Optimal Control System

[RIC 086]

[1]. Summary of the procedure for solving optimal control problems using Hamiltonian formulation of variational calculus -

Given plant equation $\dot{x}(t) = f(x, u, t)$

Given performance index $J = h(x(t_1), t_1) + \int_{t_0}^{t_1} g(x, u, t)\, dt$

Step I Form the Hamiltonian

$$H(x, u, \lambda, t) = g(x, u, t) + \lambda^T f(x, u, t)$$

Step II Solve the equation

$$\frac{\partial H(x, u, \lambda, t)}{\partial u} = 0 \quad \text{(Control equation)}$$

to obtain $u^* = u^*(x, \lambda, t)$

Step III Find Hamiltonian

$$H^*(x, \lambda, t) = H(x, u^*, \lambda, t)$$

Step IV Solve the set of $2n$ equations

$$\dot{x}(t) = \frac{\partial H^*(x, \lambda, t)}{\partial \lambda}$$

$$\dot{\lambda}(t) = -\frac{\partial H^*(x, \lambda, t)}{\partial x}$$

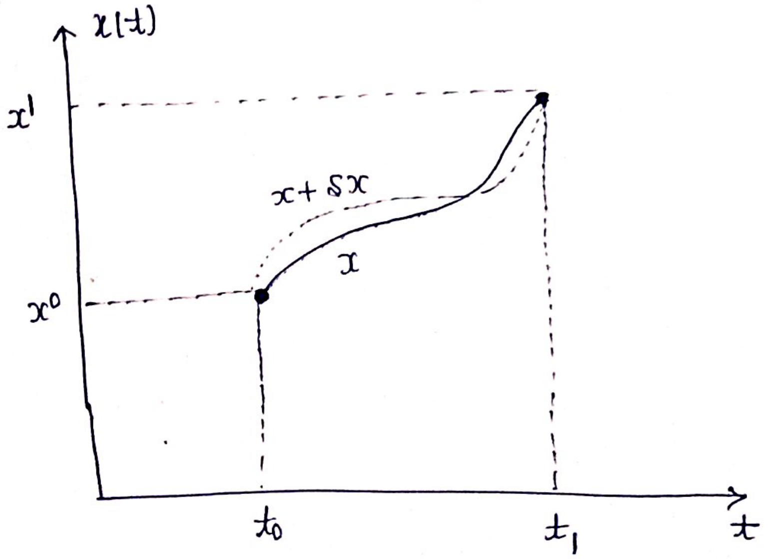
Step V Substitute the results of Step IV into the expression for u^* to obtain the optimal control.

[2] Euler-Lagrange equation Proof -

Let $x(t)$ be a scalar function in the class of functions with continuous first derivatives. In the (t, x) plane, given two points (t_0, x^0) and (t_1, x^1); it is required to find a trajectory joining (t_0, x^0) to (t_1, x^1) such that the integral along this trajectory $x = x^*(t)$, given by

$$J(x) = \int_{t_0}^{t_1} g(x, \dot{x}, t)\, dt \qquad \text{has a relative extremum;}$$

g being a function with continuous first and second partial derivatives with respect to all of its arguments.



* All continuous curves joining the points (t_0, x^0) and (t_1, x^1) are admissible curves. We are required to find the curve (if any exists) that extremizes $J(x)$.

et x be any curve in the admissible class Ω and $x+\delta x$ be the curve in its neighbourhood.

* δx represents the variation in x which is defined as an infinitesimal, arbitrary change in x for a fixed value of the variable t i.e. for $\Delta t = 0$ (δ is called the variational operator similar to differential operator d). The operation of variation is commutative with both integration and differentiation i.e;

$$\delta\left(\int g\,dt\right) = \int (\delta g)\,dt$$

and

$$\delta\left(\frac{dx}{dt}\right) = \frac{d}{dt}(\delta x).$$

The total increment in J due to variation δx in x is given by :

$$\Delta J(x, \delta x) = J(x+\delta x) - J(x)$$

④

$$= \int_{t_0}^{t_1} g(x+\delta x,\ \dot{x}+\delta\dot{x},\ t)\,dt - \int_{t_0}^{t_1} g(x,\dot{x},t)\,dt$$

$$= \int_{t_0}^{t_1} \left[g(x+\delta x,\ \dot{x}+\delta\dot{x},\ t) - g(x,\dot{x},t) \right] dt$$

