Singular Value Decomposition

(SVD)

$$A = U\ S\ V^T$$

$$A = U \; S \; V^T$$

$$m \times n = (m \times m) \cdot (m \times n) \cdot (n \times n)$$

U = Eigen vector of AA^T

V = Eigen vector of $A^T A$

$$s_i u_i = A v_i \qquad \therefore \; u_i = \frac{1}{s_i}(A v_i)$$

$$S = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & 0 & 0 \\ & - & - & - & - \end{bmatrix}_{m \times n}$$

$\underbrace{\lambda_1, \lambda_2, \lambda_3}_{\text{Positive } \& \; \lambda_1 > \lambda_2 > \lambda_3}$ are the eigen values of $A^T A$ (Positive eign values only)

$U = [u_1 \vdots u_2 \vdots u_3 \vdots \ldots]$; ($u_1, u_2, u_3 \ldots$ are eigenvector of AAT)

$V = [V_1 \vdots V_2 \vdots V_3 \vdots \ldots]$; ($V_1, V_2, V_3 \ldots$ are eigen vector of ATA)

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Assignment - 1

Q:

Find the SVD of the following symmetric matrices by hand calculation.

(a) $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & 3/2 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$ (e) $\begin{bmatrix} 0.75 & 1.25 \\ 1.25 & 0.75 \end{bmatrix}$

Solution:

(a) $A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$$A A^T = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Characterestic polynomial :.

$$(A^TA) - \lambda I = 0$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$(9-\lambda)(4-\lambda) - 0 = 0 \qquad \because |A^TA - \lambda I| = 0$$

$$\boxed{\lambda = 9, 4}$$

Eigen vector calculation:-

$$[(A^TA) - \lambda I] X = 0$$

$$\begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\boxed{\text{Put } \lambda = 9}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$0x_1 + 0x_2 = 0 \quad \text{——— (1)}$$

$$0x_1 + 4x_2 = 0 \quad \text{——— (2)} \qquad \therefore x_2 = 0$$

Eigenvector: $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\left(\sqrt{1^2+0^2} = 1\right)$

$\boxed{\text{Put } \lambda = 4}$

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$5x_1 + 0x_2 = 0 \quad \text{—— (3)} \qquad x_1 = 0$$

$$0x_1 + 0x_2 = 0 \quad \text{—— (4)}$$

Eigenvector: $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\left(\sqrt{0^2+1^2} = 1\right)$

$$V = [X_1 \,\vdots\, X_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Perform orthogonalization

$$V = \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{bmatrix} 1/\sqrt{1} & 0/\sqrt{1} \\ 0/\sqrt{1} & 1/\sqrt{1} \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we have to calculate U.

$$s_i u_i = A v_i$$

$$u_i = \frac{1}{s_i}(A\ V_i)$$

$$\boxed{u_i = \frac{1}{s_i} * A \cdot V_i} \quad \text{——— (5)}$$

$$u_1 = \frac{1}{\sqrt{9}} * \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note
$s_1 = \sqrt{9} = 3$
$s_2 = \sqrt{4} = 2$
$\because s_i = \sqrt{\lambda_i}$

$$= \frac{1}{3} * \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{4}} * \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} * \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = [u_1 : u_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Now :- $A = U\ S\ V^T$

$$\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer

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[b] $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$$

$$[A^TA - \lambda I]$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}$$

$$|A^TA - \lambda I| = 0$$ Determinant $= 0$

$$(-\lambda)(9-\lambda) = 0$$

$$\boxed{\lambda = 9, 0}$$

Eigen vector calculation

$$[A^TA - \lambda I]X = 0$$

$$\begin{bmatrix} -\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Put $\lambda = 9$ $\quad \begin{bmatrix} -9 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-9x + 0y = 0$ $\quad \therefore \quad 9x = 0 \quad$ or $\quad x = 0$

$0x + 0y = 0$

$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad$ (Let $y = 1$)

Put $\lambda = 0$ $\quad \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\left.\begin{matrix} 0x + 0y = 0 \\ 0x + 9y = 0 \end{matrix}\right\} \quad \begin{matrix} x = 1 \\ y = 0 \end{matrix}$ $\quad$ (Let $x = 1$)

