}$, columns of A = n = rows of matrix B; this makes the product matrix AB possible. Let AB = C where each entry
- **C_{ik} = $\sum_{j=1}^{j=n} a_{ij}\, b_{jk}$ where I runs over 1 to m and j runs over 1 to n while k runs over from 1 to p.**
- **Let A = $\begin{pmatrix} 4 & 5 & -1 \\ 0 & 2 & 3 \end{pmatrix}$ $and\ B = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$ $matrix\ A\ is\ of\ order$**
- **$2x3\ wh$ile matrix B of order 2x2. Columns of A = 3 ≠ rows of B =2 and so product AB is not possible.**

LALP 3.13

- **Let B=** $\begin{pmatrix} \mathbf{4} & \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} \end{pmatrix}$ $\boldsymbol{and\ A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-4} & \mathbf{3} \end{pmatrix}$ $\boldsymbol{matrix}$ Bis of 2x3 and A is of
- order 2x2. Columns of A =3not equal to rows of B. In this case product BA is not possible. Ab is possible.
- $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-4} & \mathbf{3} \end{pmatrix}$ and **B=** $\begin{pmatrix} \mathbf{4} & \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} \end{pmatrix}$ As snown, the new matrix AB = C is such that each row of C will be the ‘ sum of corresponding rows of A with corresponding columns of B.’
- First row of A with corresponding first column of B.= 1 . 4 + 2 .0 = 4
- First row of A with corresponding Second column of B = 1 .5 + 2 .2 = 9
- First row of A with corresponding third column of B = 1 . (-1) + 2 . 3= 5
- Second row of A with corresponding first column of B = (-4).4 + 3 . 0 = -16
- Second row of A with corresponding second column of B= (-4).5 + 3 .2= -14

Second row of A with corresponding third column of B (-4).(-1) + 3 . 3 = 13

LALP 3.14

- With this, the product matrix AB = C = $\begin{pmatrix} 4 & 9 & 5 \\ -16 & -14 & 13 \end{pmatrix}$.
- It is a matrix of order 2x3.= rows of A and columns of B.
- Note that: It is possible that
- 1 Product matrices AB and BA both may not be possible.
- 2 Product AB be possible but BA may not be possible.
- 3 Product AB may not be possible but BA may be possible.
- 4 Product AB be possible and BA is also possible.
- 5 For the matrix product **AB = 0**
- 6 It is possible th A $\neq$ **0** and B $\neq$ **0 [In addition to three regular cases like**
- **A = 0 and B $\neq$0, etc.]**

LALP 3.15

- 7 If both A and B are square matrices then both the products AB and BA are possible.
- 8 If both the matrices are of interchangable order [like A is 2x3 and B is3x2] then also both AB and BA are possible but they are of different order. In the above case, AB is of order 2x2 and BA of order 3x3.]
- Illustrations:1 A = $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$ and B=$\begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$, both are non null yet the product AB = **0**
- Let A = $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 5 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$, A is 2x2 and B is 3x2and so AB is not possible but BA is possible. The resultant matrix is 3x2.

LALP 3.16

- Let B = $\begin{pmatrix} 2 & 5 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$ and A = $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$. The rows of BA 3x2 matrix are
- 2.1 + 5 x(-3)   2.2 + 5 x(-6) First row   -13   -26
- 0.1 + (-1) .(-3)   0.2 + (-1).(-6) Second row   3   6 So BA = $\begin{pmatrix} -13 & -26 \\ 3 & 6 \\ -5 & -10 \end{pmatrix}$
- 1.1 + 2. (-3)   1.2 + 2.(-6) Third row   -5   -10
- Most Important point to note here is Though both product AB and BA be possible, we have, in general, **AB ≠ BA.**
- A = $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$ **and** B= $\begin{pmatrix} 7 & 2 \\ -1 & 3 \end{pmatrix}$ Find AB and BA.
- AB = BA, only if any one is a null or identity matrix or they are commutative matrices.

LALP 3.17

- * Identity Matrix: The notion of identity matrix closely resembles to that of multiplicative identity. In case of real numbers it is unity – 1; 5x1 = 1x5 = 5.

In the case of matrix system, If two matrices are conformable for matrix multiplication andtheir poduct results in to any one of them then it is an identity matrix.

For the given matri A, if we have the matrix B so that AB = BA = A, then the matrix B is an identity matrix for the binay process of matrix multipkication.

This matrix is denoted by I. We have B = I

$I_{2x2} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we simply write this as I if there is no confusion.

$I_{3x3} = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

A I = I A = A

LALP 3.18

- **Determinant of a square matrix**: We know that a matrix is an arrangement or it is a presentation of common values of two different variables.
- students
- a b
- Marks Now, one can meaningfully read each entry. There can be
- different values to rows and columns.
- Test 1 5 9
- Test 2 6 4 If write these entries in the form of a determinant then
- We have the notion of determinant. A determinant stands for a real value in the case of different order. [Only square form]
- It shows a real value and it has meaningful different value.

LALP 3.19

- D1 = $\begin{bmatrix} 2 & 1 \\ -3 & 8 \end{bmatrix}$, $D2 = \begin{bmatrix} 4 & 8 & 1 \\ 0 & 5 & 12 \\ 1 & 2 & 3 \end{bmatrix}$, D1 = (2).(8) – (1)(-3) = 19
- In the same way D2 = 4. $\begin{bmatrix} 5 & 12 \\ 2 & 3 \end{bmatrix}$ + (-8) $\begin{bmatrix} 0 & 12 \\ 1 & 3 \end{bmatrix}$ + (1) $\begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix}$
- = 4. -9 + (-8) (-12) + 1 (-5) = 55
- There are some properties associated with determinants. These properties are useful in finding the value of a determinant, We will mention some on the proper time.