Applying Taylor series (about the point $x(t), \dot{x}(t)$)

$$\Delta J(x, \delta x) = \int_{t_0}^{t_1} \Big\{ \cancel{g(x,\dot{x},t)} + \left(\frac{\partial}{\partial x} g(x,\dot{x},t)\right)\delta x + \left(\frac{\partial}{\partial \dot{x}} g(x,\dot{x},t)\right)\delta\dot{x}$$

$$+ \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} g(x,\dot{x},t)\right)\delta x^2 + \left(\frac{\partial^2}{\partial x\,\partial\dot{x}} g(x,\dot{x},t)\right)\delta x\,\delta\dot{x}$$

$$+ \frac{1}{2}\frac{\partial^2 g(x,\dot{x},t)}{\partial \dot{x}^2}\delta\dot{x}^2 + \cdots\cdots - \cancel{g(x,\dot{x},t)} \Big\} dt$$

$$\cancel{+\ \frac{1}{2}\frac{\partial^2 g}{\partial \dot{x}^2}(x,\dot{x},t)\,\delta\dot{x}^2 + \cdots\cdots = g(x,\dot{x},t)}$$

The first variation in J :-

$$\delta J(x,\delta x) = \int_{t_0}^{t_1}\left\{\left(\frac{\partial g}{\partial x}(x,\dot{x},t)\right)\delta x + \underbrace{\underbrace{\left(\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\right)}_{\mathrm{I}}\ \underbrace{\delta\dot{x}}_{\mathrm{II}}}\right\}dt$$

Apply formula ⇒

$$\int \mathrm{I}\cdot\mathrm{II}\,dt$$

$$= \int_{t_0}^{t_1}\left(\frac{\partial g}{\partial x}(x,\dot{x},t)\right)\delta x\,dt + \int_{t_0}^{t_1}\underbrace{\left(\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\right)}_{\mathrm{I}}\underbrace{\delta\dot{x}}_{\mathrm{II}}\,dt$$

$$= \int_{t_0}^{t}\frac{\partial g}{\partial x}(x,\dot{x},t)\,\delta x\,dt + \frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\,\delta x\,\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\left[\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\right)*\delta x\right]dt$$

Note:

$$\int \delta\dot{x}\,dt = \int \frac{d}{dt}(\delta x)\,dt = \delta x$$

$$= \int_{t_0}^{t_1}\frac{\partial g}{\partial x}(x,\dot{x},t)\,\delta x\,dt + 0 - \int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\right)*\delta x\,dt$$

$$= \int_{t_0}^{t_1}\frac{\partial g}{\partial x}(x,\dot{x},t)\,\delta x\,dt - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\right)\delta x\,dt$$

$$= \int_{t_0}^{t_1} \left\{ \frac{\partial g(x,\dot{x},t)}{\partial x} - \frac{d}{dt}\frac{\partial g(x,\dot{x},t)}{\partial \dot{x}} \right\} \delta x \, dt$$

The above equation maybe expressed as :-

$$\int_{t_0}^{t_1} \phi(t)\, \delta x \, dt = 0 \quad \text{—— (I)}$$

where $\phi(t)$ is a continuous function (as per the assumption made on g)

The above will only be true when ⇒

$$\phi(t) = 0$$

$$\boxed{\frac{\partial g}{\partial x}(x,\dot{x},t) - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t) = 0}$$

or

$$\boxed{\frac{\partial g}{\partial x}(x^*,\dot{x}^*,t) - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}}(x^*,\dot{x}^*,t) = 0}$$

↳ This equation is called Euler-lagrange equation.

3] Continuous-time linear state regulator / derivation of matrix Riccati Equation.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$x(t_0) = x^0$$

$$J = \frac{1}{2}x^T(t_1)Hx(t_1) + \frac{1}{2}\int_{t_0}^{t_1}\left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right)dt$$

$\left.\begin{matrix}H\\Q\end{matrix}\right\} \rightarrow$ Real symmetric positive semidefinite matrix

$R \rightarrow$ Real symmetric positive definite matrix

Use Hamilton Jacobi equation —

$$H\left(x(t), u(t), \frac{\partial J^*}{\partial x}(x(t),t), t\right) = \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t)$$

$$+\left[\frac{\partial J^*(x(t),t)}{\partial x}\right]^T\left[A(t)x(t) + B(t)u(t)\right] \quad \text{—(1)}$$

$$\frac{\partial H}{\partial u}\left(x(t), u(t), \frac{\partial J(x(t),t)}{\partial x}, t\right) = 0$$

