# Introduction to Algorithm

# CS430 Homework 1

## Question -1

**Problem-1:** Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false?

In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Answer:

False.

For Example:

Preference List for Men:

| | $1^{st}$ | $2^{nd}$ |
|---|---|---|
| x1 | y1 | X2 |
| x2 | y2 | x1 |

By following the men preference list the GS algorithm with return a stable matching (x1, y1) and (x2, y2). So, in this case x1 is pair with y1 but as per the y1's preference list x1 is not her $1^{st}$ preference.

Similarly, if x2 forms a pair with y2 but same in y2's preference list x2 is not y2's $1^{st}$ preference.

Preference list for Women:

| | $1^{st}$ | $2^{nd}$ |
|---|---|---|
| y1 | x2 | x1 |
| y2 | x1 | x2 |

By following the Women preference list the GS algorithm with return a stable matching (y1, x2) and (y2, x1)

According to the preference list for women if y1 is pair with x2 then x2 would not be in pair with y2 who is ranked first in his preference list.

In same way, if y2 pair with x1 then x2 would not be in pair with y1 as per his preference list.

Problem-2

Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. True or false?

Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

Answer:

True.

Preference List for Men:

| | $1^{st}$ | $2^{nd}$ |
|---|---|---|
| x1 | y1 | y2 |
| x2 | y2 | y1 |

Preference list for Women:

| | $1^{st}$ | $2^{nd}$ |
|---|---|---|
| y1 | x1 | x2 |
| y2 | x2 | x1 |

As per the GS Algorithm the man will always propose a woman in the order of his preferences list and if the woman is and she is already engaged, she will trade up, if she is not in pair, she will accept the proposal.

If the man is the 1st on her preference list, they will remain paired.

Proof by Contradiction:

Assume There is a stable matching pair (x1, y2) and (x2, y1) in set S.

In Contrary, as y1 is the first preference of x1, x1 will be in pair with y1 to y2 and same for x2, y2 is ranked first in his preference list so will make pair with y2 to y1.

So, its shows instability so pair (x1, y1) and (x2, y2) must belong to S.

**Problem 5:** (Stable Matching with Indifferences)

Question -1

**Does there always exist a perfect matching with no strong instability?**

**Answer:**

Yes, there always exist a Perfect matching without strong instability by resolving the indifferences.

Initially all men M∈ and women ∈ W are free

While there is a man m who is free and hasn't proposed to every woman

Choose such a man m

Let w be the highest-ranked woman in m's preference list to whom m has not yet proposed

If w is free, then

(m, w) become engaged

else

w is currently engaged to m'

If w prefers m to m' to her current partner, then

w chooses m and make pair (m, w)

m' remains free

else if w prefers her current partner m' to m

(m', w) remain engaged

m becomes free

else

w is indifferent between m and m'

(m', w) remain engaged

m becomes free as for w, m and m' are indifferent.

End if

End if

End while

Return the set S of engaged pairs

**For Example:**

**Men's preference List:**

| | $1^{st}$ | $2^{nd}$ | $3^{rd}$ |
|---|---|---|---|
| M1 | W1 | W2 | W3 |
| M2 | W1 | W2 | W3 |
| M3 | W2 | W3 | W1 |

Women's Preference List:

| | $1^{st}$ | $2^{nd}$ | $3^{rd}$ |
|---|---|---|---|
| W1 | M2 | M1 | M3 |
| W2 | M1 | M3 | M2 |
| W3 | M3 | M2 | W1 |

As per the above preference list in the GS algorithm m1 propose to w1, w1 is free so (m1, w1) become engaged.

M2 propose to W1, as M2 is ranked $1^{st}$ in w1's preference list so (m2, w1) become pair and m1 remain free.

M1 propose next w2, w2 is free so (m1, w2) become match, Next m3 propose to w2,

For w2 m1 and m3 is indifference so will be in pair with m1 and reject m3 proposal

Then M3 propose w3, w3 is free so (m3, w3) become engaged.

So stable matching by GS is (m1, w2), (m2, w1) and (m3, w3) in set S with no strong instability.

Complexity:

In this Algorithm total n number of free men will propose to n different women according to their preference list.

So, Time Complexity of the Algorithm is **$O(n^2)$**

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Question-2

Does there always exist a perfect matching with no weak instability?

Answer:

No.

Example:

For group b1, b2 and g1, g2

Preference list for boys:

| | 1st | 2nd |
|---|---|---|
| B1 | G1 | G2 |
| B2 | G1 | G2 |

As per the boy's preference list G1 and G2 are indifference for Both B1 and B2.