LALP 3.20

- **Inverse of a given matrix: As set of all square matrices on the given field of real numbers generate a vectors space with two binary operations + and ‘ . ‘, we must have a multiplicative inverse to each non-singular matrix.**
- **For a given matrix A if we another matrix B so thar the product**
- **AB =BA = I = identity matrix, then the matrix B is called the inverse of the given matrix A; it denoted = A^{-1}**
- **In fact A and B are inverses of each other. A = B^{-1} and B = A^{-1}**
- **i.e A A^{-1} = A^{-1} A = I identity matix.**
- **We have given conditions ; A must be a square matrix and it must be non-singular. A given non-singular matrix can possess inverse.**

- Non-Singular Matrix: A matrix A is non singular if its determinant value is non zero.
- A =$\begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$ $is\ a\ square\ matrix\ then\ its\ determinant\ denoted$ as |A| is $\begin{vmatrix} 2 & 4 \\ -1 & 6 \end{vmatrix}$ = 12 – (4) (-1) = 16 ≠ 0 and so the given matrix A is a non singular matrix.
- Matrx B =$\begin{pmatrix} 4 & 8 & -2 \\ 0 & 1 & 5 \\ 1 & 2 & 2 \end{pmatrix}$, its determinant $= |\mathrm{B}| = \begin{vmatrix} 4 & 8 & -2 \\ 0 & 1 & 5 \\ 1 & 2 & 2 \end{vmatrix}$=
- 4(1.2 – 5.2) – 8(0.2 -5.1) +(-2) (0.2 – 1.1) = -32 + 40 +2 =8 ≠ 0 and so B is a non-singular matrix.
- Only non-singular square matices possess inverses.

LALP 3.22

- Inverse of the given matrix A = A^{-1} = adj A / |A|; where adj A is called adjoint of the matrix A. It is obvious that $|A| \neq 0$.
- adj A is found by **transposing** the matrix of cofactors of all the elements of the given matrix A. Cofactor of the given matrix is obtained by attaching proper signs to the minor of each element.
- Signs are = $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ Note the pattern. Minors ae determinants obtained by not considering the row and the column of the given entry. We find the adjiont of the non-singular matrix that we considered.

LALP 3.23

- Let us consider $|\mathrm{B}| = \begin{vmatrix} 4 & 8 & -2 \\ 0 & 1 & 5 \\ 1 & 2 & 2 \end{vmatrix}$ minor of 4 = $\begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix}$ = -8,
- Minor of 8 = $\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix}$ = -5, minor of -2 = $\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$ = -1 ,
- minor of 0 = $\begin{vmatrix} 8 & -2 \\ 2 & 2 \end{vmatrix}$ =20
- Minor of 1 = $\begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix} =$10, minor of 5 = $\begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix}$ = 0, minor of 1 = $\begin{vmatrix} 8 & -2 \\ 1 & 5 \end{vmatrix}$ = 2
- Minor of 2 = $\begin{vmatrix} 4 & -2 \\ 0 & 5 \end{vmatrix}$ = 20, minor of 2 = $\begin{vmatrix} 4 & 8 \\ 0 & 1 \end{vmatrix}$ = 4

Matrix of cofactors = $\begin{pmatrix} +(-8) & -(-5) & +(-1) \\ -(20) & +(10) & -(0) \\ +(2) & -(20) & +(4) \end{pmatrix} = \begin{pmatrix} -8 & 5 & -1 \\ -20 & 10 & 0 \\ 2 & -20 & 4 \end{pmatrix}$

LALP 3.24

- We transpose this matrix $\begin{pmatrix} -8 & 5 & -1 \\ -20 & 10 & 0 \\ 2 & -20 & 4 \end{pmatrix}$ of cofactor and get adjoint of the given matrix A
- Adj A= $\begin{pmatrix} -8 & -20 & 2 \\ 5 & 10 & -20 \\ -1 & 0 & 4 \end{pmatrix}$ Also $|\mathrm{B}| = \begin{vmatrix} 4 & 8 & -2 \\ 0 & 1 & 5 \\ 1 & 2 & 2 \end{vmatrix}$ = 8 ---- non singular
- Therefore A^{-1} = adj A /|a|= $\begin{pmatrix} -8 & -20 & 2 \\ 5 & 10 & -20 \\ -1 & 0 & 4 \end{pmatrix}$ / 8
- So **A^{-1} =** $\begin{pmatrix} \mathbf{-1} & \mathbf{-5/2} & \mathbf{1/4} \\ \mathbf{5/8} & \mathbf{5/4} & \mathbf{-5/2} \\ \mathbf{-1/8} & \mathbf{0} & \mathbf{1/2} \end{pmatrix}$ **Find AA^{-1} ; It has to be** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

LALP 3.25

- Find the inverse of A = $\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}$
- |A|= -11 ≠ 0. It is a non-singular matrix ; inverse exists
- Sign matrix = $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$, Matrix of Minors = $\begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$
- Matrix of cofactors = $\begin{pmatrix} +(5) & -(2) \\ -(3) & +(-1) \end{pmatrix}$,
- Transpose this matrix of cofactors and get **adj. A =** $\begin{pmatrix} \mathbf{5} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{-1} \end{pmatrix}$
- **A^{-1} = adj a / |A| =** $\begin{pmatrix} \mathbf{5} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{-1} \end{pmatrix}$**/ -11 =** $\frac{\mathbf{1}}{\mathbf{-11}}\begin{pmatrix} \mathbf{5} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{-1} \end{pmatrix}$ **=** $\begin{pmatrix} \mathbf{-5/11} & \mathbf{3/11} \\ \mathbf{2/11} & \mathbf{1/11} \end{pmatrix}$

- Assignment part 2:
- 1 Find the inverse of A = $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ and check AA^{-1}
- 2 For A = $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & -2 \\ 5 & 4 \end{pmatrix}$, find (a) AB (b) $(AB)^{-1}$ (c) A^{-1}
- (d) B^{-1} , (E) $B^{-1} A^{-1}$ (F) $(A – 2B)^{-1}$
- 3 Find the inverse of A = $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$
- 4 For A = $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ find $(A')^{-1}$
- 5 Is it same as $(A^{-1})'$?

