Gerschgorin's Algorithm (गर्शगोरिथ्म)

Find the bounds on eigen values of matrix :

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 2 \end{bmatrix}$$

using
1> Gerschgorin theorem
2> Brauer's theorem
3> Both

Solution :

$$\left.\begin{aligned} R_1 &\Rightarrow \Sigma|R_1| = |1| + |1| + |-1| = 1+1+1 = 3 \\ R_2 &\Rightarrow \Sigma|R_2| = |3| + |2| + |4| = 9 \text{ (largest)} \\ R_3 &\Rightarrow \Sigma|R_3| = |-1| + |4| + |2| = 7 \end{aligned}\right\} \therefore |\lambda| \leq 9 \quad \text{—(1)}$$

$$\left.\begin{aligned} C_1 &\Rightarrow \Sigma|C_1| = |1| + |3| + |-1| = 5 \\ C_2 &\Rightarrow \Sigma|C_2| = |1| + |2| + |4| = 7 \text{ (largest)} \\ C_3 &\Rightarrow \Sigma|C_3| = |-1| + |4| + |2| = 7 \end{aligned}\right\} \therefore |\lambda| \leq 7 \quad \text{—(2)}$$

Now

$$\lambda \leq 9 \quad \cap \quad |\lambda| \leq 7$$

$$\lambda \in [-9 \quad 9] \quad \cap \quad \lambda \in (-7 \quad 7)$$

$$\Rightarrow \lambda \in [-7 \quad 7] \qquad \text{Ans}$$

$$|\lambda| \leq 7 \qquad \text{Answer}$$

Brauer's theorem :

$R_1 \Rightarrow \quad |\lambda-1| \leq |1|+|-1| \Rightarrow |\lambda-1| \leq 2$

$R_2 \Rightarrow \quad |\lambda-2| \leq |3|+|4| \Rightarrow |\lambda-2| \leq 7$

$R_3 \Rightarrow \quad |\lambda-2| \leq |-1|+|4| \Rightarrow |\lambda-2| \leq 5$

$-2 \leq (\lambda-1) \leq 2 \Rightarrow -1 \leq \lambda \leq 3$ —— (1)

$-7 \leq (\lambda-2) \leq 7 \Rightarrow -5 \leq \lambda \leq 9$ —— (2)

$-5 \leq (\lambda-2) \leq 5 \Rightarrow -3 \leq \lambda \leq 7$ —— (3)

(1) $\cup$ (2) $\cup$ (3)

$[-1, 3] \cup [-5, 9] \cup [-3, 7]$

$= [-5, 9]$ —— (4)

Along rows:

$$\left.\begin{matrix} -5 \leq \lambda \leq 9 \\ |(\lambda-2)| \leq 7 \end{matrix}\right\} \text{—— (5)}$$

$C_1 \Rightarrow \quad |\lambda-1| \leq 4 \Rightarrow -3 \leq \lambda \leq 5$

$C_2 \Rightarrow \quad |\lambda-2| \leq 5 \Rightarrow -3 \leq \lambda \leq 7$

$C_3 \Rightarrow \quad |\lambda-2| \leq 5 \Rightarrow -3 \leq \lambda \leq 7$

$[-3, 5] \cup [3, 7] \cup [-3, 7]$

$$-3 \leq \lambda \leq 7$$

Along Column

$$|\lambda - 2| \leq 5$$ ⑥

⑤ $\cap$ ⑥

$$[-5, 9] \cap [-3, 7]$$

$$[-3, 7]$$

$$|\lambda - 2| \leq 5$$

Answer

Q(1) Show by using Gerschgorin's theorems, that the symmetric matrix $A = \begin{bmatrix} -6 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 7 \end{bmatrix}$ has an eigen value

$\lambda_1 \in [5, 9]$. By finding $D_\alpha A D_\alpha^{-1}$ where $D_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$

and taking a suitable value of α, show that $\lambda_1 \in [6.4, 7.6]$

Q(2) $A = \begin{bmatrix} 1 & 0.01 & -0.01 \\ 0.01 & -2 & 0.02 \\ -0.01 & 0.02 & 3 \end{bmatrix}$

Show by using Gerschgorin's theorems that there is an eigenvalue $\lambda = -2$ to within ± 0.03. By finding $D_\alpha A D_\alpha^{-1}$ for a suitable diagonal matrix show that

$\lambda_2 = -2$ to within ± 0.00013.

Gerschgorin's Algorithm (गर्शगोरिन)

Solution (1)

$$A = \begin{bmatrix} -6 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 7 \end{bmatrix}$$

$$\left.\begin{aligned} R_1 &\Rightarrow |-6| + |1| + |1| = 8 \\ R_2 &\Rightarrow |1| + |2| + |-1| = 4 \\ R_3 &\Rightarrow |1| + |-1| + |7| = 9 \end{aligned}\right\} \text{maximum} = 9 \quad \text{—— (1)}$$

$$\left.\begin{aligned} C_1 &\Rightarrow |-6| + |1| + |1| = 8 \\ C_2 &\Rightarrow |1| + |2| + |-1| = 4 \\ C_3 &\Rightarrow |1| + |-1| + |7| = 9 \end{aligned}\right\} \text{maximum} = 9 \quad \text{—— (2)}$$

$$(1) \cap (2) \Rightarrow 9$$

$$\boxed{\lambda \leq 9}$$

$$\boxed{-9 \leq \lambda \leq 9}$$

Brauer's theorem :-

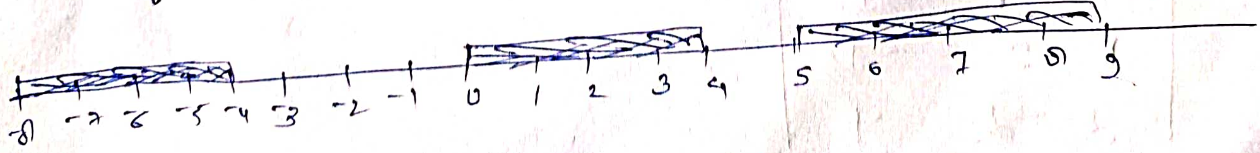
$$|\lambda + 6| \leq 2 \Rightarrow -8 \leq \lambda \leq -4 \quad \text{—— (1)}$$

$$|\lambda - 2| \leq 2 \Rightarrow 0 \leq \lambda \leq 4 \quad \text{—— (2)}$$

$$|\lambda - 7| \leq 2 \Rightarrow 5 \leq \lambda \leq 9 \quad \text{—— (3)}$$

$$(1) \cup (2) \cup (3)$$

$$[-8, -4] \cup [0, 4] \cup [5, 9] \quad \text{—— (4)}$$



$C_1 \Rightarrow \quad |\lambda+6| \leq 2 \quad \Rightarrow \quad -8 \leq \lambda \leq -4$ —— (5)

$C_2 \Rightarrow \quad |\lambda-2| \leq 2 \quad \Rightarrow \quad 0 \leq \lambda \leq 4$ —— (6)

$C_3 \Rightarrow \quad |\lambda-7| \leq 2 \quad \Rightarrow \quad 5 \leq \lambda \leq 9$ —— (7)

$\lambda \in \quad [-8,-4] \cup [0,4] \cup [5,9]$ —— (8)

Final answer $\Rightarrow$ (5) $\cap$ (8)

