ALIAS METHOD

Assumptions :

Let $X \sim p$ where $p = (p_1\ p_2\ \cdots\ p_n)$ is a PMF

$P(X=i) = p_i \qquad i = 1, 2 \cdots n$

Goal :

To express $p = \frac{1}{n-1} \sum_{k=1}^{n-1} q^{(k)}$ where each $q^{(k)}$ having at most two nonzero components and are valid PMFs (Probability mass function)

Lemma :

Let $p = (p_1, p_2, \cdots, p_n)$ be a PMF. Then

* there exist j such that $p_j < \frac{1}{(n-1)}$
* For this j there exist i where $j \neq i$ such that

$$\boxed{p_i + p_j \geq \frac{1}{(n-1)}}$$

Note

$$p_k = \frac{1}{(n-1)}\left[q_k^1 + q_k^2 + \cdots + q_k^{n-1}\right]$$

Let $n=3$

$$p_k = \frac{1}{2}\left[q_k^1 + q_k^2\right]$$

Let $n=4$

$$p_k = \frac{1}{3}\left[q_k^1 + q_k^2 + q_k^3\right]$$

$n=3$

	1	2
p_1	q_1^1	q_1^2
p_2	q_2^1	q_2^2
p_3	q_3^1	q_3^2

$p_1 = \frac{1}{2}\left(q_1^1 + q_1^2\right)$

$p_2 = \frac{1}{2}\left(q_2^1 + q_2^2\right)$

$p_3 = \frac{1}{2}\left(q_3^1 + q_3^2\right)$

where $\begin{cases} q^1 = (q_1^1, q_2^1, q_3^1) \\ q^2 = (q_1^2, q_2^2, q_3^2) \end{cases}$

$n=4$

	1	2	3
p_1	q_1^1	q_1^2	q_1^3
p_2	q_2^1	q_2^2	q_2^3
p_3	q_3^1	q_3^2	q_3^3
p_4	q_4^1	q_4^2	q_4^3

$p_1 = \frac{1}{3}\left[q_1^1 + q_1^2 + q_1^3\right]$

$p_2 = \frac{1}{3}\left[q_2^1 + q_2^2 + q_2^3\right]$

$p_3 = \frac{1}{3}\left[q_3^1 + q_3^2 + q_3^3\right]$

$p_4 = \frac{1}{3}\left[q_4^1 + q_4^2 + q_4^3\right]$

where : $\begin{cases} q^1 = (q_1^1, q_2^1, q_3^1, q_4^1) \\ q^2 = (q_1^2, q_2^2, q_3^2, q_4^2) \\ q^3 = (q_1^3, q_2^3, q_3^3, q_4^3) \end{cases}$

Three point distribution : (3- Point distribution)

Let $p = \left(\frac{7}{16}, \frac{1}{2}, \frac{1}{16}\right)$

$= (0.437, 0.5, 0.062)$

Using lemma choose $j = 3$ that is $p_3 < \frac{1}{2}$ and $i = 2$

such that

$$p_2 + p_3 > \frac{1}{2}$$

$p_2 + p_3 = \frac{1}{2} + \frac{1}{16} = \frac{8+1}{16} = \frac{9}{16} > \frac{1}{2}$ (Consider this)

$p_1 + p_3 = \frac{7}{16} + \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$ (We will not consider)

Now

We have to distribute this in $q^1 = (\_, \_, \_)$

and $q^2 = (\_, \_, \_)$ such that

$$p_K = \frac{1}{2}\left(q^1_K + q^2_K\right)$$

* $q^1 = (\_, \_, \_)$ and $q^2 = (\_, \_, \_)$ can have at most two non zero ~~entries~~ entries.

* q^1 has two nonzero entries for points 2 and 3 (because $p_2 + p_3 > \frac{1}{2}$).

* p_3 is lesser than p_2 so we may consider $q^2_3 = 0$

$$p_3 = \frac{1}{2}\left[q^1_3 + q^2_3\right]$$

$$\frac{1}{16} = \frac{1}{2}\left[q^1_3 + 0 \text{ (let)}\right]$$

$$q^1_3 = \frac{1}{8}$$

* q^1 has non zero entries and points 2 and 3 and we can say $q^1_1 = 0$

[Because $p_2 + p_3 > \frac{1}{2}$ and p_1 is out from game)

* $q^1_2 = 1 - q^1_3 = 1 - \frac{1}{8} = 7/8$

* $q^2_3 = 0$ (Discussed earlier)

* $q^2_2 = ?$

$$p_K = \frac{1}{2}\left[q^1_K + q^2_K\right]$$

$$p_2 = \frac{1}{2}\left[q^1_2 + q^2_2\right]$$

$$\frac{1}{2} = \frac{1}{2}\left[\frac{7}{8} + q^2_2\right]$$

$$1 = \frac{7}{8} + q^2_2$$

$$q^2_2 = \frac{1}{8}$$

$$q^2_3 = 0 \quad \therefore \quad q^2_1 = 1 - \frac{1}{8} = 7/8$$

Thus $q^1 = \left(0, \; 7/8, \; \frac{1}{8}\right)$

$q^2 = \left(7/8, \; \frac{1}{8}, \; 0\right)$

and $p = \frac{1}{2}(q^1 + q^2)$ is the desired distribution

Summary:

	1	2
$p_1 = \frac{7}{16}$	$q^1_1 = 0$	q^2_1 $7/8$
$p_2 = \frac{1}{2}$	q^1_2 $7/8$	q^2_2 $1/8$
$p_3 = \frac{1}{16}$	q^1_3 $\frac{1}{8}$	$q^2_3 = 0$