LALP unit 4

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LALP 4.1

- System of linear Equations:
- We have a system of two linear equations in R^2 which shows two lines. Either they are parallel or they intersect at one point only. This is the point of intersection of two lines. It means that the coordinate of that point will satify equation of both lines.
- We would like to identify such situations using the notion of ‘ inverse of a matrix’.
- We consider two examples in order to verify existence of point of intersection.
- General approach:
- Consider $a1x + b1y = c1$ …(1)
- and $a2x + b2y = c2$ . .. (2)

LALP 4.2

- The system in matrix equation form can be written as
- $\begin{pmatrix} a1 & b1 \\ a2 & b2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c1 \\ c2 \end{pmatrix}$ check this; say **AX = B**
- **A X B**
- **If** the solution of the system; i.e. values of x, and y which can satisfy the equation.
- **So X =** $\begin{pmatrix} x \\ y \end{pmatrix}$ **= A^{-1} B where A^{-1} = adj A/ |A| if |A| ≠0,** I,e. A is a non- singular matrix.
- Example 1: Solve the system x +2y =5, 3x + 4y = 11,
- In the matrix notation the system is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$
- A X = B

LALP 4.3

- A $=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $, B = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$, |A| = 4 -6 = -2 ≠ 0 , A is non – singular matrix. Therefore A^{-1} exists. So the system AX = B has a solution
- ∴ X = A^{-1}B, ….(1)
- Matrix of cofactors = $\begin{pmatrix} +(4) & -(3) \\ -(2) & +(1) \end{pmatrix}$ Transpose this; adj A = $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$
- Using (1) X = $\begin{pmatrix} x \\ y \end{pmatrix}$ = $\frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \end{pmatrix}$ = $\begin{pmatrix} -2\ 5\ +(1)(11) \\ \left(\frac{3}{2}\right).5 - \left(\frac{1}{2}\right).11 \end{pmatrix}$ $=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- ∴ x =1 and y = 2 = (1/-2) $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $=\begin{pmatrix} x \\ y \end{pmatrix}$

- Row operations : There are three types of row operations on the rows of given matrix.
- 1 Interchange of any two rows. R1 $\longleftrightarrow$ R2
- 2 Multiplying i th row by a constant k and adding the result to the corresponding members of j th row. R_{ij} (k)
- 3 Multiplying all the members of a row by a constant c. Denoted as cR
- Let A = $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 12 & 2 \\ 6 & 7 & 5 \end{pmatrix}$ perform r2 to r3, ∴ A' = $\begin{pmatrix} 1 & 2 & -3 \\ 6 & 7 & 5 \\ 0 & 12 & 2 \end{pmatrix}$
- Multiply r1 by -6 and add to corresponding members of the second row r2.
- A'' = $\begin{pmatrix} 1 & 2 & -3 \\ -6 & 0 & 20 \\ 6 & 7 & 5 \end{pmatrix}$ Multiply r3 by ½ . A''' = $\begin{pmatrix} 1 & 2 & -3 \\ -6 & 0 & 20 \\ 3 & 7/2 & 5/2 \end{pmatrix}$

LALP 4.5

- x +2y =5, 3x + 4y = 11, The system is A =$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$ The system is
- $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 11 \end{pmatrix}$ The central idea is to apply row operations on the given matrix A so that it is converted in to identity matrix and hence the augmented matrix (identity matrix) becomes inverse of the given matrix and the resource matrx will become the solution of the system.
- Multiply the first row by 3 and subtract the corresponding members from second row. $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \end{pmatrix}$ R_{12} (-3) , Add second row to first row.
- $\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, Divde second row by -2,
- $\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ So x = 1 and y =2, inverse of A= A^{-1}= $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

- Ex-2 Solve the system of equations.
- 2x – 3y = 2 Which in the matrix form $\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$
- 1x + 2y = -6 A X = B (1) Solution is
- X = A^{-1} B
- 1 |A| = 7≠ 0 ∴ |A| is a non-zero (non- singularmatrix. ∴ Imverse exists.
- Matrix of cofactors = $\begin{pmatrix} +(2) & -(1) \\ -(-3) & +(2) \end{pmatrix}$ = $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$. Transpose it.
- ∴ Adj . A = $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$. From (1) X = $\begin{pmatrix} x \\ y \end{pmatrix}$ = A $^{-1}$ B= ((adj A)/ |a|) B
- X = $\begin{pmatrix} x \\ y \end{pmatrix}$ = (1/7) $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ = (1/7)$\begin{pmatrix} -14 \\ -14 \end{pmatrix}$= $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ = $\begin{pmatrix} x \\ y \end{pmatrix}$
- Now find the solution.

LALP 4.7

- $\begin{pmatrix} 2 & -3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -6 \end{pmatrix}$, Apply row operations to convert A in to identity.
- Divide first row by 2 and subtract it from second row.

$$\begin{pmatrix} 2 & -3 & 1 & 0 \\ 0 & 7/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

- Multiply the second row by 6/7 and add to the first row.
- $\begin{pmatrix} 2 & 0 & -3/7 & 6/7 \\ 0 & 7/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ -7 \end{pmatrix}$
- Make the first matrix an identity mtrix.
- Divide the first row by 2 and multiply the second row by 2/7.
- $\begin{pmatrix} 1 & 0 & -3/14 & 3/7 \\ 0 & 1 & -1/7 & 1/7 \end{pmatrix}\begin{pmatrix} -2 \\ -2 \end{pmatrix}$. ∴ x = -2, and y = -2, $A^{-1} = \begin{pmatrix} -3/14 & \frac{3}{7} \\ -1/7 & 1/7 \end{pmatrix}$

LALP 4.8

Solve the system :

x + y + z = 0

2x –y +z =1

y +2z =0

In the matrix notation, $\begin{pmatrix}1 & 1 & 1\\ 2 & -1 & 1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$

We write it as $\begin{pmatrix}1 & 1 & 1 & 1 & 0 & 0\\ 2 & -1 & 1 & 0 & 1 & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$

Multiply r1 by 2 and subtract from r2, $\begin{pmatrix}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -3 & -1 & -2 & 1 & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$

Subtract r3 from r1, $\begin{pmatrix}1 & 0 & -1 & 1 & 0 & -1\\ 0 & -3 & -1 & -2 & 1 & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$

LALP 4.9

- Solve by matrix inversion and also by Gauss Jordan method.
- Ex 1 Solve 2x + 3y = 14 and 4x + y = 8 $\begin{pmatrix} 2 & 3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 8 \end{pmatrix}$ r1- (3/2) r2
- Ex-2 Solve 4x -3y = 11, and 2x + y = 3 $\begin{pmatrix} 4 & -3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 3 \end{pmatrix}$
- Ex-3 x + 2y –z = 6, 2x + y + z = 3, 3x –y +z =3
- Ex – 4 2x + y + z =3, 4x + 3y +z = 9, x + 2y + 2z = 3
-
- $\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}$ ½ r1, r2 -4r1, r3- r1, ………, r3. (1/4),
- r3 (3) +r2, **r1- (1/2) r2,** **r1- (1/2) r3**
-

LALP SET 5

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LALP 5.1

- Linear Programming: (L.P.)
- When the resources like money, manpower, material, and time are to be allocated to different types of activities on certain given constraints wth an objective like maximizing the profit or minimizing the loss, we have **problems**
- **. A**ctivities consume resources which are given to perform activities like production, sale, marketing, and parallel work like these but all these have an **objective. Objectives may be like profit / production maximization or may be of cost minimization.**
- Conditions imposed on execution of activities on consumption of resources are called **constraints.**

In this way, the number of units of the activities of each type will be called **Decision variable.**

LALP 5.2

- In most of the cases the decision variables are required to be non-negative.