Preference list for Girls:

| | 1st | 2nd |
|---|---|---|
| G1 | B1 | B2 |
| G2 | B1 | B2 |

With the above preference there is a matching pair (B1, G2) and (B2, G1)

For 1st pair (B1, G2):

B1 is engaged with G2 and as per G2's preference list B1 is her 1st preference.

However, For 2nd match (B2, G1):

B2 is engaged with G1 .As per the G1's preference list she prefers B1 to B2 and B2 is not interested in either G1 or G2 means B2 does not care.

Similarly,

B1 propose G1, G1 is free so B1 create a pair with g1 (B1, G1).

B2 propose G1, G1 is already in pair with B1 who her 1st preference is so reject B2,

Next B2 propose G2, G2 is free and make pair (B2, G2)

So, for the pair (B1, G1) and (B2, G2) in set S.

B2 is paired with G2 but for G2, B2 is not her 1st preference, she prefer B1 to B2 whereas for B1 both G1 and G2 are indifferent.

Which prove that there always exists a weak instability in perfect matching.

Problem -3

Truncating Schedule

shipping company that owns **n ships** and provides service to **n ports**.

Each **ship has a schedule** for each day of the month

Month has **m days**

**Constrain:**

1. Each ship visits each port for exactly one day during the month
2. No two ships can be in the same port on the same day
3. for each ship Si, there will be some day when it arrives in its scheduled port and simply remains there for the rest of the month (for maintenance)

**Answer:**

**Data:**

Ship: s

Port: p

Total days in month: m

set s=p=n

Initially All ports are free: initialize the Port Array PA [] to 0

Valid stopping port P is port not in Port Array PA []

get no of days **d**

**set m = d**

**while port Array PA==0**

```
  for (m=d, i=1; m! =0 && i <=n; i++) //for each ship S
          {
                   If   ship Si is at port valid stopping port P
                             Truncate (Si, P)     //Truncate ship Si to Port P
                             PA[i]+=P ;  //Add port P to Port Array PA
                              m = d ;      //reset m to no of days

                  else
                             ship S is at Sea  or  at port P for scheduled visit
                             m=m – 1;
           }
}
```

Return ship and port pairs

### Description:

Initially all ports are free.

Each Ship visit the ports as per its schedule for each day of month.

Each ship will mark the ports from its scheduled no of days to its 1st visit in reverse order. While traversing according day schedule if ship is at port which is not visited or occupied by some ship then ship will be truncated at that port else ship continue its visit to as per its schedule.

### For Example:

No of ship n =2

No of Ports n = 2

No of days m=4

Schedule for Ships:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| S1 | P1 | Sea | P2 | Sea |
| S2 | Sea | P1 | Sea | P2 |

Port Array=0

To truncate ship at port we need to find the valid stopping port for each ship. Port is a valid stopping port if port P is not subset of Port Array PA [].

So, on 4th day ship S1 will be at ship, on 3rd it will be at Port P2 and make pair (S1, P2). Port P2 is not listed in Port Array so P2 is a Valid Stopping port for Ship S1 so ship S1 will remain there for maintenance form 3rd day to rest of the month.

For ship S2 cannot be at port P2 on 4th day of month as Port P2 is not valid stopping port for other any ship after day 3rd of month. So, ship S2 visit sea on 3rd and on 2nd day ship s2 will be at valid stopping port P1 and truncate there for rest of days.

### Time complexity of Algorithm:

N ships will check for the valid stopping port for m days.

So complexity is O(mn)

Problem-4

Assume the group of three girls g1, g2, g3 and three boys b1, b2 and b3 with the following preference list for Girls:

[ G3 true preference and G3' False preference which pretend G3 prefer B3 to B1]

| | 1st | 2nd | 3rd |
|---|---|---|---|
| G1 | B1 | B2 | B3 |
| G2 | B1 | B2 | B3 |
| G3 | B2 | B1 | B3 |
| G3' | B2 | B3 | B1 |

Preference list For Boys:

| | 1st | 2nd | 3rd |
|---|---|---|---|
| B1 | G3 | G1 | G2 |
| B2 | G1 | G3 | G2 |
| B3 | G3 | G1 | G2 |

Execution of GS algorithms with actual preference list of G3 forms a matching pair (B1, G3), (B2, G1) and (B3, G2).

Here G3 is paired with B1 who her 2nd choice in preference list.

Execution of GS Algorithm with G3 's fake preference list where B1 propose to G3, B2 to G1 and lastly B3 to G3.

G3 accept the proposal and pair with B3 and leave her current partner B1 so B1 become free.

Then B1 propose to G1, G1 choose B1 to her partner B2. Next B2 propose to G3 (which is her 1$^{st}$ choice in her preference list) and make pair (B2, G3).

So G3 is finally got the Boy who is truly her favourite which prove by switching order of preference a girl (or Boy ) may be able to get her wanted partner in GS Algorithm.