⑧

$$R(t)\,u(t) + B^T(t)\cdot \frac{\partial J^*}{\partial x}(x(t),t) = 0 \quad \text{——} (2)$$

now::

$$\frac{\partial^2 H}{\partial^2 u}\left(x(t),\, u(t)\, \frac{\partial J}{\partial x}(x(t),t)\, ,t\right) = R(t)$$

$$= \text{Positive definite matri}$$

∴ From equation(2)

$u^*(t)$ will given the optimum value.

$$\boxed{u^*(t) = -\,R^{-1}(t)\; B^T(t)\; \frac{\partial J^*}{\partial x}(x(t),t)}$$

now substituting the value of $u^*(t)$ for $u(t)$ in equation (1)

$$H^*\left(x(t),\, \frac{\partial J^*}{\partial x}(x(t),t)\, ,t\right) = \frac{1}{2}\, x^T(t)\; Q(t)\; x(t)$$

$$+\frac{1}{2}\,\frac{\partial J^*}{\partial x}(x(t),t)\;\; B(t)\, R^{-1}(t) * R(t) * R^{-1}(t)\cdot B^T(t)\cdot \frac{\partial J}{\partial x}(x(t),t)$$

$$+\left(\frac{\partial J^*}{\partial x}(x(t),t)\right)^T A(t)\, x(t) - \left(\frac{\partial J^*}{\partial x}(x(t),t)\right)^T \left[B(t)\cdot R^{-1}(t)\, B^T(t)\; \frac{\partial J^*}{\partial x}(x(t),t)\right]$$

$$= \frac{1}{2} x^T(t)\, Q\, x(t) - \frac{1}{2}\left(\frac{\partial J^*}{\partial x}(x(t),t)\right)^T B(t)\cdot R^{-1}(t)\; B^T(t)\, \frac{\partial J^*}{\partial x}(x(t),t)$$

$$+\left(\frac{\partial J^*}{\partial x}(x(t),t)\right)^T A(t)\, x(t)$$

The Hamilton - Jacobi equation is $\Rightarrow$

$$H^*\left(x, \frac{\partial J}{\partial x}, t\right) + \frac{\partial J^*}{\partial t} = 0$$

$$\therefore \quad \frac{\partial J^*}{\partial t}(x(t),t) + \frac{1}{2} x^T(t)\, Q\, x(t) - \frac{1}{2}\left(\frac{\partial J}{\partial x}(x(t),t)\right)^T \left[B(t)\, R^{-1}(t)\, B^T(t)\, \frac{\partial J^*}{\partial x}(x(t),t)\right]$$

$$+\left(\frac{\partial J}{\partial x}(x(t),t)\right)^T A(t)\, x(t) = 0 \qquad \text{———} \; (3)$$

Boundary Condition is :-

$$J^*(x(t_1), t_1) = \frac{1}{2} x^T(t_1)\, H\, x(t_1)$$

We can take

$$J^*(x(t),t) = \frac{1}{2} x^T(t)\, P(t)\, x(t) \qquad \text{———} \; (4)$$

$$P(t_1) = H$$

Substituting eq (4) into equation (3).

$$\frac{1}{2} x^T(t)\, \dot{P}(t)\, x(t) + \frac{1}{2} x^T(t)\, Q(t)\, x(t)$$

$$-\frac{1}{2} x^T(t)\, P(t)\, B(t)\, R^{-1}(t)\, B^T(t)\, P(t)\, x(t) + x^T(t)\, P(t)\, A(t)\, x(t) = 0$$

or

$$x^T(t)\left[\dot{P}(t) + Q(t) - P(t)\, B(t)\, R^{-1}(t)\, B^T(t)\, P(t) + 2\, P(t) \cdot A(t)\right] x(t) = 0 \qquad (5)$$

$$x^T(t)\left[\qquad " \qquad " \qquad " \qquad \right] x(t) = 0$$

The above equation will be satisfied only when quantity inside the brackets equal to zero. However, we know that in the scalar function $Z^T W Z$, only the symmetric part of the matrix W,

$$W_s = \frac{W + W^T}{2} \quad \text{is of importance.}$$

In equation (5), all the terms within brackets are already symmetric except the last term.

$$\text{Symmetric part of } 2P(t)A(t) = 2\,\frac{P(t)A(t) + A^T(t)P(t)}{2}$$

$$= P(t)A(t) + A^T(t)P(t) \qquad (6)$$

Put the eq (6) in eqn (5).

$$\boxed{\begin{aligned}&\dot{P}(t) + Q(t) - P(t)\,B(t)\,R^{-1}(t)\,B^{T}(t)\,P(t)\\ &+ P(t)\,A(t) + A^{T}(t)\,P(t) = 0\end{aligned}} \quad \text{—(7)}$$

$\because$ $P(t) \rightarrow$ symmetric

& $H \rightarrow$ symmetric $\quad (H = P(t_1))$

$\therefore$ Matrix differential equation $\Rightarrow$ Symmetric

$\therefore$ Solution $P(t)$ for all $t_0 \le t \le t_1$ must be symmetric.

* This symmetry will often be used especially when computing $P(t)$.

We know that: Scalar differential equation:

$$\boxed{\frac{dy}{dx} + \alpha(x)\,y + \beta(x)\,y^2 = \gamma(x)} \quad \text{—(8)}$$

Eq^n (7) and (8) are similar.

* Equation (7) is referred to as Matrix Riccati Equation.

Once $P(t)$ has been determined, the optimal control law is given by :-

$$u^*(t) = -R^{-1}(t)\, B^T(t)\, P(t)\, x(t)$$

$$u^*(t) = K(t)\, x(t)$$

where $K(t) = -R^{-1}(t)\, B^T(t)\, P(t)$

Thus by assuming a solution of the form

$J^*(x(t), t) = \frac{1}{2} x^T(t)\, P(t)\, x(t)$, the optimal control law is linear, time varying state feedback.

Q:[4]

$$\dot{x} = -x + u$$

$$x(0) = x^0$$

$$x(2) = x^1$$

$$J = \int_0^2 (x^2 + u^2)\, dt$$

Find u^* that minimizes J.

Solution:

$$\dot{x} = -x + u$$

$$u = \dot{x} + x$$

$$J = \int_0^2 \left[x^2 + (\dot{x} + x)^2\right] dt$$

$$= \int_0^2 \left[x^2 + \dot{x}^2 + x^2 + 2x\dot{x}\right] dt$$

$$= \int_0^2 \left[2x^2 + 2x\dot{x} + \dot{x}^2\right] dt$$

$$g(x, \dot{x}) = 2x^2 + 2x\dot{x} + \dot{x}^2$$

We know that:

$$\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t) = 0$$

$$[4x + 2\dot{x} + 0] - \frac{d}{dt}[0 + 2x + 2\dot{x}] = 0$$

$$4x + \cancel{2\dot{x}} - \cancel{2\dot{x}} - 2\ddot{x} = 0$$

$$4x - 2\ddot{x} = 0$$

$$2x - \ddot{x} = 0$$

$$\ddot{x} - 2x = 0$$

$$\frac{d^2x}{dt^2} - 2x = 0$$

$$(D^2 - 2)x = 0 \qquad \left(D = \frac{d}{dt}\right)$$

$$D^2 = 2$$

$$D = \pm\sqrt{2}$$

$$x(t) = k_1 e^{-\sqrt{2}t} + k_2 e^{+\sqrt{2}t}$$

$$u(t) = x(t) + \dot{x}(t)$$

$$u^*(t) = x^*(t) + \dot{x}^*(t)$$

$$= \left[k_1 e^{-\sqrt{2}t} + k_2 e^{+\sqrt{2}t}\right] - \sqrt{2}k_1 e^{-\sqrt{2}t} + \sqrt{2}k_2 e^{\sqrt{2}t}$$

(16)

$$\boxed{u^*(t) = K_1 (1-\sqrt{2}) e^{-\sqrt{2}t} + K_2 (1+\sqrt{2}) e^{\sqrt{2}t}}$$

Q[5]: $\dot{x}(t) = -2x(t) + u(t)$

$$J = \frac{1}{4} \int_0^{t_1} \left(3x^2 + \frac{1}{4}u^2\right) dt$$

$$p(t_1) = 0$$

It is desired to find the control law that minimizes the performance index.