- All the decision variables, wherever they appear (in objective function or in the constraints) are always **linear.**
- **Programming does not mean computer programming but it means 'planning '.** Planning of allocation of resources in to the different activities with specific objective.

LALP 5.3

- A designer plans to share two departments A and B available for 160 and 180 hours respectively. He thinks of manufacturing **tables** and **chairs** which will be sold in the market with a profit of 10 rs. and 15 rs. Respectively. One table takes 4 hours in dept. A and the chair takes 2 hours in dept. B. A chair takes 3 hours in dept. A and 6 hours in dept. B. Deprtment A and B are available for 160 and 180 hours respectively. Design allocations so tht he gets maximum profit.
-
- Resources: Dept A: 160 hours and dept. B : 180 hours.
- Activities : Tables   Chairs

	Dept A	Dept. B	profit per unit
Table:	4	2	rs. 10
Chair	2	6	rs. 15

LALP 5.4

- Resources: Dept A: 160 hours and dept. B : 180 hours.
- Activities : Tables Chairs

	Dept A	Dept. B	profit per unit	Plan
Table:	4	2	rs. 10	x units
Chair	2	6	rs. 15	y units

- Find x and y so that $F(x,y) = 10x + 15y$ is maximized
- With constraint on resources (1) $4x + 2y \leq 160$, (2) $2x + 6y \leq 180$
- Non negativity constraints (2) $x, y \geq 0$

LALP 5.5

- General (LPP)Model:
- Find x_1 , x2,……,xn (Decision variables)
- In order to **optimize** z = c1x1 + c2x2 + c3x3 + …………………+ cnxn (1)
- Subject to a11 x1 + a12 x2 + ………………………..+ a1n xn ≤, =, ≥ b1
- a21 x1 + a22 x2 + ………………………..+ a2n xn ≤ ,= , ≥ b2

- am1 x1 + am2 x2 + ………………………..+ amn xn ≤ = ≥ b_m
- (Where in each each constraint one and only one sign holds true)
- With all xi ≥ **0 [There are n D.V. and m constraints, m** ≤ n]
- **All ci ≥ 0 are cost / profit factors. All bi ≥ 0 are the resources**
- **A = (a_{ij}) is a mxn matrix.**

LALP 5.6

- Ex. 2 Planning a menu. Type P and Type Q.

	fat	Carbohydrate	Protein (content in 10 gram)	cost/10gram
Type P.	2	3	5	10
Type Q	3	4	2	20
Requirement	20	24	16	

- (minimum)
- Let x1 and x2 be amount of type p and type q of the food to be planned to satisfy the requirement and minimize the cost.
- Construct the model. Find x1 and x2 so as to **minimize** the total purchase cost: $10x_1 + 20x_2$ subject to $2x_1 + 3x_2 \geq 20$ fat const.
- $3x_1 + 4x_2 \geq 24$, $5x_1 + 2x_2 \geq 16$,
- With $x_1, x_2 \geq 0$
-

LALP 5.7

- Ex-3 investment of 200,000 in two types of bonds aa and bb.
- aa pays 7% and bb pays 9 % interest. Not more than 60.000 to be invested in bb. The amount invested in aa must be at least twice of the one invested in bb. Find the solution .
- Say x rs. In aa type and y rs. In bb type.
- To maximize : $0.07x + 0.09y$ interest from both aa and bb
- $x + y = 200000$
- $y \le 60000$
- $x \ge 2y$
- with $x, y \ge 0$

- General (LPP)Model:
- Find x_1 , x2,……,xn (Decision variables)
- In order to **optimize** z = c1x1 + c2x2 + c3x3 + …………………+ cnxn (1)
- Subject to a11 x1 + a12 x2 + ………………………..+ a1n xn ≤, =, ≥ b1
- a21 x1 + a22 x2 + ………………………..+ a2n xn ≤ ,= , ≥ b2
- am1 x1 + am2 x2 + ………………………..+ amn xn ≤ = ≥ b_m
- (Where in each each constraint one and only one sign holds true)
- With all xi ≥ **0 [There are n D.V. and m constraints, m** ≤ n]
- **All ci ≥ 0 are cost / profit factors. All bi ≥ 0 are the resources**
- **A = (a_{ij}) is a mxn matrix.**

LALP 5.8

- General form of the model, Matrix form:
- Now we write the matrix form of the model: Find the matrix X so as to
- Optimize Z = C'X (1) [**standard form has max. with all ≤ or = type,**
- **and mini. With all** ≥ or = type]
- Subject to AX ≤ = ≥ b (2)
- With X ≥ **0** **(3)** where **C'=** $(\boldsymbol{c_1} \quad \boldsymbol{c_2} \cdots \quad \cdots \quad \boldsymbol{c_n})_{\boldsymbol{1xn}}$ X = $\begin{pmatrix} x_1 \\ x_2 \\ \dots. \\ \dots \\ x_n \end{pmatrix}_{nx1}$

b = $\begin{pmatrix} b_1 \\ b_2 \\ \dots. \\ \dots \\ b_m \end{pmatrix}_{mx1}$ **, and A =** $\begin{pmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \dots\dots\dots\dots\dots. & \boldsymbol{a_{1n}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \dots\dots\dots\dots & \boldsymbol{a_{2n}} \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} \dots\dots\dots\dots & \boldsymbol{a_{3n}} \\ \dots. & \dots.. & \dots\dots. \\ \boldsymbol{a_{m1}} & \boldsymbol{a_{m2}} \dots\dots\dots\dots & \boldsymbol{a_{mn}} \end{pmatrix}$

LALP 5.9

- E.g. Find x and y so as to maximize z = 4x + 6y subject to
- 2x + 4y ≤ 7
- 3x + 7y ≤ 12 with x , y ≥ 0
- Find x =$\begin{pmatrix} x \\ y \end{pmatrix}$, C'= $\begin{pmatrix} 4 & 6 \end{pmatrix}$, A = $\begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}$, b = $\begin{pmatrix} 7 \\ 12 \end{pmatrix}$
- Maximize z = C'X , subject to AX ≤ b with X =$\begin{pmatrix} x \\ y \end{pmatrix}$ ≥ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- **Find X =$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$ so as to maximize Z = $\begin{pmatrix} \mathbf{4} & \mathbf{6} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$**
- **Subject to $\begin{pmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{7} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$ ≤ $\begin{pmatrix} \mathbf{7} \\ \mathbf{12} \end{pmatrix}$ with $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$ ≥ $\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$**

A step towards solution:

When there are exactly two variables we can find the graphical soluton of the problem.

