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Floor and ceilings functions: The floor function assign to the real number x the larger integer that is less than or equal to x. the value of floor function is denoted by $\lfloor x \rfloor$.

The ceiling function assign to a real number x, the smallest integer greater than or equal to x. the value of ceiling function is denoted by $\lceil x \rceil$

eg $\lfloor 2.1 \rfloor = 2$ $\lceil 2.1 \rceil = 3$

$\lfloor -0.1 \rfloor = -1$ $\lfloor 2/3 + \lceil 1/2 \rceil \rfloor = 1$

Theorems:

i) $\lfloor x \rfloor = n$ iff $n \le x < n+1$

ii) $\lfloor x \rfloor = n$ iff $x-1 < n \le x$

iii) $\lceil x \rceil = n$ iff $n-1 < x \le n$

iv) $\lceil x \rceil = n$ iff $x \le n < x+1$

$x+1$	$\lceil x \rceil$	n	$n-1$
1.5	1		
2.5			
3			
5			

check for $\lfloor -x \rfloor = -\lceil x \rceil$ True

$\lceil -x \rceil = -\lfloor x \rfloor$ True

eg: $\lfloor x+n \rfloor = \lfloor x \rfloor + n \quad \forall\, x \in \mathbb{R}\ \&\ n \in \mathbb{Z}$

Assume that $\lfloor x \rfloor = m$ iff $m \le x < m+1$ (Cond 1)

$m+n \le x+n < m+n+1$

$\lfloor m+n \rfloor \le \lfloor x+n \rfloor < \lfloor m+n+1 \rfloor$

$m+n \le \lfloor x+n \rfloor < (m+n)+1$

LHS: $\lfloor x+n \rfloor = m+n$ iff $m+n \le \lfloor x+n \rfloor < m+n+1$

RHS: $\lfloor x \rfloor + n = m+n$

$\therefore$ LHS = RHS

** $\lceil x+n \rceil = \lceil x \rceil + n$

Assume that $\lceil x \rceil = m$ iff $m-1 < x \le m$

$m-1+n < x+n \le m+n$

$\lceil m+n-1 \rceil < \lceil x+n \rceil \le \lceil m+n \rceil$

$(m+n)-1 < \lceil x+n \rceil \le m+n$

LHS $\lceil x+n \rceil = m+n$

RHS $\lceil x \rceil + n = m+n$ $\because \lceil x \rceil = m$

Hence LHS = RHS

eg Data stored in a computer disks or transmitted over on a data notebook are usually represented as string of bytes. Each byte is made up of 8 bits How many bytes are required to encode 100 bits

No. of bytes required to encode the message

$\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13.$

** Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

let $x = 2.2$

$\lfloor x+0.5 \rfloor = \lfloor 2.2+0.5 \rfloor = \lfloor 2.7 \rfloor = 2$

So $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

$= \lfloor 2.2 \rfloor + 2$

$\lfloor 2 \times 2.2 \rfloor = 2+2$

$\lfloor 4.4 \rfloor = 4$

LHS = RHS

soln To prove this statement, we let $x = n + \epsilon$ where n is any positive integer & x is any real no. & where $0 \le \epsilon < 1$

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There are 2 cases to consider, depending on whether ϵ is less than or greater than or equal to $1/2$

take two situations.

1) $0 \le \epsilon < 1/2$

we know that $x = n + \epsilon$

$2 \cdot x = 2n + 2\epsilon$ $0 \le \epsilon < 1/2$

$\lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor$ $0 \times 2 \le 2\epsilon < 1/2 \times 2$

$0 \le 2\epsilon < 1$

LHS $\lfloor 2x \rfloor = 2n$

RHS $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor$ $x + 1/2 = n + \epsilon + 1/2$

$\lfloor x + 1/2 \rfloor + \lfloor x \rfloor$ $\lfloor x + 1/2 \rfloor = \lfloor n + \epsilon + 1/2 \rfloor$

$\lfloor n + \epsilon + 1/2 \rfloor + \lfloor n + \epsilon \rfloor$

$n + n$ $0 \le \epsilon < 1/2$

$\Rightarrow 2n$ $0 + 1/2 \le \epsilon + 1/2 < 1/2 + 1/2$

$0 < 1/2 < \epsilon + 1/2 < 1$

LHS = RHS $0 \le \epsilon + 1/2 < 1$

2) $1/2 \le \epsilon < 1$ $1/2 \le \epsilon < 1$

$x = n + \epsilon$ $1/2 \times 2 \le 2\epsilon < 1 \times 2$

$2x = 2n + 2\epsilon$ $1 \le 2\epsilon < 2$

$\lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor$ $0 \le 2\epsilon - 1 < 1$