Sol ⇒ $A = -2$

$B = 1$

$H = 0$

$Q = \frac{3}{2}$

$R = \frac{1}{8}$

$$\dot{P}(t) + Q(t) - \underline{P(t)\, B(t)}\, R^{-1}(t)\, \underline{B^T(t)\, P(t)} + P(t)\, A(t) + A^T(t)\, P(t) = 0$$

$$\dot{P}(t) + \frac{3}{2} - P(t)\,(1)\,(8)\cdot(1)\,P(t) + P(t)\cdot(-2) + (-2)\cdot P(t) = 0$$

$$\dot{P}(t) + \frac{3}{2} - 8P^2(t) - 4P(t) = 0$$

$$\dot{P}(t) = 8P^2(t) + 4P(t) - \frac{3}{2}$$

$$= 8\left[P^2(t) + \frac{1}{2}P(t) - \frac{3}{16}\right]$$

$$\dot P(t) = 8\left[P^2(t) + \frac{3}{4}P(t) - \frac{1}{4}P(t) - \frac{3}{16}\right]$$

$$\frac{3}{16} = \frac{3}{4} * \frac{1}{4}$$

$$\frac{3}{4} - \frac{1}{4} = +\frac{2}{4} = +\frac{1}{2}$$

$$\dot P(t) = 8\left[P(t)\left(P(t) + \frac{3}{4}\right) - \frac{1}{4}\left(P(t) + \frac{3}{4}\right)\right]$$

$$\dot P(t) = 8\left[P(t) + \frac{3}{4}\right]\left[P(t) - \frac{1}{4}\right]$$

$$\frac{dP}{dt} = 8\left(P(t) + \frac{3}{4}\right)\left(P(t) - \frac{1}{4}\right)$$

$$\int_t^{t_1} \frac{1}{8}\,\frac{dP}{\left(P(t) + \frac{3}{4}\right)\left(P(t) - \frac{1}{4}\right)} = \int_t^{t_1} dt$$

$$\frac{1}{8}\left[\int_t^{t_1} \frac{-1\,dP}{\left[P(t) + \frac{3}{4}\right]} + \int_t^{t_1} \frac{1\,dP}{\left[P(t) - \frac{1}{4}\right]}\right] = \int_t^{t_1} dt$$

$$-\frac{1}{8}\log_e\left(P(t) + \frac{3}{4}\right)\Bigg]_t^{t_1} + \frac{1}{8}\log_e\left(P(t) - \frac{1}{4}\right)\Bigg]_t^{t_1} = (t_1 - t)$$

$$-\frac{1}{8}\log_e\left[\frac{P(t_1) + \frac{3}{4}}{P(t) + \frac{3}{4}}\right] + \frac{1}{8}\log_e\left[\frac{P(t_1) - 1/4}{P(t) - \frac{1}{4}}\right] = (t_1 - t)$$

$$\log_e\left(\frac{P(t_1) - 1/4}{P(t) - 1/4}\right) * \left(\frac{P(t) + 3/4}{P(t_1) + 3/4}\right) = 8(t_1 - t)$$

(18)

$$\left(\frac{P(t_1) - \frac{1}{4}}{P(t) - \frac{1}{4}}\right) * \left(\frac{P(t) + 3/4}{P(t_1) + 3/4}\right) = e^{\theta(t_1 - t)}$$

$$P(t_1) = 0$$

$$\frac{(0 - 1/4)}{(P(t) - 1/4)} * \frac{(P(t) + 3/4)}{(0 + 3/4)} = e^{\theta(t_1 - t)}$$

$$-\frac{1}{3}\left[\frac{P(t) + 3/4}{P(t) - 1/4}\right] = e^{\theta(t_1 - t)}$$

$$\frac{P(t) + 3/4}{P(t) - 1/4} = \frac{-3e^{\theta(t_1 - t)}}{1}$$

$$P(t) + \frac{3}{4} = -3e^{\theta(t_1 - t)} P(t) + \frac{3}{4} e^{\theta(t_1 - t)}$$

$$P(t)\left[1 + 3e^{\theta(t_1 - t)}\right] = \frac{3}{4}\left(e^{\theta(t_1 - t)} - 1\right)$$

$$P(t) = \frac{\frac{3}{4}\left[e^{\theta(t_1 - t)} - 1\right]}{\left[3e^{\theta(t_1 - t)} + 1\right]}$$

$$\boxed{P(t) = \frac{-\frac{3}{4}\left[1 - e^{\theta(t_1 - t)}\right]}{\left[1 + 3e^{\theta(t_1 - t)}\right]}}$$