Find X to Maximize z = C'X (1)

subject to AX ≤ b (2)

with X = $\begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (3)

Any set of values of x and y which satisfy (1) and (2) is called a **solution**.

If the same satisfies all (1), (2), and (3) is called a **feasible solution** of the system.

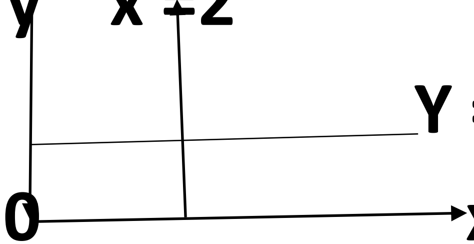
- Graphical Solution:
- 1 When there are exactly two variables one can use graphical solution.
- 2 Consider inequality of the constraint as equality. Each equality is now an equation of a line. $Ax + by + c = 0$
- 3 Considering the last non-negativity constraint we shall find our feasible solution in first quadrant only.
- 4 Draw each constraint line and find region corresponding to inequality.
- 5 Do the same for all constraints.
- 6 Find the most common region to all inequality constraints.
- 7 If the common region is a **convex polygon** or an **open region bounded below** then only the optimal solution exists.

LALP 5.13

- 8 The optimal solution exists on at least one (may be more) of the vertices.
- 9 Find all the vertices and evaluate the objective function and make selection of optimal value.
- --
- Some points:
- X = 0 is an equation of y axis. Y = 0 is an equation of x axis.
- From one point there passes infinite number of lines but from any two points there passes exactly one unique line.
- Two non parallel lines intersect at one point only.
- Equation of a line passing through two points (x1, y1) and (x2, y2) is
- $\left(\frac{x - x1}{x1 - x2}\right) = \left(\frac{y - y1}{y1 - y2}\right)$ or $\begin{vmatrix} x & y & 1 \\ x1 & y1 & 1 \\ x2 & y2 & 1 \end{vmatrix} = 0$

LALP 5.14

- **Two intercept Form:** **y x =2**
- **X = constant is a line parallel to y axis.** **Y = 1**
- **Y = constant is a line parallel to x axis.** **0 x**



- If ‘a’ and ‘b’ are the intercepts on x and y axis then intercept form of the line
- $x/a + y/b = 1$ where a and b are intercepts other than zero. $2x + 4y = 8$
- , $x/4 + y/2 = 1$
- General equation of a line is $ax + by + c = 0$
- Find the equation of line passing through (1 , 3) and (-2, 4),
- $(x - 1)/(1 - (-2)) = (y - (3)/(3 - 4)$
- Find the intercepts of $4x + 3y = 8$ $4x/8 + 3y/8 = 1$, $x/2 + y/(8/3) = 1$

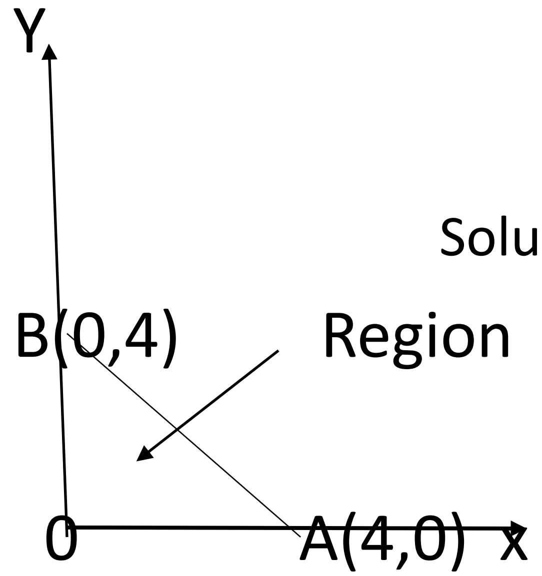
LALP 5.15

Find x and y so as to maximize z = 5x +7y

subject to x + y ≤ 4 (1), 3x + 8y ≤ 24 (2) with x, y ≥ 0

Solution: Consider x +y = 4 (1) for y = 0, x = 4 Point (4,0)

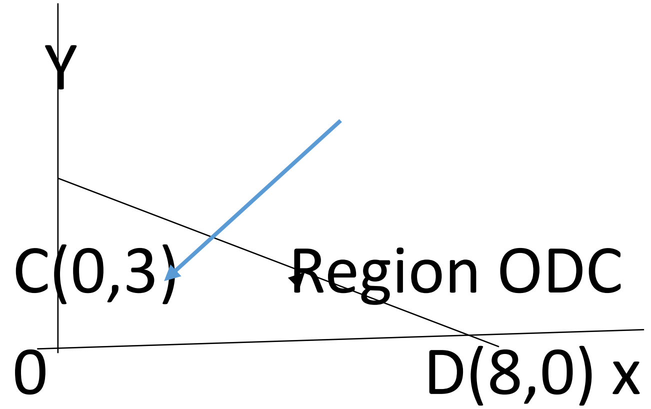
for x = 0, y = 4 Point (0,4)

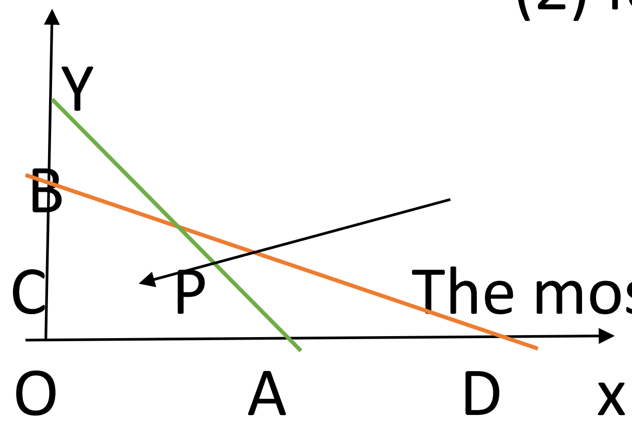


The regon OAB is the required region ≤

Consider 3x + 8y = 24 (1) for y = 0 , x = 8, point (8,0)

