Control System - (II)

Digital to analog converter (DAC)

1. Weighted resistor D/A converter

2. R-2R ladder D/A converter

Analog to digital converter (ADC)

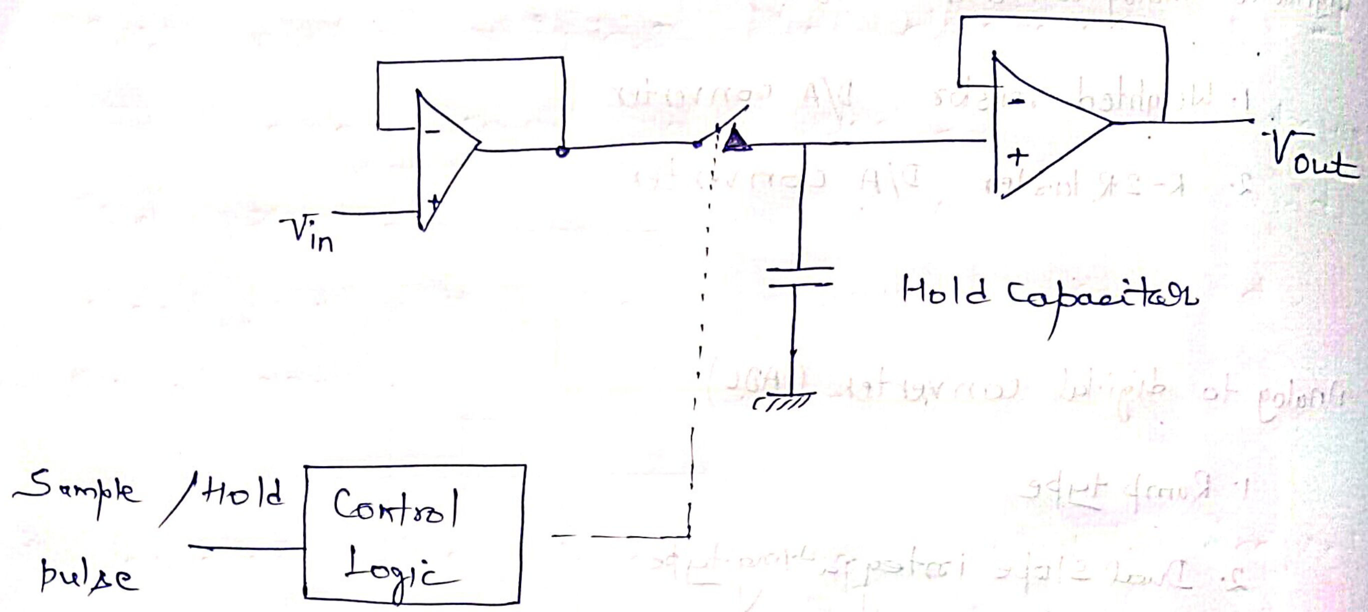
1. Ramp type

2. Dual slope integrating type

3. Successive approximation type

4. Flash type ADC

Sample and hold operations - A sampler is a device which converts an analog signal into a train of amplitude modulated pulses. A hold device simply maintains the value of the pulse for a prescribed time duration. In a majority of the practical digital operations sample and hold functions are performed by a single device, commercially known as sample-and-hold (S/H). The S/H operates by storing input signal voltage as charge on a small, high quality capacitor.

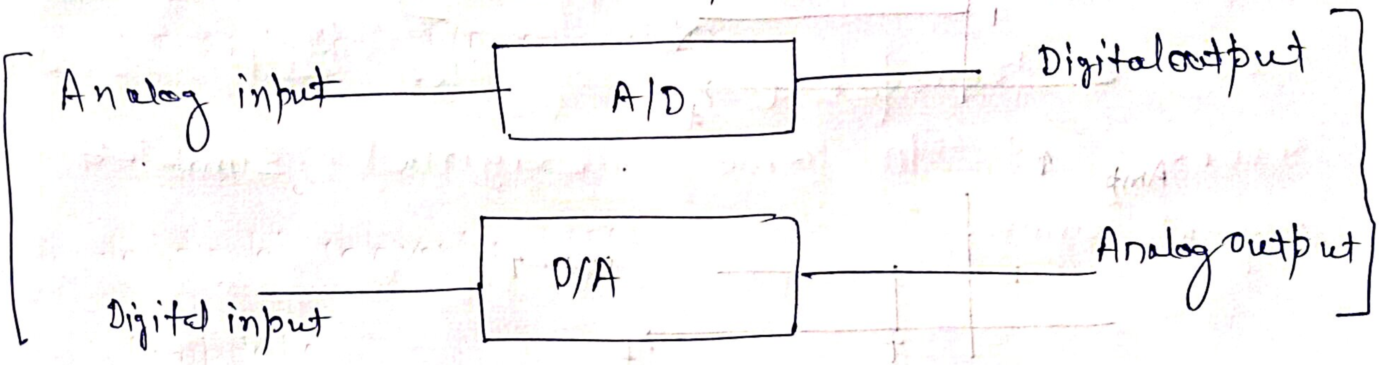


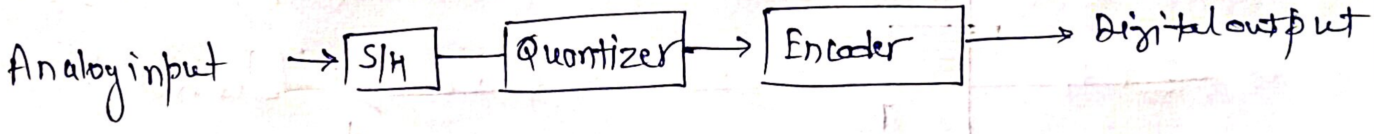
* When the switch is closed (Logic '1' input sample pulse, the capacitor rapidly charges to Vin and Vout is equal to Vin approximately.

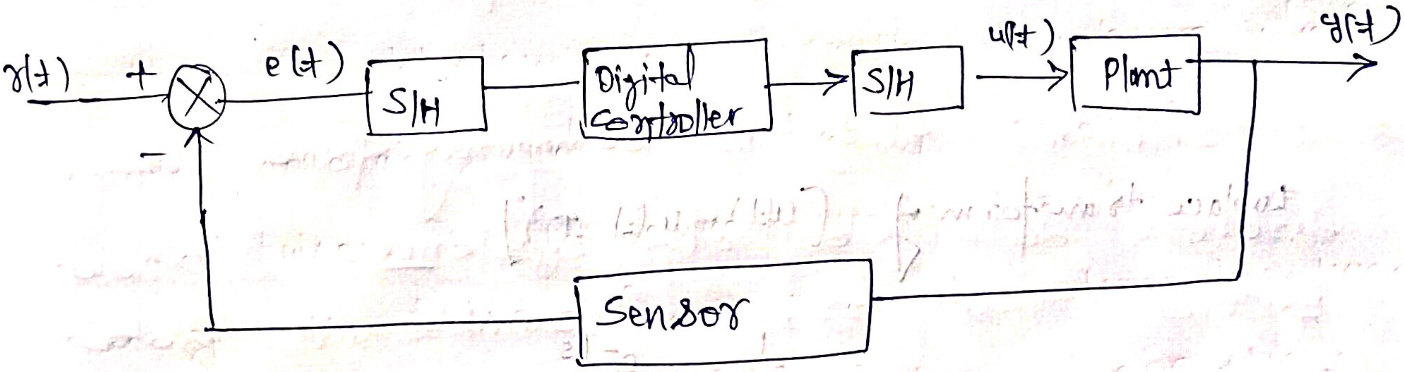
* When the switch opens (Logic '0' input pulse) the capacitor retains its charge, the output holds at a value of Vin.

* If the input voltage changes rapidly while the switch is closed, the capacitor can follow this voltage because the charging time constant is very short. If the switch is suddenly opened, the capacitor voltage represents a sample of input voltage at the

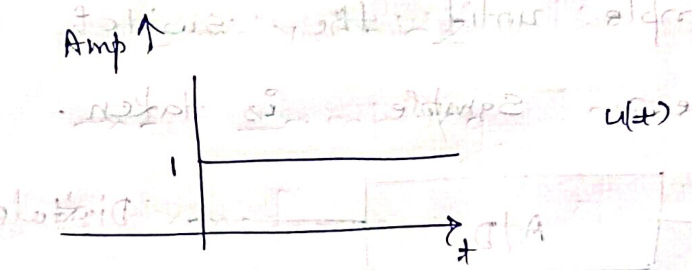
instant switch was opened. The capacitor then holds this sample until the switch is again closed and a new sample is taken.

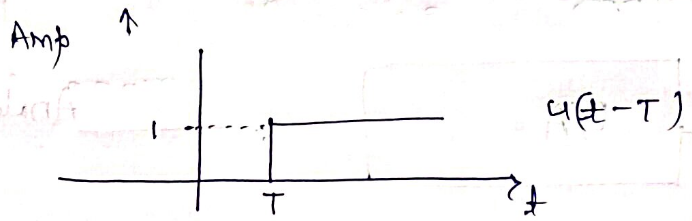


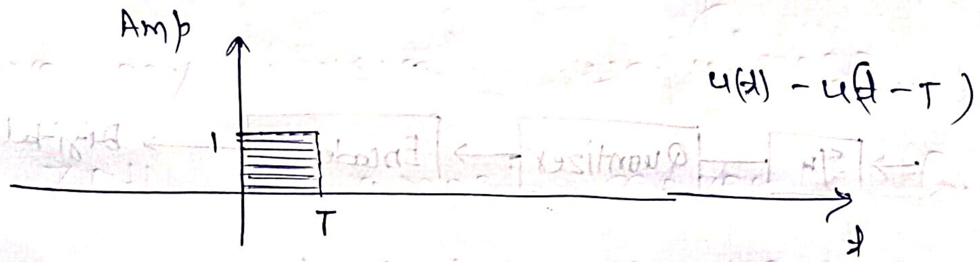




Zero order hold (ZOH)



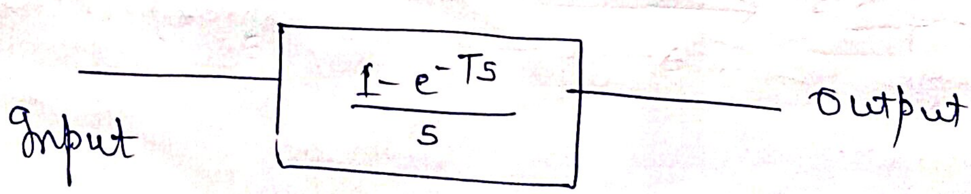




Laplace transform of $[u(t) - u(t-T)]$

$$= \frac{1}{s} - \frac{e^{-Ts}}{s}$$

$$= \frac{1-e^{-Ts}}{s} \qquad (1)$$



Zero Order hold

Since the term e^{-Ts} in equation (1) represents a delay of one sampling period, then considering the right shift theorem of the Z-transform

$$G_{ho}(z) = Z \text{ transform of } \left[\frac{1-e^{-Ts}}{s}\right]$$

Put $z = e^{Ts}$

$$= Z \text{ transform of } \left\{\frac{1-z^{-1}}{s}\right\}$$

$$= (1-z^{-1}) \quad Z\left(\frac{1}{s}\right)$$

$$= \left(1-\frac{1}{z}\right) \quad Z\left(\frac{1}{s}\right)$$

$$= \left(\frac{z-1}{z}\right) * \left(\frac{z}{z-1}\right)$$

$$= 1$$

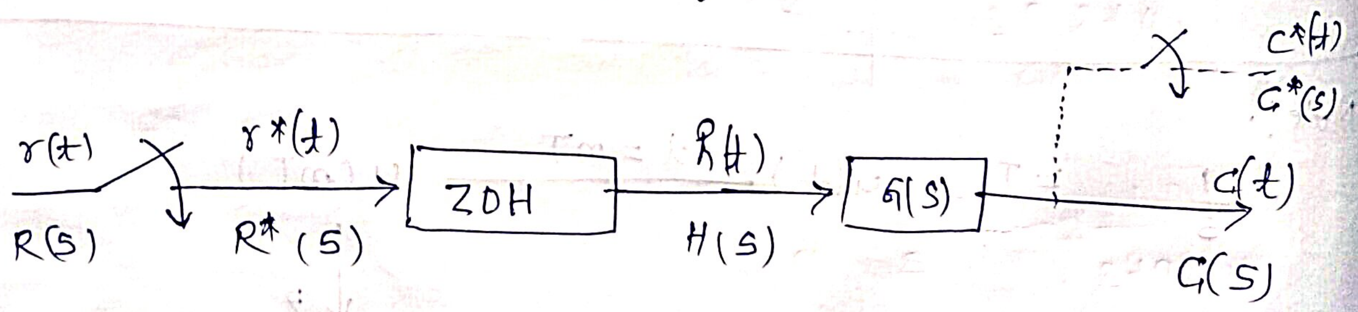
Note:

$$\frac{1}{s} \xrightarrow{L^{-1}} u(t) \xrightarrow{t=nT} u(nT) \xrightarrow{Z} \left(\frac{z}{z-1}\right)$$

* This result is expected since ZOH (zero order hold) simply holds the discrete signal for one sampling period, and taking the z transform of the ZOH would revert it to the original sampled signal again.

* However, the above thing simply does not have any physical usefulness, since a ZOH is almost never found just situated by itself.

* A situation that is commonly found in discrete data control systems is shown below, that is, a sample and hold (S/H) unit is followed by a linear system with transfer function G(S).



*Question: $r(t) = e^{-at}\ u_s(t)$ Determine its Z transform.

Sol^n: Put $t = nT$

$$r(nT) = e^{-anT}\ u_s(nT)$$

$$r(nT) = \left(e^{-aT}\right)^n\ u_s(nT)$$

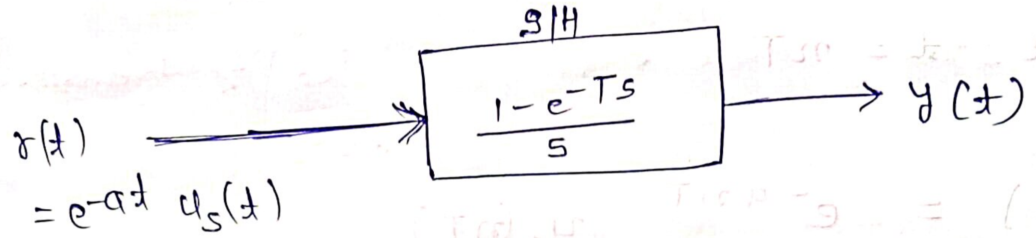
taking Z- transform

$$\Rightarrow \frac{Z}{Z - e^{-aT}}$$

Q: Let input $r(t) = e^{-at}\ u_s(t)$ is given to S/H circuit. Prove that if $T \to 0$ output of sample and hold circuit is same as input.

*

Solution:



Output $Y(s) =$ T.F. * Laplace transform of input

$$Y(s) = \left(\frac{1-e^{-Ts}}{s}\right) * \left(\frac{1}{s+a}\right)$$

$$Y(s) = (1-e^{-Ts}) * \frac{1}{s(s+a)}$$

$$\lim_{T\to 0} Z[Y(s)] = \lim_{T\to 0} Z\left[(1-e^{-Ts}) \frac{1}{s(s+a)}\right]$$

$$= (1-z^{-1}) \; Z\left[\frac{1}{s(s+a)}\right]$$

$$= (1-z^{-1}) \; Z\left[\frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)\right]$$

$$= \left(\frac{z-1}{z}\right)\left(\frac{1}{a}\right) Z\left[u(t) - e^{-at}u(t)\right]$$

$$= \left(\frac{z-1}{z}\right)\left(\frac{1}{a}\right) Z\left[u(nT) - e^{-anT}u(nT)\right]$$

$$= \left(\frac{z-1}{z}\right)\left(\frac{1}{a}\right)\left[\frac{z}{z-1} - \frac{z}{z-e^{-aT}}\right]$$

$$= \left(\frac{z-1}{z}\right) * \left(\frac{1}{a}\right)\left(\frac{z}{z-1}\right)\left[1 - \frac{z-1}{z-e^{-aT}}\right]$$

$$= \frac{1}{a}\left[1 - \frac{z-1}{z-e^{-aT}}\right]$$

$$\because \left[\begin{aligned} z &= e^{Ts} \\ &= 1+Ts \end{aligned}\right]$$

$$= \lim_{T \to 0} \frac{1}{a}\left[1 - \frac{1+Ts-1}{1+Ts-1+aT}\right]$$

$$= \lim_{T \to 0} \frac{1}{a}\left[1 - \frac{Ts}{T(s+a)}\right]$$

$$= \frac{1}{a}\left[1 - \frac{s}{s+a}\right]$$

$$= \frac{1}{a}\left[\frac{s+a-s}{s+a}\right]$$

$$= \left(\frac{1}{a}\right)\left(\frac{a}{s+a}\right)$$

$$= \left(\frac{1}{s+a}\right)$$

$$= \text{Input}$$

$$\boxed{\lim_{T \to 0} \left[\text{Output of Sample and hold circuit}\right] = \text{Input of the S/H unit}}$$

Q: Transfer function of the system is $G(s) = \frac{k}{s(s+a)}$ and zero order hold $G_{ho}(s) = \frac{1-e^{-Ts}}{s}$

(a) Evaluate z transform of $[G(s) \cdot G_{ho}(s)]$

(b) Prove that $\lim_{T \to 0} Z[G(s)\ G_{ho}(s)] = G(s)$

Solution: (a) $G(S) = \frac{K}{s(s+a)}$

K, a are constants

$$G_{ho} = \frac{1-e^{-Ts}}{s}$$

$$\therefore G_{ho} * G(S) = \left(\frac{1-e^{-Ts}}{s}\right) * \left(\frac{k}{s(s+a)}\right)$$

$$\therefore Z\left[G_{ho}(S) * G(S)\right] = (1-z^{-1})\, Z\left[\frac{K}{s^2(s+a)}\right] \quad \text{—— (1)}$$

As we know from the derivation:

$$\frac{k}{s^2(s+a)} \xrightarrow{I.L.T.} \frac{-k}{a^2}u(t) + \frac{k}{a}t\,u(t) + \frac{k}{a^2}e^{-at}u(t)$$

To convert this continuous signal into discrete signal we will have to put, $t = nT$

$$\Rightarrow \frac{-k}{a^2}u(nT) + \frac{K}{a}(nT)\,u(nT) + \frac{K}{a^2}e^{-a(nT)}u(nT)$$

$$\Rightarrow \frac{-k}{a^2}\left(\frac{z}{z-1}\right) + \frac{K}{a}\frac{z}{(z-1)^2}T + \frac{K}{a^2}\left(\frac{z}{z-e^{-aT}}\right) \quad \text{—— (2)}$$

Now put the value of equation (2) in equation (1)

$$= (1-z^{-1})\left[\frac{-k}{a^2}\left(\frac{z}{z-1}\right) + \frac{k}{a}\left(\frac{zT}{(z-1)^2}\right) + \frac{k}{a^2}\left(\frac{z}{z-e^{-aT}}\right)\right]$$

$$= \frac{-k}{a^2} + \frac{k}{a}\frac{T}{(z-1)} + \frac{k}{a^2}\frac{z-1}{z-e^{-aT}} \quad \text{--- (3)}$$

$$= \frac{kT}{a(z-1)} - \frac{k}{a^2}\left[1 - \frac{z-1}{z-e^{-aT}}\right] = \frac{kT}{a(z-1)} - \frac{k}{a^2}\left[\frac{z-e^{-aT}-z+1}{z-e^{-aT}}\right]$$