$$u^*(t) = K(t) \cdot x(t)$$

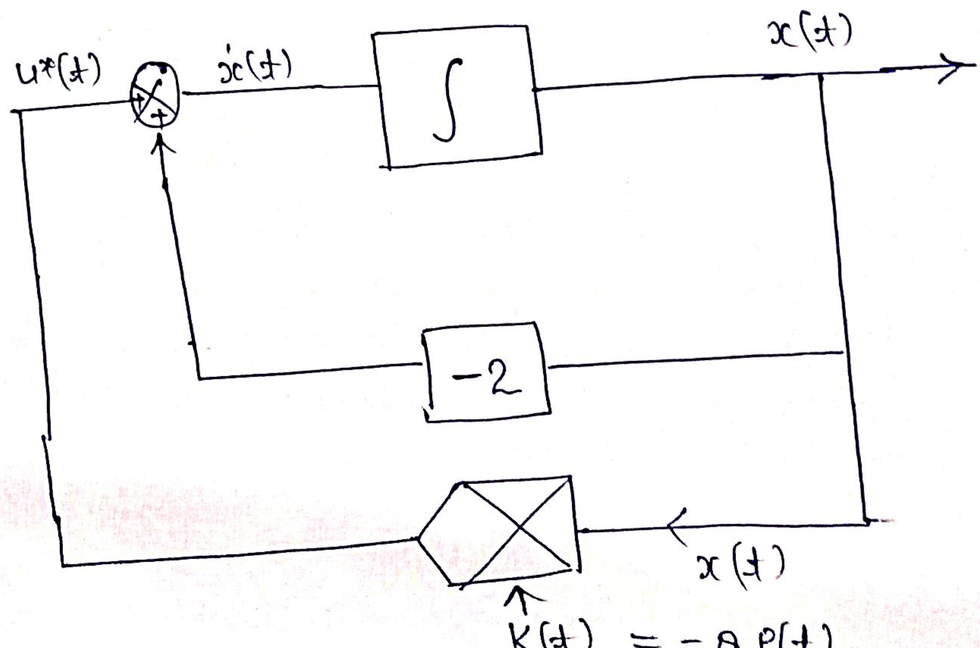
$$K(t) = -R^{-1}(t)\ B^T(t) \cdot P(t)$$

$$\therefore K(t) = -8 * 1 * \left[-\frac{3}{4} \left\{ \frac{1-e^{8(t_1-t)}}{1+3e^{8(t_1-t)}} \right\} \right]$$

$$= +6 \left[\frac{1-e^{8(t_1-t)}}{1+3e^{8(t_1-t)}} \right]$$

$$u^*(t) = K(t) \cdot x(t)$$

$$\boxed{u^*(t) = 6 \left[\frac{1-8e^{8(t_1-t)}}{1+3e^{8(t_1-t)}} \right] * x(t)}$$



ELECTRONICS AND INSTRUMENTATION ENGINEERING

CLASS TEST-1 EXAMINATION 2019-2020

RIC-086 OPTIMALCONTROL SYSTEM

TIME: 1 HOURS **MAX MARKS: 20**

Note attempt all questions

Q	Question Paper Based On Course Outcomes According To Bloom's Cognitive Level	Marks	CO	BL
1	**Give** the summary of the procedure for solving optimal control problem using Hamiltonian formulation of variational calculus.	4	CO 1	BL 3
2	**Derive** the Euler-Lagrange equation (only mathematical expressions).	4	CO 1	BL 4
3	**Derive** the expression of matrix *Riccati equation* for continuous time linear state regulator (only mathematical expressions).	4	CO 2	BL 4
4	$\dot{x} = -x + u$ $x(0) = x^0$ $x(2) = x^1$ **Find** u^* that minimizes $J = \int_0^2 (x^2 + u^2)\ dt$	4	CO 1	BL 4
5	$\dot{x}(t) = -2x(t) + u(t)$ Performance index: $J = \frac{1}{4}\int_0^{t_1} \left(3x^2 + \frac{1}{4}u^2\right) dt;$ t_1 is specified, assume $p(t_1) = 0$, **Find** the control law that minimizes the performance index.	4	CO 2	BL 4