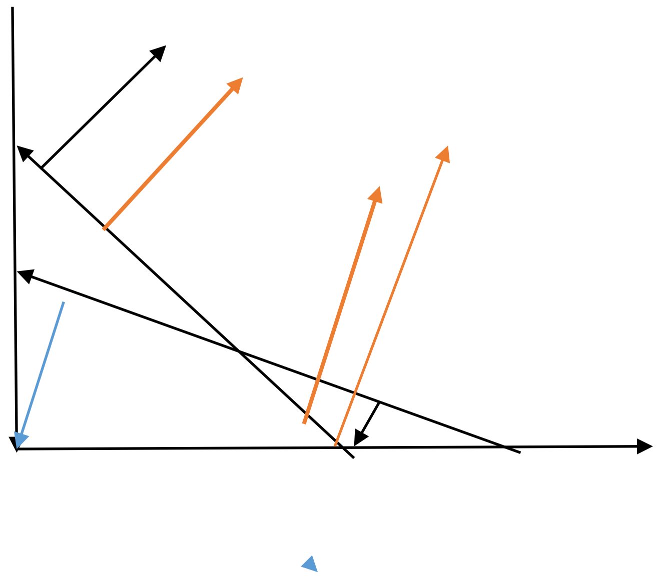
(2) for x = 0, y = 3 , Point (0,3)

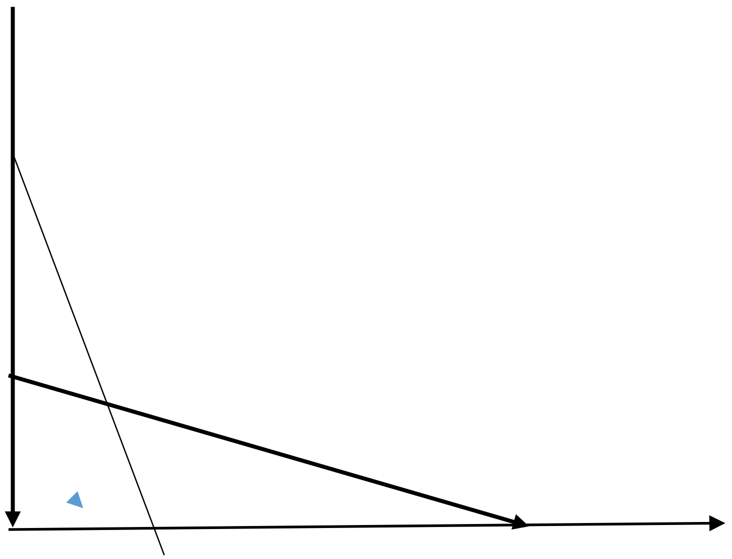




The most common region is OAPC

O(0,0), A(4,0), P(8/5,12/5), C(0,3)





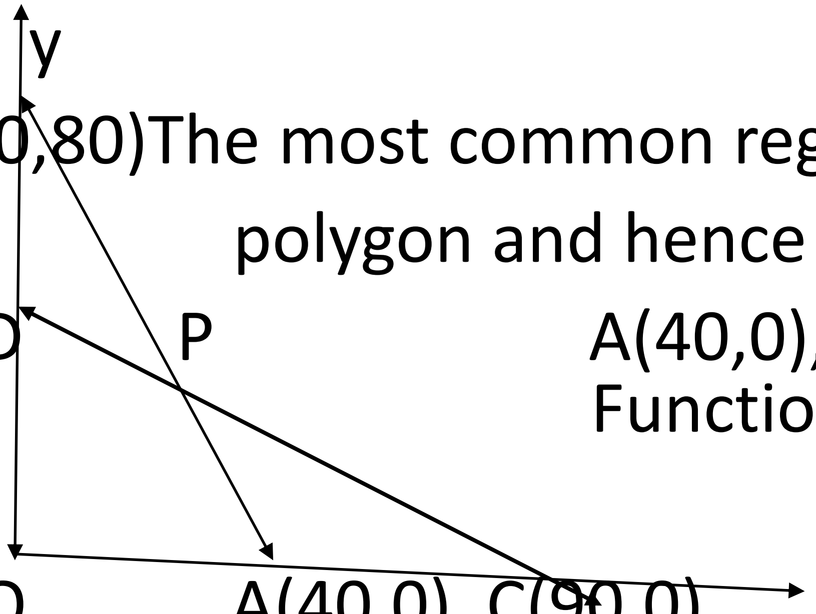
LALP 5.16

- Example: Resources: Dept A: 160 hours and dept. B : 180 hours.
- Activities : Tables Chairs

	Dept A	Dept. B	profit per unit	Plan
Table:	4	2	rs. 10	x units
Chair	2	6	rs. 15	y units

- Find x and y so that $F(x,y) = 10x + 15y$ is maximized
- With constraint on resources (1) $4x + 2y \le 160$, (2) $2x + 6y \le 180$
- Non negativity constraints (2) $x , y \ge 0$

- 1 Consider $4x + 2y \le 160$, for y = 0 , x =40 , Point A (40,0,)
- for x = 0,y = 80 point B (0 , 80)
- $2x + 6y \le 180$ for y =0, x= 90, point C (90, 0) for x = 0, y = 30 ,Point D(0,30)



- y

B(0,80)The most common region to **OAB** and **CDO** is **APDO. This is a convex**

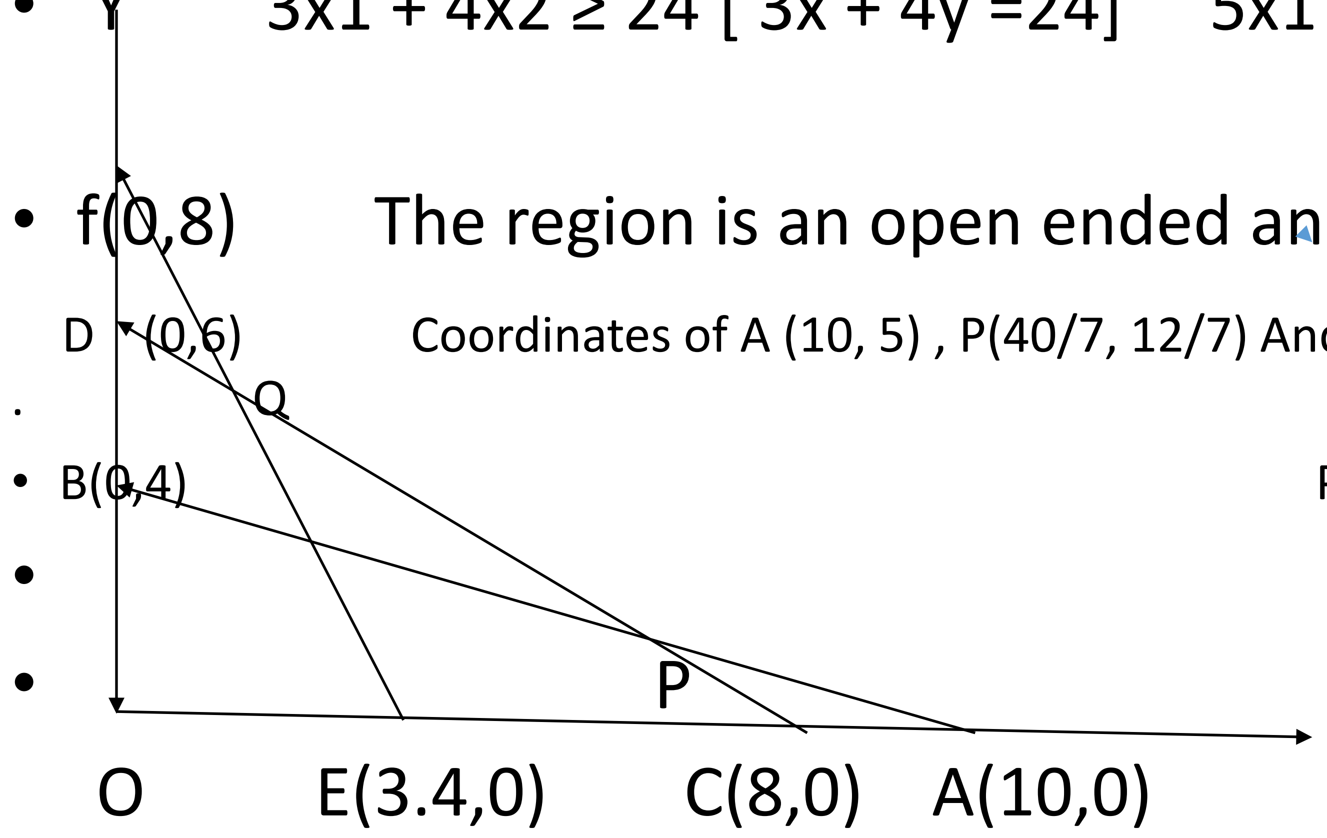
- polygon and hence solution lies on at least one of the vertex.
- D P A(40,0), P (,) , D(0,30) and O(0,0), Evaluate objective . Function on vertices.

Points	Value of 10x + 15 y
A (40,0)	10(40) + 15(0) = 400
P(30,20)	10(30) + 15(20) **=600 so x =30, y = 20**
D(0,30)	0(10) + 15(30) = 450

- O A(40,0) C(90,0) x

LALP 5.18

- Let x1 and x2 be amount of type p and type q of the food to be planned to satisfy the requirement and minimize the cost.
- Construct the model. Find x1 and x2 so as to **minimize** the total purchase cost: 10 x1 + 20 x2 subject to **2x1 + 5x2 ≥ 20** fat const.(2x + 5y = 20)
- 3x1 + 4x2 ≥ 24 [3x + 4y =24] 5x1 + 2x2 ≥ 16, With x1, x2 ≥ 0



The region is an open ended and bounded below. It is **XAPQEY**

Coordinates of A (10, 5) , P(40/7, 12/7) And Q(8/7, 36/7) and E(0,8) Now evaluate objective f

Point	value of obj. fun= 10x1 + 20x2
A(10,5)	100+100 = 200
P (40/7 , 12/7)	400/7 + 240/7 = 640/7 = **91**
Q (8/7, 36/7)	80/7 + 720/7 = 800/7 = 114

LALP 5.19

- Going to Simplex: In the case of linear programming problems in two variables, possibly we can find graphical solution to a given problem with finite number of constraints. The case in which the number of decision variables exceed two or many constraints of different types then the graphical method is not applicable and we have Simplex method with us. There are certain conditions before a problem can be solved using Simplex method.
- 1 Converting inequality constraints in to equality:
- (a) Slack variable: If the given constraint is of the ≤ type then we follow this routine.
- Let $a_{11} x_1 + a_{12} x_2 + \dots\dots\dots + a_{1n} x_n \le b_1$ be a constraint. We know that the quantity on the right side shows resource. The expression on the left shows actual usage of the resource.

LALP 5.20

- E.g. say 2x + **3y ≤ 20 x = number of tables and y = number of chairs to be manufactured. 20 is the amount of wood given.**
- **2x + 3y = actual use of wood in making x number of tables and y number of chairs. [consumption can be at the most 20 units. If all units used thn fine if some amount of wood remains unused than it is not going to add in the total profit. The unutilized resource [leftover resource] is called slack variable.Let us denote it by s1. [the first slack variable]**
- **We have 2x + 3y + s1 = 20. S1 is a slack variable.**
- **Contribution of slack variable to the objective function is zero.**
- **2 SURPLUS VARIBLE: Consider inequality** of the type ≥.
- say 2x + 5y ≥ 8. Right sideis the resource and the left side is the actual usage. The usage in this case is greater then the resource. To make equality we subtract a variable say s2 from left side. 2x + 5y - S2 = 8. This variable S2 is called a surplus variable.
- Contribution of surplus variable to the objective function is zero.
-

LALP 5.21

- After converting **inequality in to equality** the next step is to have a basic variable in each **equality constraint.**
- **A basic variable is that which appears with its coefficient +1 in each equation. [In remaining equation its coeff. Is zero.]**
- **$2x + 3y$** ≤ 8 constraint with introduction of slack variable becomes
- **$2x + 3y + 1S1 = 8$. In this case coeff. Of S1 is +1. This slack variable can be called a basic variable.**
- **$2x + 3y$** ≥ 8 becomes, on introduction of surplus variable becomes
- **$2x + 3y – 1s = 8$. This surplus variable has coeff. = -1. It cannot be treated as a basic variable. [Why ?]**
- **$3x + 4y = 9$ has no basic variable.**

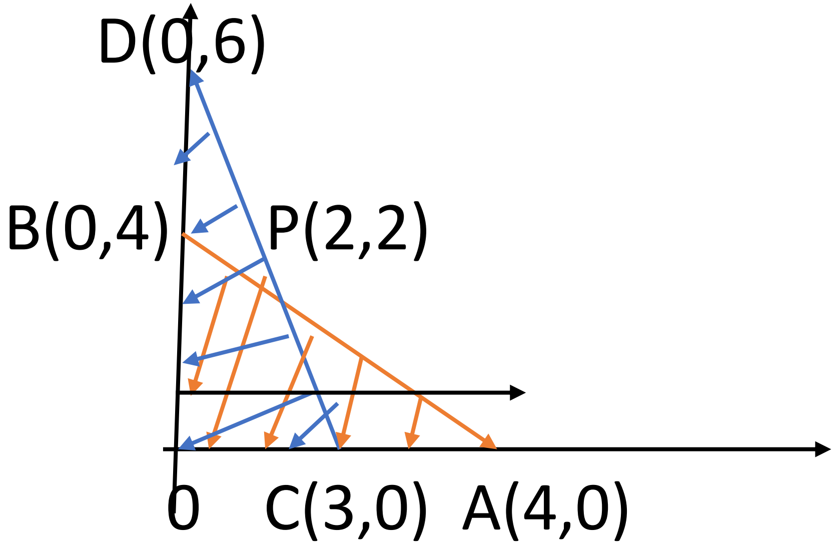
LALP 5.23

- Find x and y to maximize,
- $Z = 6x + 9y$
- Subject to $6x + 8y \le 20$ $\quad 6x + 8y + 1s1 = 20$ $\quad 6x + 8y + 1s1 \qquad = 20$
- $4x + 7y \ge 25$ $\quad 4x + 7y - 1s2 = 25$ $\quad 4x + 7y - 1s2 + 1A1 \qquad = 25$
- $9x + 6y = 50$ $\quad 9x + 6y \qquad = 50$ $\quad 9x + 6y \qquad + 1A2 = 50$
- $x, y \ge 0$ $\qquad x, y, s1, s2, A1, A2 \ge 0$

- Maximize, $z = 6x + 9y + 0s1 + 0s2 - MA1 - MA2$
- $6x + 8y + 1s1 + 0s2 + 0A1 + 0A2 = 20$
- $4x + 7y + 0s1 - 1s2 + 1A1 + 0A2 = 25$
- $9x + 6y + 0s1 + 0s2 + 0A1 + 1A2 = 50$, $x, y, s1, s2, A1, A2 \ge 0$

LALP 5.24

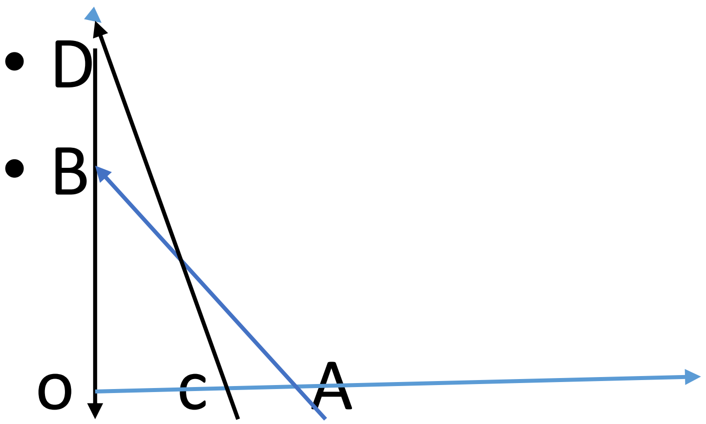
- Problem1:
- Find x and y to maximize z =5x + 6y
- S.t. $x + y \le 4$, $2x + y \le 6$, $x,y \ge 0$



- Consider x + y = 4, and 2x + y = 6
- Draw lines and regions.
- Most common region to both constraint is OCPB. Solve the equations of lines. Get the point P (2 ,2).
- Point Value of 5x + 6y
- O(0 , 0) 5(0)+ 6(0) = 0
- C(3,0) 5(3) + 6(0) =15
- P(2 , 2) 5(2) + 6(2) =22
- B (0 ,4) 5(0) +6(4) =24

Max. occurs at x = 0, y = 4, Max. Profit = 24

- Solving with Simplex.
- Find x and y to maximize $z = 5x + 6y$
- S.t. $x + y \le 4$, $2x + y \le 6$, $y \le 3$ $x, y \ge 0$
- 1 Make equality. 2 A basic variable in each 3 Rhs > 0
- Restructure the problem.
- problem 2 Find x and y to maximize $z = 5x + 6y$
- S.t. $x + y \le 4$, $2x + y \le 6$, $y \le 3$, $x, y \ge 0$



LALP 5.26

- Find x and y to maximize $z = 5x + 8y$ is an objective fun.
- Subject to $1x + 5y \le 11$, $2x + 8y \le 12$ $x, y \ge 0$
- --
- Max. $5x + 8y + 0s1 + 0s2$
- $1x + 5y + 1s1 + 0s2 = 11$
- $2x + 8y + 0s1 + 1s2 = 12$, $x, y, s1, s2 \ge 0$ s1 and s2 are slack.

Cj		5	8	0	0		
	VAR Basis	x	y	s1	s2	Rhs	R.R.
0	s1	1	5	1	0	11	
0	s2	2	8	0	1	12	
	cj-zj						

Find x and y to minimize

$z = 4x - 1y$

s.t. $3x + 5y \leq 20$

$x + 4y \leq 5$

$2x \leq 8,\ x, y \geq 0$

- Min.

Cj		4	-1	0	0	0		
	VAR Basis	x	y	s1	s2	s3	Rhs	R.R.
0	s1	3	5	1	0	0	20	
0	s2	1	4	0	1	0	5	
0	s3	2	0	0	0	1	8	

$z = 4x - 1y$

s.t. $3x + 5y \leq 20$

$x + 4y \leq 5$

$2x \leq 8, \qquad x, y \geq 0$