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Assignment - 1

Q:

Find the SVD of the following symmetric matrices by hand calculation.

(a) $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & 3/2 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$ (e) $\begin{bmatrix} 0.75 & 1.25 \\ 1.25 & 0.75 \end{bmatrix}$

Solution:

(a) $A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$$A A^T = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Characteristic polynomial:

$$(A^TA) - \lambda I = 0$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$(9-\lambda)(4-\lambda) - 0 = 0 \qquad \because |A^TA - \lambda I| = 0$$

$$\boxed{\lambda = 9, 4}$$

Eigen vector calculation:-

$$[(A^TA) - \lambda I]X = 0$$

$$\begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\boxed{\text{Put } \lambda = 9}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$0x_1 + 0x_2 = 0 \longrightarrow (1)$$

$$0x_1 + 4x_2 = 0 \longrightarrow (2) \qquad \therefore x_2 = 0$$

Eigenvector: $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\left(\sqrt{1^2 + 0^2} = 1 \right)$

$\boxed{\text{Put } \lambda = 4}$

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$5x_1 + 0x_2 = 0 \longrightarrow (3) \qquad x_1 = 0$$

$$0x_1 + 0x_2 = 0 \longrightarrow (4)$$

Eigenvector: $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\left(\sqrt{0^2 + 1^2} = 1 \right)$

$$V = [X_1 \,\vdots\, X_2] = \left[\begin{array}{c:c} 1 & 0 \\ 0 & 1 \end{array}\right]$$

Perform orthogonalization

$$V = \begin{array}{cc} V_1 & V_2 \\ \left[\begin{array}{c:c} 1/\sqrt{1} & 0/\sqrt{1} \\ 0/\sqrt{1} & 1/\sqrt{1} \end{array}\right] \end{array} = \left[\begin{array}{c:c} 1 & 0 \\ 0 & 1 \end{array}\right]$$

$$V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we have to calculate U.

$$s_i u_i = A v_i$$

$$u_i = \frac{1}{s_i}(A \; v_i)$$

$$\boxed{u_i = \frac{1}{s_i} * A \cdot v_i} \quad \text{———} \quad (5)$$

$$u_1 = \frac{1}{\sqrt{9}} * \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note
$s_1 = \sqrt{9} = 3$
$s_2 = \sqrt{4} = 2$
$\because s_i = \sqrt{\lambda_i}$

$$= \frac{1}{3} * \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{4}} * \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} * \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = [u_1 : u_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Now :- $A = U\ S\ V^T$

$$\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer

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[6] $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$$

$[A^TA - \lambda I]$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}$$

$|A^TA - \lambda I| = 0$ Determinant $= 0$

$(-\lambda)(9-\lambda) = 0$

$\boxed{\lambda = 9, 0}$

Eigen vector calculation

$$[A^TA - \lambda I]X = 0$$

$$\begin{bmatrix} -\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Put $\lambda = 9$ $\quad \begin{bmatrix} -9 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-9x + 0y = 0$ $\quad \therefore \quad 9x = 0 \quad$ or $\quad x = 0$

$0x + 0y = 0$

$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (Let $y = 1$)

Put $\lambda = 0$ $\quad \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\left.\begin{matrix} 0x + 0y = 0 \\ 0x + 9y = 0 \end{matrix}\right\} \quad \begin{matrix} x = 1 \\ y = 0 \end{matrix}$ (Let $x = 1$)

$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$V = [X_1 \; \vdots \; X_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Perform orthogonalization

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$s_i u_i = A V_i$$

$$u_i = \frac{1}{s_i}(A \cdot V_i)$$

$$u_1 = \frac{1}{\sqrt{9}} * \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because $\lambda = 0$ so we have to assume $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$U = [u_1 : u_2]$$

$$= \begin{bmatrix} 0 & \vdots & 1 \\ 1 & \vdots & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \qquad (\because \lambda_1 = \sqrt{9} = 3)$$

$$A = U \; S \; V^T$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2} * \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

<u>Answer</u>

[C]

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$A A^T = \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{4} + \frac{1}{4} & -\frac{3}{4} - \frac{3}{4} \\ -\frac{3}{4} - \frac{3}{4} & +\frac{1}{4} + \frac{9}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{4} & \frac{-6}{4} \\ \frac{-6}{4} & \frac{10}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{5}{2} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ \frac{-3}{2} & \frac{5}{2} \end{bmatrix}$$