$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$V = [X_1 \mid X_2] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right]$$

Perform orthogonalization

$$V = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right]$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$s_i u_i = A V_i$$

$$u_i = \frac{1}{s_i}(A \cdot V_i)$$

$$u_1 = \frac{1}{\sqrt{9}} * \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because $\lambda = 0$ so we have to assume $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$U = [u_1 : u_2]$$
$$= \begin{bmatrix} 0 & \vdots & 1 \\ 1 & \vdots & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$ $(\because \lambda_1 = \sqrt{9} = 3)$

$$A = U \; S \; V^T$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2} * \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

Answer

[C]

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$A\,A^T = \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{4} + \frac{1}{4} & -\frac{3}{4} - \frac{3}{4} \\ -\frac{3}{4} - \frac{3}{4} & +\frac{1}{4} + \frac{9}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{4} & -\frac{6}{4} \\ -\frac{6}{4} & \frac{10}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

Characteristic equation $\Rightarrow$

$$(A^T A) - \lambda I$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} - \lambda \end{bmatrix}$$

Determinant $= 0$ $\qquad\qquad |(A^T A) - \lambda I| = 0$

$$\left(\frac{5}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) - \left(\frac{-3}{2}\right)\left(\frac{-3}{2}\right) = 0$$

$$\left(\frac{5}{2} - \lambda\right)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$$\left(\frac{5}{2} - \lambda + \frac{3}{2}\right)\left(\frac{5}{2} - \lambda - \frac{3}{2}\right) = 0$$

$$(4 - \lambda)\,(1 - \lambda) = 0 \qquad \therefore \boxed{\lambda = 4, 1}$$

Eigen vector calculation $\Rightarrow$

$$(A^TA - \lambda I)\begin{bmatrix} X \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{5}{2}-\lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\boxed{\lambda = 4}$

$$\begin{bmatrix} \frac{5}{2}-4 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \frac{3}{2}x_1 + \frac{3}{2}x_2 = 0$$

$$x_1 + x_2 = 0 \quad \text{or} \quad x_1 = -x_2$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\boxed{\lambda = 1}$

$$\begin{bmatrix} \frac{5}{2}-1 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & +\frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} \frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \\ -\frac{3}{2}x_1 + \frac{3}{2}x_2 &= 0 \end{aligned}\right\} \quad x_1 = x_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = [X_1 : X_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$V = \begin{array}{cc} v_1 & v_2 \\ \left[\begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{matrix}\right. & \left.\begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{matrix}\right] \\ \downarrow & \downarrow \\ v_1 & v_2 \end{array}$$

(orthogonalization)

$$VT = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now we have to calculate U

$$s_i u_i = A v_i$$

$$u_i = \frac{1}{s_i}(A\ v_i)$$

$$\boxed{u_i = \frac{1}{s_i} * (A * v_i)}$$

$$u_1 = \frac{1}{\sqrt{4}} * \underset{2\times 2}{\begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix}} * \underset{2\times 1}{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$= \frac{1}{2} * \begin{bmatrix} \frac{3}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1}} * \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2\sqrt{2}} \frac{-1}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = [u_1 \vdots u_2]$$

$$= \begin{bmatrix} 1/\sqrt{2} & \vdots & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \vdots & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = U S V^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer

(d) $A = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$

$$A\,A^T = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} * \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{4} + \frac{1}{4} & -\frac{3}{4} - \frac{3}{4} \\ -\frac{3}{4} - \frac{3}{4} & \frac{1}{4} + \frac{9}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{4} & -\frac{6}{4} \\ -6/4 & \frac{10}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$\Rightarrow (A^T A) - \lambda I$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2}-\lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-\lambda \end{bmatrix}$$

$$|A^T A - \lambda I| = 0$$

$$\left(\frac{5}{2}-\lambda\right)\left(\frac{5}{2}-\lambda\right) - \left(-\frac{3}{2}\right)^2 = 0$$

$$\left(\frac{5}{2}-\lambda\right)^2 - \left(\frac{3}{2}\right)^2 = 0 \rightarrow \left(\frac{5}{2}-\lambda+\frac{3}{2}\right)\left(\frac{5}{2}-\lambda-\frac{3}{2}\right) = 0$$

$$(4-\lambda)\,(1-\lambda) = 0 \quad \therefore \boxed{\lambda = 4, 1}$$

Eigen vector calculation :

$$(A^TA - \lambda I)X = 0$$

$$\begin{bmatrix} (\frac{5}{2}-\lambda) & -\frac{3}{2} \\ -\frac{3}{2} & (\frac{5}{2}-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\underline{\lambda = 4}$

$$\begin{bmatrix} \frac{5}{2}-4 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -\frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \\ -\frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \end{aligned}\right\}$$

$x_1 + x_2 = 0$

$\therefore x_1 = -x_2$

$\left(\therefore \frac{x_1}{x_2} = \frac{-1}{1} = \frac{1}{-1}\right)$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\underline{\lambda = 1}$

$$\begin{bmatrix} \frac{5}{2}-1 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} \frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \\ -\frac{3}{2}x_1 + \frac{3}{2}x_2 &= 0 \end{aligned}\right\}$$

$x_1 = x_2$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = [X_1 : X_2]$$

$$= \left[\begin{array}{c:c} 1 & 1 \\ -1 & 1 \end{array}\right]$$

$$\sqrt{1^2 + (-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

$$V = \begin{array}{cc} V_1 & V_2 \\ \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{array}\right. & \left.\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right] \end{array}$$

(orthogonalization)

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$s_i u_i = A V_i$$

$$u_i = \frac{1}{s_i} \quad A * V_i$$

$$u_1 = \frac{1}{\sqrt{4}} * \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -\frac{3}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1}} * \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = [U_1 : U_2] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$A = U * S * V^T$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer

(c)

$$A = \begin{bmatrix} 0.75 & 1.25 \\ 1.25 & 0.75 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ 5/4 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{9}{16} + \frac{25}{16} & \frac{15}{16} + \frac{15}{16} \\ \frac{15}{16} + \frac{15}{16} & \frac{25}{16} + \frac{9}{16} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \frac{34}{16} & \frac{30}{16} \\ \frac{30}{16} & \frac{34}{16} \end{bmatrix} = \begin{bmatrix} \frac{17}{8} & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \frac{17}{8} & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} \end{bmatrix}$$

Characteristic Equation ⇒

$$= (A^TA - \lambda I)$$

$$= \begin{bmatrix} 17/8 & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{17}{8} - \lambda\right) & \frac{15}{8} \\ \frac{15}{8} & \left(\frac{17}{8} - \lambda\right) \end{bmatrix}$$

$$|A^TA - \lambda I| = 0$$

$$\left(\frac{17}{8} - \lambda\right)\left(\frac{17}{8} - \lambda\right) - \frac{15}{8} \times \frac{15}{8} = 0$$

$$\left(\frac{17}{8} - \lambda\right)^2 - \left(\frac{15}{8}\right)^2 = 0$$

$$\left[\frac{17}{8}-\lambda+\frac{15}{8}\right]\left[\frac{17}{8}-\lambda-\frac{15}{8}\right]=0$$

$$(4-\lambda)\left(\frac{1}{4}-\lambda\right)=0$$

$$\boxed{\lambda=4,\ \frac{1}{4}}$$

Eigen vector calculation $\Rightarrow$

$$\begin{bmatrix}\left(\frac{17}{8}-\lambda\right) & \frac{15}{8}\\ \frac{15}{8} & \left(\frac{17}{8}-\lambda\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0 \quad \text{---} (1)$$

Put $\lambda=4$

$$\begin{bmatrix}\left(\frac{17}{8}-4\right) & \frac{15}{8}\\ \frac{15}{8} & \frac{17}{8}-4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$$\begin{bmatrix}-15/8 & 15/8\\ 15/8 & -\frac{15}{8}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$$-\frac{15}{8}x_1+\frac{15}{8}x_2=0$$

$$x_1=x_2$$

$$\frac{x_1}{x_2}=\frac{1}{1}$$

$$X_1=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Put $\lambda = \frac{1}{4}$ in equation (1)

$$\begin{bmatrix} \frac{17}{8}-\frac{1}{4} & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8}-\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{15}{8} & 15/8 \\ \frac{15}{8} & \frac{15}{8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\frac{15}{8}x_1 + \frac{15}{8}x_2 = 0$$

$$x_1 = -x_2$$

$$\frac{x_1}{x_2} = \frac{-1}{1}$$

or $\frac{x_1}{x_2} = \frac{1}{-1}$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore V = [X_1 \;\vdots\; X_2]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(orthogonalization)

$$VT = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Calculation of matrix U

$$s_i u_i = A * V_i$$

$$u_i = \frac{1}{s_i} * A * V_i$$

$$u_1 = \frac{1}{\sqrt{4}} * \begin{bmatrix} \frac{3}{4} & 5/4 \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{3}{4\sqrt{2}} + \frac{5}{4\sqrt{2}} \\ \frac{5}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1/4}} \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \sqrt{4} \begin{bmatrix} \frac{3}{4\sqrt{2}} - \frac{5}{4\sqrt{2}} \\ \frac{5}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} \end{bmatrix}$$

$$= 2 \begin{bmatrix} \frac{-2}{4\sqrt{2}} \\ \frac{2}{4\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = [u_1 : u_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{\frac{1}{4}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$A = U * S * V^T$$

$$\begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

<u>Answer</u>

Q:(d) $A = \begin{bmatrix} -4 & -12 \\ 12 & 11 \end{bmatrix}$ find SVD.

Solⁿ $A = \begin{bmatrix} -4 & -12 \\ 12 & 11 \end{bmatrix}$ $A^T = \begin{bmatrix} -4 & 12 \\ -12 & 11 \end{bmatrix}$

$AA^T = \begin{bmatrix} 160 & -180 \\ -180 & 265 \end{bmatrix}$ $A^TA = \begin{bmatrix} 160 & 180 \\ 180 & 265 \end{bmatrix}$

$|A^TA - \lambda I| = 0$

$\begin{bmatrix} 160 & 180 \\ 180 & 265 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 160-\lambda & 180 \\ 180 & 265-\lambda \end{bmatrix}$; $\begin{vmatrix} 160-\lambda & 180 \\ 180 & 265-\lambda \end{vmatrix} = 0$

$(160-\lambda)(265-\lambda) - 180^2 = 0 \Rightarrow \lambda^2 - 425\lambda + 10000 = 0$

$\lambda^2 - 400\lambda - 25\lambda + 10000 = 0$

$\lambda(\lambda - 400) - 25(\lambda - 400) = 0$

$(\lambda - 400)(\lambda - 25) = 0$

$\boxed{\lambda = 400, 25}$

Eigen vector calculation:.

Put $\lambda = 400$

$\begin{bmatrix} -240 & 180 \\ 180 & -135 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-240x_1 + 180x_2 = 0$ —— (1)

$180x_1 - 135x_2 = 0$ —— (2)

from (1)

$240x_1 = 180x_2$

$\frac{x_1}{x_2} = \frac{180}{240} = \frac{6}{8} = \frac{3}{4}$

from eqⁿ (2)

$\frac{x_1}{x_2} = \frac{135}{180} = \frac{3}{4}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Put $\lambda = 25$

$\begin{bmatrix} 135 & 180 \\ 180 & 240 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$135x_1 + 180x_2 = 0$ —— (3)

$180x_1 + 240x_2 = 0$ —— (4)

$\frac{x_1}{x_2} = \frac{-180}{135} = -\frac{4}{3} = \frac{4}{-3}$

$\frac{x_1}{x_2} = \frac{-240}{180} = -\frac{4}{3} = \frac{4}{-3}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$V = [V_1 : V_2] = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$ (orthogonalization)

$\sqrt{3^2+4^2} \quad | \quad \sqrt{4^2+-3^2}$

$u_i = \frac{1}{s_i} * A \cdot V_i$

$= \frac{1}{\sqrt{400}} * \begin{bmatrix} -4 & -12 \\ 12 & 11 \end{bmatrix} * \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$= \begin{bmatrix} -3/5 \\ +4/5 \end{bmatrix}$