Short cut method : 3-Point distribution

$$p = \left(\frac{7}{16}, \frac{1}{2}, \frac{1}{16}\right)$$

$$(0.437, \underline{0.5}, \underline{0.0632})$$

Step(I)

	1	2
7/16	0	
$\frac{1}{2}$	7/8	
$\frac{1}{16}$	$\frac{1}{8}$	0

Step(II)

	1	2
7/16	0	$\frac{7}{8}$
$\frac{1}{2}$	7/8	$\frac{1}{8}$
$\frac{1}{16}$	1/8	0

$2 \times \frac{7}{16} = 7/8$

↓ Now

$= 1 - 7/8 = \frac{1}{8}$

$$\begin{cases} q^1 = \left(0, 7/8, \frac{1}{8}\right) \\ q^2 = \left(7/8, \frac{1}{8}, 0\right) \end{cases}$$

Answer

4 - point distribution

$$p = \frac{7}{16}, \frac{1}{4}, \frac{1}{8}, \frac{3}{16}$$

0.43, 0.25, 0.125, 0.18

Step (I)

$\frac{1}{3} = 0.3$

$p_3 < \frac{1}{3}$ & $p_1 + p_3 > \frac{1}{3}$

$\therefore q'_2 = 0$ and $q'_4 = 0$

* $p_3 < p_1$ $(0.125 < 0.43)$

So $q^2_3 = 0$ & $q^3_3 = 0$

	1	2	3
$p_1 = 7/16 = 0.43$			
$p_2 = 1/4 = 0.25$	0		
$p_3 = \frac{1}{8} = 0.125$		0	0
$p_4 = \frac{3}{16} = 0.18$	0		

Step II

$$p_3 = \frac{1}{3}\left(q_3^1 + q_3^2 + q_3^3\right)$$

$$\frac{1}{8} = \frac{1}{3}\left[q_3^1 + 0 + 0\right]$$

$$q_3^1 = \frac{3}{8}$$

$$q_1^1 = 1 - \frac{3}{8} = \frac{5}{8}$$

	1	2	3
$p_1 = 7/16$	$5/8$		
$p_2 = \frac{1}{4}$	0		
$p_3 = \frac{1}{8}$	$\frac{3}{8}$	0	0
$p_4 = \frac{3}{16}$	0		

Step III

$$p_1 = \frac{1}{3}\left[q_1^1 + q_1^2 + q_1^3\right]$$

or

$$p_K = \frac{1}{3}\left[q_K^1 + q_K^2 + q_K^3\right]$$

$$3p_K = q_K^1 + q_K^2 + q_K^3$$

$$q_K^2 + q_K^3 = 3p_K - q_K^1 \quad \text{———} \; (1)$$

$$q_1^2 + q_1^3 = 3 * 7/16 - \frac{5}{8} = \frac{21}{16} - \frac{10}{16} = \frac{11}{16}$$

$$q_2^2 + q_2^3 = 3 * \left(\frac{1}{4}\right) - 0 = \frac{3}{4}$$

$$q_3^2 + q_3^3 = 3 * \left(\frac{1}{8}\right) - \frac{3}{8} = 0$$

$$q_4^2 + q_4^3 = 3 * \left(\frac{3}{16}\right) - 0 = \frac{9}{16}$$

$$\Bigg\} \text{—— (2)}$$

$$\because \frac{11}{16} + \frac{3}{4} + 0 + \frac{9}{16} = \frac{11 + 12 + 0 + 9}{16} = \frac{32}{16} = 2$$

** So to convert 4-point distribution into 3 point point distribution we have to devide equation(2) by '2'.

Therefore $p = \left(\frac{11}{32}, \frac{3}{8}, 0, \frac{9}{32}\right)$ —— (1)

** We can also use formula as given below:

$$\boxed{p^* = \frac{3p - q'}{2}}$$ Imp

$$\therefore p^* = \left(\frac{11}{32}, \frac{3}{8}, 0, \frac{9}{32}\right)$$

Now we have to solve $p = \left(\frac{11}{32}, \frac{3}{8}, 0, \frac{9}{32}\right)$ **considering a case of** **<u>3-point distribution</u>.**

$$p = \left(\frac{11}{32}, \frac{3}{8}, \frac{9}{32}\right)$$

$.34, \quad .37 \quad .28$

Case (i)

We will solve this as a 3-point distribution case

$$\frac{9}{32} < \frac{1}{2}$$

$$\frac{9}{32} + \frac{3}{8} > \frac{1}{2}$$

Step I

	1	2
11/32	0	
3/8	7/16	
9/32	$\frac{9}{16}$	0

Step II:

	1	2
11/32	0	??
3/8	7/16	?
9/32	$\frac{9}{16}$	0

$$\frac{3}{8} = \frac{1}{2}\left[\frac{7}{16} + ?\right]$$

$$? = \frac{6}{8} - \frac{7}{16} = \frac{5}{16}$$

$$?? = 1 - \frac{5}{16} = \frac{11}{16}$$

<u>Finally :</u>

	1	2
	0	$\frac{11}{16}$
	$7/16$	$5/16$
	$\frac{9}{16}$	0

$$q^1 = \left(0, \frac{7}{16}, \frac{9}{16}\right) \qquad q^2 = \left(\frac{11}{16}, \frac{5}{16}, 0\right) \quad \text{—— (2)}$$

Case (ii)

$$\frac{11}{32} < \frac{1}{2}$$

$$\frac{11}{32} + \frac{3}{8} > \frac{1}{2}$$

	1	2
$\frac{11}{32}$		0
$\frac{3}{8}$		
$\frac{9}{32}$	0	

$\Rightarrow$

	1	2
	$\frac{11}{16}$	0
	$\frac{5}{16}$	$\frac{7}{16}$
	0	$\frac{9}{16}$

$$q^1 = \left(\frac{11}{16}, \frac{5}{16}, 0\right) \qquad q^2 = \left(0, \frac{7}{16}, \frac{9}{16}\right)$$

—— (3)

Case(iii)

$p = \frac{11}{32}, \frac{3}{8}, \frac{9}{32}$

0.34 0.37 0.28.

$\frac{9}{32} < \frac{1}{2}$

$\frac{9}{32} + \frac{11}{32} > \frac{1}{2}$

Step I

	1	2
11/32		
3/8	0	
9/32		0

Step II

$\Downarrow$

	1	2
$\frac{11}{32}$	7/16	4/16
$\frac{3}{8}$	0	$\frac{12}{16}$ =
$\frac{9}{32}$	$\frac{9}{16}$	0

$q^1 = \left(\frac{7}{16}, 0, \frac{9}{16}\right)$

$q^2 = \left(\frac{4}{16}, \frac{12}{16}, 0\right)$

$\simeq \left(\frac{1}{4}, \frac{3}{4}, 0\right)$

— 4

Now we have to use this solution for 4-point distribution.

	1	2	3
$7/16$	$\frac{5}{8}$	$\frac{11}{16}$	0
$\frac{1}{4}$	0	$\frac{5}{16}$	$7/16$
$\frac{1}{8}$	$\frac{3}{8}$	0	0
$\frac{3}{16}$	0	0	$\frac{9}{16}$

Note: Values written with red colour, came from the solution of 3 point distribution (here I have used the case-ii solution).

The desired distribution:

$$q^1 = \left(\frac{5}{8}, 0, \frac{3}{8}, 0\right)$$

$$q^2 = \left(\frac{11}{16}, \frac{5}{16}, 0, 0\right)$$

$$q^3 = \left(0, \frac{7}{16}, 0, \frac{9}{16}\right)$$

$$p = \frac{1}{3}\left(q^1 + q^2 + q^3\right) \qquad \text{(condition)}$$

Answer

Q: Give a complete distribution for p, a four point distribution as $p_1 = \frac{1}{2}$ $p_2 = \frac{1}{4}$ $p_3 = \frac{3}{16}$ $p_4 = \frac{1}{16}$ are given.

Solution:- 4 point distribution :-

$$p = \left(\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{16}\right)$$

$$= (0.5, 0.25, 0.18, 0.0625)$$

$n = 4$

$$\frac{1}{(4-1)} = \frac{1}{3} = 0.33$$

$$\frac{1}{16} < \frac{1}{3}$$

$$\frac{1}{16} + \frac{1}{2} > \frac{1}{3}$$

Step (1)

	1	2	3
$\frac{1}{2}$	$\frac{13}{16}$		
$\frac{1}{4}$	0		
$\frac{3}{16}$	0		
$\frac{1}{16}$	$\frac{3}{16}$	0	0

— (1)

$q_4^1 = ?$

$$\frac{1}{16} = \frac{1}{3}\left[q_4^1 + q_4^2 + q_4^3 \right]$$

$$\frac{1}{16} = \frac{1}{3}\left[q_4^1 + 0 + 0 \right] \qquad (\text{Let } q_4^2 \text{ \& } q_4^3 = 0)$$

$$\therefore q_4^1 = \frac{3}{16}$$

$$q_1^1 = 1 - \frac{3}{16} = \frac{16-3}{16} = \frac{13}{16}$$

Step (II)

$$p_K = \frac{1}{3}\left[q_K^1 + q_K^2 + q_K^3 \right]$$

$$q_K^2 + q_K^3 = 3p_K - q_K^1$$

$$\left.\begin{aligned} 3 * \left(\frac{1}{2}\right) - \frac{13}{16} &= \frac{24-13}{16} = \frac{11}{16} \\ 3 * \left(\frac{1}{4}\right) - 0 &= \frac{3}{4} \\ 3 * \left(\frac{3}{16}\right) - 0 &= \frac{9}{16} \\ 3 * \left(\frac{1}{16}\right) - \frac{3}{16} &= 0 \end{aligned}\right\} \text{—} (2)$$

$$\frac{11}{16} + \frac{3}{4} + \frac{9}{16} + 0 = \frac{11 + 12 + 9}{16} = \frac{32}{16} = 2$$

To convert 4 point distribution into 3-point distribution we will devide equation [2] by 2.

$\therefore\ p = \left(\frac{11}{32}, \frac{3}{8}, \frac{9}{32}, 0\right)$ —— (3)

$(0.34,\ 0.37\ \ 0.28,\ 0)$

Now we will solve equation (3) considering a problem (case) of 3 point distribution

3 point distribution :-

$$p = \left(\frac{11}{32}, \frac{3}{8}, \frac{9}{32}\right)$$

$0.34 \qquad 0.37 \qquad 0.28$

Step(I):-

$\frac{9}{32} < \frac{1}{2}$ $\qquad$ $\frac{9}{32} + \frac{3}{8} > \frac{1}{2}$

Step (II)

11/32	0	
3/8	7/16	
9/32	9/16	0

Step(II)

11/32	0	??
3/8	7/16	?
9/32	9/16	0

$\frac{3}{8} = \frac{1}{2}\left(\frac{7}{16} + ?\right)$

$? = \frac{6}{8} - \frac{7}{16} = \frac{5}{16}$

$?? = 1 - \frac{5}{16} = \frac{11}{16}$

finally :-

	1	2
11/32	0	$\frac{11}{16}$
3/8	7/16	$\frac{5}{16}$
9/32	$\frac{9}{16}$	0

$$\left.\begin{array}{l} q^1 = \left(0, \; 7/16, \; \frac{9}{16}\right) \qquad q^2 = \left(\frac{11}{16}, \frac{5}{16}, 0\right) \end{array}\right\} \quad (4)$$