$\lambda \in \quad [-8,-4] \cup [0,4] \cup [5,9]$

(Here eqn (5) and eq (8) are same because matrix is symmetric) **Answer**

$D_\alpha \; A \; D_\alpha^{-1}$

$$D_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \qquad D_\alpha^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{bmatrix}$$

$D_\alpha \quad A \quad D_\alpha^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} -6 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} -6 & 1 & \alpha \\ 1 & 2 & -\frac{1}{\alpha} \\ 1 & -1 & 7/\alpha \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 & 1/\alpha \\ 1 & 2 & -1/\alpha \\ \alpha & -\alpha & 7 \end{bmatrix}$$

$$|\lambda+6| \leq \left(1+\frac{1}{\alpha}\right)$$

$$|\lambda-2| \leq 1+\left|\frac{-1}{\alpha}\right|$$

$$|\lambda-7| \leq |\alpha|+|-\alpha|$$

$\therefore$

$$|\lambda+6| \leq 1+\frac{1}{\alpha} \quad \text{—— (9)}$$

$$|\lambda-2| \leq 1+\frac{1}{\alpha} \quad \text{—— (10)}$$

$$|\lambda-7| \leq 2\alpha \quad \text{—— (11)}$$

$$-2\alpha+7 \leq \lambda \leq 2\alpha+7$$

from equation (7) we are getting $5 \leq \lambda \leq 9$ when $|\lambda-7| \leq 2$
So corresponding to equation (7) we will use equation (11) so that with the help of α we would change the eigen value range from 5 to 9 to 6.4 to 7.6 (new range)

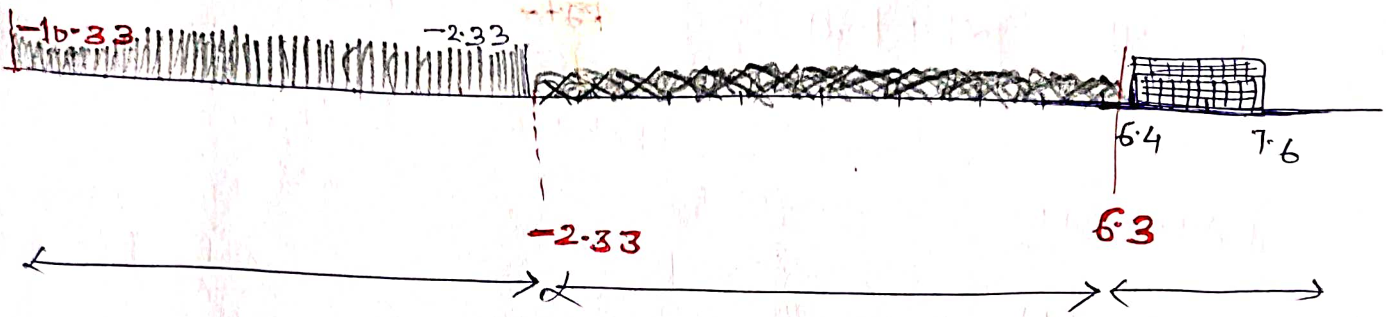
$$-2\alpha+7 \leq \lambda \leq 2\alpha+7$$

$$6.4 \leq \lambda \leq 7.6$$

$2\alpha+7=7.6$	$-2\alpha+7=6.4$
$2\alpha=0.6$	$-2\alpha=-0.6$
$\alpha=0.3$	$2\alpha=0.6$
	$\alpha=0.3$

So if we will use $\alpha=0.3$ then new range will be [6.4 7.6].

Location of new eigen value ⇒



$[-10.33, -2.33] \cup [-2.33, 6.3] \cup [6.3, 7.6]$

Q(2) $A = \begin{bmatrix} 1 & 0.01 & -0.01 \\ 0.01 & -2 & 0.02 \\ -0.01 & 0.02 & 3 \end{bmatrix}$

Show by using Gerschgorin's theorems that there is an eigenvalue $\lambda = -2$ to within ± 0.03. By finding $D_\alpha A D_\alpha^{-1}$ for a suitable diagonal matrix show that $\hat{\lambda}_2 = -2$ to within ± 0.00013.

Solution

$$A = \begin{bmatrix} 1 & 0.01 & -0.01 \\ 0.01 & -2 & 0.02 \\ -0.01 & 0.02 & 3 \end{bmatrix}$$

D_1 centered at 1, radus 0.02

D_2 centered at -2, radius 0.03

D_3 centered at 3 radius 0.03

$|\lambda - 1| \leq 0.02$ $\qquad$ $-0.02 \leq (\lambda - 1) \leq 0.02$

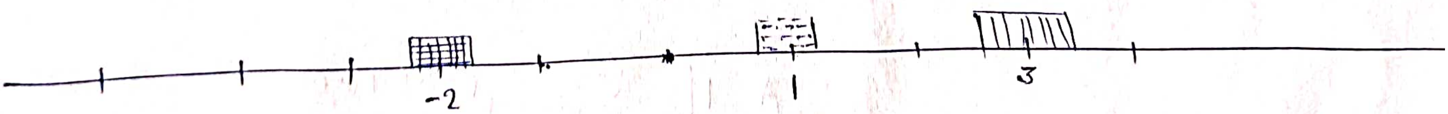
$|\lambda + 2| \leq 0.03$ $\qquad$ $-0.03 \leq (\lambda + 2) \leq 0.03$

$|\lambda - 3| \leq 0.03$ $\qquad$ $-0.03 < \lambda - 3 \leq 0.03$

$0.98 \leq \lambda \leq 1.02$

$-2.03 \leq \lambda \leq -1.97$

$2.97 \leq \lambda \leq 3.03$



From the figure above it is clear that matrix A has eigen value $\lambda_2 = -2 \pm 0.3$

Now we have to make $\lambda_2 = -2 \pm 0.00013$

$$D_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_\alpha \quad A \quad D_\alpha^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.01 & -0.01 \\ 0.01 & -2 & 0.02 \\ -0.01 & 0.02 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{0.01}{\alpha} & -0.01 \\ 0.01 & \frac{-2}{\alpha} & 0.02 \\ -0.01 & \frac{0.02}{\alpha} & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{0.01}{\alpha} & -0.01 \\ 0.01\alpha & -2 & 0.02\alpha \\ -0.01 & \frac{0.02}{\alpha} & 3 \end{bmatrix}$$

$$|\lambda - 1| \leq 0.01 + \frac{0.01}{\alpha} \quad \text{——— (4)}$$

$$|\lambda + 2| \leq 0.03\alpha \quad \text{——— (5)}$$

$$|\lambda - 3| \leq 0.01 + \frac{0.02}{\alpha} \quad \text{——— (6)}$$

$$1 - 0.01 - \frac{0.01}{\alpha} \le \lambda \le 1 + 0.01 + \frac{0.01}{\alpha} \quad \text{—— (8)} : D_1^\alpha$$

$$-2 - 0.03\alpha \le \lambda \le -2 + .03\alpha \quad \text{—— (9)} : D_2^\alpha$$

$$3 - 0.01 - \frac{0.02}{\alpha} \le \lambda \le 3 + .01 + \frac{.02}{\alpha} \quad \text{—— (10)} : D_3^\alpha$$