Characteristic equation $\Rightarrow$

$$(A^T A) - \lambda I$$

$$= \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{5}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{5}{2} - \lambda & \frac{-3}{2} \\ -\frac{3}{2} & \frac{5}{2} - \lambda \end{bmatrix}.$$

Determinant $= 0$ $\quad\quad |(A^T A) - \lambda I| = 0$

$$\left(\frac{5}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) - \left(\frac{-3}{2}\right)\left(\frac{-3}{2}\right) = 0$$

$$\left(\frac{5}{2} - \lambda\right)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$$\left(\frac{5}{2} - \lambda + \frac{3}{2}\right)\left(\frac{5}{2} - \lambda - \frac{3}{2}\right) = 0$$

$$(4 - \lambda)(1 - \lambda) = 0 \quad\quad \therefore \boxed{\lambda = 4, 1}$$

Eigen vector calculation $\Rightarrow$

$$(A^TA - \lambda I)[X] = 0$$

$$\begin{bmatrix} \frac{5}{2}-\lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\boxed{\lambda = 4}$

$$\begin{bmatrix} \frac{5}{2}-4 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \frac{3}{2}x_1 + \frac{3}{2}x_2 = 0$$

$$x_1 + x_2 = 0 \quad \text{or} \quad x_1 = -x_2$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\boxed{\lambda = 1}$

$$\begin{bmatrix} \frac{5}{2}-1 & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2}-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & +\frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} \frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \\ -\frac{3}{2}x_1 + \frac{3}{2}x_2 &= 0 \end{aligned}\right\} \quad x_1 = x_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = [X_1 : X_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$V = \begin{array}{cc} v_1 & v_2 \\ \left[\begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{matrix}\right. & \left.\begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{matrix}\right] \\ \downarrow & \downarrow \\ v_1 & v_2 \end{array}$$

(orthogonalization)

$$VT = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now we have to calculate U

$$s_i u_i = A V_i$$

$$u_i = \frac{1}{s_i}(A V_i)$$

$$\boxed{u_i = \frac{1}{s_i} * (A * V_i)}$$

$$u_1 = \frac{1}{\sqrt{4}} * \underset{2\times 2}{\begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix}} * \underset{2\times 1}{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$= \frac{1}{2} * \begin{bmatrix} \frac{3}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1}} * \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2\sqrt{2}} \frac{-1}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = [u_1 \vdots u_2]$$

$$= \begin{bmatrix} 1/\sqrt{2} & \vdots & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \vdots & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = U S V^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer

(d)

$$A = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$A\,A^T = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} * \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{4} + \frac{1}{4} & -\frac{3}{4} - \frac{3}{4} \\ -\frac{3}{4} - \frac{3}{4} & \frac{1}{4} + \frac{9}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{4} & -\frac{6}{4} \\ -6/4 & \frac{10}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$$\Rightarrow (A^T A) - \lambda I$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} - \lambda \end{bmatrix}$$

$$|A^T A - \lambda I| = 0$$

$$\left(\frac{5}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) - \left(-\frac{3}{2}\right)^2 = 0$$

$$\left(\frac{5}{2} - \lambda\right)^2 - \left(\frac{3}{2}\right)^2 = 0 \rightarrow \left(\frac{5}{2} - \lambda + \frac{3}{2}\right)\left(\frac{5}{2} - \lambda - \frac{3}{2}\right) = 0$$

$$(4 - \lambda)(1 - \lambda) = 0 \quad \therefore \boxed{\lambda = 4,}$$

Eigen vector calculation :

$$(A^TA - \lambda I)X = 0$$

$$\begin{bmatrix} (\frac{5}{2}-\lambda) & \frac{-3}{2} \\ \frac{-3}{2} & (\frac{5}{2}-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\lambda = 4$

$$\begin{bmatrix} \frac{5}{2}-4 & \frac{-3}{2} \\ \frac{-3}{2} & \frac{5}{2}-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -\frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \\ -\frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \end{aligned}\right\} \quad x_1 + x_2 = 0$$

$\therefore x_1 = -x_2$

$\left(\therefore \frac{x_1}{x_2} = \frac{-1}{1} = \frac{1}{-1}\right)$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} \frac{5}{2}-1 & \frac{-3}{2} \\ \frac{-3}{2} & \frac{5}{2}-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} \frac{3}{2}x_1 - \frac{3}{2}x_2 &= 0 \\ -\frac{3}{2}x_1 + \frac{3}{2}x_2 &= 0 \end{aligned}\right\} \quad x_1 = x_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = [\, X_1 : X_2 \,]$$

$$= \left[\begin{array}{c|c} 1 & 1 \\ -1 & 1 \end{array}\right]$$

$$\sqrt{1^2 + (-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

$$V = \begin{array}{cc} V_1 & V_2 \\ \left[\begin{array}{c|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] \end{array}$$