Now from equation (1) (figure) and equation (3) and equation (4) (figure), final solution will be :→

	1	2	3
$\frac{1}{2}$	$\frac{13}{16}$	0	$\frac{11}{16}$
$\frac{1}{4}$	0	7/16	$\frac{5}{16}$
$\frac{3}{16}$	0	$\frac{9}{16}$	0
$\frac{1}{16}$	$\frac{3}{16}$	0	0

The desired solution/desired distribution :-

$$q^1 = \left(\frac{13}{16}, 0, 0, \frac{3}{16}\right)$$

$$q^2 = \left(0, \frac{7}{16}, \frac{9}{16}, 0\right)$$

$$q^3 = \left(\frac{11}{16}, \frac{5}{16}, 0, 0\right)$$

Answer

Answer verification ⇒

$$p_1 = \frac{1}{3}\left(\frac{13}{16} + 0 + \frac{11}{16}\right) = \frac{1}{3} \times \frac{24}{16} = \frac{1}{2}$$

$$p_2 = \frac{1}{3}\left(0 + \frac{7}{16} + \frac{5}{16}\right) = \frac{1}{3} \times \frac{12}{16}) = \frac{4}{16} = \frac{1}{4}$$

$$p_3 = \frac{1}{3}\left(0 + \frac{9}{16} + 0\right) = \frac{3}{16}$$

$$p_4 = \frac{1}{3}\left(\frac{3}{16} + 0 + 0\right) = \frac{1}{16}$$

$$p = (p_1, p_2, p_3, p_4)$$

$$= \left(\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{16}\right)$$

which is same as given in the question.

Note

$$\Sigma p = \frac{1}{2} + \frac{1}{4} + \frac{3}{16} + \frac{1}{16} = \frac{8+4+3+1}{16} = \frac{16}{16} = 1$$

$$\Sigma q^1 = \frac{13}{16} + 0 + 0 + \frac{3}{16} = \frac{16}{16} = 1$$

$$\Sigma q^2 = 0 + \frac{7}{16} + \frac{9}{16} + 0 = \frac{16}{16} = 1$$

$$\Sigma q^3 = \frac{11}{16} + \frac{5}{16} + 0 + 0 = \frac{16}{16} = 1$$

Shortcut method :- 4 Point distribution :-

$$p = \left(\frac{7}{16}, \frac{1}{4}, \frac{1}{8}, \frac{3}{16}\right)$$

0.43 , 0.25 , 0.125 , 0.18

Step I

$7/16$	$\overline{3/8} = \frac{5}{8}$		
$\frac{1}{4}$	0		
$\frac{1}{8}$	$\frac{3}{8}$	0	0
$\frac{3}{16}$	0		

"Table - 1"

Step II : Convert 4 point distribution into 3 point distribution

$$\boxed{p^* = \frac{3p - q'}{2}}$$

$$p^* = \frac{1}{2}\left[3 \times \frac{7}{16} - \frac{5}{8}\right] = \frac{1}{2}\left[\frac{21}{16} - \frac{10}{16}\right] = \frac{11}{32}$$

$$p^* = \frac{1}{2}\left[3 \times \frac{1}{4} - 0\right] = \frac{3}{8}$$

$$p^* = \frac{1}{2}\left[3 \times \frac{1}{8} - \frac{3}{8}\right] = 0$$

$$p^* = \frac{1}{2}\left[3 \times \frac{3}{16} - 0\right] = \frac{9}{32}$$

$$p^* = \left\{\frac{11}{32}, \frac{3}{8}, 0, \frac{9}{32}\right\}$$

Let $p^* = \left\{\frac{11}{32}, \frac{3}{8}, \frac{9}{32}\right\}$ (3 Point)

Step III : Find of the distribution of above 3 point using short cut for 3 point.

$0.34 = 11/32$	0	$\frac{11}{16}$	$\xrightarrow{I^{st}}$ $2 * \frac{11}{32} = \frac{11}{16}$
$0.37 = \frac{3}{8}$	$7/16$	$\frac{5}{16}$	$\overline{(11/16)} = 1 - \frac{11}{16} = \frac{5}{16}$
$0.28 = \frac{9}{32}$	$\frac{9}{16}$	0	

step ④ : Fill the element in table (1) using the solution of step ③

	1	2	3
$7/16$	$5/8$	0	$\frac{11}{16}$
$\frac{1}{4}$	0	$7/16$	$\frac{5}{16}$
$\frac{1}{8}$	$\frac{3}{8}$	0	0
$\frac{3}{16}$	0	$\frac{9}{16}$	0

from above :-

$$q^1 = \left(\frac{5}{8}, 0, \frac{3}{8}, 0\right)$$

$$q^2 = \left(0, \frac{7}{16}, 0, \frac{9}{16}\right)$$

$$q^3 = \left(\frac{11}{16}, \frac{5}{16}, 0, 0\right)$$

Answer

Q: Give a complete distribution for p, a four point distribution as $p_1 = \frac{2}{5}$ $p_2 = \frac{3}{10}$ $p_3 = \frac{5}{20}$ and $p_4 = \frac{1}{20}$ are given.

