Householder's Method
(Reduction to tridiagonal form)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Tridiagonal form of $A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$

Step I Let initial matrix $V_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$

Step II: $s_1 = \sqrt{a_{12}^2 + a_{13}^2}$

Step III: Find $T_1 = I - 2 V_1 V_1^T$

where $x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{s_1}\right]$ $\quad x_3 = \frac{a_{13}}{2 x_2 s_1}$

Here T_1 is both symmetric and orthogonal

Step 4 :: Calculate $A_1 = T_1 A T_1$

and find tridiagonal form.

Q: Transform the matrix in tridiagonal form using house holder method.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 1 \end{bmatrix}$$

Solⁿ: Let $v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$ $\qquad S_1 = \sqrt{a_{12}^2 + a_{13}^2}$

$$S_1 = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{S_1}\right] = \frac{1}{2}\left[1 + \frac{3}{5}\right] = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}$$

$$\therefore \boxed{x_2 = \frac{2}{\sqrt{5}}}$$

$$x_3 = \frac{a_{13}}{2 x_2 S_1} = \frac{4}{2 * \left(\frac{2}{\sqrt{5}}\right) * 5} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$$\boxed{x_3 = \frac{1}{\sqrt{5}}} \qquad \therefore v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Now ⇒

$$T_1 = I - 2 V_1 V_1'$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4/5 & \frac{2}{5} \\ 0 & 2/5 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8/5 & 4/5 \\ 0 & \frac{4}{5} & \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & -4/5 \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$A_1 = T_1 \quad A \, T_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ 3 & \frac{-2}{5} & \frac{-11}{5} \\ 4 & \frac{-1}{5} & \frac{7}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 \\ -5 & \frac{2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{13}{5} \end{bmatrix}$$

Q: Reduce the following symmetric matrix to tridiagonal form using Householder's method.

$$A = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Solⁿ: Let $v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$

$$S_1 = \sqrt{a_{12}^2 + a_{13}^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{S_1}\right] = \frac{1}{2}\left[1 + \frac{3}{5}\right] = \frac{1}{2} \times \frac{8}{5} = \frac{4}{5}$$

$$\therefore x_2 = \frac{2}{\sqrt{5}}$$

$$x_3 = \frac{a_{13}}{2 x_2 S_1} = \frac{4}{2 \times \frac{2}{\sqrt{5}} \times 5} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$$\therefore v_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$T_1 = I - 2 v_1 v_1'$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & -4/5 & +3/5 \end{bmatrix}$$

$$A_1 = T_1 \quad A \quad T_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/5 & \frac{-4}{5} \\ 0 & -4/5 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ 3 & -\frac{2}{5} & -\frac{11}{5} \\ 4 & -1/5 & 7/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 \\ -5 & 2/5 & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{13}{5} \end{bmatrix}$$

QR - Decomposition

(using rotation matrix)

$$\boxed{A = Q \cdot R}$$

$A \Rightarrow$ Square matrics.

$Q \Rightarrow$ Orthogonal matrix.

$R \Rightarrow$ Upper triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 1 & 2 \\ \boxed{3} & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$i=2, j=1$ we have to kill $a_{21} = 3$.

$C = \frac{TOP}{HYP}$ c_{ii} c_{jj} $i \rightarrow$ Row

$S = \frac{KILL}{HYP}$ $-s_{ij}$ s_{ji} $\longrightarrow$ $j \rightarrow$ column

$$c = \frac{4 \xrightarrow{} \text{Top}}{\sqrt{4^2+3^2}} = \frac{4}{5}$$

$$s = \frac{3 \xrightarrow{} \text{kill element}}{\sqrt{4^2+3^2}} = \frac{3}{5}$$

$$G_1 = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Put negative sign where you want to kill.

$$G_1 = \begin{bmatrix} 4/5 & \frac{3}{5} & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 5 & 2 & 3.4 \\ 0 & 1 & 1.2 \\ \boxed{2} & 3 & 2 \end{bmatrix}$$

We have to remove this element $i=3 \quad j=1$

$$A_1 = G_1 A = \begin{bmatrix} 5 & 2 & 3.4 \\ 0 & 1 & 1.2 \\ 2 & 3 & 2 \end{bmatrix}$$