From eqn (9)

$$\lambda = -2 \pm 0.03\alpha$$

$$\lambda = -2 \pm 0.00013$$

$$\therefore \quad 0.03\alpha = 0.00013$$

$$\boxed{\alpha = 0.00433}$$

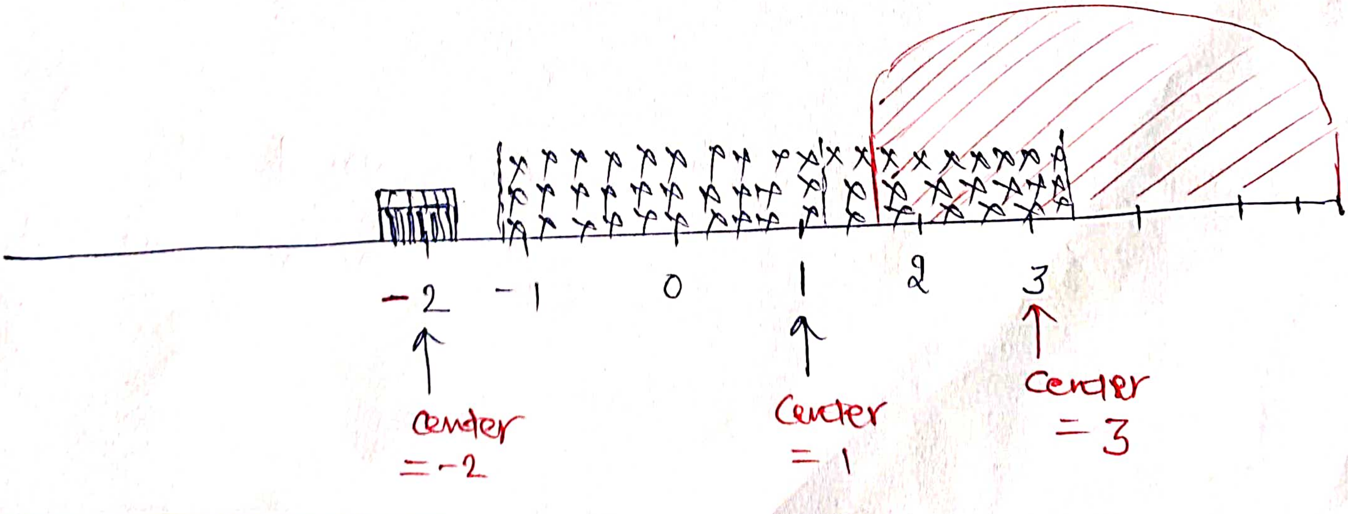
Note: In order to decrease D_2^α decrease α, such that D_2^α doesnot overlap either D_1^α or D_3^α.

Put the value of α in equation (8), (9) and (10)

$$-1.3194 \le \lambda \le 3.3194 \iff \lambda = 1 \pm 2.3355$$

$$-2.0001299 \le \lambda \le -1.9998 \iff \lambda = -2 \pm 0.0001299$$

$$-1.6209 \le \lambda \le 7.6209 \iff \lambda = 3 \pm 4.6209$$



Conjugate Gradient Method
(Fletcher Reeves)

Q: Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1 x_2 + x_2^2$ starting from the point $X_1 = (0, 0)$

Sol^n: The gradient of f is

$$\Delta f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix} \quad \text{--- (1)}$$

$$A = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \qquad \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22}. \end{matrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

Iteration 1 : $X_1 = \overset{x_1, x_2}{(0, 0)}$ Put $x_1 = 0$, $x_2 = 0$ in eq^n (1)

Step 1 $\Rightarrow$ $\Delta f_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad S_1 = -\nabla f_1 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$

$\because S_1 = -\nabla f_1 + \beta_1 S$

$S_1 = -\nabla f_1$ $\quad$ 0

Step 2 $\Rightarrow$

$$\lambda_1 = \frac{\nabla f_1^T \cdot \nabla f_1}{S_1^T \, A \, S_1} = \frac{[1 \ \ -1] \begin{bmatrix} +1 \\ -1 \end{bmatrix}}{[-1 \ \ 1] \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \frac{2}{[-1 \ \ 1] \begin{bmatrix} -2 \\ 0 \end{bmatrix}} = \frac{2}{2} = 1$$

Hence new point ⇒

$$X_2 = X_1 + \lambda_1 S_1$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step III   Check the optimum ⇒

$$\nabla f_2 = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix} \quad \text{Put} \begin{pmatrix} x_1 = -1 \\ x_2 = 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 - 4 + 2 \\ -1 - 2 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So X_2 is not optimum so move next iteration.

Iteration 2:   At X_2

Step 1

$$\nabla f_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \text{ and } \beta_2 = \frac{|\nabla f_2|^2}{|\nabla f_1|^2} \rightarrow (-1)^2 + (-1)^2$$

$$= \frac{2}{2} = 1$$

$$S_2 = -\nabla f_2 + \beta_2 S_1$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Step II

$$\lambda_2 = \frac{\nabla f_2^T \cdot \nabla f_2}{S_2^T A S_2} = \frac{\begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$= \frac{2}{\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix}} = \frac{2}{8} = \frac{1}{4}$$

Thus new point is ⇒

$$X_3 = X_2 + \lambda_2 S_2$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$$

Step III

Check the optimum

$$\nabla f_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus X_3 is optimum point.

$$\text{Ans} = X_3 = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$$

$$\boxed{x_1 = -1, \quad x_2 = 1.5}$$

Answer

Note If you want to calculate the next iteration then you also get the same result as;

Iteration 3 At X_3

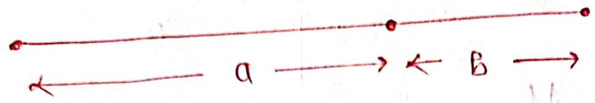
$$\nabla f_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\beta_3 = \frac{|\nabla f_3|^2}{|\nabla f_2|^2} = \frac{0}{2} = 0$$

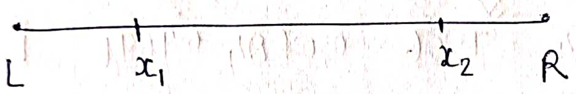
$$S_3 = -\nabla f_3 + \beta_3 S_2 = -\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This shows that there is no search direction to reduce f further, and hence X_3 is optimum.

Golden section search method



$$\frac{a}{b} = \frac{(a+b)}{a} = 1.618$$



$$x_2 = L + 0.618\,(R-L) \qquad \text{①}$$

$$x_1 + x_2 = L + R \qquad \therefore\ x_1 = L + R - x_2 \qquad \text{②}$$

Step

① * First calculate x_1 and x_2 from equation ① and ②.