Solution:

4 Point distribution

$$p = \left(\frac{2}{5}, \frac{3}{10}, \frac{5}{20}, \frac{1}{20} \right)$$

$$= (0.40, 0.30, 0.25, 0.05)$$

$n = 4$

$$\frac{1}{n-1} = \frac{1}{4-1} = \frac{1}{3} = 0.33$$

$$\frac{1}{20} + \frac{2}{5} = 0.05 + .40 = .45 > 0.33$$

Step I

$n = 4$

$n-1 = 4-1 = 3$

	1	2	3
$\frac{2}{5}$	$\frac{17}{20}$		
$\frac{3}{10}$	0		
$\frac{5}{20}$	0		
$\frac{1}{20}$	$3/20$	0	0

$1 - \frac{3}{20} = \frac{17}{20}$

$3 * p_4 = 3 \times \frac{1}{20} = \frac{3}{20}$

(1)

Step II   Convert 4 point distribution into 3 point distribution.

$$\left.\begin{aligned} 3*\frac{2}{5}-\frac{17}{20} &= \frac{24}{20}-\frac{17}{20}=\frac{7}{20} \\ 3*\frac{3}{10}-0 &= \frac{9}{10} \\ 3*\frac{5}{20}-0 &= \frac{15}{20} \\ 3*\frac{1}{20}-\frac{3}{20} &= 0 \end{aligned}\right\} \longrightarrow (2)$$

$$\frac{7}{20}+\frac{9}{10}+\frac{15}{20}+0 = \frac{7+18+15}{20} = \frac{40}{20} = 2$$

To convert 4 point distribution into 3 point distribution we will devide equation 2 by 2.

$$p = \left(\frac{7}{40}, \frac{9}{20}, \frac{15}{40}, 0\right) \longrightarrow (3)$$

Now we will sove equation (3) considering a problem (case) of 3 point distribution.

3 Point distribution

$$p = \frac{7}{40}, \frac{9}{20}, \frac{15}{40} \longrightarrow (4)$$

Note: last element is zero in equation (3). In equation (4) we have not used last element of equation (3) because it is zero.

Step I

$$\frac{7}{40}, \quad \frac{9}{20}, \quad \frac{15}{40}$$

$$.175, \quad .45, \quad .375$$

$$\frac{7}{40} + \frac{9}{20} = .175 + 0.45 = 0.625 > \frac{1}{(3-1)}$$

$$0.625 > \frac{1}{2}$$

and $\frac{7}{40} < \frac{1}{2}$

Step II

	1	2
$7/40$	$\frac{14}{40}$	0
$\frac{9}{20}$	$\frac{26}{40}$	
$\frac{15}{40}$	0	

Step III

	1	2
$7/40$	$\frac{14}{40}$	0
$\frac{9}{20}$	$\frac{26}{40}$	$\frac{10}{40}$
$\frac{15}{40}$	0	$\frac{30}{40}$

$\frac{10}{40} \rightarrow = 1 - \frac{30}{40} = \frac{10}{40}$

$\frac{30}{40} \rightarrow 2 \times \frac{15}{40}$ —— (5)

$$\begin{bmatrix} q^1 = \left(\frac{14}{40}, \frac{26}{40}, 0\right) \\ q^2 = \left(0, \frac{10}{40}, \frac{30}{40}\right) \end{bmatrix} \quad \text{—— (6)}$$

Step(IV) Fill the element in table (1) using the solution of step (3) [Solution is given in equation (6)]

finally :

	1	2	3
$\frac{2}{5}$	$\frac{17}{20}$	$\frac{14}{40}$	0
$\frac{3}{10}$	0	$\frac{26}{40}$	$\frac{10}{40}$
$\frac{5}{20}$	0	0	$\frac{30}{40}$
$\frac{1}{20}$	$\frac{3}{20}$	0	0

from above :- The desired distribution / desired solution $\Rightarrow$

$$q^1 = \left(\frac{17}{20}, 0, 0, \frac{3}{20}\right)$$

$$q^2 = \left(\frac{14}{40}, \frac{26}{40}, 0, 0\right)$$

$$q^3 = \left(0, \frac{10}{40}, \frac{30}{40}, 0\right)$$

Answer

Answer verification $\rightarrow$

$$p_1 = \frac{1}{3}\left[\frac{17}{20} + \frac{14}{40} + 0\right) = \frac{1}{3}\left(\frac{34+14}{40}\right) = \frac{1}{3}\left(\frac{48}{40}\right) = \frac{16}{40} = \frac{2}{5}$$

$$p_2 = \frac{1}{3}\left(0 + \frac{26}{40} + \frac{10}{40}\right) = \frac{1}{3}\left(\frac{36}{40}\right) = \frac{12}{40} = \frac{3}{10}$$

$$p_3 = \frac{1}{3}\left(0 + 0 + \frac{30}{40}\right) = \frac{10}{40} = \frac{5}{20}$$

$$p_4 = \frac{1}{3}\left(\frac{3}{20} + 0 + 0\right) = \frac{1}{20}$$

$p = (p_1, p_2, p_3, p_4)$

$p = \left(\frac{2}{5}, \frac{3}{10}, \frac{5}{20}, \frac{1}{20}\right)$ which is same as

given in question.

$\Sigma q^1 = \frac{17}{20} + 0 + 0 + \frac{3}{20} = \frac{20}{20} = 1$

$\Sigma q^2 = \frac{14}{40} + \frac{26}{40} + 0 + 0 = \frac{40}{40} = 1$

$\Sigma q^3 = 0 + \frac{10}{40} + \frac{30}{40} + 0 = \frac{40}{40} = 1$

Inverse Transform Method

(Inverse CDF transform)

* CDF : Comulative distribution function or continuous distribution function
* PDF : Probability desity function
* PMF : Probability mass function

PDF

A function $f(x)$ is pdf if

① $f(x) \geq 0 \quad -\infty < x < \infty$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

In PDF x is continuous random variable.

CDF

Let X be a continuous random variable having pdf $f(x)$ then $F_X(x)$ will be a continuous distribution function of X if

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\frac{d}{dx} F_X(x) = f(x)$$

PMF : Probability mass function is the probability distribution of a discrete random variable and provides the possible values and their associated probabilities.