$j=1 \; i=3$

$$c = \frac{5}{\sqrt{5^2+2^2}} = \frac{5}{\sqrt{29}} = 0.9284 \qquad ii,\ jj$$

$$s = \frac{2}{\sqrt{5^2+2^2}} = \frac{2}{\sqrt{29}} = .3713$$

$$G_2 = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

$$G_2 A_1 = \begin{bmatrix} \frac{5}{\sqrt{29}} & 0 & \frac{2}{\sqrt{29}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{29}} & 0 & \frac{5}{\sqrt{29}} \end{bmatrix} \begin{bmatrix} 5 & 2 & 3.4 \\ 0 & 1 & 1.2 \\ 2 & 3 & 2 \end{bmatrix}$$

$$G_2 * (G_1 A) = \begin{bmatrix} 5.385 & 2.9711 & 3.0996 \\ 0 & 1 & 1.2 \\ 0 & 2.0426 & 0.5942 \end{bmatrix}$$

$$= A_2$$

$$A_2 = \begin{bmatrix} 5.385 & 2.9711 & 3.0996 \\ 0 & 1 & 1.2 \\ 0 & \boxed{2.0426} & 0.5942 \end{bmatrix}$$

$i = 3, j = 2$

We have to make zero this element

$$A_{2,(2\times2)} = \begin{bmatrix} 1 & 1.2 \\ \boxed{2.0426} & 0.5942 \end{bmatrix}$$

kill it

$$c = \frac{1}{\sqrt{1^2 + 2.0426^2}} = 0.4397 \quad s = \frac{2.0406}{\sqrt{1^2 + 2.0426^2}} = 0.8972$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{G_{3,2\times 2}}} \\ 0 & & \end{bmatrix}$$

$i=2 \quad j=1$

$$G_{3,2\times2} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \quad \text{or} \quad i=3\ j=2 \quad G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

$$G_3 * A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4397 & .8972 \\ 0 & -.8972 & .4397 \end{bmatrix} \begin{bmatrix} 5.385 & 2.9711 & 3.8996 \\ 0 & 1 & 1.2 \\ 0 & 2.0426 & 0.5942 \end{bmatrix}$$

$$G_3(G_2 G_1 A) = \begin{bmatrix} 5.385 & 2.9711 & 3.8996 \\ 0 & 2.2723 & 1.0607 \\ 0 & 0 & -0.0153 \end{bmatrix}$$

$$G_3 G_2 G_1 A \simeq \begin{bmatrix} 5.39 & 2.97 & 3.9 \\ 0 & 2.27 & 1.06 \\ 0 & 0 & -0.02 \end{bmatrix}$$

$= R$

$\Rightarrow$ $\boxed{R = G_3 G_2 G_1 A}$

$$A = QR$$

$$Q^{-1} A = Q^{-1} QR$$

$$Q^{-1} A = R$$

$$\Rightarrow \quad Q^T A = R$$

$$Q^T = (G_3 \; G_2 \; G_1)$$

$$Q = (G_3 \, G_2 \, G_1)^T$$

$$\boxed{Q = G_1^T \; G_2^T \; G_3^T}$$

Note:

$$G_1 = \begin{bmatrix} 0.8 & .6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.9284 & 0 & 0.3713 \\ 0 & 1 & 0 \\ -0.3713 & 0 & 0.9284 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4397 & 0.8972 \\ 0 & -0.8972 & 0.4397 \end{bmatrix}$$

$$Q = G_1^T \cdot G_2^T \cdot G_3^T$$

$$=$$

QR Decomposition
(using Gram Schmidt process)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Here I will drop T notation for simplicity, but we have to remember that all vectors are column vectors.

$$u_1 = a_1 = (1, 1, 0)$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_2 = a_2 - (a_2 \cdot e_1)\, e_1 = (1, 0, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3/2}}\left(\frac{1}{2}, \frac{-1}{2}, 1\right) = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$u_3 = a_3 - (a_3 \cdot e_1)\, e_1 - (a_3 \cdot e_2)\, e_2$$

$$= (0, 1, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$= \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$e_3 = \frac{u_3}{\|u_3\|} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Thus :-

$$Q = \left[e_1 \;\vdots\; e_2 \;\vdots\; e_3 \;\vdots\; \dots \right]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

$$\boxed{A = Q * R}$$