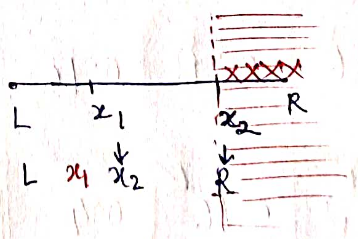
② * If $f(x_1) < f(x_2)$

$R \leftarrow x_2$

$x_2 \leftarrow x_1$

Rreserve L

$x_1 = L + R - x_2$



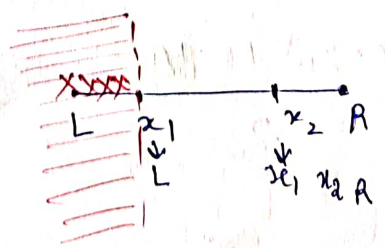
* If $f(x_1) > f(x_2)$

$L \leftarrow x_1$

$x_1 \leftarrow x_2$

Rreserve R

$x_2 = L + R - x_1$

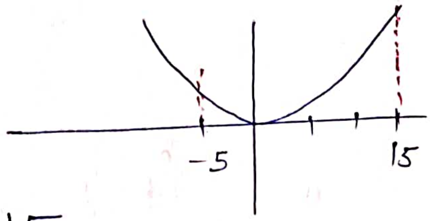


* Now repeat the step ② for next iterations.

[1] Minimize $f(x) = x^2$ over $[-5, 15]$ using golden section search method. (Take $n = 7$)

<u>Sol</u>

Sol^n $f(x) = x^2$



$[-5 \quad 15]$

$L = -5 \qquad R = 15$

$x_2 = L + 0.618(R-L) = -5 + 0.618\,[15+5] = 7.36$

$x_1 = L + R - x_2 = -5 + 15 - 7.36 = 2.64$

K	L	x_1	x_2	R	$f(x_1)$	$f(x_2)$	L/R	Comment
1	-5	2.64	7.36	15	6.9696	54.17	L	$f(x_1) < f(x_2)$
2	-5	0.28 $(L+R-x_2)$	2.64	7.36	0.0784	6.9696	L	$f(x_1) < f(x_2)$
3	-5	-2.08 $(L+R-x_2)$	0.28	2.64	4.3264	0.0784	R	$f(x_1) > f(x_2)$
4	-2.08	-0.28	0.84 $(L+R-x_1)$	2.64	0.0784	0.7056	L	$f(x_1) < f(x_2)$
5	-2.08	-0.96	-0.28	0.84	0.9216	0.0784	R	$f(x_1) > f(x_2)$
6	-0.96	-0.28	0.16	0.84	0.0784	0.0256	R	$f(x_1) > f(x_2)$
7	$\boxed{-0.28}$	0.16	0.4	$\boxed{0.84}$	0.0256	0.16		

$$x^* = \frac{(-0.28) + (0.84)}{2} = \frac{0.56}{2} = 0.28$$

$$f(0.28) = (0.28)^2 = 0.0784$$

Ans

[2] Minimize $f(x) = x(x-1.5)$ in $[0, 1]$ by golden search rule with interval of uncertainity as 0.3

$L = 0 \qquad R = 1$

$$\begin{bmatrix} x_2 = L + 0.618\,(R-L) \\ x_1 = L + R - x_2 \end{bmatrix}$$

K	L	x_1	x_2	R	$f(x_1)$	$f(x_2)$	L/R	Comment
1	0	0.382	0.618	1	-0.42708	-.54508	R	$f(x_1) > f(x_2)$
2	0.382	0.618	0.764	1	-.54508	-.5623	R	$f(x_1) > f(x_2)$
3	.618	0.764	0.854	1	-.5623	-.5516	L	$f(x_1) < f(x_2)$
4	.618	0.708	0.764	0.854	-.5607	-.5623	R	$f(x_1) > f(x_2)$
5								

At K=4:

Since $0.854 - 0.618 = 0.236 < \epsilon = 0.3$ So we stop

Hence $x^* = \dfrac{0.618 + 0.854}{2} = 0.736$

$f(x^*) = 0.736 \times (.736 - 1.5) = -.5623$

Simple and inverse iterative method

Power method for finding dominant eigen value calculator: [Rayleigh's Power Method]

Q:
Find the largest eigen value of the matrix $A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Solution: let initial eigen vector $\Rightarrow X^0 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$$Y^1 = AX^0 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -4 \\ 3 \end{bmatrix}$$

now we will keep numerically largest value outside of the matrix.

$$Y^1 = 4 \begin{bmatrix} 3/4 \\ 1 \\ -4 \\ 3 \end{bmatrix} \text{———} (1)$$

$$Y^2 = AX_1$$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1 \\ -1 \\ 3/4 \end{bmatrix} = \begin{bmatrix} +\frac{10}{4} \\ -15/4 \\ 15/4 \\ -\frac{10}{4} \end{bmatrix}$$

$$Y^2 = \frac{15}{4} \begin{bmatrix} 2/3 \\ -1 \\ 1 \\ -2/3 \end{bmatrix}$$

$$Y^3 = AX_2 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -1 \\ 1 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 11/3 \\ -11/3 \\ 7/3 \end{bmatrix} = \frac{11}{3} \begin{bmatrix} -\frac{7}{11} \\ 1 \\ -1 \\ \frac{7}{11} \end{bmatrix}$$

$$Y^4 = AX_3 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -7/11 \\ 1 \\ -1 \\ \frac{7}{11} \end{bmatrix} = \begin{bmatrix} 25/11 \\ -40/11 \\ 40/11 \\ -25/11 \end{bmatrix} = \frac{40}{11} \begin{bmatrix} 25/40 \\ -1 \\ 1 \\ -\frac{25}{40} \end{bmatrix}$$

$$Y^5 = AX_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 25/40 \\ -1 \\ 1 \\ -25/40 \end{bmatrix} = \begin{bmatrix} -90/40 \\ 145/40 \\ -145/40 \\ 90/40 \end{bmatrix} = \frac{145}{40} \begin{bmatrix} \frac{-90}{145} \\ 1 \\ -1 \\ \frac{90}{145} \end{bmatrix}$$

$$Y^6 = AX_5 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -90/145 \\ 1 \\ -1 \\ 90/145 \end{bmatrix} = \begin{bmatrix} \frac{325}{145} \\ -\frac{525}{145} \\ +\frac{525}{145} \\ -\frac{325}{145} \end{bmatrix} = \begin{bmatrix} 65/29 \\ -105/29 \\ 105/29 \\ -65/29 \end{bmatrix}$$

$$Y^6 = \frac{105}{29} \begin{bmatrix} \frac{65}{105} \\ -1 \\ 1 \\ -\frac{65}{105} \end{bmatrix}$$

$$Y^7 = AX_6 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{65}{105} \\ -1 \\ 1 \\ -\frac{65}{105} \end{bmatrix} = \begin{bmatrix} -2.2381 \\ 3.619 \\ -3.619 \\ 2.2381 \end{bmatrix} = 3.619 \begin{bmatrix} -0.6184 \\ 1 \\ -1 \\ 0.6184 \end{bmatrix}$$

$$Y^8 = AX_7 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -0.6184 \\ 1 \\ -1 \\ 0.6184 \end{bmatrix} = \begin{bmatrix} 2.2368 \\ -3.6184 \\ 3.6184 \\ -2.2368 \end{bmatrix} = 3.6184 \begin{bmatrix} 0.6182 \\ -1 \\ 1 \\ -0.6182 \end{bmatrix} \quad -(8)$$

$$Y^9 = AX_8 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} .6182 \\ -1 \\ 1 \\ -0.6182 \end{bmatrix} = \begin{bmatrix} -2.2364 \\ 3.6182 \\ -3.6182 \\ 2.2364 \end{bmatrix} = 3.6182 \underbrace{\begin{bmatrix} -0.6181 \\ 1 \\ -1 \\ 0.6181 \end{bmatrix}}_{X_9}$$

In eq (8) and (9) vectors are same so we will stop here. -(9)

Dominant eigen value $= \lambda = 3.6182 \simeq 3.62$

Dominant eigen vector $= \begin{bmatrix} -0.6181 \\ 1 \\ -\lambda \\ 0.6181 \end{bmatrix} \cong \begin{bmatrix} -0.62 \\ 1 \\ -1 \\ 0.62 \end{bmatrix}$

Method (II) to calculate eigen value:

From Step ⑧ and ⑨

Since: $X_9 - X_8 = 0$

Hence the eigen value is ⇒

$$\lambda_i = \frac{Y_9}{X_8} = \frac{(-2.2364,\ 3.6182,\ -3.6182,\ 2.2364)}{(0.6182,\ -1,\ 1,\ -0.6182)}$$

$$= -3.6176,\ -3.6182,\ -3.6182,\ -3.6176$$

Eigen value $= -3.6182$

Eigen vector $= \begin{bmatrix} -0.6181 \\ 1 \\ -1 \\ 0.6181 \end{bmatrix}$ [from eqⁿ ⑨]

Method (III) (Short trick)

$$Y^9 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0.6182 \\ \boxed{-1} \\ 1 \\ -0.6182 \end{bmatrix} = \begin{bmatrix} -2.2364 \\ \boxed{3.6182} \\ -3.6182 \\ 2.2364 \end{bmatrix}$$

Largest in number (donot consider sign)

Correspoing value

$$\lambda = \frac{3.6182}{-1}$$

$$= -3.6182$$

* This technique is mostly used when number of iterations is given So we donot check the stop criteria.

This method III is known as simple and inverse iterative method.

Q: Find the eigen value nearest to 3 for the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ using power method.

Perform five iteration and take initial Vector as (1, 1, 1)

Solution: Step I

$$A - 3I = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

Eigen value of A which is nearest to 3 = smallest eigen (magnitude) value of (A - 3I)

Step II

$$\therefore \; B = (A-3I)^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

(We will calculate largest eigen value of Matrix B)

Find the Largest eigen value of B :·

Initial matrix $X^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Y^1 = BX^0 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$Y^2 = BX^1 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$Y^3 = BX^2 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 0.6667 \\ -1 \\ 0.6667 \end{bmatrix}$$

$$Y^4 = BX^3 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0.6667 \\ -1 \\ .6667 \end{bmatrix} = \begin{bmatrix} 1.6667 \\ -2.3334 \\ 1.6667 \end{bmatrix} = 2.3334\begin{bmatrix} 0.7143 \\ -1 \\ 0.7143 \end{bmatrix}$$

$$Y5 = BX^4 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} 0.7143 \\ -1 \\ 0.7143 \end{bmatrix}}_{X^4} = \underbrace{\begin{bmatrix} 1.7143 \\ -2.4286 \\ 1.7143 \end{bmatrix}}_{Y^5} = 2.4286\begin{bmatrix} 0.7058 \\ -1 \\ 0.7058 \end{bmatrix}$$

After the five iteration, we obtain $\Rightarrow$

$$\lambda = \frac{Y5}{X^4} = \frac{(1.7143, \ -2.4286, \ 1.7143)}{(.7143, \ -1, \ .7143)}$$

$$= (2.40, \ \underline{\underline{2.43}}, \ 2.40)$$

Hence the largest eigen value of $B = (A-3I)^{-1}$
$= 2.43$

Smallest eigen value of $(A - 3I) = \frac{1}{2\cdot 43}$

Eigen value of A near to 3 is $= 3 + \frac{1}{2\cdot 43}$

$= \underline{3\cdot 42}$ Ans

Householder's Method

(Reduction to tridiagonal form)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Tridigonal form of $A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$

Step I: Let initial matrix $V_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$

Step II: $s_1 = \sqrt{a_{12}^2 + a_{13}^2}$

Step III: Find $T_1 = I - 2 V_1 V_1^T$

where $x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{s_1}\right]$ $\qquad x_3 = \frac{a_{13}}{2 x_2 s_1}$

Here T_1 is both symmetric and orthogonal

Step 4 :: Calculate $A_1 = T_1 A T_1$

and find tridiagonal form.

Q: Transform the matrix in tridiagonal form using house holder method.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 1 \end{bmatrix}$$

Sol^n :

Let $v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$ $S_1 = \sqrt{a_{12}^2 + a_{13}^2}$

$$S_1 = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{S_1}\right] = \frac{1}{2}\left[1 + \frac{3}{5}\right] = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}$$

$$\therefore \boxed{x_2 = \frac{2}{\sqrt{5}}}$$

$$x_3 = \frac{a_{13}}{2 x_2 S_1} = \frac{4}{2 * \left(\frac{2}{\sqrt{5}}\right) * 5} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$$\boxed{x_3 = \frac{1}{\sqrt{5}}}$$

$$\therefore v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Now ⇒

$$T_1 = I - 2 V_1 V_1'$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4/5 & \frac{2}{5} \\ 0 & 2/5 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8/5 & 4/5 \\ 0 & \frac{4}{5} & \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & -4/5 \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$A_1 = T_1 \quad A \; T_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ 3 & \frac{-2}{5} & \frac{-11}{5} \\ 4 & \frac{-1}{5} & \frac{7}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 \\ -5 & \frac{2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{13}{5} \end{bmatrix}$$

Q: Reduce the following symmetric matrix to tridiagonal form using Householder's method.

$$A = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Sol^n: Let $v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$

$$S_1 = \sqrt{a_{12}^2 + a_{13}^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{S_1}\right] = \frac{1}{2}\left[1 + \frac{3}{5}\right] = \frac{1}{2} \times \frac{8}{5} = \frac{4}{5}$$

$\therefore x_2 = \frac{2}{\sqrt{5}}$

$$x_3 = \frac{a_{13}}{2 x_2 S_1} = \frac{4}{2 \times \frac{2}{\sqrt{5}} \times 5} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$$\therefore v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$T_1 = I - 2 v_1 v_1'$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & -4/5 & +3/5 \end{bmatrix}$$

$$A_1 = T_1 \quad A \quad T_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -\frac{4}{5} \\ 0 & -4/5 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ 3 & -\frac{2}{5} & -\frac{11}{5} \\ 4 & -1/5 & 7/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 \\ -5 & 2/5 & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{13}{5} \end{bmatrix}$$

QR - Decomposition

(using rotation matrix)

$$\boxed{A = Q \cdot R}$$

$A \Rightarrow$ Square matrics.

$Q \Rightarrow$ Orthogonal matrix.