(1) $p_X(x) \geq 0$

(2) $\sum_x p_X(x) = 1$

Exponential distribution :-

CDF: $F(x) = \int_{-\infty}^{x} f(t)\, dt = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

We need to solve $y = 1 - e^{-\lambda x}$ for x where y is uniformly distributed on $(0, 1)$.

Solution:-

$$y = \text{CDF} = 1 - e^{-\lambda x}$$ (CDF for exponential distribution)

$$y = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - y$$

$$-\lambda x = \ln(1-y)$$

$$\boxed{x = \frac{-1}{\lambda} \ln(1-y)} \quad \text{—— (1)}$$

$$x = F^{-1}[y]$$

* Generate the uniform random numbers $y_1, y_2, y_3 \ldots y_n$ and compute the value of random variates x from the equation (1) i.e $\boxed{x_i = \frac{-1}{\lambda} \ln(1 - y_i)}$

Q Lead time have been exponentially distributed with mean 3.7 days. Generate lead time variate from this distribution using inverse transform technique. Choose random numbers 0.01, 0.13, 0.35, 0.65 and 0.53.

Solution:

First we have to find the equation of CDF. But in this equation function is exponentially distributed which is a standard function and its CDF is $\boxed{F(x) = 1-e^{-\lambda x}}$

$$CDF \Rightarrow F(x) = 1-e^{-\lambda x}$$

$$y = 1-e^{-\lambda x}$$

$$x = -\frac{1}{\lambda}\log_e(1-y)$$

$$\boxed{x_i = -\frac{1}{\lambda}\log_e(1-y_i)}$$

$$\text{mean} = \frac{1}{\lambda} = 3.7$$

$$x_1 = -3.7\log_e(1-0.01) = -3.7 * (-0.01) = 0.037$$

$$x_2 = -3.7\log_e(1-0.13) = -3.7 * (-0.139) = 0.514$$

$$x_3 = -3.7\log_e(1-0.35) = -3.7 * (-.430) = 1.5938$$

$$x_4 = -3.7\log_e(1-0.65) = -3.7 * (-1.049) = 3.881$$

$$x_5 = -3.7\log_e(1-0.53) = -3.7 * (-.755) = 2.793$$

Ans

Uniform distribution :

* Used to generate uniform random variate which is continuous over (a, b)

Procedure :

find CDF $= F(x) = \frac{x-a}{b-a}$

$$y = \frac{x-a}{b-a}$$

$$y(b-a) = x-a$$

$$\boxed{x = a + y(b-a)}$$

or

$$\boxed{x_i = a + y_i(b-a)}$$

* Now generate random no's $y_1, y_2, y_3 - - y_n$ and find the corresponding random variate x_1 x_2 - - - x_n using $x_i = a + y_i(b-a)$

Q: Generate 5 random variates which are uniformly distributed in the interval (3, 5) take random numbers, 0.779, 0.921, 0.186, 0.289 and 0.653.

$$(\underset{a}{3}, \underset{b}{5})$$

Solution

Step (I) CDF of uniformly distributed function:

$$F(x) = \frac{x-a}{b-a}$$

$$y = \frac{x-a}{b-a}$$

$$x = a + y(b-a)$$

$$\boxed{x_i = a + y_i(b-a)}$$

Step (II)

$x_1 = 3 + 0.779(5-3) = 4.558$

$x_2 = 3 + 0.921(5-3) = 4.842$

$x_3 = 3 + 0.186(5-3) = 3.372$

$x_4 = 3 + 0.289(5-3) = 3.578$

$x_5 = 3 + 0.653(5-3) = 4.306$

Q: Use inverse transform method to generate a random variable having comulative distribution function (CDF)

$$F(x) = \frac{x^2+2x}{3} \quad \text{for} \quad 0 < x < 1$$

Solution:

$$CDF = F(x) = \frac{x^2+2x}{3}$$

$$y = \frac{x^2+2x}{3}$$

$$x^2+2x-3y=0$$

$$x^2+2x+c=0 \qquad \text{Let } c = -2y$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{4-4(1)(c)}}{2\times 1}$$

$$= \frac{-2 \pm \sqrt{4-4c}}{2}$$

$$= \frac{-2 \pm 2\sqrt{(1-c)}}{2}$$

$$x = -1 \pm \sqrt{1+2y} \qquad (\text{put } c = -2y)$$

$$\begin{cases} x = -1 - \sqrt{1+2y} & \times \\ x = -1 + \sqrt{1+2y} & \checkmark \end{cases}$$

$x_i = -1 + \sqrt{1+2y_i}$

$\rightarrow y_i$ = Random number Between 0 to 1

$y_1 = 0.5$ (Let)

$x_1 = -1 + \sqrt{1+2y_1} = -1+\sqrt{1+2(0.5)} = -1+\sqrt{2}$

$= -1+1.414$

$= 0.414$

$y_2 = 0.1$ (Let)

$x_2 = -1+\sqrt{1+2(0.1)} = -1+\sqrt{1.2} = -1+1.095 = 0.095$

$y_3 = 0.3$ (Let)

$x_3 = -1+\sqrt{1+2\times.3} = -1+\sqrt{1+0.6} = -1+\sqrt{1.6} = -1+1.264$

$= 0.264$

$y_4 = 0.6$ Let

$x_4 = -1+\sqrt{1+2\times.6} = -1+\sqrt{1+1.2} = -1+\sqrt{2.2} = -1+0.4832$

$= 0.4832$

$y_5 = 0.7$ (Let)

$x_5 = -1+\sqrt{1+2\times.7} = -1+\sqrt{1+1.4} = -1+\sqrt{2.4} = -1+1.549$

$= 0.549$

$y_6 = 0.9$ Let

$x_6 = -1+\sqrt{1+2\times0.9} = -1+\sqrt{1+1.8} = -1+\sqrt{2.8} = -1+1.673$

$= 0.673$

0.5	0.1	0.5	0.6	0.7	0.9	← y_i (Random no)
0.414	0.095	0.264	0.4832	0.549	0.673	← x_i

The Rejection Method or

Acceptance-Rejection Method

Q : Let use use the rejection method to generate a random variable having density function $f(x) = 20x(1-x)^3$; $0 < x < 1$.

Since this random variable is concentrated in the interval $(0,1)$, Let us consider the rejection method with

$$g(x) = 1 \qquad 0 < x < 1$$

Solution :

$$\frac{f(x)}{g(x)} \leq C$$

$$\frac{f(x)}{g(x)} = \frac{20x(1-x)^3}{1} = 20x(1-x)^3$$

Now we have to find maximum value of $\frac{f(x)}{g(x)}$.

$$\frac{d}{dx}\left(\frac{f(x)}{g x}\right) = \frac{d}{dx}\left[20x(1-x)^3\right]$$

$$= 20\left[(1-x)^3 + x \cdot 3(1-x)^2 * (-1)\right]$$

$$0 = 20\left[(1-x)^3 - 3x(1-x)^2\right]$$

$$(1-x)^3 - 3x(1-x)^2 = 0$$

$$(1-x)^2\left[1-x-3x\right] = 0$$

$$(1-x)^2(1-4x) = 0$$

$$x = 1, \frac{1}{4}$$

We will get maximum when $x = \frac{1}{4}$.

$$\frac{f(x)}{g(x)} \leq 20 * \left(\frac{1}{4}\right)\left(1-\frac{1}{4}\right)^3$$

$$\frac{f(x)}{g(x)} \leq \frac{20}{4} * \frac{3\times3\times3}{4\times4\times4}$$

$$\frac{f(x)}{g(x)} \leq \frac{135}{64} \equiv C \qquad \left(\frac{135}{64} = 2.109\right)$$

$$\frac{f(x)}{C\,g(x)} = \frac{20\,x(1-x)^3}{(135/64)}$$

$$\frac{f(x)}{C\,g(x)} = \frac{64\times20}{135}\,x(1-x)^3$$

$$\boxed{\frac{f(x)}{C\,g(x)} = \frac{256}{27}\,x(1-x)^3} \quad \text{——— (2)}$$

And thus the rejection producedure is as follows.

Step 1: Generate random numbers U_1 and U_2.

Step 2: If $U_2 \leq \frac{256}{27} U_1 * (1-U_1)^3$ Stop and

Set $X = U_1$. otherwise return to step 1.

Q:

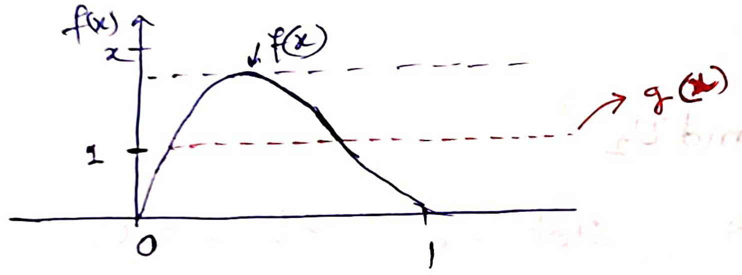
Example: Use A-R technique to generate random variable x with pdf:

pdf: $f(x) = 30x(1-x)^4 \quad 0<x<1$

Solution:

* We need to find $g(x)$: known density from which we know how to sample.

* $\because f(x) = 30x(1-x)^4 \quad 0<x<1$



Since $f(x)$ is concentrated in $(0,1)$ we can take

$g(x) = 1 \quad 0<x<1$

* find smallest constant c such that $\frac{f(x)}{g(x)} \leq c$

$$\frac{f(x)}{g(x)} = \frac{30x(1-x)^4}{1}$$

$$\frac{f(x)}{g(x)} = 30x(1-x)^4$$

$$d\left(f(x)/g(x)\right) = 30\left[x\,4(1-x)^3(-1) + (1-x)^4 * 1\right] = 0$$

$\boxed{x = \frac{1}{5}}$ maximum value of $\frac{f(x)}{g(x)}$ at $x = \frac{1}{5}$

$$\frac{f(x)}{g(x)} \leq 30 * \left(\frac{1}{5}\right)\left(1-\frac{1}{5}\right)^4$$

$$\therefore \; C = \frac{1536}{625} = 2.45$$

$$\boxed{\frac{f(x)}{c\,g(x)} = \frac{3125}{256} \; x(1-x)^4}$$

A. R. Steps :

1. Generate random numbers U_1 and U_2

2. If $U_2 \leq \frac{3125}{256} U_1 (1-U_1)^4$ stop and set $X = U_1$.

   otherwise return to step 1.

Note : Average number of steps will be executed $= c = \frac{1536}{625} = 2.45$

Q: Illustrate acceptance - rejection method (AR-Method) to generate a random variable X whose density function is $f(x) = \frac{1}{2} x^2 e^{-x}$ $x > 0$; by using an exponential density having rate λ.

$\rightarrow g(x)$

Solution:

pdf : $f(x) = \frac{1}{2} x^2 e^{-x}$ Given

pdf : $g(x) = e^{-\lambda x}$ (Assume)

$g(x) = e^{-2x}$ $\Longrightarrow$ Let $\lambda = 2$

or

$g(x) = e^{-\frac{x}{2}}$ $\Longrightarrow$ Let $\lambda = \frac{1}{2}$

$$\frac{f(x)}{g(x)} = \frac{\frac{1}{2} x^2 e^{-x}}{e^{-2x}} = \frac{1}{2} x^2 e^{x}$$

$$d\left[\frac{f(x)}{d(x)}\right] = \frac{1}{2}\left(x^2 \cdot e^{x} + e^{x} \cdot 2x\right)$$

$$= \frac{1}{2}\left[x^2 + 2x\right] e^{x}$$

$$= \frac{1}{2} x(x+2) e^{x}$$

$x = 0, -2$ It is not defined, because $x > 0$

Let $\boxed{\lambda = \frac{1}{2}}$ $g(x) = e^{-\frac{x}{2}}$

$$\frac{f(x)}{g(x)} = \frac{\frac{1}{2} x^2 e^{-x}}{e^{-x/2}} = \frac{1}{2} x^2 \cdot e^{-\frac{x}{2}}$$

maximum $\left[\frac{f(x)}{g(x)}\right] = ?$

$$\frac{d}{dx}\left[\frac{x^2 e^{-x/2}}{2}\right] = \frac{1}{2}\left[x^2 \cdot \left(-\frac{1}{2}\right) e^{-\frac{x}{2}} + e^{-\frac{x}{2}} * 2x\right]$$

$$x^2\left(-\frac{1}{2}\right) e^{-\frac{x}{2}} + e^{-\frac{x}{2}} 2x = 0$$

$$\frac{e^{-x/2}}{2}\left[-x^2 + 4x\right) = 0$$

$$-x^2 + 4x = 0$$

$$-(x^2 - 4x) = 0$$

$$-x(x-4) = 0$$

$$\boxed{x(x-4) = 0} \quad \therefore \; x = 0, \; x = 4$$

At $x = 4$ function $\rightarrow \frac{f(x)}{g(x)}$ will be maximum.

* $$\frac{f(x)}{g(x)} = \frac{1}{2}(4)^2 * e^{-\frac{4}{2}} = \frac{1}{2}(16)\, e^{-2} = \frac{8}{2.71 \times 2.71} = 1.082$$

$$\frac{8}{7.344}$$

$$\boxed{C = 1.082}$$

* $$\frac{f(x)}{c\,g(x)} = \frac{\frac{1}{2} x^2 e^{-x/2}}{(1.082)} = \frac{0.924}{2} x^2 e^{-\frac{x}{2}}$$

$$= 0.4621\, x^2 e^{-\frac{x}{2}}$$

$$\boxed{\frac{f(x)}{c\,g(x)} = 0.4621\, x^2 e^{-x/2}}$$

A-R Steps :

Step(I) Generate random number U_1 and U_2

Step(II) If $U_2 \leq 0.4621 \; U_1^2 \, e^{-U_1/2}$ stop and set $X = U_1$. Otherwise return to step I.

Q: Give an algorithm for simulating a random Variable having density function, $f(x) = 30[x^2 - 2x^3 + x^4]$ where $0 < x < 1$.

Solution:

pdf: $f(x) = 30(x^2 - 2x^3 + x^4) \qquad 0 < x < 1$

Assume pdf: $g(x) = 1 \qquad 0 < x < 1$

* Find smallest constant **c** such that:

$$\frac{f(x)}{g(x)} \leq c$$

$$\frac{f(x)}{g(x)} = \frac{30(x^2 - 2x^3 + x^4)}{1} = 30(x^2 - 2x^3 + x^4) \quad \text{—— (1)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{gx}\right) = \frac{d}{dx}\left[30(x^2 - 2x^3 + x^4)\right]$$

$$= 30 \times \left[2x - 6x^2 + 4x^3\right]$$

$$\frac{d}{dx}\left(\frac{f(x)}{gx}\right) = 0$$

$$30\left[2x - 6x^2 + 4x^3\right] = 0$$

$$2x - 6x^2 + 4x^3 = 0$$

$$2x\left[1 - 3x + 2x^2\right] = 0$$

$$\therefore x = 0 \quad \text{or} \quad 1 - 3x + 2x^2 = 0$$

$$2x^2 - 3x + 1 = 0$$

$$x = \frac{+3 \pm \sqrt{9 - 4 \times 2 \times 1}}{2 \times 2}$$

$$= \frac{3 \pm \sqrt{9 - 8}}{4}$$

$$= \frac{3 \pm 1}{4}$$

$$= \frac{3+1}{4}, \quad \frac{3-1}{4}$$

$$x = 1, \quad \frac{1}{2} \qquad \text{———} (2)$$

$\therefore$ $x = 0, \frac{1}{2}, 1$ but $x = \frac{1}{2}$ is only valid $\because$ $0 < x < 1$

$$\left[\frac{f(x)}{g(x)}\right]_{\text{at } x=\frac{1}{2}} = 30\left[\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right] \quad \text{[From (1) and (2)]}$$

$$= 30\left[\frac{1}{4} - \frac{2}{8} + \frac{1}{16}\right]$$

$$= 30\left[\frac{1}{4} - \frac{1}{4} + \frac{1}{16}\right]$$

$$= 30 * \frac{1}{16} = \frac{15}{8}$$

$$\therefore \quad C = \frac{15}{8}$$

$$\boxed{\frac{f(x)}{C\, g(x)} = \frac{30\,(x^2 - 2x^3 + x^4)}{\left(\frac{15}{8}\right)} = 16\,(x^2 - 2x^3 + x^4)}$$

Acceptance - rejection steps:

Step [1] Generate random numbers U_1 and U_2

Step [2] If $U_2 \leq 16\,(U_1^2 - 2U_1^3 + U_1^4)$ stop and set $X = U_1$

otherwise return to step 1.