(orthogonalization)

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$s_i u_i = A V_i$$

$$u_i = \frac{1}{s_i} \quad A * V_i$$

$$u_1 = \frac{1}{\sqrt{4}} * \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -\frac{3}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1}} * \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = [U_1 : U_2] = \left[\begin{array}{c:c} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}\right]$$

$$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{bmatrix}_{2\times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$A = U * S * V^T$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer

(c)

$$A = \begin{bmatrix} 0.75 & 1.25 \\ 1.25 & 0.75 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ 5/4 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{9}{16} + \frac{25}{16} & \frac{15}{16} + \frac{15}{16} \\ \frac{15}{16} + \frac{15}{16} & \frac{25}{16} + \frac{9}{16} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \frac{34}{16} & \frac{30}{16} \\ \frac{30}{16} & \frac{34}{16} \end{bmatrix} = \begin{bmatrix} \frac{17}{8} & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \frac{17}{8} & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} \end{bmatrix}$$

Characteristic Equation $\Rightarrow$

$$= (A^TA - \lambda I)$$

$$= \begin{bmatrix} 17/8 & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{17}{8} - \lambda\right) & \frac{15}{8} \\ \frac{15}{8} & \left(\frac{17}{8} - \lambda\right) \end{bmatrix}$$

$$|A^TA - \lambda I| = 0$$

$$\left(\frac{17}{8} - \lambda\right)\left(\frac{17}{8} - \lambda\right) - \frac{15}{8} \times \frac{15}{8} = 0$$

$$\left(\frac{17}{8} - \lambda\right)^2 - \left(\frac{15}{8}\right)^2 = 0$$

$$\left[\frac{17}{8}-\lambda+\frac{15}{8}\right]\left[\frac{17}{8}-\lambda-\frac{15}{8}\right]=0$$

$$(4-\lambda)\left(\frac{1}{4}-\lambda\right)=0$$

$$\boxed{\lambda = 4, \frac{1}{4}}$$

<u>Eigen vector calculation</u> ⇒

$$\begin{bmatrix} \left(\frac{17}{8}-\lambda\right) & \frac{15}{8} \\ \frac{15}{8} & \left(\frac{17}{8}-\lambda\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \text{———} (1)$$

Put $\lambda = 4$

$$\begin{bmatrix} \left(\frac{17}{8}-4\right) & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8}-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -15/8 & 15/8 \\ 15/8 & -\frac{15}{8} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-\frac{15}{8}x_1 + \frac{15}{8}x_2 = 0$$

$$x_1 = x_2$$

$$\frac{x_1}{x_2} = \frac{1}{1}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Put $\lambda = \frac{1}{4}$ in equation (1)

$$\begin{bmatrix} \frac{17}{8} - \frac{1}{4} & \frac{15}{8} \\ \frac{15}{8} & \frac{17}{8} - \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{15}{8} & 15/8 \\ \frac{15}{8} & \frac{15}{8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\frac{15}{8} x_1 + \frac{15}{8} x_2 = 0$$

$$x_1 = -x_2$$

$$\frac{x_1}{x_2} = \frac{-1}{1}$$

or $\frac{x_1}{x_2} = \frac{1}{-1}$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore V = [X_1 \; \vdots \; X_2]$$

$$= \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right]$$

$$V = \left[\begin{array}{c|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}\right]$$

(orthogonalization)

$$VT = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Calculation of matrix U

$$s_i u_i = A * V_i$$

$$u_i = \frac{1}{s_i} * A * V_i$$

$$u_1 = \frac{1}{\sqrt{4}} * \begin{bmatrix} \frac{3}{4} & 5/4 \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{3}{4\sqrt{2}} + \frac{5}{4\sqrt{2}} \\ \frac{5}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1/4}} \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \sqrt{4} \begin{bmatrix} \frac{3}{4\sqrt{2}} - \frac{5}{4\sqrt{2}} \\ \frac{5}{4\sqrt{2}} - \frac{3}{4\sqrt{2}} \end{bmatrix}$$

$$= 2 \begin{bmatrix} \frac{-2}{4\sqrt{2}} \\ \frac{2}{4\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = [u_1 : u_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{\frac{1}{4}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0{\cdot}5 \end{bmatrix}$$

$$A = U * S * V^T$$

$$\begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ 0 & 0{\cdot}5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer