

Control System - (II)

The following notes are based on the notes given by Prof. S. R. Rao.

1. Block Diagram

Block diagram is a graphical representation of a system showing the input, output, and internal components. It consists of blocks connected by lines representing signals. The blocks represent different parts of the system, such as sensors, actuators, and controllers.

2. Block Diagram Reduction

Block diagram reduction is the process of simplifying a complex block diagram into a smaller one. This is done by combining blocks that are in series or parallel, or by using feedback loops to reduce the number of blocks.

3. Block Diagram Representation of Transfer Functions

Block diagram representation of transfer functions is a way of representing a system using blocks and lines. Each block represents a transfer function, which is a ratio of the output signal to the input signal. The blocks are connected in a specific way to represent the overall system behavior.

4. Block Diagram Representation of State Space Equations

Block diagram representation of state space equations is a way of representing a system using state variables. The state variables are represented by blocks, and the relationships between them are represented by lines. This representation is useful for analyzing the stability and performance of the system.

5. Block Diagram Representation of Difference Equations

Block diagram representation of difference equations is a way of representing a system using difference equations. The difference equations are represented by blocks, and the relationships between them are represented by lines. This representation is useful for analyzing the stability and performance of the system.

6. Block Diagram Representation of Differential Equations

Block diagram representation of differential equations is a way of representing a system using differential equations. The differential equations are represented by blocks, and the relationships between them are represented by lines. This representation is useful for analyzing the stability and performance of the system.

7. Block Diagram Representation of Nonlinear Systems

Block diagram representation of nonlinear systems is a way of representing a system using nonlinear blocks. These blocks represent nonlinear relationships between the input and output signals. The block diagram can be reduced to a linear form by using appropriate linearization techniques.

8. Block Diagram Representation of Discrete-time Systems

Block diagram representation of discrete-time systems is a way of representing a system using discrete-time blocks. These blocks represent discrete-time relationships between the input and output signals. The block diagram can be reduced to a continuous-time form by using appropriate sampling and reconstruction techniques.

9. Block Diagram Representation of Multivariable Systems

Block diagram representation of multivariable systems is a way of representing a system using multivariable blocks. These blocks represent multivariable relationships between the input and output signals. The block diagram can be reduced to a single-variable form by using appropriate coordinate transformations.

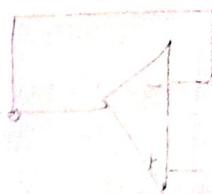
10. Block Diagram Representation of Time-varying Systems

Block diagram representation of time-varying systems is a way of representing a system using time-varying blocks. These blocks represent time-varying relationships between the input and output signals. The block diagram can be reduced to a time-invariant form by using appropriate time-invariant representations.

Digital to analog converter (DAC)

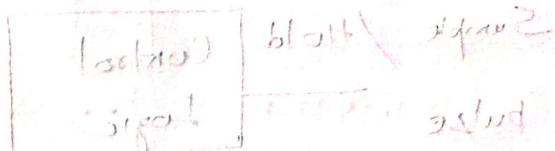
1. Weighted resistor D/A converter
2. R-2R ladder D/A converter

Weighted resistor



Analog to digital converter (ADC)

1. Ramp type
2. Dual slope integrating type
3. Successive approximation type



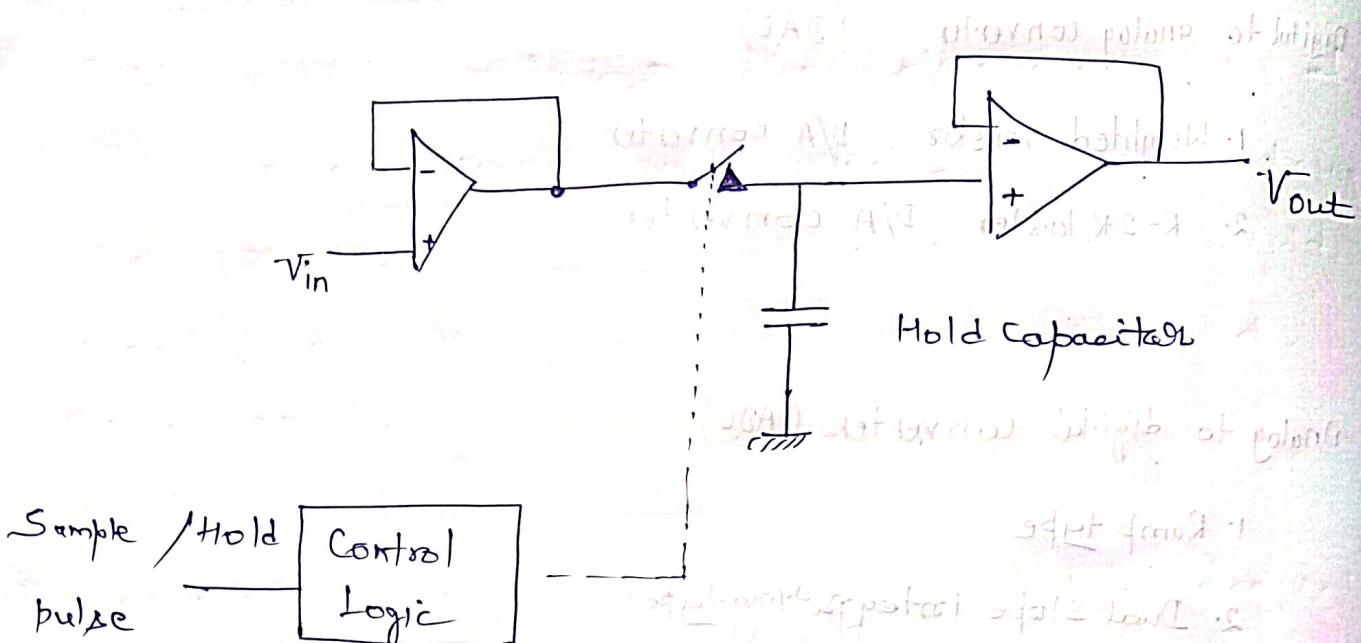
4. Flash type ADC

Sample and hold operations - A sampler is a device which

converts an analog signal into a train of amplitude modulated pulses. A hold device simply maintains the value of the pulse for a prescribed time duration.

In a majority of the practical digital operations sample and hold functions are performed by a single device commercially known as sample-and-hold (S/H). The S/H operates by storing input signal voltage as charge on a small, high quality capacitor.

Op-amps can be used to implement the sample-and-hold circuit for storage in a capacitor.

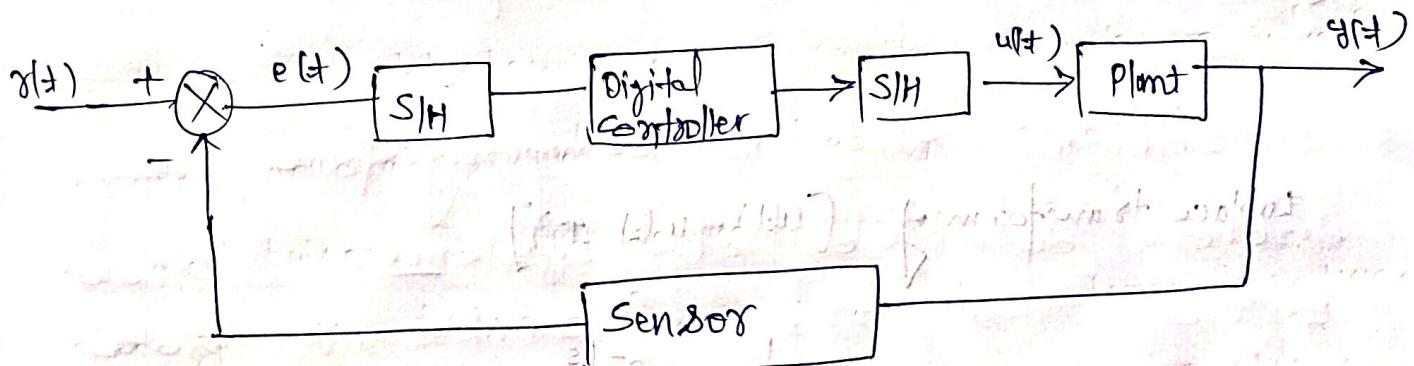
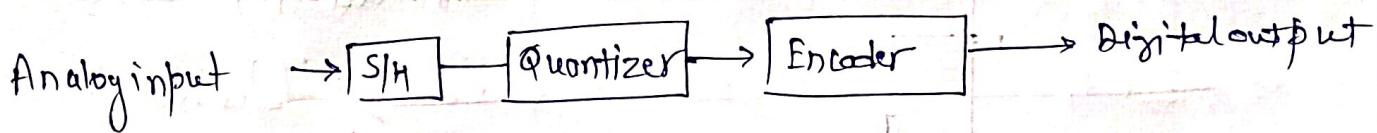
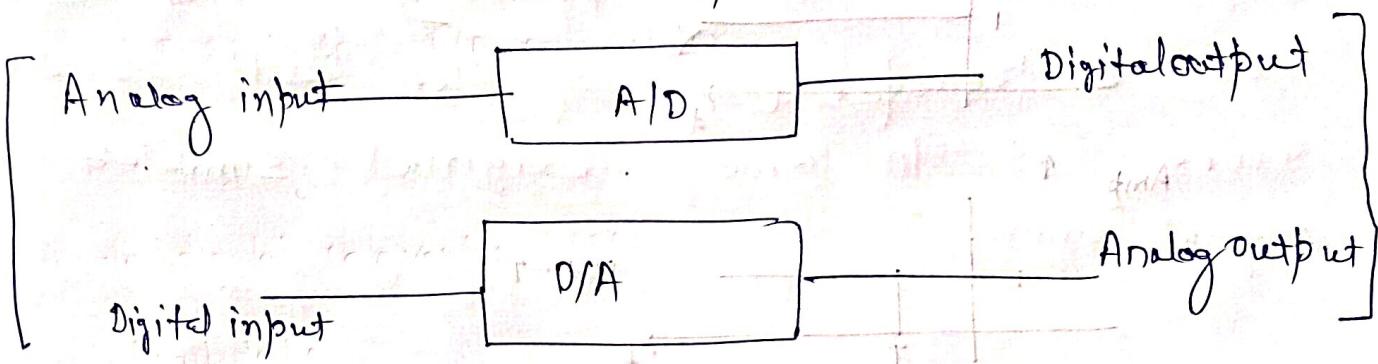


* When the switch is closed ('Logic '1' input Sample pulse), the capacitor rapidly charges to V_{in} and V_{out} is equal to V_{in} approximately.

* When the switch is open ('Logic '0' input Hold pulse), the capacitor retains its charge, the output holds at a value of V_{in} .

* If the input voltage changes rapidly while the switch is closed, the capacitor can follow this voltage because the charging time constant is very short. If the switch is suddenly opened, the capacitor voltage represents a sample of input voltage at the

instant switch was opened.) The capacitor then holds this sample until the switch is again closed and a new sample is taken.



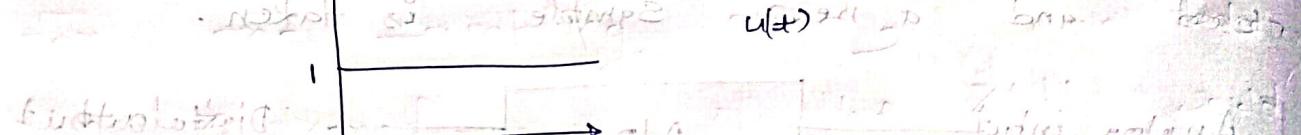
$$\frac{2T_s - 1}{2}$$

Zero order hold (ZOH)

Input signal $u(t)$ is held constant during the interval $[t, t+T]$.

Amp

$u(t)$



Amp

$u(t-T)$

t

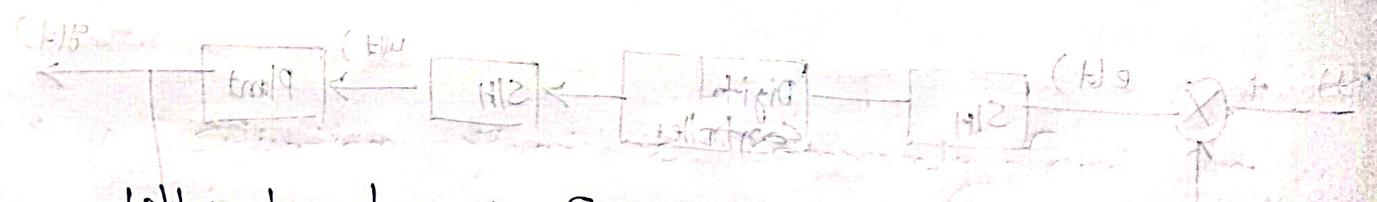
T

Amp

$u(t) - u(t-T)$

t

T

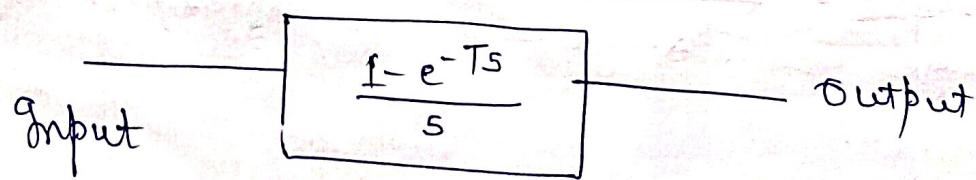


Laplace transform of $[u(t) - u(t-T)]$

$$= \frac{1}{s} - \frac{e^{-Ts}}{s}$$

$$= \frac{1 - e^{-Ts}}{s}$$

①



Zero Order Hold

Since the term e^{-Ts} in equation ① represents a delay of one sampling period, then considering the right shift theorem of the z-transform

$$G_{H_0}(z) \text{ is } z \text{ transform of } \left[\frac{1-e^{-Ts}}{s} \right]$$

$$\text{Put } z = e^{Ts}$$

$$= z \text{ transform of } \left\{ \frac{1-z^{-1}}{s} \right\}$$

$$= (1-z^{-1}) \cdot z \cdot \left(\frac{1}{s} \right)$$

$$= \left(1 - \frac{1}{z} \right) \cdot z \cdot \left(\frac{1}{s} \right)$$

$$= \left(\frac{z-1}{z} \right) * \left(\frac{z}{s} \right)$$

$$\Rightarrow 1$$

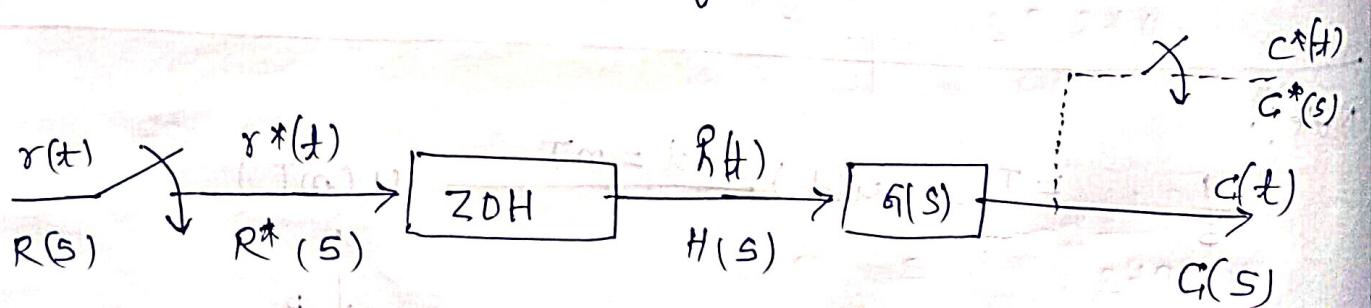
Note:

$$\begin{aligned} \left(\frac{1}{s} \right) &\xrightarrow{T} u(t) \xrightarrow{t=nT} u(nT) \\ &\downarrow z \\ &\left(\frac{z}{z-1} \right) \end{aligned}$$

* This result is expected since ZOH (zero order hold) simply holds the discrete signal for one sampling period and taking the Z transform of the ZOH would revert it to the original sampled signal again.

* However, the above thing simply does not have any physical usefulness, since a ZOH is almost never found just situated by itself.

* A situation that is commonly found in discrete data control systems is shown below, that is, a sample and hold (S/H) unit is followed by a linear system with transfer function $G(s)$.



Question: $r(t) = e^{-at} u_s(t)$ Determine its z transform.

Sol): Put $t = nT$

$$\text{Block diagram: } \begin{array}{c} \text{Input } r(t) \\ \xrightarrow{\text{H}(t)} \boxed{\frac{e^{-at}-1}{t}} \xrightarrow{\text{H}(t)} \text{Output } r(nT) \end{array}$$

$$r(nT) = e^{-ant} u_s(nT)$$

$$\text{but } r(nT) = (e^{-aT})^n u_s(nT)$$

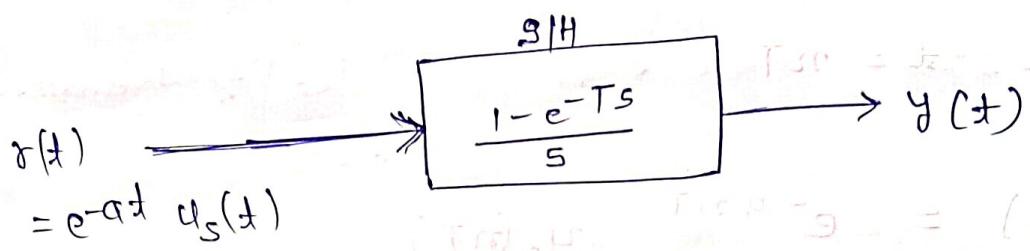
taking z-transform

$$\Rightarrow \frac{(z^{-a}-1)}{(z-e^{-aT})} = (z)y$$

$$\left| \frac{1}{(z-e^{-aT})} \right| \leq \left| (z)y \right| \text{ for small } z$$

Q: Let input $r(t) = e^{-at} u_s(t)$ is given to S/H circuit. Prove that if $T \rightarrow 0$ output of sample and hold circuit is same as input.

Solution:



Output $Y(s) = \text{T.F.} * (\text{Laplace transform of input})$

$$Y(s) = \left(\frac{1 - e^{-Ts}}{s} \right) * \left(\frac{1}{s+a} \right)$$

$$Y(s) = (1 - e^{-Ts}) * \frac{1}{s(s+a)}$$

$$\lim_{T \rightarrow 0} Z[Y(s)] = \lim_{T \rightarrow 0} Z \left[(1 - e^{-Ts}) \cdot \frac{1}{s(s+a)} \right]$$

$$= (1 - z^{-1}) Z \left[\frac{1}{s(s+a)} \right]$$

$$= (1 - z^{-1}) Z \left[\frac{1}{a} \left(\frac{1}{s} - \frac{1}{s+a} \right) \right]$$

$$= \left(\frac{z-1}{z} \right) \left(\frac{1}{a} \right) - z \left[u(t) - e^{-at} u(t) \right]$$

$$= \left(\frac{z-1}{z} \right) \left(\frac{1}{a} \right) - z \left[u(nT) - e^{-ant} u(nT) \right]$$

$$= \left(\frac{z-1}{z} \right) \left(\frac{1}{a} \right) \left[\frac{z}{z-1} - \frac{z}{z-e^{-aT}} \right]$$

$$= \left(\frac{z-1}{z} \right) * \left(\frac{1}{a} \right) \left(\frac{z}{z-1} \right) \left[1 - \frac{z-1}{z-e^{-aT}} \right]$$

$$= \frac{1}{a} \left[1 - \frac{z-1}{z-e^{-aT}} \right]$$

$$= \text{Limit}_{T \rightarrow 0} \frac{1}{a} \left[1 - \frac{1+Ts-1}{1+Ts-1+aT} \right] \quad \because [z = e^{Ts} = 1+Ts]$$

$$= \text{Limit}_{sT \rightarrow 0} \frac{1}{a} \left[1 - \frac{sT}{s+aT} \right]$$

$$= e^{-j\omega_0 t} \frac{1}{a} \left[1 - \frac{s}{s+a} \right]$$

$$= \frac{1}{a} \left[\frac{s+a-\delta}{s+a} \right]$$

$$= \left(\frac{1}{q} \right) \cdot \left(\frac{q}{s+a} \right)$$

$$= \left(\frac{1}{s+a} \right)$$

= Input

Limit [Output of Sample and hold circuit] = Output of
 $T \rightarrow 0$ the S/H unit

$$\left[\frac{1}{T+1} \right]$$

$$\left[\frac{1-T}{T+1} \right]$$

$$\left[\frac{1-2T+1}{T+1-2T+1} \right] = 1$$

limit
 $0 \leftarrow T$

$$\left[\frac{1-2T}{T+1-2T} \right] = 1$$

limit
 $0 \leftarrow T$

Q: Transfer function of the system is $G(s) = \frac{k}{s(s+a)}$

and zero order hold $G_{ho}(s) = \frac{1-e^{-Ts}}{s}$

(a) Evaluate Z transform of $[G(s) \cdot G_{ho}(s)]$

(b) Prove that $\lim_{T \rightarrow 0} Z[G(s) \cdot G_{ho}(s)] = G(s)$

Solution: (9) $G(s) = \left[\frac{K}{s^2 + \frac{s(s-a)}{s_p}} \right] = \left[\frac{K}{s^2 + \frac{s(s-a)}{s_p}} \right] = \left[\frac{K}{s^2 + \frac{s(s-a)}{s_p}} \right] = \left[\frac{K}{s^2 + \frac{s(s-a)}{s_p}} \right]$

K, a are constants

$$G_{HD} = \frac{1 - e^{-Ts}}{s}$$

$$\therefore G_{HD} * G(s) = \left(\frac{1 - e^{-Ts}}{s} \right) * \left(\frac{K}{s^2 + \frac{s(s-a)}{s_p}} \right)$$

$$\therefore Z \left[G_{HD}(s) * G(s) \right] = (z-1) + z \left[\frac{T K}{s^2 + \frac{s(s-a)}{s_p}} \right]$$

As we know from the derivation:

$$\frac{K}{s^2 + \frac{s(s-a)}{s_p}} \xrightarrow{\text{I.L.T.}} \frac{-K}{a^2} u(t) + \frac{k}{a} t u(t) + \frac{k}{a^2} e^{-at} u(t)$$

To convert this continuous signal into discrete signal we will have to put $t = nT$

$$\Rightarrow \frac{-k}{a^2} u(nT) + \frac{k}{a} (nT) u(nT) + \frac{k}{a^2} e^{-a(nT)} u(nT)$$

$$\Rightarrow \frac{-k}{a^2} \left(\frac{z}{z-1} \right)^n + \frac{k}{a} \left(\frac{z}{z-1} \right)^n T + \frac{k}{a^2} \left(\frac{z}{z-e^{-aT}} \right)^n$$

Now put the value of equation (2) in equation (1)

$$= (1-z^{-1}) \left[-\frac{K}{a^2} \left(\frac{z}{z-1} \right) + \frac{K}{a} \left(\frac{zT}{(z-1)^2} \right) + \frac{K}{a^2} \left(\frac{z}{z-e^{-aT}} \right) \right]$$

$$= -\frac{K}{a^2} + \frac{K}{a} \frac{T}{(z-1)} + \frac{K}{a^2} \frac{z-1}{z-e^{-aT}} \quad \text{--- (3)}$$

$$= \frac{KT}{a(z-1)} - \frac{K}{a^2} \left[1 - \frac{z-1}{z-e^{-aT}} \right] = \frac{KT}{a(z-1)} - \frac{K}{a^2} \left[\frac{z-e^{-aT}-z+1}{z-e^{-aT}} \right]$$

$$\text{***} = \frac{KT}{a(z-1)} - \frac{K}{a^2} \left[\frac{1-e^{-aT}}{z-e^{-aT}} \right]$$

$$(B) = \lim_{T \rightarrow 0} -\frac{K}{a^2} + \frac{KT}{a(z-1)} + \frac{K}{a^2} \left(\frac{z-1}{z-e^{-aT}} \right) \quad [\text{From } (3)]$$

$$= \lim_{T \rightarrow 0} -\frac{K}{a^2} + \frac{KT}{a[1+TS-1]} + \frac{K}{a^2} \left[\frac{1+TS-1}{(1+TS) - (1-aT)} \right]$$

$$= -\frac{K}{a^2} + \frac{Ks}{as} + \frac{K}{a^2} \left[\frac{Ts}{Ts+a} \right]$$

$$= -\frac{K}{a^2} + \frac{K}{a\infty} + \frac{K}{a^2} \left(\frac{s}{s+a} \right)$$

$$= \frac{K}{a\infty} - \frac{K}{a^2} + \frac{K}{a^2} \left(\frac{s}{s+a} \right)$$

$$= \frac{\kappa}{\alpha s} - \frac{\kappa}{\alpha^2} \left[1 - \frac{s}{s+a} \right]$$

$$= \frac{\kappa}{\alpha s} - \frac{\kappa}{\alpha^2} \left[\frac{a}{s+a} \right]$$

$$= \frac{\kappa}{\alpha s} - \frac{\kappa}{\alpha^2} \left[\frac{a}{s+a} \right]$$

$$= \frac{\kappa}{\alpha s} - \frac{\kappa}{\alpha^2} \frac{a(s+a)}{a(s+a)}$$

$$= \frac{\kappa(s+a) - \kappa s}{\alpha s(s+a)}$$

$$= \frac{\kappa a}{\alpha s(s+a)}$$

$$= \frac{\kappa}{s(s+a)}$$

$$= G(s)$$

Thus

$$\lim_{T \rightarrow 0} G_{ho}(s) G(s) = G(s)$$

Proved

Note

$$\frac{K}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+a)}$$

$$\frac{K}{s^2(s+a)} = \frac{As(s+a) + Bs(s+a) + Cs^2}{s^2(s+a)}$$

$$K = A(s^2 + as) + (Bs + asB) + Cs^2$$

$$K = s^2(A+C) + s[A+B] + aB$$

$$[A+C] = 0$$

$$A = -C$$

$$aA + B = 0$$

$$aA = -B$$

$$A = -\frac{B}{a}$$

$$aB = K$$

$$B = \frac{K}{a}$$

$$\therefore A = -\frac{B}{a} = -\frac{1}{a}\left(\frac{K}{a}\right) = -\frac{K}{a^2}$$

$$C = -A = -\left(-\frac{K}{a^2}\right) = +\frac{K}{a^2}$$

$$\frac{K}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s+a} + \frac{C}{s^2(s+a)}$$

$$= \frac{(k/a^2)}{s} + \frac{(k/a)}{s+a} + \frac{(k/a^2)}{s^2(s+a)}$$

$$\frac{K}{s^2(s+a)} = \frac{-k}{a^2} \left(\frac{1}{s} \right) + \frac{k}{a} \left(\frac{1}{s+a} \right) + \frac{k}{a^2} \left(\frac{1}{s^2(s+a)} \right)$$

Taking inverse Laplace transform:

$\frac{K}{s^2(s+a)} \xrightarrow{\text{I. L. T. (Inverse Laplace transform)}}$

$$-\frac{k}{a^2} u(t) + \frac{k}{a} e^{-at} u(t) + \frac{k}{a^2} \frac{e^{-at}}{s} u(t)$$

$$\Sigma = (s)Y + \frac{1}{s} + (s)Y - \frac{1}{s} - \frac{1}{s} = (s)Y - \frac{s}{s}$$

$$\Sigma + V = (s)Y \left[\frac{1}{s} + \frac{V}{s} - \frac{1}{s} \right] = (s)V$$

Q: Solve the given differential equation:

$$x(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = u(k) \quad \text{--- (1)}$$

$$y(0) = 0$$

$$y(1) = 1$$

Sol: Taking Z transform of eqⁿ (1)

$$\left[z^2 Y(z) - z y(1) - z^2 y(0) \right] - \frac{3}{4} \left[z Y(z) - z y(0) \right]$$

↓ ↓ ↓

1 0 0

$$+\frac{1}{8} Y(z) = \frac{z + z^2}{z-1}$$

$$z^2 Y(z) - z(1) - z^2(0) - \frac{3}{4}[zY(z) - z(0)] \\ + \frac{1}{8} Y(z) = \frac{z}{z-1}$$

$$z^2 Y(z) - z - \frac{3}{4}zY(z) + \frac{1}{8} Y(z) = \frac{z}{z-1}$$

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8} \right] Y(z) = z + \frac{z}{z-1}$$

$$Y(z) = \frac{z}{\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)} + \frac{z}{\left(z^2 - \left(\frac{3}{4}z + \frac{1}{8}\right)(z-1)\right)}$$

$$= \frac{z}{\left(z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}\right)} + \frac{z}{\left(z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}\right)(z-1)}$$

$$= \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} + \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)(z-1)}$$

$$Y(z) = Y_1(z) + Y_2(z) \quad \text{--- (1)}$$

$$Y_1(z) = \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{1}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} = \frac{A}{\left(z - \frac{1}{4}\right)} + \frac{B}{\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{A}{\left(z - \frac{1}{4}\right)} + \frac{B}{\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{-4}{\left(z - \frac{1}{4}\right)} + \frac{4}{\left(z - \frac{1}{2}\right)}$$

$$Y_1(z) = \frac{-4z}{\left(z - \frac{1}{4}\right)} + \frac{4z}{\left(z - \frac{1}{2}\right)}$$

→ ②

$$Y_2(z) = \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)\left(z - 1\right)}$$

$$\frac{Y_2(z)}{z} = \frac{1}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)\left(z - 1\right)}$$

$$\frac{Y_2(z)}{z} = \frac{A}{\left(z - \frac{1}{4}\right)} + \frac{B}{\left(z - \frac{1}{2}\right)} + \frac{C}{\left(z - 1\right)}$$

$$\frac{Y_2(z)}{z} = \frac{16/3}{\left(z - \frac{1}{4}\right)} + \frac{(-8)}{\left(z - \frac{1}{2}\right)} + \frac{8/3}{\left(z - 1\right)}$$

$$Y_2(z) = \frac{16/3 z}{\left(z - \frac{1}{4}\right)} - \frac{8z}{\left(z - \frac{1}{2}\right)} + \frac{8/3 z}{\left(z - 1\right)}$$

→ ③

Put eqn ② and ③ in eqn ①

$$Y(z) = \frac{-4z}{(z-\frac{1}{4})} + \frac{4z}{(z-\frac{1}{2})} + \frac{(16/3)z}{(z-\frac{1}{4})} - \frac{8z}{(z-\frac{1}{2})} + \frac{8/3z}{(z-1)}$$

Taking inverse Z transform :

$$y(k) = \left[-4\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{2}\right)^k \right] + \left[\frac{16}{3}\left(\frac{1}{4}\right)^k - 8\left(\frac{1}{2}\right)^k \right] + \frac{8}{3}(1)^k$$

$\underbrace{\hspace{10em}}$ $\underbrace{\hspace{10em}}$

Forced response zero-state (initial condition) response

$k \geq 0$

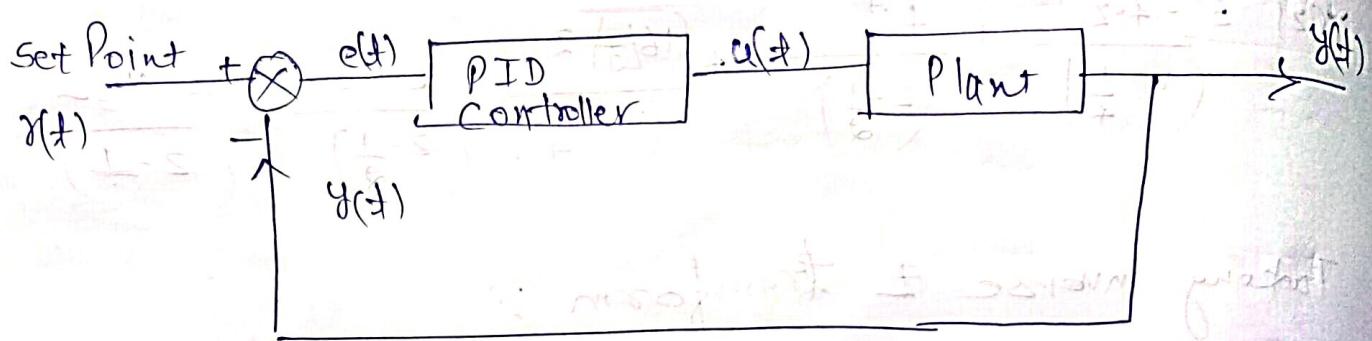
$$y(k) = \left[-4\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{2}\right)^k \right] + \left[\frac{16}{3}\left(\frac{1}{4}\right)^k - 8\left(\frac{1}{2}\right)^k \right] + \frac{8}{3}(1)^k$$

\downarrow \downarrow \downarrow

Caused by initial conditions Caused by excitation (input) Forced Response

Transient response

Conventional PID Controller



$r(t)$ \rightarrow Set point

$y(t)$ = Process output

$e(t)$ = Error Signal $= r(t) - y(t)$

$u(t)$ = Controller output

$$u(t) = K_C \left[e(t) + \frac{1}{T_I} \int_0^t e(\tau) d\tau + T_D \frac{de(t)}{dt} \right]$$

$$U(s) = K_C \left[E(s) + \frac{1}{T_I s} E(s) + \sigma T_D E(s) \right]$$

$$U(s) = \left[K_C + \frac{K_C}{T_I s} + K_C T_D \sigma \right] E(s)$$

$$U(s) = [K_P + \frac{K_I}{s} + K_D s] E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

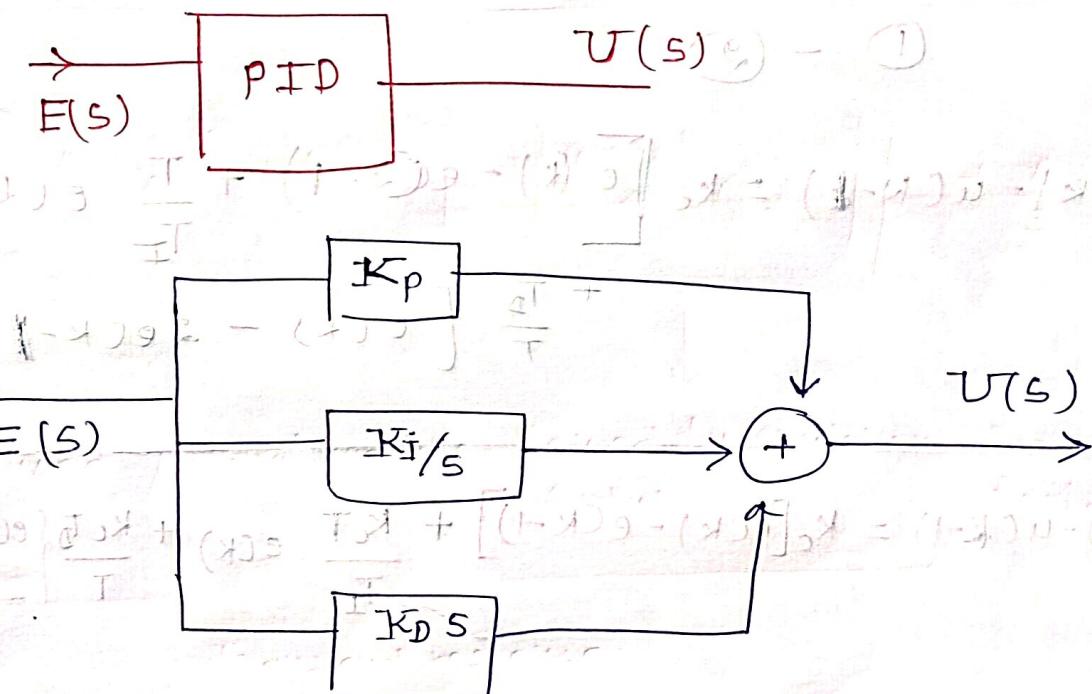
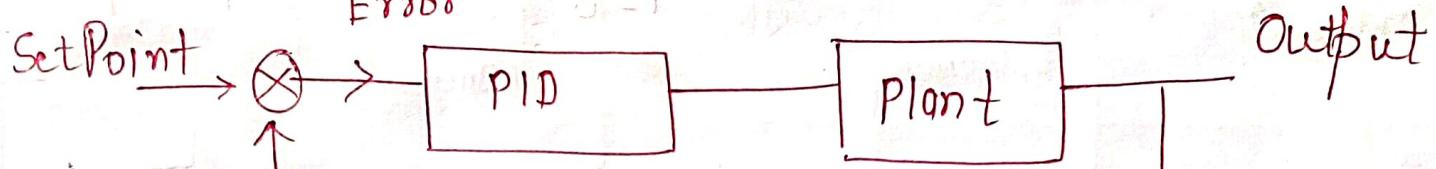
Note ::

$$K_P = K_C$$

$$K_I = \frac{K_C}{T_I} = \frac{K_C}{\tau_I}$$

$$K_D = K_C T_D$$

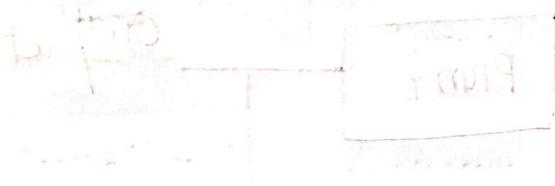
Digital PID controller



$$u(t) = K_c \left[e(t) + \frac{1}{T_I} \int_0^t e(t) dt + T_D \frac{de(t)}{dt} \right]$$

In discrete form

$$u(k) = K_c \left[e(k) + \frac{T}{T_I} \sum_{i=0}^{i=k} e(i) + T_D \left\{ \frac{e(k) - e(k-1)}{T} \right\} \right]$$



(Note $dt = T$)

$$u(k-1) = K_c \left[e(k-1) + \frac{T}{T_I} \sum_{i=0}^{k-1} e(i) + T_D \left\{ \frac{e(k-1) - e(k-2)}{T} \right\} \right]$$

②

① - ②

$$u(k) - u(k-1) = K_c \left[e(k) - e(k-1) + \frac{T}{T_I} e(k) \right] + \frac{T_D}{T} \left[e(k) - 2e(k-1) + e(k-2) \right]$$

$$u(k) - u(k-1) = K_c [e(k) - e(k-1)] + \frac{K_c T}{T_I} e(k) + \frac{K_c T_D}{T} [e(k) - 2e(k-1) - e(k-2)]$$

Velocity form of PID Controller algorithm:

Note In some books discrete form of PID controller is given as:

$$u(k) = K_c [e(k) + \frac{\Delta t}{T_I} \sum_{i=0}^{i=k} e(i) + \frac{T_D}{\Delta t} \Delta e(k)]$$

$$u(k) = K_p e(k) + K_I \sum_{i=0}^{i=k} e(i) + K_D \Delta e(k)$$

where $K_p = K_c$

$$K_I = K_c \cdot \frac{\Delta t}{T_I} = K_p \cdot \frac{\Delta t}{T_I}$$

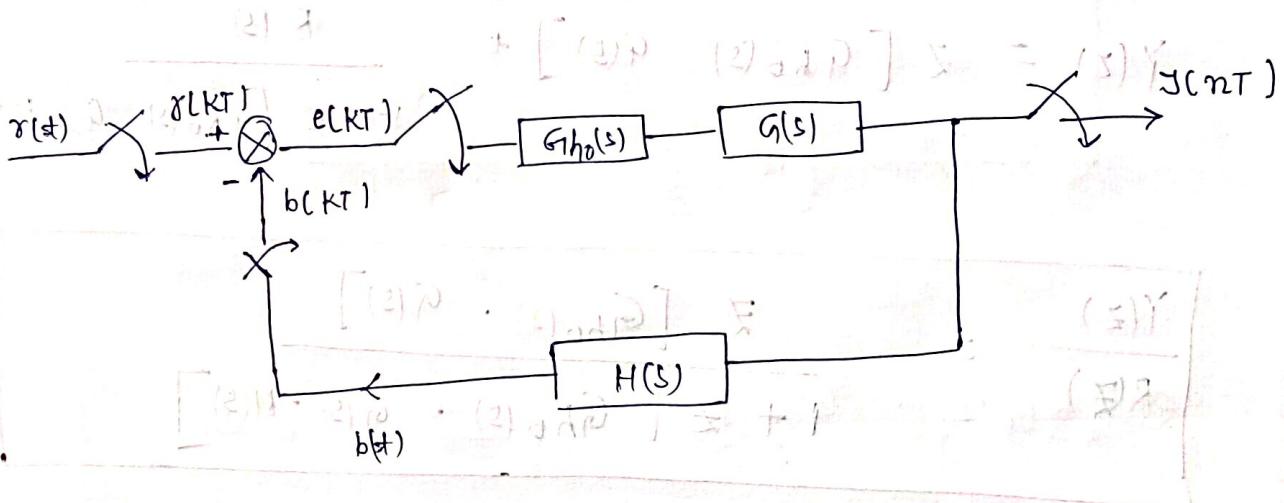
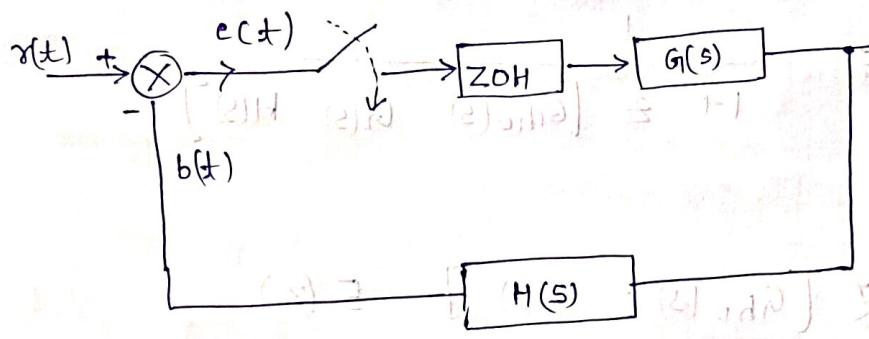
$$K_D = K_c \frac{T_D}{\Delta t} = K_p \cdot \frac{T_D}{\Delta t}$$

	Overshoot	Risetime	Settlingtime	Stability
Proportional (P)	\uparrow	\downarrow	\downarrow	\downarrow (good)
Integral (I)	\uparrow (Increase)	\uparrow (Increase)	\uparrow (Increase)	\downarrow decrease
Derivative (D)	\downarrow decrease	\downarrow decrease	\downarrow decrease	\uparrow increase

$$\frac{d^2y}{dt^2} + \frac{2\zeta}{T} \frac{dy}{dt} + \frac{1}{T^2} y = \frac{1}{T^2} u$$

$$\frac{d^2y}{dt^2} + \frac{2\zeta}{T} \frac{dy}{dt} + \frac{1}{T^2} y = \frac{1}{T^2} u$$

Error sampled feedback system



$$Y(z) = \mathcal{Z} [G_{ho}(s) \quad G(s)] * E(z)$$

$$B(z) = \mathcal{Z} [G_{ho}(s) \quad G(s) \quad H(s)] * E(z)$$

where $B(z) = \mathcal{Z}[b(kT)]$

$$E(z) = \mathcal{Z}[e(kT)]$$

$$\text{Since } e(kT) = r(kT) - b(kT)$$

$$E(z) = R(z) - B(z)$$

$$E(z) = R(z) - \mathcal{Z} [G_{ho}(s) \cdot H(s) \cdot H(s)] E(z)$$

$$E(z) \left[1 + z [G_{ho}(s) \cdot G(s) H(s)] \right] = R(z)$$

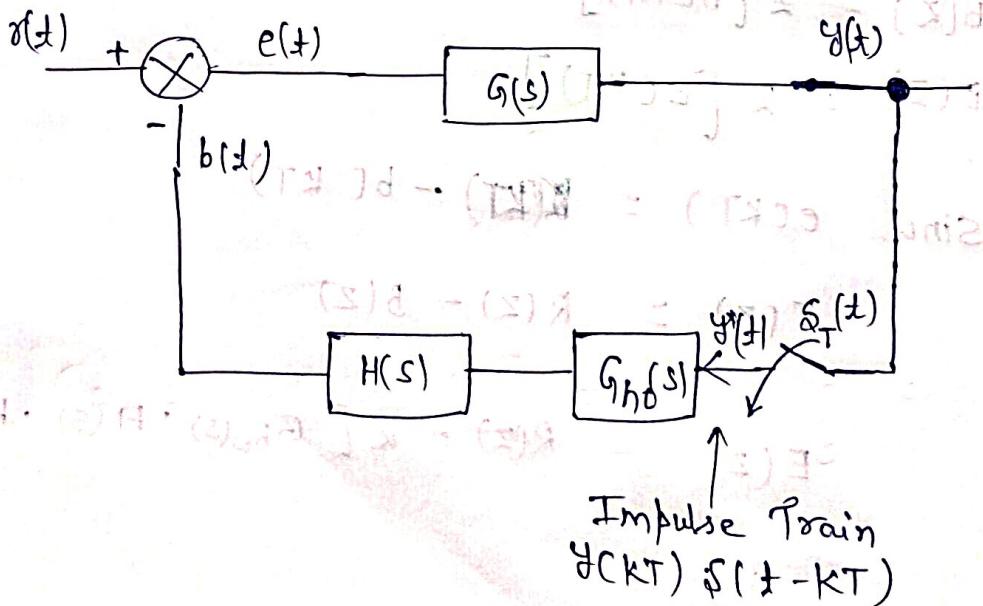
$$\frac{E(z)}{R(s)} = \frac{1}{1 + z [G_{ho}(s) \cdot G(s) H(s)]}$$

$$Y(z) = z [G_{ho}(s) \cdot G(s)] E(z)$$

$$Y(z) = z [G_{ho}(s) \cdot G(s)] + \frac{R(s)}{1 + z [G_{ho}(s) \cdot G(s) H(s)]}$$

$$\boxed{\frac{Y(z)}{R(z)} = \frac{z [G_{ho}(s) \cdot G(s)]}{1 + z [G_{ho}(s) \cdot G(s) \cdot H(s)]}}$$

Sampled data system with sampler and ZOH in feed back path



$$E(s) = R(s) - G_{ho}(s) H(s) Y^*(s)$$

$$Y(s) = [R(s) - B(s)] G(s) = [R(s) - G_{ho}(s) H(s) Y^*(s)] G(s)$$

$$Y(s) = -G_{ho}(s) G(s) H(s) Y^*(s) + G(s) R(s)$$

Taking Z Transform :-

$$\therefore Y(z) = z \left[-G_{ho}(s) G(s) H(s) Y^*(s) \right] + z \left[G(s) R(s) \right]$$

$$Y(z) = -G_{ho} G H(z) Y(z) + G R(z)$$

$$Y(z) = \frac{G R(z)}{1 + G_{ho} G H(z)}$$

$$\left[\frac{1}{(1+z)^2} \right] \times (z-1) =$$

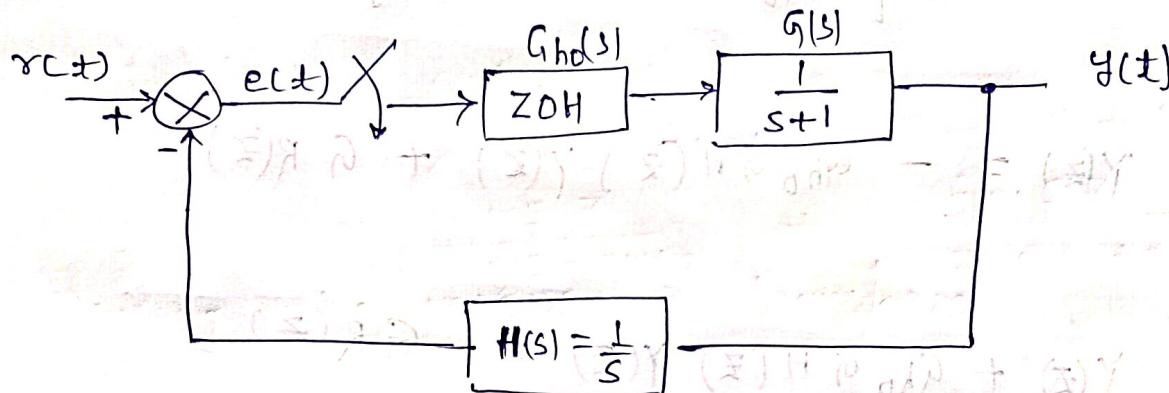
$$\left[\frac{(z-1)}{(z-3)(z-5)} \right] =$$

$$\left[\frac{(z-1)}{(z-3-\Sigma)} \right] =$$

$$\left[\frac{1}{z} + \frac{1}{(z-3-\Sigma)} \right] = \left[\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right]$$

Q: For the sampled data system shown below find the output $y(z)$ for $r(t) = \text{unit step}$.

$T = 1 \text{ second}$



$$Z[G_{ho}(s) \cdot G(s)] = Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{1}{s+1}\right] = (1-z^{-1}) Z\left[\frac{1}{s(s+1)}\right]$$

$$= [1-z^{-1}] \left[\frac{z(1-e^{-1})}{(z-1)(z-e^{-1})} \right] = \frac{(1-e^{-1})}{(z-e^{-1})}$$

$$Z[G_{ho}(s) \cdot G(s) H(s)] = Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{1}{s+1} \cdot \frac{1}{s}\right]$$

$$\begin{aligned}
 &= (1-z^{-1}) - z \left[\frac{1}{s^2(s+1)} \right] = (S) \\
 &= (1-z^{-1}) \left[\frac{z}{(z-1)^2(z+1)} - \frac{z(1-e^{-1})}{(z-1)(z-e^{-1})} \right] \\
 &= \left[\frac{1}{z-1} - \frac{1-e^{-1}}{z-e^{-1}} \right]
 \end{aligned}$$

$$\frac{e^{-1}z - 2e^{-1} + 1}{(z-1)(z-e^{-1})} = (+)$$

$$\frac{\gamma(z)}{R(z)} = \frac{z [G_{ho}(s) \quad G(s)]}{1 + z [G_{ho}(s) \quad G(s) \quad H(s)]}$$

$$R(z) = \frac{z}{(z-1)}$$

$$\frac{\gamma(z)}{R(z)} = \frac{(1-e^{-1})/(z-e^{-1})}{1 + \left[\frac{e^{-1}z - 2e^{-1} + 1}{(z-1)(z-e^{-1})} \right]}$$

$$\frac{\gamma(z)}{R(z)} = \frac{(1-e^{-1})(z-1)}{z^2 - z + 1 - e^{-1}}$$

$$Y(z) = \frac{(1-e^{-1})(z-1)}{z^2 - z + (1-e^{-1})} * \left(\frac{z}{z-1}\right)$$

$$(1-e^{-1})z$$

$$\frac{(1-e^{-1})z}{z^2 - z + 1 - e^{-1}}$$

$$= \frac{0.632z}{(z - 0.5 - j0.62)(z - 0.5 + j0.62)}$$

$$= \frac{-j0.51z}{z - 0.5 - j0.62} + \frac{j0.51z}{z - 0.5 + j0.62}$$

Taking the inverse z -transform:

$$y(k) = -j0.51(0.5 + j0.62)^k + j0.51(0.5 - j0.62)^k$$

Ans

Jury Stability Test

$$\Delta(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1$$

Necessary conditions for stability are:

$$\Delta(1) = 2 + 7 + 10 + 4 + 1 = 24 > 0 \quad \text{Satisfied}$$

$$(-1)^4 \Delta(-1) = 2 - 7 + 10 - 4 + 1 = 2 > 0 \quad \text{Satisfied}$$

To check the requirement of sufficiency, we construct the jury table:

Row	z^0	z^1	z^2	z^3	z^4
1	1 α_1	4 α_3	10 α_2	7 α_1	2 α_0
2	2 α_0	7 α_1	10 α_2	4 α_3	1 α_4
3	-3 b_3	-10 b_1	-10 b_1	-10 b_0	-3 b_3
4	-1 b_0	-10 b_1	-10 b_2	-3 b_3	
5	8 a_2	20 a_1	20 a_0		

(i) $|1| < |2|$

Satisfied $[|2| < |1|]$

(ii) $|-3| > |-1|$

Satisfied $[|-3| > |-1|]$

(iii) $|8| > |20|$

Not Satisfied $(|8| > |20|)$

Therefore system is unstable.

$$b_0 = \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} = 7 - 8 = -1$$

$$b_1 = \begin{vmatrix} 1 & 10 \\ 2 & 10 \end{vmatrix} = 10 - 20 = -10$$

$$b_2 = \begin{vmatrix} 1 & 7 \\ 2 & 4 \end{vmatrix} = 4 - 14 = -10$$

$$b_3 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

Thus necessary conditions for stability are:

$$\Delta(1) > 0$$

$$(-1)^n \Delta(-1) > 0$$

Sufficient conditions for stability are:

$$\left. \begin{array}{l} |a_n| < |a_0| \\ |b_{n-1}| > |b_0| \\ |c_{n-1}| > |c_0| \\ |a_2| > |a_0| \end{array} \right\} (n-1) \text{ constraints}$$

* The Jury table is continued until $(2n-3)^{\text{rd}}$ row is reached which will exactly three elements.

$$\text{let } n = 4$$

$$(2n-3) = 2 \times 4 - 3 = 8 - 3 = 5$$

Total number of rows in Jury table will be 5. And in the last row of the Jury table there will be exactly three elements.

Q: Consider the characteristic polynomial:

$$\Delta(z) = z^3 + 3 \cdot 3z^2 + 4z + 0.8$$

using Jury table, determine whether system is stable or not (show all steps).

Soln

$$\Delta(z) = z^3 + 3 \cdot 3z^2 + 4z + 0.8$$

$$\Delta(1) = 1 + 3 \cdot 3 + 4 + 0.8 = 19.1$$

$$(-1)^3 \Delta(-1) = (-1)(-1 + 3 \cdot 3 - 4 + 0.8)$$

$$= (-1)(-0.9) = +0.9$$

$\Delta(1) > 0$ Linearly satisfied

$(-1)^3 \Delta(-1) > 0$ Non linearly satisfied

Row	z^0	z^1	z^2	z^3	$\lambda = 15$	$\lambda = 21$
1	0.8 α_3	-4 α_2	3.3 α_1	1 α_0		
2	0.8 α_3	3.3 α_1	4 α_2	0.8 α_0		
3	-0.36 α_2	-0.1 α_1	-1.36 α_0			

$$q_0 = \begin{vmatrix} 0.8 & 4 \\ 1 & 3.3 \end{vmatrix} = 0.8 \times 3.3 - 1 \times 4 = 2.64 - 4 = -1.36$$

$$q_1 = \begin{vmatrix} 0.8 & 3.3 \\ 1 & 4 \end{vmatrix} = 0.8 \times 4 - 1 \times 3.3 = 3.2 - 3.3 = -0.1$$

$$q_2 = \begin{vmatrix} 0.8 & 1 \\ 1 & 0.8 \end{vmatrix} = (0.8)(0.8) - 1 \times 1 = .64 - 1 = -0.36$$

(ii) $|0.8| < 1.1$; $|\alpha_3| < |\alpha_0|$ satisfied

(ii) $|-0.36| \neq |1.36|$; $|\alpha_2| > \alpha_0$

This condition is not satisfied

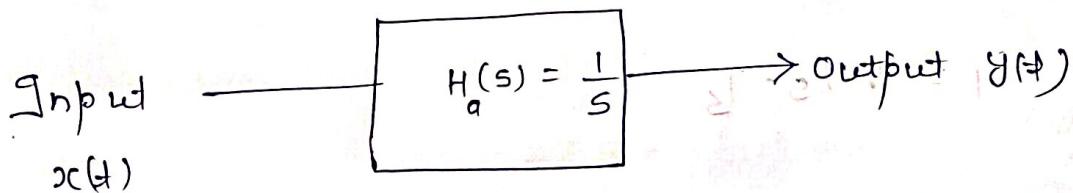
Therefore system is unstable.

Bilinear Transformation

- * This method is used to convert analog filter / analog transfer function to digital filter / discrete transfer function.

- * S domain \longrightarrow Z domain
- * Analog filter \longrightarrow Digital filter

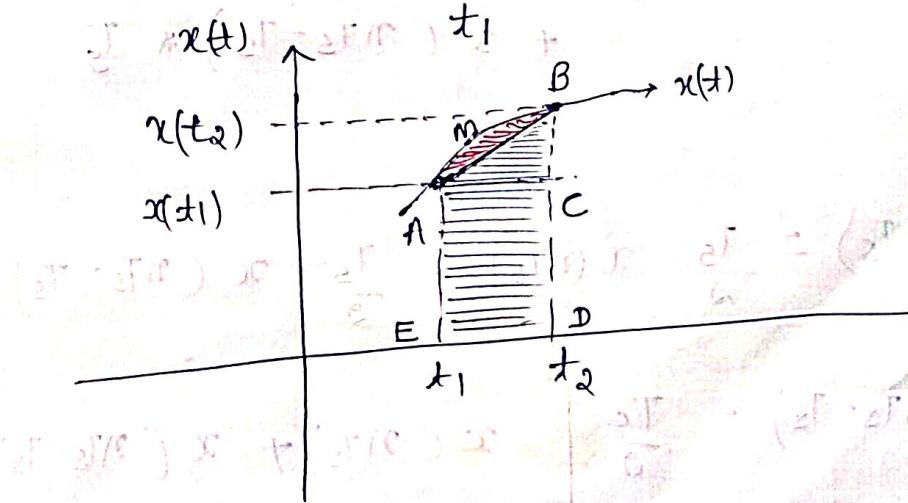
Consider an analog integrator:



For the time period T, the difference in the output is given by

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(\tau) d\tau$$

$$y(t) = \int x(t) dt$$



* area ABM is very small so it can be neglected.

* We have assumed two input positions $x(t_1)$ and $x(t_2)$ corresponding output is denoted by $y(t_1)$ and $y(t_2)$.

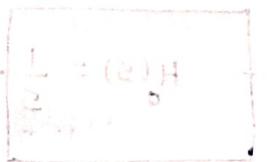
* Area under the curve = Area of triangle ABC + area of rectangle ACDE + Area of A BM

$$\approx \text{area of } \triangle ABC + \text{area of rectangle ACDE}$$

$$= \frac{1}{2} (t_2 - t_1) * [x(t_2) - x(t_1)] + x(t_1)(t_2 - t_1)$$

$$\text{Let } t_1 + t_2 = nT_s$$

$$t_1 = nT_s - T_s$$



$$t_2 - t_1 = nT_s - (nT_s - T_s)$$

$$t_2 - t_1 = T_s$$

$$y(nT_s) - y(nT_s - T_s) = \frac{1}{2} T_s [x(nT_s) - x(nT_s - T_s)]$$

$$+ x(nT_s - T_s) * T_s$$

$$y(nT_s) - y(nT_s - T_s) = \frac{T_s}{2} x(nT_s) + \frac{T_s}{2} x(nT_s - T_s)$$

$$y(nT_s) - y(nT_s - T_s) = \frac{T_s}{2} [x(nT_s) + x(nT_s - T_s)]$$

Taking Z transform of both the sides

$$Y(z) - z^{-1} Y(z) = \frac{T_s}{2} [x(z) + z^{-1} x(z)]$$

$$Y(z) [1 - z^{-1}] = \frac{T_s}{2} x(z) (1 + z^{-1})$$

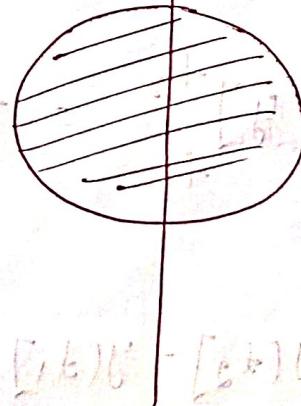
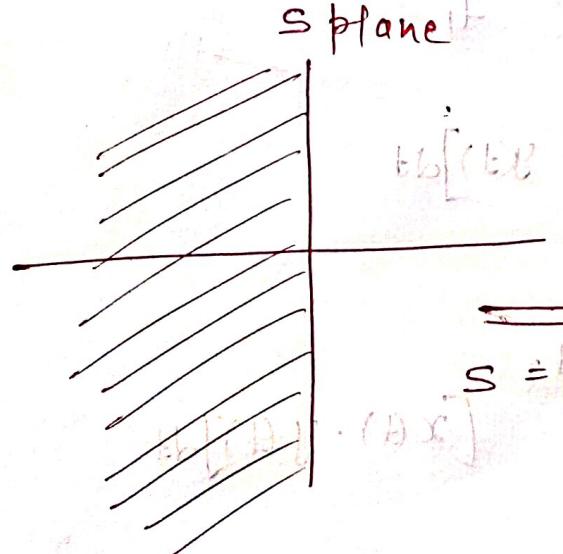
$$\frac{Y(z)}{X(z)} = \frac{T_s}{2} \left[\frac{1 + z^{-1}}{1 - z^{-1}} \right] = H(z) = \frac{\frac{1}{2} X(z)}{\frac{2}{T_s} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)}$$

We have $\frac{Y(s)}{X(s)} = H_a(s) = \frac{1}{s}$

$$\therefore \frac{1}{s} = \frac{T_s}{2} \frac{1 + z^{-1}}{1 - z^{-1}}$$

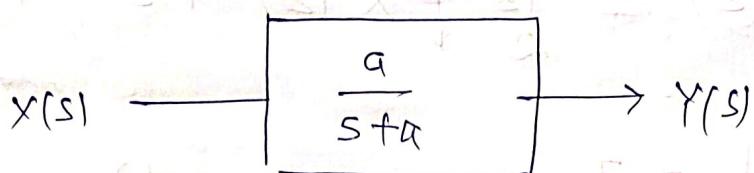
$$s = \frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$s = \frac{2}{T_s} \frac{z-1}{z+1}$$



$$s = \frac{2}{T_s} \frac{z-1}{z+1}$$

Note: Let transfer function of analog filter is $\frac{a}{s+a}$.



$$\frac{Y(s)}{X(s)} = H_a(s) = \frac{a}{s+a}$$

$$sY(s) + aY(s) = aX(s)$$

$$\therefore \frac{dy}{dt} + a y(t) = a x(t)$$

$$\frac{dy}{dt} = -a y(t) + a x(t)$$

$$dy = -a y(t) dt + a x(t) dt$$

$$\int_{t_1}^{t_2} dy = -a \int_{t_1}^{t_2} y(t) dt + a \int_{t_1}^{t_2} x(t) dt$$

$$[y]_{t_1}^{t_2} = +a \int_{t_1}^{t_2} [x(t) - y(t)] dt$$

$$y(t_2) - y(t_1) = +a \int_{t_1}^{t_2} [x(t) - y(t)] dt$$

$$y(nT_s) - y(nT_s - T_s) = a \left[\frac{x(nT_s) + x(nT_s - T_s)}{2} * \frac{T_s}{2} \right]$$

$$\neq \left[\frac{y(nT_s) + y(nT_s - T_s)}{2} * \frac{T_s}{2} \right]$$

$$y(nT_s) - y(nT_s - T_s) = \frac{\alpha T_s}{2} x(nT_s) + \frac{\alpha T_s}{2} x(nT_s - T_s) - \frac{\alpha T_s}{2} y(nT_s) - \frac{\alpha T_s}{2} y(nT_s - T_s)$$

$$\left(1 + \frac{\alpha T_s}{2}\right) y(nT_s) - \left(1 - \frac{\alpha T_s}{2}\right) y(nT_s - T_s) = \frac{\alpha T_s}{2} x(nT_s) + \frac{\alpha T_s}{2} x(nT_s - T_s) -$$

$$\left(1 + \frac{\alpha T_s}{2}\right) Y(z) - \left(1 - \frac{\alpha T_s}{2}\right) z^{-1} Y(z) = \frac{\alpha T_s}{2} X(z) + \frac{\alpha T_s}{2} z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{\alpha T_s}{2} (1 + z^{-1})}{\left(1 + \frac{\alpha T_s}{2}\right) - \left(1 - \frac{\alpha T_s}{2}\right) z^{-1}}$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{\alpha T_s}{2} (1 + z^{-1})}{(1 - z^{-1}) + \frac{\alpha T_s}{2} (1 + z^{-1})}$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{\alpha}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}}}{1 + \frac{\alpha}{T_s} z^{-1}} + \alpha$$

$$\frac{Y(s)}{X(s)} = \frac{\alpha}{s + \alpha}$$

$$T_s \left[(e^{fD})s + (e^{-fD})\bar{s} \right] D \xrightarrow{T_s} \frac{1-z^{-1}}{1+z^{-1}} e^{fD} B = (e^{fD})B$$

$$\boxed{s = \frac{2}{T_s} \frac{z-1}{z+1}}$$

$$(e^{fD})s + (e^{-fD})\bar{s} = (e^{fD} - e^{-fD})s \xrightarrow{\text{cancel}} (e^{fD} + e^{-fD})s = (e^{2fD})s$$

$$(e^{fD}s) \times \frac{e^{fD}}{s} + (e^{-fD}s) \times \frac{e^{-fD}}{s} = (e^{fD} - e^{-fD})s \left(\frac{e^{fD}}{s} - 1 \right) = (e^{2fD})s$$

$$(s)X \times \frac{e^{fD}}{s} + (s)X \times \frac{e^{-fD}}{s} = (s)\gamma' \times s \left(\frac{e^{fD}-1}{s} \right) - (s)\gamma' \left(\frac{e^{-fD}}{s} \right)$$

$$\frac{(s+1) \frac{e^{fD}}{s}}{(s+1) \frac{e^{-fD}}{s}}$$

$$\frac{s \left(\frac{e^{fD}-1}{s} \right)}{\left(\frac{e^{-fD}}{s} \right)} = \frac{\left(e^{fD} - 1 \right)}{e^{-fD}}$$

$$\frac{(s+1) \frac{e^{fD}}{s}}{(s+1) \frac{e^{-fD}}{s}}$$

$$\frac{(s+1) \frac{e^{fD}}{s}}{(s+1) \frac{e^{-fD}}{s} + (s+1)}$$

$$= (s+1) \frac{e^{fD}}{e^{-fD} + s+1}$$

$$= \frac{(s+1) e^{fD}}{(s+1) e^{-fD} + 1}$$