Ans

$$** = \frac{kT}{a(z-1)} - \frac{k}{a^2}\left[\frac{1-e^{-aT}}{z-e^{-aT}}\right]$$

(b) [From eq (3)]

$$= \lim_{T\to 0} \frac{-k}{a^2} + \frac{kT}{a(z-1)} + \frac{k}{a^2}\left(\frac{z-1}{z-e^{-aT}}\right)$$

$$= \lim_{T\to 0} \frac{-k}{a^2} + \frac{kT}{a[1+Ts-1]} + \frac{k}{a^2}\left[\frac{1+Ts-1}{(1+Ts)-(1-aT)}\right]$$

$$= \frac{-k}{a^2} + \frac{k}{as} + \frac{k}{a^2}\left[\frac{Ts}{T(s+a)}\right]$$

$$= \frac{-k}{a^2} + \frac{k}{as} + \frac{k}{a^2}\left(\frac{s}{s+a}\right)$$

$$= \frac{k}{as} - \frac{k}{a^2} + \frac{k}{a^2}\left(\frac{s}{s+a}\right)$$

$$= \frac{K}{as} - \frac{K}{a^2}\left[1 - \frac{s}{s+a}\right]$$

$$= \frac{K}{as} - \frac{K}{a^2}\left[\frac{a}{s+a}\right]$$

$$= \frac{K}{as} - \frac{K}{a^2}\left[\frac{a}{s+a}\right]$$

$$= \frac{K}{as} - \frac{K}{a(s+a)}$$

$$= \frac{K(s+a) - Ks}{as(s+a)}$$

$$= \frac{Ka}{as(s+a)}$$

$$= \frac{K}{s(s+a)}$$

$$= G(s)$$

Thus

$$\boxed{\lim_{T \to 0} G_{ho}(s)\, G(s) = G(s)}$$

Proved

Note

$$\frac{K}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+a)}$$

$$\frac{K}{s^2(s+a)} = \frac{A\,s(s+a) + B(s+a) + Cs^2}{s^2(s+a)}$$

$$K = A(s^2+as) + (Bs + aB) + Cs^2$$

$$K = s^2(A+C) + s\,[aA+B] + aB$$

$[A+C] = 0 \qquad A = -C$

$aA + B = 0 \qquad aA = -B$

$A = -\frac{B}{a}$

$aB = K \qquad \therefore\; B = \frac{K}{a}$

$$\therefore\; A = -\frac{B}{a} = -\frac{1}{a}\left(\frac{K}{a}\right) = -\frac{K}{a^2}$$

$$\boxed{C = -A = -\left(\frac{-K}{a^2}\right) = +\frac{K}{a^2}}$$

$$\frac{K}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{\cancel{s}(s+a)}$$

$$= \frac{(-K/a^2)}{s} + \frac{(K/a)}{s^2} + \frac{(K/a^2)}{s(s+a)}$$

$$\frac{K}{s^2(s+a)} = \frac{-K}{a^2}\left(\frac{1}{s}\right) + \frac{K}{a}\left(\frac{1}{s^2}\right) + \frac{K}{a^2}\left(\frac{1}{(s+a)}\right)$$

Taking inverse Laplace transform :-

$$\boxed{\frac{K}{s^2(s+a)} \xrightarrow{\text{I. L. T. (Inverse laplace transform)}} -\frac{K}{a^2}u(t) + \frac{K}{a}t\,u(t) + \frac{K}{a^2}e^{-at}u(t)}$$

Q: Solve the given differential equation :

$$y(k+2) - \frac{3}{4} y(k+1) + \frac{1}{8} y(k) = u(k) \quad \text{--- (1)}$$

$$y(0) = 0$$

$$y(1) = 1$$

Sol^n: Taking Z transform of eq^n (1)

$$\left[Z^2 Y(z) - Z y(1) - z^2 y(0)\right] - \frac{3}{4}\left[zY(z) - Z y(0)\right]$$

(with $y(1) \to 1$, $y(0) \to 0$, $y(0) \to 0$)

$$+ \frac{1}{8} Y(z) = \frac{z}{z-1}$$

$$z^2 Y(z) - z(1) - z^2(0) - \frac{3}{4}\left[zY(z) - z(0)\right] + \frac{1}{8} Y(z) = \frac{z}{z-1}$$

$$z^2 Y(z) - z - \frac{3}{4} zY(z) + \frac{1}{8} Y(z) = \frac{z}{z-1}$$

$$\left[z^2 - \frac{3}{4} z + \frac{1}{8}\right] Y(z) = z + \frac{z}{z-1}$$

$$Y(z) = \frac{z}{\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)} + \frac{z}{\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)(z-1)}$$

$$= \frac{z}{\left(z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}\right)} + \frac{z}{\left(z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}\right)(z-1)}$$

$$= \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} + \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)(z-1)}$$

$$Y(z) = Y_1(z) + Y_2(z) \quad \text{--- } 1$$

$$Y_1(z) = \frac{z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{1}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{A}{\left(z - 1/4\right)} + \frac{B}{\left(z - \frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{-4}{\left(z - 1/4\right)} + \frac{4}{\left(z - 1/2\right)}$$

$$Y_1(z) = \frac{-4z}{\left(z-\frac{1}{4}\right)} + \frac{4z}{\left(z-\frac{1}{2}\right)} \quad \text{—②}$$

$$Y_2(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-1)}$$

$$\frac{Y_2(z)}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-1)}$$

$$\frac{Y_2(z)}{z} = \frac{A}{(z-1/4)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{(z-1)}$$

$$\frac{Y_2(z)}{z} = \frac{16/3}{\left(z-\frac{1}{4}\right)} + \frac{(-8)}{\left(z-\frac{1}{2}\right)} + \frac{8/3}{(z-1)}$$

$$Y_2(z) = \frac{16/3\, z}{(z-1/4)} - \frac{8z}{\left(z-\frac{1}{2}\right)} + \frac{8/3\, z}{(z-1)} \quad \text{—③}$$

Put eqn ② and ③ in eqn ①

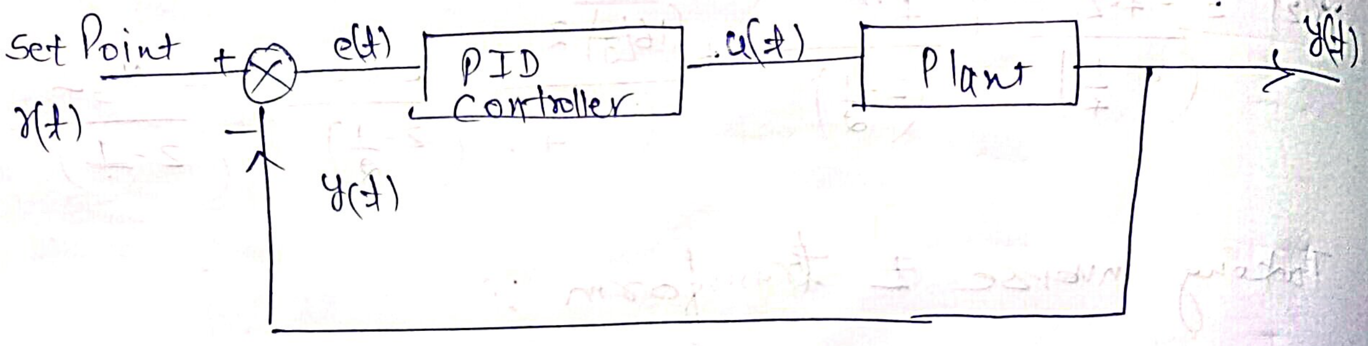
$$Y(z) = \frac{-4z}{\left(z-\frac{1}{4}\right)} + \frac{4z}{\left(z-\frac{1}{2}\right)} + \frac{(16/3)\,z}{\left(z-1/4\right)} - \frac{8z}{\left(z-\frac{1}{2}\right)} + \frac{8/3\,z}{\left(z-1\right)}$$

Taking inverse z transform :

$$\boxed{y(k) = \underbrace{\left[-4\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{2}\right)^k\right]}_{\text{Zero input response}} + \underbrace{\left[\frac{16}{3}\left(\frac{1}{4}\right)^k - 8\left(\frac{1}{2}\right)^k\right] + \frac{8}{3}(1)^k}_{\text{zero state (initial condition) response}} \qquad k \geq 0}$$

$$y(k) = \underbrace{\left\{\underbrace{\left[-4\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{2}\right)^k\right]}_{\text{Caused by initial conditions}} + \underbrace{\left[\frac{16}{3}\left(\frac{1}{4}\right)^k - 8\left(\frac{1}{2}\right)^k\right]}_{\text{Caused by excitation (input)}}\right\}}_{\text{Transient response}} + \underbrace{\frac{8}{3}(1)^k}_{\text{Forced Response}}$$

Conventional PID Controller



$r(t) \rightarrow$ Set point

$y(t) =$ Process output

$e(t) =$ Error Signal $= r(t) - y(t)$

$u(t) =$ Controller output

$$u(t) = K_c\left[e(t) + \frac{1}{T_I}\int_0^t e(t)\,dt + T_D\frac{de(t)}{dt}\right]$$

$$U(s) = K_c\left[E(s) + \frac{1}{T_I s}E(s) + s T_D E(s)\right]$$

$$U(s) = \left[K_c + \frac{K_c}{T_I s} + K_c T_D s\right]E(s)$$

$$U(s) = \left[K_P + \frac{K_I}{s} + K_D s\right]E(s)$$

$$\boxed{\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s}$$

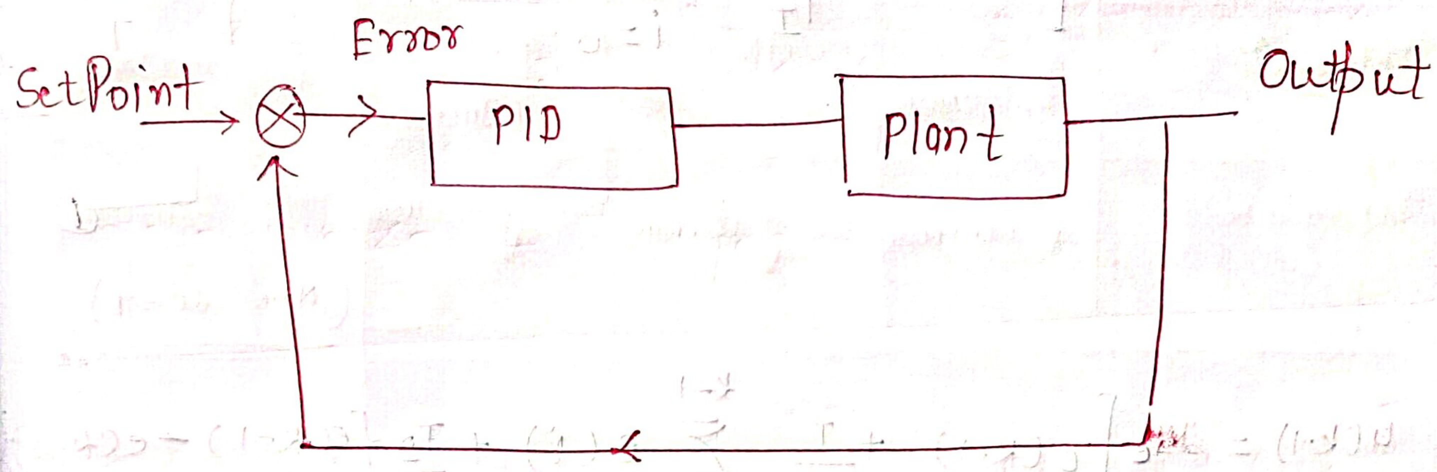
Note:

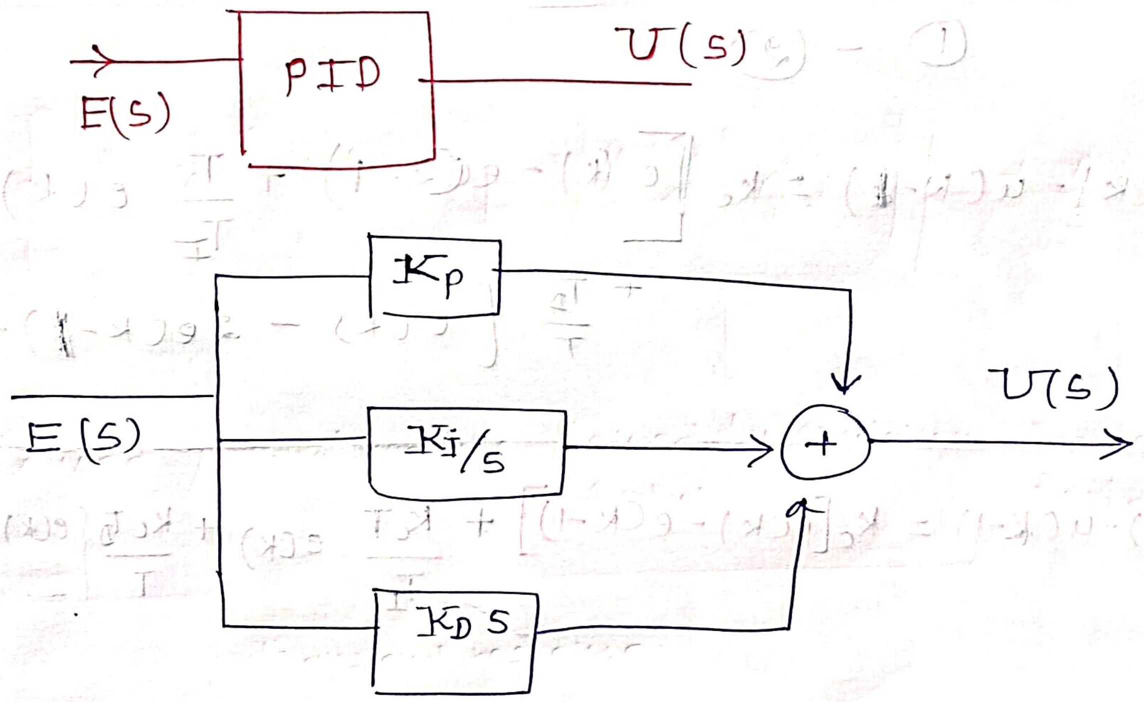
$K_P = K_c$

$K_I = \frac{K_c}{T_I} = \frac{K_P}{T_I}$

$K_D = K_c T_D = K_P T_D$

Digital PID controller





$$u(t) = K_c \left[e(t) + \frac{1}{T_I} \int_0^t e(t)\,dt + T_D \frac{d\,e(t)}{dt} \right]$$

$$\Downarrow$$

In discrete form

$$u(k) = K_c\left[e(k) + \frac{T}{T_I}\sum_{i=0}^{i=k} e(i) + T_D\left\{\frac{e(k)-e(k-1)}{T}\right\}\right] \quad \text{①}$$