Kalman filter

Gain algorithm

$$M(k+1) = -FP(k|k-1)C^T\left(R + CP(k|k-1)C^T\right)^{-1}$$

Variance algorithm :

$$P(k+1|k) = FP(k|k-1)F^T + Q - FP(k|k-1)C^T\left(R + CP(k|k-1)C^T\right)^{-1} CP(k|k-1)F^T$$

$$P(0|0) = P_0$$

Q: Determine the Kalman gain $K(k)$ for $k = 1$ to 2 for the following estimation problem :

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + w(k)$$

$$y(k) = x_1(k) + v(k)$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$R = 1$$

$$P_0 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

P(1|0)

Solution -

$$x(k+1) = F\,x(k) + G\,u(k) + w(k)$$

$$y(k) = C\,x(k) + v(k)$$

$$\therefore \quad F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$M(1) = -F\,P(0|-1)\,C^T\left[R + C\,P(0|-1)\,C^T\right]^{-1}$$

$$= -F\,P(0|0)\,C^T\left[R + C\,P(0|0)\,C^T\right]^{-1}$$

$$= -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left\{\left(1 + \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right\}^{-1}$$

$$= -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix}\left[1 + \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right]^{-1}$$

$$= -\begin{bmatrix} 10 \\ 5 \end{bmatrix} \cdot \left[1 + 10\right]^{-1}$$

$$= -\begin{bmatrix} 10 \\ 5 \end{bmatrix}\left[11\right]^{-1}$$

$$= -\begin{bmatrix} 10 \\ 5 \end{bmatrix} \cdot \left(\frac{1}{11}\right)$$

$$M(1) = \begin{bmatrix} -10/11 \\ -5/11 \end{bmatrix}$$

Ans

$$P(1|0) = F\,P(0|-1)\,F^T + Q - F\,\underbrace{P(0|-1)\,C^T\left[R + C P(1|-1)\,C^T\right]^{-1}}_{\text{This is } M(1)} \cdot C\,P(0|-1)\,F^T$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} + \underbrace{\begin{bmatrix} -10/11 \\ -5/11 \end{bmatrix}}_{\text{Value of } M(1)} \cdot \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} -10/11 \\ -5/11 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} -10/11 \\ -5/11 \end{bmatrix}\begin{bmatrix} 10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} -\frac{100}{11} & -\frac{50}{11} \\ -\frac{50}{11} & -\frac{25}{11} \end{bmatrix}$$

$$= \begin{bmatrix} 10+0-\frac{100}{11} & 5+0-\frac{50}{11} \\ 5+0-\frac{50}{11} & 5+0.5-\frac{25}{11} \end{bmatrix}$$

$$= \begin{bmatrix} +\frac{10}{11} & +\frac{5}{11} \\ +\frac{5}{11} & +\frac{71}{22} \end{bmatrix}$$

$$M(2) = -F P(1|0) C^T \left[R + C P(1|0) C^T\right]^{-1}$$

$$= -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{10}{11} & \frac{5}{11} \\ \frac{5}{11} & \frac{71}{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 22/133 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15/11 \\ 81/22 \end{bmatrix} \begin{bmatrix} 22/133 \end{bmatrix}$$

$$= -\begin{bmatrix} 111/22 \\ 81/22 \end{bmatrix} \begin{bmatrix} 22/133 \end{bmatrix}$$

$$= -\begin{bmatrix} +\frac{111}{22} * \frac{22}{133} \\ \frac{81}{22} * \frac{22}{133} \end{bmatrix}$$

$$= \begin{bmatrix} -111/133 \\ -81/133 \end{bmatrix}$$

Note:

$$\left[R + C P(1|0) C^T\right]^{-1}$$

$$\left\{1 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{10}{11} & \frac{5}{11} \\ \frac{5}{11} & \frac{71}{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^{-1}$$

$$\left\{1 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 15/11 \\ 81/22 \end{bmatrix} \right\}^{-1}$$

$$= \left\{1 + \left(\frac{15}{11} + \frac{81}{22}\right)\right\}^{-1}$$

$$= \left\{1 + \frac{111}{22}\right\}^{-1}$$

$$= \left[\frac{133}{22}\right]^{-1}$$

$$= 22/133$$

Ans:

$$M(1) = \begin{bmatrix} -10/11 \\ -\frac{5}{11} \end{bmatrix}$$

$$M(2) = \begin{bmatrix} -111/133 \\ -81/133 \end{bmatrix}$$