$u_i = \frac{1}{s_i} \quad A \cdot V_i$

$u_2 = \frac{1}{\sqrt{25}} * \begin{bmatrix} -4 & -12 \\ 12 & 11 \end{bmatrix} * \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$

$= \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

$U = [u_1 : U_2] = \begin{bmatrix} -3/5 & 4/5 \\ +4/5 & 3/5 \end{bmatrix}$

$S = \begin{bmatrix} \sqrt{400} & 0 \\ 0 & \sqrt{25} \end{bmatrix}$

$\therefore S = \begin{bmatrix} 20 & 0 \\ 0 & 5 \end{bmatrix}$

$A = U \; S \; V^T$

$A = \begin{bmatrix} -\frac{3}{5} & 4/5 \\ +\frac{4}{5} & 3/5 \end{bmatrix} \begin{bmatrix} \sqrt{400} & 0 \\ 0 & \sqrt{25} \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$

$A = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$

Ans

Q:
(c) $A = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ find SVD.

$\therefore A^T = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$

Solution:

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$|A^TA - \lambda I| = 0$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix}$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0$ $\Rightarrow (1-\lambda)(4-\lambda) = 0$

$\boxed{\lambda = 4, 1}$

Eigen vector calculation: $\lambda = 4$

$\begin{bmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} 1-4 & 0 \\ 0 & 4-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-3x_1 + 0x_2 = 0$

$x_1 = 0$

$x_2 = \text{free}$ Let $x_2 = 1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda = 1$

$\begin{bmatrix} 1-1 & 0 \\ 0 & 4-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$0x_1 + 0x_2 = 0$

$0x_1 + 3x_2 = 0 \therefore x_2 = 0$

$x_1 = \text{free}$

$x_1 = 1$ (Let)

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$V = [V_1 : V_2] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right]$

$u_i = \frac{1}{S_i} A * V_i$

$= \frac{1}{\sqrt{4}} \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$u_i = \frac{1}{S_i} * A * V_i$

$= \frac{1}{\sqrt{1}} * \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= 1 * \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$U = [U_1 : U_2]$

$= \left[\begin{array}{c|c} -1 & 0 \\ 0 & -1 \end{array}\right]$

$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$A = U \; S \; V^T$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Ans

Q: Find SVD :-

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Soln.

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}_{2\times 2}$$

$$A^T A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}_{3\times 3}$$

characteristic polynomial ⇒

$$= A^T A - \lambda I$$

$$= \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{bmatrix}$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad \text{—(1)}$$

S_1 = Trace of (A^TA) = $10+10+2 = 22$

S_2 = Sum of minor of diagonal

$$\begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}$$

$$= (20-16) + (20-4) + 100-0$$

$$= 4 + 16 + 100 = 120$$

S_3 = Determinant $(A^TA) = 0$

From equation (1)

$$\lambda^3 - 22\lambda^2 + 120\lambda - 0 = 0$$

$$\lambda(\lambda^2 - 22\lambda + 120) = 0$$

$$\lambda(\lambda^2 - 12\lambda - 10\lambda + 120) = 0$$

$$\lambda(\lambda - 12)(\lambda - 10) = 0$$

$$\boxed{\lambda = 12, 10, 0}$$

Put $\lambda = 12$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Cramer's Rule

$$\frac{x_1}{\begin{vmatrix} -2 & 4 \\ 4 & -10 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 4 \\ 2 & -10 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -2 \\ 2 & 4 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\lambda = 10$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} 0 & 4 \\ 4 & -8 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 4 \\ 2 & -8 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix}}$$

$$\frac{x_1}{(0-16)} = \frac{-x_2}{(0-8)} = \frac{x_3}{0-0}$$

$$\frac{x_1}{-16} = \frac{x_2}{8} = \frac{x_3}{0}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{0} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} +2 \\ -1 \\ 0 \end{bmatrix}$$

$\lambda = 0$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{(20-16)} = \frac{-x_2}{(0-8)} = \frac{x_3}{0-20}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{-20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-5}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}$$

$$V = [\, V_1 \vdots V_2 \vdots V_3 \,]$$

$$= [\, X_1 \vdots X_2 \vdots X_3 \,]$$

$$= \left[\begin{array}{c:c:c} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & -5 \end{array}\right]$$

$$\sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\sqrt{2^2+1^2+0^2} = \sqrt{5}$$

$$\sqrt{1^2+2^2+(-5)^2} = \sqrt{30}$$

$$V = \left[\begin{array}{c:c:c} 1/\sqrt{6} & +\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ 2/\sqrt{6} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ 1/\sqrt{6} & 0 & \frac{-5}{\sqrt{30}} \end{array}\right]$$

orthogonalization

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{+2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

Calculation of U :

Characteristic polynomial ⇒

$$(AA^T) - \lambda I$$

$$= \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix}$$

$$|AA^T - \lambda I| = 0$$

$$(11-\lambda)(11-\lambda) - 1 = 0$$

$$(11-\lambda)^2 - 1^2 = 0$$

$$(11-\lambda+1)(11-\lambda-1) = 0$$

$$(12-\lambda)(10-\lambda) = 0$$

$$\boxed{\lambda = 12, 10}$$

Calculation of eigen vector:

$$[(AA^T) - \lambda I][X] = 0$$

$$\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\lambda = 12$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\}$$

$x_1 = +x_2$

$\frac{x_1}{x_2} = +1$ or $\frac{x_1}{x_2} = \frac{1}{+1}$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$$

$\lambda = 10$

$$\begin{bmatrix} 11-10 & 1 \\ 1 & 11-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$\frac{x_1}{x_2} = \frac{1}{-1}$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U = [U_1 : U_2]$$

$$= [X_1 : X_2] = \left[\begin{array}{c:c} 1 & 1 \\ +1 & -1 \end{array}\right]$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(Orthogonalization)

$\sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\sqrt{1^2 + -1^2} = \sqrt{2}$

Calculation of S:

$$S_{2\times 3} = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

Note:-

$\because \lambda = 12, 10$ are common for A^TA and AA^T

$$A = U\, S\, V^T$$

Largest value will be at this place.

$$\underset{2\times 3}{\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}} * \underset{2\times 3}{\begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}} * \underset{3\times 3}{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ +\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}}$$

Method (II) Calculation of U by using formula:-

$$u_1 = \frac{1}{s_1} A V_1$$

$$u_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{12}} \begin{bmatrix} 6/\sqrt{6} \\ \frac{6}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{s_2} A V_2 \Rightarrow$$

$$u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \frac{5}{\sqrt{5}} \\ \frac{-5}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Q:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2}$$

Find SVD

Solution:

$$AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{3\times 3}$$

$$A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2}$$

Calculation of V :

first calculate eigenvalue of A^TA.

$A^TA - \lambda I \Rightarrow$

$$= \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$$

$$|A^TA - \lambda I| = 0$$

$$(2-\lambda)(3-\lambda) = 0$$

$$\boxed{\lambda = 3, 2}$$

Eigen vector calculation $\Rightarrow$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\lambda = 3$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{aligned} -x_1 + 0x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned} \Rightarrow \cancel{x_2 = 0}$$

$x_1 = 0$

$x_2 = 1$ (Let)

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} 0x_1 + 0 \cdot x_2 &= 0 \\ 0x_1 + 1 x_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 1 \text{ (Let)} \\ x_2 &= 0 \end{aligned}$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V = [X_1 : X_2]$$

$$= [V_1 : V_2]$$

$$V = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right]$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Calculation of U:

$$(AA^T) - \lambda I = 0$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} (2-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 1 \\ 0 & 1 & (2-\lambda) \end{bmatrix} = (2-\lambda)\left[(1-\lambda)(2-\lambda)-1\right] - 1\left[(2-\lambda)-0\right] - 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)^2 - (2-\lambda) - (2-\lambda) = 0$$

$$(2-\lambda)\left\{(1-\lambda)(2-\lambda) - 1 - 1\right\} = 0$$

$$(2-\lambda)\left\{2 - 2\lambda - \lambda + \lambda^2 - 2\right\} = 0$$

$$(2-\lambda)\left\{\lambda^2 - 3\lambda\right\} = 0$$

$$(2-\lambda)\lambda(\lambda-3) = 0$$

$$\boxed{(\lambda-3)(\lambda-2)\lambda = 0} \qquad \boxed{\lambda = 3, 2, 0}$$

Eigen vector calculation ⇒

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ 0x_1 + x_2 - x_3 &= 0 \end{aligned}\right\}$$

$$\therefore \left.\begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 \end{aligned}\right\} \quad \therefore x_1 = x_2 = x_3$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ K \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\underline{\lambda = 2}}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-0} = \frac{x_3}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\boxed{\lambda = 0}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$U = [U_1 : U_2 : U_3]$$

$$U = [X_1 : X_2 : X_3]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\sqrt{1^2+1^2+1^2} \qquad \sqrt{1^2+0^2+-1^2} \qquad \sqrt{1^2+-2^2+1^2}$$

$$\sqrt{3} \qquad \sqrt{2} \qquad \sqrt{6}$$

$$U = \begin{bmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1/\sqrt{3} & 0 & \frac{-2}{\sqrt{6}} \\ 1/\sqrt{3} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Calculation of **S** matrix :-

$$S = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

Because Eigen value 3 and 2 are common for AA^T and A^TA. We will use eigen value 3 and 2 for Calculation of Matrix S.

Note:

We can also apply formula (as given below) to calculate U:

$$s_i u_i = A v_i$$

$$\boxed{u_1 = \frac{1}{s_1} * A * V_1}$$

$$u_1 = \frac{1}{s_1} * AV_1$$

$$u_2 = \frac{1}{s_2} * A * V_2$$

$$u_3 = ?$$

Because s_3 is not available

$$\mathbf{U} = [u_1 : u_2 : u_3]$$

* To calculate u_3 we will use null space matrix rule:

$$u_3 = \frac{\text{Null space}(A^T)}{|\text{Null space}(A^T)|} = \frac{NS(A^T)}{|NS(A^T)|}$$

$$\Rightarrow \boxed{A^T X = 0}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

x_3 is free
[let $x_3 = 1$]

$$\left.\begin{aligned} x_1 + 0x_2 - x_3 &= 0 \\ 0x_1 + x_2 + 2x_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}\right\} \text{ — (2)}$$

Let $x_3 = 1$

$x_1 = 1$

$x_2 = -2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$|X| = \sqrt{1^2 + -2^2 + 1^2} = \sqrt{6}$$

$$u_3 = \begin{bmatrix} 1/\sqrt{6} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$A = U \cdot S \cdot V^T$$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Answer

Calculation of u_1 and u_2 using formula:

$$u_1 = \frac{1}{\sqrt{3}} A \cdot V_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Answer

$$u_2 = \frac{1}{\sqrt{2}} AV_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Q: Find SVD

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{5 \times 3}$$

<u>Sol^n</u>:

$$A^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{3 \times 5}$$

$$A \cdot A^T = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix}_{5 \times 5}$$

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}_{3 \times 3}$$

<u>Calculation of V :</u>

$$|A^T A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$$

$$\therefore \boxed{\lambda = 5, 2, 1}$$

Eigen vector calculation :-

$\lambda = 5$

$$\begin{bmatrix} 2-5 & 1 & 1 \\ 1 & 4-5 & 1 \\ 1 & 1 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{3-1} = \frac{-x_2}{-3-1} = \frac{x_3}{1+1}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 2-2 & 1 & 1 \\ 1 & 4-2 & 1 \\ 1 & 1 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{0-1} = \frac{-x_2}{0-1} = \frac{x_3}{1-2}$$

$$\frac{x_1}{-1} = \frac{x_2}{+1} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{\underline{\lambda = 1}}$

$$\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 4-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{3-1} = \frac{-x_2}{1-1} = \frac{x_3}{1-3}$$

$$\frac{x_1}{2} = \frac{-x_2}{0} = \frac{x_3}{-2}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\therefore V = [V_1 : V_2 : V_3]$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

$$V = [V_1 \vdots V_2 \vdots V_3]$$

$$= \begin{bmatrix} 1/\sqrt{6} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{8}} \\ 2/\sqrt{6} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{8}} \end{bmatrix}$$