$\lfloor 2x \rfloor = \lfloor 2n + 1 + 2\epsilon - 1 \rfloor$

$\lfloor 2n + 1 + 0 \rfloor$

$2n + 1$

LHS $= 2n + 1$ $x = n + \epsilon$

$\lfloor x \rfloor + \lfloor x + 1/2 \rfloor$ $\lfloor x + 1/2 \rfloor = \lfloor n + \epsilon + 1/2 \rfloor$

$\lfloor x + 1/2 \rfloor + \lfloor x \rfloor$ $= \lfloor n + \epsilon + 1 - 1/2 \rfloor$

$\lfloor n + \epsilon + 1 - 1/2 \rfloor + \lfloor n + \epsilon \rfloor$ $\frac{1}{2} \le \epsilon < 1$

$= n + 1 + n$

$2n + 1$ $0 \le \epsilon - 1/2 < 1/2$

$0 \le \epsilon - 1/2 < 1$

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eg $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil \quad \forall x \in \mathbb{R},\ y \in \mathbb{R}$

$x = 0.6, \quad y = 0.7$

$\lceil .6 + .7 \rceil = \lceil .6 \rceil + \lceil .7 \rceil \qquad \lceil .4 \rceil + \lceil .3 \rceil \quad \lceil .4 + .3 \rceil$

$\lceil 1.3 \rceil \neq 1+1 \qquad 1+1 \quad \lceil 0.7 \rceil = 1$

$2 = 2$ true $\qquad 2 \neq 1$ false

$x = n + \epsilon_1 \qquad \lceil x \rceil + \lceil y \rceil = \lceil n + \epsilon_1 \rceil + \lceil n + \epsilon_2 \rceil$

$y = m + \epsilon_2 \qquad = n + 1 + m + 1$

$\lceil x+y \rceil = \lceil n + \epsilon_1 + m + \epsilon_2 \rceil$

$\lceil m + n + \epsilon_1 + \epsilon_2 \rceil \quad \epsilon_1 + \epsilon_2$ may be less than 1

sometimes true sometimes false

$\lceil 1/2 + 1/2 \rceil = \lceil 1/2 \rceil + \lceil 1/2 \rceil$

$\lceil 1 \rceil \qquad \lceil 0.5 \rceil + \lceil 0.5 \rceil$

$1 \neq 2$ (False)

Polynomial functions : A polynomial function is a function that involves only non-negative integer power or only the $\mathbb{Z}$ exponents of a variable in a given equation called a poly. func.

eg: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$

$f(x) = 2x^2 + 3x + 1 \ ; \ 2x - 1 \ ; \ x^3 + 2x$

Types of Polynomial functions :

1) Zero degree poly. fn : / Constant fn 2, 100 : $f(x) = a$

2) Linear poly fn: (1 degree) $f(x) = 2x+3$ OR $ax+b$

3) Quadratic poly fn: (2 degree) $f(x) = ax^2+bx+c$

4) Cubic $f(x) = ax^3+bx^2+cx+d$

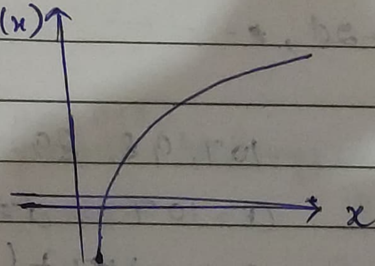
5) exponential fn: $f(x) = 2^x$ (OR) a^x where $a \in \mathbb{R}^+ - \{1\}$

6) Factorial of any variable $f(x) = x!$ OR $n!$

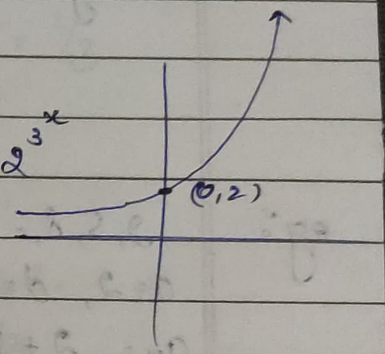
* How many zero does 100! ends with

$$\frac{100}{5} + \frac{100}{5^2} + \frac{100}{5^3} + \cdots = 20+4+0+\cdots = 24$$