$R \Rightarrow$ Upper triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 1 & 2 \\ \boxed{3} & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$i=2, j=1$ we have to kill $a_{21} = 3$,

$C = \frac{TOP}{HYP}$ c_{ii} c_{jj} $i \rightarrow$ Row

$S = \frac{KILL}{HYP}$ $-s_{ij}$ s_{ji} $\longrightarrow$ $j \rightarrow$ column

$$c = \frac{\overset{\rightarrow \text{Top}}{4}}{\sqrt{4^2+3^2}} = \frac{4}{5}$$

$$s = \frac{\overset{\rightarrow \text{kill element}}{3}}{\sqrt{4^2+3^2}} = \frac{3}{5}$$

$$G_1 = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Put negative sign where you want to kill.

$$G_1 = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 5 & 2 & 3.4 \\ 0 & 1 & 1.2 \\ \boxed{2} & 3 & 2 \end{bmatrix}$$

We have to remove this element

$i=3 \quad j=1$

$$A_1 = G_1 A = \begin{bmatrix} 5 & 2 & 3.4 \\ 0 & 1 & 1.2 \\ 2 & 3 & 2 \end{bmatrix}$$

$j=1 \quad i=3$

$$c = \frac{5}{\sqrt{5^2+2^2}} = \frac{5}{\sqrt{29}} = 0.9284 \qquad ii, jj$$

$$s = \frac{2}{\sqrt{5^2+2^2}} = \frac{2}{\sqrt{29}} = .3713$$

$$G_2 = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

$$G_2 A_1 = \begin{bmatrix} \frac{5}{\sqrt{29}} & 0 & \frac{2}{\sqrt{29}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{29}} & 0 & \frac{5}{\sqrt{29}} \end{bmatrix} \begin{bmatrix} 5 & 2 & 3.4 \\ 0 & 1 & 1.2 \\ 2 & 3 & 2 \end{bmatrix}$$

$$G_2 * (G_1 A) = \begin{bmatrix} 5.385 & 2.9711 & 3.0996 \\ 0 & 1 & 1.2 \\ 0 & 2.0426 & 0.5942 \end{bmatrix}$$

$$= A_2$$

$$A_2 = \begin{bmatrix} 5.385 & 2.9711 & 3.0996 \\ 0 & 1 & 1.2 \\ 0 & \boxed{2.0426} & 0.5942 \end{bmatrix}$$

$i = 3, j = 2$

We have to make zero this element

$$A_{2,(2\times2)} = \begin{bmatrix} 1 & 1.2 \\ \boxed{2.0426} & 0.5942 \end{bmatrix}$$

kill it

$$c = \frac{1}{\sqrt{1^2 + 2.0426^2}} = 0.4397 \quad s = \frac{2.0406}{\sqrt{1^2 + 2.0426^2}} = 0.8972$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$G_{3,2\times 2}$}} \\ 0 & & \end{bmatrix}$$

$i=2 \quad j=1$

$$G_{3,\,2\times2} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \quad \text{or} \quad i=3\ j=2 \quad G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

$$G_3 * A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4397 & .8972 \\ 0 & -.8972 & .4397 \end{bmatrix} \begin{bmatrix} 5.385 & 2.9711 & 3.8996 \\ 0 & 1 & 1.2 \\ 0 & 2.0426 & 0.5942 \end{bmatrix}$$

$$G_3(G_2 G_1 A) = \begin{bmatrix} 5.385 & 2.9711 & 3.8996 \\ 0 & 2.2723 & 1.0607 \\ 0 & 0 & -0.0153 \end{bmatrix}$$

$$G_3 G_2 G_1 A \simeq \begin{bmatrix} 5.39 & 2.97 & 3.9 \\ 0 & 2.27 & 1.06 \\ 0 & 0 & -0.02 \end{bmatrix}$$

$$= R$$

$$\rightarrow \boxed{R = G_3 G_2 G_1 A}$$

$$A = QR$$

$$Q^{-1}A = Q^{-1}QR$$

$$Q^{-1}A = R$$

$$\Rightarrow \quad Q^{T}A = R$$

$$Q^{T} = (G_3\ G_2\ G_1)$$

$$Q = (G_3\ G_2\ G_1)^{T}$$

$$\boxed{Q = G_1^{T}\ G_2^{T}\ G_3^{T}}$$

Note:

$$G_1 = \begin{bmatrix} 0.8 & .6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.9284 & 0 & 0.3713 \\ 0 & 1 & 0 \\ -0.3713 & 0 & 0.9284 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4397 & 0.8972 \\ 0 & -0.8972 & 0.4397 \end{bmatrix}$$

$$Q = G_1^{T} \cdot G_2^{T} \cdot G_3^{T}$$

$$=$$

QR Decomposition
(using Gram Schmidt process)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Here I will drop T notation for simplicity, but we have to remember that all vectors are column vectors.

$$u_1 = a_1 = (1, 1, 0)$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_2 = a_2 - (a_2 \cdot e_1)\, e_1 = (1, 0, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3/2}}\left(\frac{1}{2}, \frac{-1}{2}, 1\right) = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$u_3 = a_3 - (a_3 \cdot e_1)\, e_1 - (a_3 \cdot e_2)\, e_2$$

$$= (0, 1, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$= \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$e_3 = \frac{u_3}{\|u_3\|} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Thus :

$$Q = \left[e_1 \;\vdots\; e_2 \;\vdots\; e_3 \;\vdots\; \ldots \right]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

$$\boxed{A = Q * R}$$

Sturm Sequence Method

*

*
$$A = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & 0 \\ \beta_2 & \alpha_2 & \beta_3 & 0 \\ 0 & \beta_3 & \alpha_3 & \beta_4 \\ 0 & 0 & \beta_4 & \alpha_4 \end{bmatrix}$$

* A is tridiagonal, symmetric matrix.

$$p_1(\lambda) = \text{Det}[A' - \lambda I] = \text{Det}[\alpha_1 - \lambda] = \alpha_1 - \lambda$$

$$p_2(\lambda) = \text{Det}\begin{bmatrix} \alpha_1 - \lambda & \beta_2 \\ \beta_2 & \alpha_2 - \lambda \end{bmatrix}$$

$$p_2(\lambda) = (\alpha_1 - \lambda)(\alpha_2 - \lambda) - \beta_2^2$$

$$p_3(\lambda) = \text{Det}\begin{bmatrix} \alpha_1 - \lambda & \beta_2 & 0 \\ \beta_2 & \alpha_2 - \lambda & \beta_3 \\ 0 & \beta_3 & \alpha_3 - \lambda \end{bmatrix}$$

$$= (\alpha_3 - \lambda) * p_2(\lambda) - \beta_3 \; \det\begin{bmatrix} \alpha_1 - \lambda & \beta_2 \\ 0 & \beta_3 \end{bmatrix}$$

$$= (\alpha_3 - \lambda)\; p_2(\lambda) - \beta_3^2\; (\alpha_1 - \lambda)$$

$$= (\alpha_3 - \lambda)\; p_2(\lambda) - \beta_3^2\; p_1(\lambda)$$

$$p_4(\lambda) = (\alpha_4 - \lambda)\; p_3(\lambda) - \beta_4^2\; p_2(\lambda)$$

* Now find sturm sequence $\Rightarrow p_0(\hat{\lambda}), p_1(\hat{\lambda}), p_2(\hat{\lambda}), p_3(\hat{\lambda}), p_4(\hat{\lambda})$
* Calculate the number of times sign changes (N_1).
* It means there are N_1 Eigen values which are less than or equal to $\hat{\lambda}$.

Q: Use the sturm sequence property to deduce that matrix

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$ has one eigen value in $(-\infty, -6)$.

Use newton's method to find this eigen value to six decimal places, starting with $\mu_0 = -7$.