(After orthogonalization)

$$\sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\sqrt{1^2+(-1)^2+(1)^2} = \sqrt{3}$$

$$\sqrt{2^2+0^2+(-2)^2} = \sqrt{8}$$

Calculation of U ⇒:

$$u_i = \frac{1}{s_i} A * V_i$$

$\boxed{\lambda = 5}$

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{5\times3} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}_{3\times1}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 3/\sqrt{6} \\ \frac{2}{\sqrt{6}} \\ \frac{3}{\sqrt{6}} \\ 2/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}_{5\times1}$$

$$u_1 = \begin{bmatrix} 3/\sqrt{30} \\ 2/\sqrt{30} \\ 3/\sqrt{30} \\ 2/\sqrt{30} \\ 2/\sqrt{30} \end{bmatrix}_{5\times1}$$

$\boxed{\lambda = 2}$

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{5\times 3} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} 0 \\ -1/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}_{5\times 1}$$

$\boxed{\lambda = 1}$

$$u_3 = \frac{1}{\sqrt{1}} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{8} \\ 0 \\ -2/\sqrt{8} \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{8} \\ 0 \\ 2/\sqrt{8} \\ 0 \\ 0 \end{bmatrix}_{5\times 1}$$

$$U = [u_1 : u_2 : u_3 : u_4 : u_5]$$

$u_4 = ?$

$u_5 = ?$

$$|A \cdot A^T - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 & 0 \\ 1 & 1 & 2-\lambda & 1 & 1 \\ 1 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \boxed{\lambda = 0, 0, 1, 2, 5}$$

$$\boxed{A^T X = 0}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times 1} = 0$$

$$\left[\begin{array}{ll} x_3 + x_5 = 0 & \text{——— } (1) \\ x_1 + x_2 + x_3 + x_4 = 0 & \text{——— } (2) \\ x_1 + x_5 = 0 & \text{——— } (3) \end{array}\right.$$

Case (1) Let $x_5 = 1$ $\quad \therefore x_1 = -1$

$$x_3 = -1$$

$$x_2 + x_4 = -(x_1 + x_3) = -(-1-1) = +2$$

Let $x_4 = 1$

$\therefore x_2 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \\ +1 \end{bmatrix}$$

$$u_4 = \begin{bmatrix} -1/\sqrt{5} \\ +1/\sqrt{5} \\ -1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Case(2)

Let $x_5 = 0$

$x_1 = 0$
$x_3 = 0$

$x_2 + x_4 = 0$

$x_2 = -x_4$

$x_2 = -1$

Let $x_4 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_5 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$U = [u_1 : u_2 : u_3 : u_4 : u_5]$$

$$= \begin{bmatrix} \frac{3}{\sqrt{30}} & 0 & -2/\sqrt{8} & -\frac{1}{\sqrt{5}} & 0 \\ 2/\sqrt{30} & -1/\sqrt{6} & 0 & 1/\sqrt{5} & \frac{-1}{\sqrt{2}} \\ 3/\sqrt{30} & 0 & +2/\sqrt{8} & -1/\sqrt{5} & 0 \\ 2/\sqrt{30} & -1/\sqrt{6} & 0 & 1/\sqrt{5} & \frac{1}{\sqrt{2}} \\ 2/\sqrt{30} & 2/\sqrt{6} & 0 & 1/\sqrt{5} & 0 \end{bmatrix}_{5\times5}$$

Solution

$$A = U * S * V^T$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{30} & 0 & -2/\sqrt{8} & -1/\sqrt{5} & 0 \\ 2/\sqrt{30} & -1/\sqrt{6} & 0 & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{2}} \\ 3/\sqrt{30} & 0 & 2/\sqrt{8} & \frac{-1}{\sqrt{5}} & 0 \\ 2/\sqrt{30} & -1/\sqrt{6} & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ 2/\sqrt{30} & 2/\sqrt{6} & 0 & \frac{1}{\sqrt{5}} & 0 \end{bmatrix}_{5\times5} * \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{5\times3} * \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & \frac{1}{\sqrt{6}} \\ 1/\sqrt{3} & \frac{-1}{\sqrt{3}} & 1/\sqrt{3} \\ \frac{2}{\sqrt{8}} & 0 & -\frac{2}{\sqrt{8}} \end{bmatrix}_{3\times3}$$