$f(n) = \lg x$



$f(n) = 2^{3^x}$



Monotonicity:

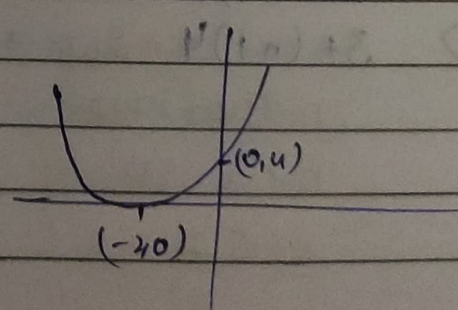
⇒ $f(x) = 3x+3$

$f'(n) = 3$ $f'(n) > 0 \uparrow$

⇒ $f(n) = \frac{x^4}{4} - \frac{4x^3}{6}$ $f'(n) = x^3 - 2x^2$ $f(n) > 0$ when $n > 2$

⇒ $f(n) = x^2+4x+4$ $[-4, 0]$

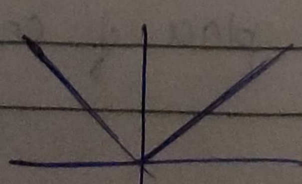
$f'(n) = 2x+4$



⇒ $f(x) = 1/x$ $[-1, 3]$

$f'(n) = -x^{-2}$

⇒ $f(n) = |x|$



$f(n) > 0$ for $n > 0$

$f(n) < 0$ for $x < 0$

Sequences and Summations:

A sequence is a discrete structure used to represent an ordered list.

eg $1, 2, 3, 4 \ldots$ $\qquad a_n = n$

$2, 4, 6, 8 \ldots$ $\qquad a_n = 2n$

$1, 1, 2, 3, 5, 8 \ldots$ $\qquad a_{n+1} = a_n + a_{n-1}$ where $n >$

$1, 1/2, 1/3, 1/4 \ldots$ $\qquad a_n = 1/n$

eg: $1, 2, 3, 4, \ldots$

$a = 1, \ d = 1 \ (a_{n+1} - a_n)$

$\Rightarrow a, a+d, a+2d, \ldots$

eg: $2, 5, 8, \ldots$

$a = 2, \ d = 3$

$a_n = 2 + (n-1)3$

$= a + (n-1)d$

$= 2 + 3n - 3$

$\Rightarrow \boxed{3n - 1}$

$107, 98, 89 \ldots$

$a = 107, \ d = -9$

$a_n = 107 + (n-1)(-9)$

$107 - 9n + 9$

$a_n = \boxed{-9n + 116}$

Ques: Does the number 203 belongs to the given Arith seq.

$3, 7, 11, \ldots$

$a = 3, \ d = 4$

$a + (n-1)d$

$\Rightarrow 3 + (n-1)4 = 203$

$(n-1)4 = 200$

$n-1 = 50$

$n = 51$

Geometric Sequence: A geometric sequence has the form a, ar, ar^2, ar^3. Here we find the common ratio in place of common difference eg $1, 2, 4, 8, 32 \ldots \Rightarrow 1, 2^1, 2^2, 2^3, 2^4 \ldots$

$a = 1, r = 2$

$$\boxed{a_n = ar^{n-1}}$$

eg $3, -6, 12, -24, 48, -96$

$a = 3, \quad r = (-2) \qquad a_n = 3r^{n-1}$

eg: $2, 6, 18, \ldots$

$2, 2\times 3, 2\times 3^2, 2\times 3^3 \ldots$

$a = 2, \quad r = 3$

eg: $486, 162, 54$

$a = 486, \quad r = 1/3$

Ques: Does the no. (48) ~~sequence~~ belongs in the Geometric sequence $3072, 1536, 768 \ldots$ ar^{n-1}

$a = 3072, \quad r = 1/2$

$\frac{3072}{2}$

ro

1536

$\frac{48}{3072} = (1/2)^{n-1}$

$2^{n-1} = 64$

$n-1 = 6$

$n = 7$

* $S_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$2S_n = (n+1) + (2 + (n-1)) + \ldots + (n+1)$

$= (n+1) + (n+1) + \ldots + (n+1)$

$S_n = \frac{n(n+1)}{2}$

for $n = 1 \quad \sum_{r=1}^{1} r^2 = 1^2 = 1 \qquad \frac{1\cdot 2\cdot 3}{6} = 1$

$n = 2 \quad \sum_{r=1}^{2} r^2 = 1^2 + 2^2 = 5 \qquad \frac{2\cdot 3\cdot 5}{6} = 5$

$n = 3 \quad \sum_{r=1}^{3} r^2 = 1^2 + 2^2 + 3^2 = 14 \qquad \frac{3\times 4\times 7}{6} = 14$

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let it is true for $n=k$; will prove for $n=k+1$

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \cdot \frac{6}{6}$$

$$= \frac{(k+1)}{6}\left\{k(2k+1) + 6(k+1)\right\}$$

$$= \frac{(k+1)}{6}\left\{2k^2 + 7k + 6\right\}$$

$$= \frac{(k+1)}{6}\left\{2k^2 + 4k + 3k + 6\right\}$$

$$= \frac{(k+1)}{6}\left(2k(k+2) + 3(k+2)\right)$$

$$\frac{1}{6}(k+1)\left\{((k+1)+1)(2(k+1)+1)\right\}$$

$$= \frac{1}{6} n \cdot (n+1)(2n+1)$$

Hence $\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$

Ques $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

another method: $\sum r^2 = \frac{n(n+1)(2n+1)}{6}$

$$n^3 - (n-1)^3 = n^3 - (n^3 - 3n^2 + 3n - 1) = 3n^2 - 3n + 1$$

$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$

$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$

$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$; $2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$; $1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$

$$n^3 - 0^3 = 3 \sum_{r=1}^{n} r^2 - 3 \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \text{ OR } (n)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{3}\left(n^3 + 3\sum_{r=1}^{n} r - n\right) \Rightarrow \frac{1}{3}\left(n^3 + 3\frac{n(n+1)}{2} - n\right)$$

$$n(n+1)\left\{n-1+\frac{3}{2}\right\} \Rightarrow n(n+1)\left\{\frac{2n-2+3}{2}\right\}$$

$$3\sum_{r=1}^{n} r^2 = n(n+1)\left\{\frac{2n+1}{2}\right\}$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

eg $$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \simeq \frac{n^4}{4}$$ $$\int_1^n r^3 = \frac{n^4}{4} - \frac{1}{4}$$

$$\int_1^n r = \frac{n^2}{2} - \frac{1}{2}$$

Sets

A set is an unordered collection of objects. The object in a set are called elements or members of the set.

eg The set V of all the vowels in english alphabet:

$V = \{e, a, i, o, u\}$

eg: Set of all +ve integers less than 10

$V = \{1, 2, 4, 5, 6, 7, 8, 9\}$

odd integers $O = \{1, 3, 5, 7, 9\}$

OR

$O = \{x \mid x$ is an odd positive int less than $10\}$

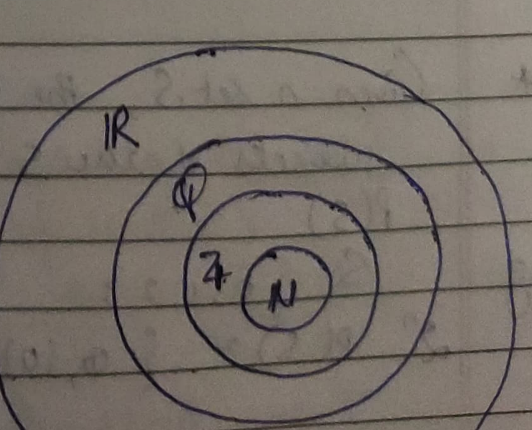
$O = \{x \in \mathbb{Z}^+ \mid x$ is odd and $x < 10\}$

Facts: $\mathbb{Q}^+ = \{x \in \mathbb{R} \mid x = p/q$ for some +ve $\mathbb{Z}$ p & $q\}$

$\mathbb{N} = \{0, 1, 2, \ldots n \mid$ set of Natural no.$\}$

$\mathbb{Q} = \{p/q \mid$ where $p \in \mathbb{Z}$; $q \in \mathbb{Z}$ & $q \neq 0\}$ Set of rational

$\mathbb{R}$ = Set of real no.