Solution:

$$A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & 0 \\ \beta_2 & \alpha_2 & \beta_3 & 0 \\ 0 & \beta_3 & \alpha_3 & \beta_4 \\ 0 & 0 & \beta_4 & \alpha_4 \end{bmatrix}$$

$$\boxed{p_1(\lambda) = \text{Det}[\alpha_1 - \lambda] = (-3-\lambda)} \quad \text{—— (1)}$$

$$p_2(\lambda) = \text{Det}\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = (-3-\lambda)(-3-\lambda) - 1^2$$

$$= (3+\lambda)^2 - 1$$

$$= 9 + \lambda^2 + 6\lambda - 1$$

$$\boxed{p_2(\lambda) = \lambda^2 + 6\lambda + 8} \quad \text{—— (2)}$$

$$p_3(\lambda) = \text{Det}\begin{bmatrix} -3-\lambda & 1 & 0 \\ 1 & -3-\lambda & 2 \\ 0 & 2 & -4-\lambda \end{bmatrix}$$

$$= \left[(-3-\lambda)(-3-\lambda) - 1\right](-4-\lambda) - 2\left[2(-3-\lambda) - 0\right]$$

$= (\lambda^2+6\lambda+8)(-4-\lambda) + 4(3+\lambda)$

$$\boxed{p_3(\lambda) = -[\lambda^3+10\lambda^2+28\lambda+20]} \quad \text{——— (3)}$$

$$p_4(\lambda) = \text{Det}\begin{bmatrix} -3-\lambda & 1 & 0 & 0 \\ 1 & -3-\lambda & 2 & 0 \\ 0 & 2 & -4-\lambda & 2 \\ 0 & 0 & 2 & -5-\lambda \end{bmatrix}$$

$p_4(\lambda) = (-\lambda^3-10\lambda^2-36\lambda-44)(-5-\lambda) - 2^2[\lambda^2+6\lambda+8]$

$$\boxed{p_4(\lambda) = \lambda^4+15\lambda^3+74\lambda^2+136\lambda+68} \quad \text{——— (4)}$$

$\lambda \rightarrow$	$-\infty$	-8	-6	0	∞	Comment
Initial $p_0(\lambda)$	+	+	+	+	+	Initial setting
$p_1(\lambda) = (-3-\lambda)$	+	+	+	−	−	Calculate sign by putting $\lambda=-\infty,-8,-6,0,\infty$ in $p_1(\lambda)$
$p_2(\lambda) = \lambda^2+6\lambda+8$	+	+	+	+	+	Calculate sign by putting λ in $p_2(\lambda)$
$p_3(\lambda) = -(\lambda^3+10\lambda^2+28\lambda+20)$	+	+	+	−	−	Calculate sign by putting λ in $p_3(\lambda)$
$p_4(\lambda) = \lambda^4+15\lambda^3+74\lambda^2+136\lambda+68$	+	+	−	+	+	Calculate sign by putting λ in $p_4(\lambda)$
Number of changes of sign	0	0	1	4	4	Calculate the number of sign change.

Hence real roots $\Rightarrow (-\infty$ to $+\infty)$

$= 4 - 0 = 4$

Number of +ve roots $\Rightarrow$ 0 to ∞

$= 4 - 4 = 0$

Number of negative roots $\Rightarrow -\infty$ to 0

$= 4 - 0 = 4$

Number of roots between -8 to -6 $\Rightarrow$

$= 1 - 0 = 1$

Thus from sturm sequence method we can say that in given question matrix has one eigen value in $(-8$ to $-6)$.

NOTE $p_4(\lambda) = \lambda^4 + 15\lambda^3 + 74\lambda^2 + 136\lambda + 68$

Eigen Values $= -0.78, -2.83, -4.35, -7.029$

(from calculator)

Part II

From sturm sequence we have found that matrix A has one eigen value inbetween -8 to -6. To find exact value we will use Newton's method.

as per question initial guess = -7

$$\lambda^4 + 15\lambda^3 + 74\lambda^2 + 136\lambda + 68 = 0$$

Let :-

$f(x) = x^4 + 15x^3 + 74x^2 + 136x + 68$

$f'(x) = \frac{d}{dx} f(x) = 4x^4 + 45x^2 + 148x + 136$

1st Iteration :

$x_0 = -7$

$\therefore f(-7) = -2$

$f'(-7) = -67$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= -7 - \frac{(-2)}{(-67)}$

$= -7.029851$

$f(-7) = -2$

$f'(-7) = -67$

2nd Iteration

$f(x_1) = f(-7.029851) = 0.047573$

$f'(x_1) = f'(-7.029851) = -70.199037$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -7.029851 - \frac{0.047573}{-70.199037}$

$= -7.029173$

3rd iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -7.029173 - \frac{0.000025}{-70.125635}$$

$$= -7.029173$$

$f(-7.029173)$
$= 0.000028$

$f'(-7.029173)$
$= -70.125635$

4th iteration..

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= -7.029173 - \frac{0}{-70.125597}$$

$$= -7.029173$$

Approximate root of the equation $x^4 + 15x^3 + 74x^2 + 136x + 68 = 0$ using Newton Raphson Method is -7.029173 (after 4 iterations).

- Simple and inverse iterative method

- Power method for finding dominant eigen value calculator: [Rayleigh's Power Method]

Q: Find the largest eigen value of the matrix $A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Solution:

Let initial eigen vector $\Rightarrow X_0 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Note: $\|X_0\|_\infty = 1$

$$Y^1 = AX_0 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -4 \\ 3 \end{bmatrix}$$

now we will keep numerically largest value outside of the matrix.

$$Y^1 = 4 \begin{bmatrix} 3/4 \\ 1 \\ -1 \\ 3 \end{bmatrix} \quad \text{——— (1)}$$

$$Y^2 = AX_1$$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1 \\ -1 \\ 3/4 \end{bmatrix} = \begin{bmatrix} +\frac{10}{4} \\ -15/4 \\ 15/4 \\ -\frac{10}{4} \end{bmatrix}$$

$$Y^2 = \frac{15}{4} \begin{bmatrix} 2/3 \\ -1 \\ 1 \\ -2/3 \end{bmatrix}$$

$$Y^3 = AX_2 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -1 \\ 1 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 11/3 \\ -11/3 \\ 7/3 \end{bmatrix} = \frac{11}{3} \begin{bmatrix} -\frac{7}{11} \\ 1 \\ -1 \\ \frac{7}{11} \end{bmatrix}$$

$$Y^4 = AX_3 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -7/11 \\ 1 \\ -1 \\ \frac{7}{11} \end{bmatrix} = \begin{bmatrix} 25/11 \\ -40/11 \\ 40/11 \\ -25/11 \end{bmatrix} = \frac{40}{11} \begin{bmatrix} 25/40 \\ -1 \\ 1 \\ \frac{-25}{40} \end{bmatrix}$$

$$Y5 = AX_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 25/40 \\ -1 \\ 1 \\ -25/40 \end{bmatrix} = \begin{bmatrix} -90/40 \\ 145/40 \\ -145/40 \\ 90/40 \end{bmatrix} = \frac{145}{40} \begin{bmatrix} \frac{-90}{145} \\ 1 \\ -1 \\ \frac{90}{145} \end{bmatrix}$$

$$Y6 = AX_5 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -90/145 \\ 1 \\ -1 \\ 90/145 \end{bmatrix} = \begin{bmatrix} \frac{325}{145} \\ -\frac{525}{145} \\ +\frac{525}{145} \\ -\frac{325}{145} \end{bmatrix} = \begin{bmatrix} 65/29 \\ -105/29 \\ 105/29 \\ -65/29 \end{bmatrix}$$

$$Y6 = \frac{105}{29} \begin{bmatrix} \frac{65}{105} \\ -1 \\ 1 \\ -\frac{65}{105} \end{bmatrix}$$

$$Y7 = AX_6 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{65}{105} \\ -1 \\ 1 \\ \frac{-65}{105} \end{bmatrix} = \begin{bmatrix} -2.2381 \\ 3.619 \\ -3.619 \\ 2.2381 \end{bmatrix} = 3.619 \begin{bmatrix} -0.6184 \\ 1 \\ -1 \\ 0.6184 \end{bmatrix}$$

$$Y8 = AX_7 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -0.6184 \\ 1 \\ -1 \\ 0.6184 \end{bmatrix} = \begin{bmatrix} 2.2368 \\ -3.6184 \\ 3.6184 \\ -2.2368 \end{bmatrix} = 3.6184 \underbrace{\begin{bmatrix} 0.6182 \\ -1 \\ 1 \\ -0.6182 \end{bmatrix}}_{x_8} \quad -(8)$$

$$Y9 = AX8 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} .6182 \\ -1 \\ 1 \\ -0.6182 \end{bmatrix} = \begin{bmatrix} -2.2364 \\ 3.6182 \\ -3.6182 \\ 2.2364 \end{bmatrix} = 3.6182 \underbrace{\begin{bmatrix} -0.6181 \\ 1 \\ -1 \\ 0.6181 \end{bmatrix}}_{x_9}$$

In eq (8) and (9) vectors are same so we will stop here. — (9)

Dominant eigen value $= \lambda = 3.6182 \simeq 3.62$

Dominant eigen vector $= \begin{bmatrix} -0.6181 \\ 1 \\ -\lambda \\ 0.6181 \end{bmatrix} \cong \begin{bmatrix} -0.62 \\ 1 \\ -1 \\ 0.62 \end{bmatrix}$

Method (II) To calculate eigen value :-

from step (8) and (9)

Since: $X_9 - X_8 = 0$

Hence the eigen value is ⇒

$$\lambda_i = \frac{Y_9}{X_8} = \frac{(-2.2364, \; 3.6182, \; -3.6182, \; 2.2364)}{(0.6182, \; -1, \; 1, \; -0.6182)}$$

$$= -3.6176, \; -3.6182, \; -3.6182, \; -3.6176$$

[Largest in number
do not consider sign for comparision]

Eigen Value $= -3.6182$

[Because this is largest in magnitude]

Eigen vector $= \begin{bmatrix} -0.6181 \\ 1 \\ -1 \\ 0.6181 \end{bmatrix}$ [from eqn (9)]

Method (III) (Short trick) [Simple and Inverse Iterative method]

$$Y^9 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0.6182 \\ \boxed{-1} \\ 1 \\ -0.6182 \end{bmatrix} = \begin{bmatrix} -2.2364 \\ \boxed{3.6182} \\ -3.6182 \\ 2.2364 \end{bmatrix}$$

Largest in number (donot consider sign)

Corresponding value

$$\lambda = \frac{3.6182}{-1}$$

$$= -3.6182$$

* This technique is mostly used when number of iterations is given so we donot check the stop criteria.

This method III is known as simple and inverse Iterative method.

Answer verification: From the calculator

Eigen value of matrix $A \Rightarrow \lambda = -3.6180, -2.6180, -1.3819, -0.3819$

Eigen vector corresponding to Eigen value λ }

$$\begin{bmatrix} -1 \\ 1.6180 \\ -1.6180 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.6180 \\ -0.6180 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -0.6180 \\ 0.6180 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.6180 \\ 1.6180 \\ 1 \end{bmatrix}$$

- Simple and inverse iterative method

- Power method for finding dominant eigen value calculator: [Rayleigh's Power Method]

Q: Find the largest eigen value of the matrix $A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Solution:

let initial eigen vector $\Rightarrow X^0 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Note

$\|X^0\|_\infty = 1$

$$Y^1 = AX^0 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -4 \\ 3 \end{bmatrix}$$

now we will keep numerically largest value outside of the matrix.

$$Y^1 = 4 \begin{bmatrix} 3/4 \\ 1 \\ -1 \\ 3 \end{bmatrix} \quad \text{———} (1)$$

$$Y^2 = AX^1$$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1 \\ -1 \\ 3/4 \end{bmatrix} = \begin{bmatrix} +\frac{10}{4} \\ -15/4 \\ 15/4 \\ -\frac{10}{4} \end{bmatrix}$$

$$Y^2 = \frac{15}{4} \begin{bmatrix} 2/3 \\ -1 \\ 1 \\ -2/3 \end{bmatrix}$$

$$Y^3 = AX^2 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -1 \\ 1 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 11/3 \\ -11/3 \\ 7/3 \end{bmatrix} = \frac{11}{3} \begin{bmatrix} -\frac{7}{11} \\ 1 \\ -1 \\ \frac{7}{11} \end{bmatrix}$$

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Dominant eigen value $= \lambda = 3.6182 \simeq 3.62$

Dominant eigen vector $= \begin{bmatrix} -0.6181 \\ 1 \\ -\lambda \\ 0.6181 \end{bmatrix} \cong \begin{bmatrix} -0.62 \\ 1 \\ -1 \\ 0.62 \end{bmatrix}$

Method (II) To calculate eigen value :-

from Step (8) and (9)

Since: $X^{9} - X^{8} = 0$

Hence the eigen value is $\Rightarrow$

$$\lambda_i = \frac{Y^9}{X^8} = \frac{(-2.2364, \; 3.6182, \; -3.6182, \; 2.2364)}{(0.6182, \; -1, \; 1, \; -0.6182)}$$

$$= -3.6176, \; -3.6182, \; -3.6182, \; -3.6176$$

[Largest in number do not consider sign for comparision]

[Because this is largest in magnitude]

$\therefore$ Eigen value $= -3.6182$

Eigenvector $= \begin{bmatrix} -0.6181 \\ 1 \\ -1 \\ 0.6181 \end{bmatrix}$ [from eqn (9)]

Method (III) (Short trick) [Simple and Inverse Iterative Method]

$$Y^9 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0.6182 \\ \boxed{-1} \\ 1 \\ -0.6182 \end{bmatrix} = \begin{bmatrix} -2.2364 \\ \boxed{3.6182} \\ -3.6182 \\ 2.2364 \end{bmatrix}$$

Largest in number (donot consider sign)

Corresponding value

$$\lambda = \frac{3.6182}{-1}$$

$$= -3.6182$$

* This technique is mostly used when number of iterations is given So we donot check the stop criteria.

This method III is known as simple and inverse iterative method.

Answer verification: From the calculator

Eigen value of matrix $A \Rightarrow \lambda = -3.6180, -2.6180, -1.3819, -0.3819$

Eigen vector corresponding to Eigen value λ }

$$\begin{bmatrix} -1 \\ 1.6180 \\ -1.6180 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.6180 \\ -0.6180 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -0.6180 \\ 0.6180 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1.6180 \\ 1.6180 \\ 1 \end{bmatrix}$$