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eg Two sets $A = \{1, 3, 5\}$, $B = \{5, 3, 1\}$
then A equals B

$A = \{1, 3, 5\}$ or $B = \{1, 3, 3, 5, 5, 5, 5\}$
A equals B
$|A| = 3$; $|B| = 7$

vowels = $\{a, e, i, o, u\}$
Consonants = $\{b, c, d, f, j, k \ldots\}$

empty set ϕ

Null Set = $A \cap B = \phi$

eg $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$
$A \subseteq B$

* The Set A is said to be subset of B iff every elt of A is also an elt of B

$\forall x (x \in A \rightarrow x \in B)$

* Cardinality of a Set is denoted by | |
$|A| = 3$ OR $|B| = 8$

* Given a set S, the power set of S is the set of all subsets of the S. The power set is denoted by P(S)

eg $S = \{0, 1, 2\}$
2^n $P(S) = \{\phi, \{0\}, \{1\}, \{2\}, \{1, 2\}, \{0, 1\}, \{0, 2\}$
$\{0, 1, 2\}\}$

$S = \phi$
$P(S) = \{\phi, \{\phi\}\}$

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Def: Let A and B are two sets. The cartesion product of A and B denoted by $A \times B$, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

eg $A = \{1,2\}$ and $B = \{a, b, c\}$

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$

$$B \times A = \{(a,1), (a,2), (b,2), (b,1), (c,1), (c,2)\}$$

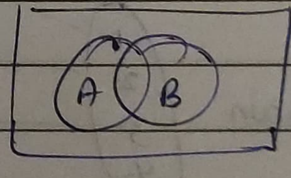
$$A \times B \neq B \times A$$

eg What is the cartesian product of $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$, $C = \{0,1,2\}$

Set operations: $A = \{1,2,3\}$ $B = \{3,4,5\}$

$A \cup B = \{1,2,3,4,5\}$

$A \cap B = \{3\}$



* **Disjoint Set:**

$A = \{1,3,5,7,9\}$ $A \cup B = A + B$

$B = \{2,4,6,8,10\}$ $A \cap B = \phi$

* **Difference (−) / minus:**

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

$A = \{1,3,5\}$ $B = \{4,5,6\}$

$A - B = \{1,3\}$ $B - A = \{4,6\}$

* **Complement of a Set A:** $\bar{A} = U - A$

$A = \{a,e,i,o,u\}$ $\bar{A} = \{b,c,d,f \dots z\}$

eg: The bit string for set {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are

1 1 1 1 1 0 0 0 0 and 1 0 1 0 1 0 1 0 1 0 respectively use bit string to find the union and intersection of these sets.

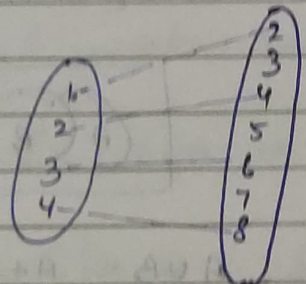
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{1 1 1 1 1 0 1 0 1\}$

$A \cap B = \{1, 3, 5, \} = \{1 0 1 0 1 0 1 0 1\}$

eg: We have seen that the bit string for $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $A = \{1, 2, 3, 4, 5\}$ is $\{1 1 1 1 1 0 0 0 0\}$ then what is the complement of A using bitwise string

$\bar{A} = \{6, 7, 8, 9\} = \{0 0 0 0 0 1 1 1 1\}$

* Types of function (as per the mapping) $f: X \to Y$



$y = f(x)$

$f: X \to Y$

Let $f(x) = 2x$

elts of range

$f\{(1, 2), (2, 4), (3, 6), (4, 8)\}$

Set of elts of domain

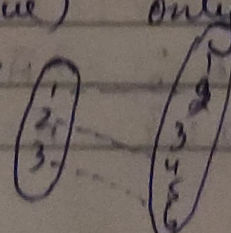
Set of elts of f(x) — Hence range = y 2, 4, 6, 8

Codomain = y

* So Now according to the mapping we will classify the fn:

1 One to One (Injective): every elt in domain has one & only one image in Co-domain (unique)

$f(x) = 2x$ X



1-1 map

(ii) Onto (Surjective): A fn $f: X \to Y$ is stb onto if every elt of Y has pre image in Domain

i.e when Range = Codomain

OR for $y \in Y$ $\exists$ $x \in X$ s.t $f(x) = y$

(iii) Bijective: (Injective + Surjective)