(Note $dt = T$)

$$u(k-1) = K_c\left[e(k-1) + \frac{T}{T_I}\sum_{i=0}^{k-1} e(i) + \frac{T_D}{T}\left\{e(k-1) - e(k-2)\right\}\right] \quad \text{②}$$

① − ②

$$u(k) - u(k-1) = K_c\left[e(k) - e(k-1) + \frac{T}{T_I} e(k) + \frac{T_D}{T}\left\{e(k) - 2e(k-1) + e(k-2)\right\}\right]$$

$$\boxed{u(k) - u(k-1) = K_c\left[e(k) - e(k-1)\right] + \frac{K_c T}{T_I} e(k) + \frac{K_c T_D}{T}\left\{e(k) - 2e(k-1) + e(k-2)\right\}}$$

Velocity form of PID Controller algorithm.

Note In some books discrete form of PID controller is given as:

$$u(k) = K_c \left[e(k) + \frac{\Delta t}{T_I} \sum_{i=0}^{i=k} e(i) + \frac{T_D}{\Delta t} \Delta e(k) \right]$$

or

$$u(k) = K_p e(k) + K_I \sum_{i=0}^{i=k} e(i) + K_D \Delta e(k)$$

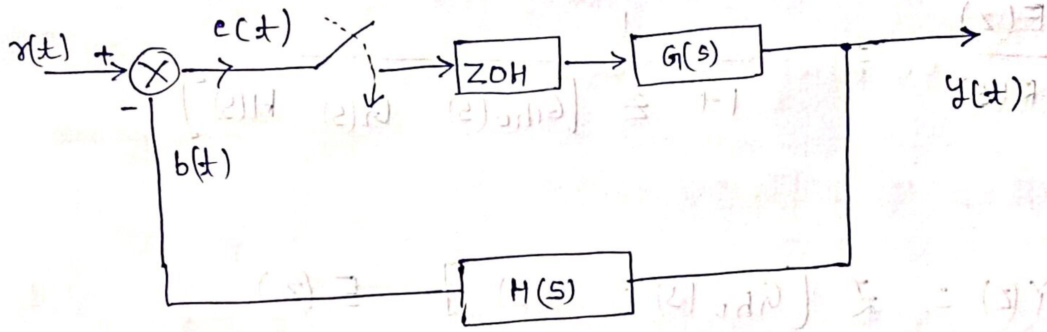
where $K_p = K_c$

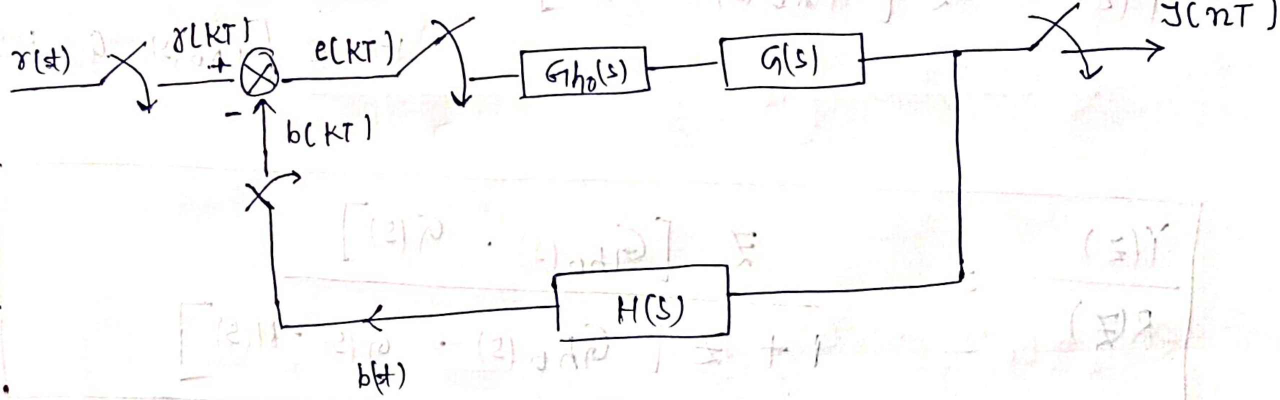
$$K_I = K_c \cdot \frac{\Delta t}{T_I} = K_p \cdot \frac{\Delta t}{T_I}$$

$$K_D = K_c \frac{T_D}{\Delta t} = K_p \cdot \frac{T_D}{\Delta t}$$

	% Overshoot	Risetime	Settlingtime	Stability
Proportional (P)				
Integral (I)	↑ (Increase)	↑ (Increase)	↑ (Increase)	↓ decrease
Derivative (D)	↓ decrease	↓ decrease	↓ decrease	↑ Increase

Error sampled feedback system





$$Y(z) = \mathcal{Z}\left[G_{ho}(s)\ G(s)\right] * E(z)$$

$$B(z) = \mathcal{Z}\left[G_{ho}(s)\ G(s)\ H(s)\right] * E(z)$$

where $B(z) = \mathcal{Z}\left[b(KT)\right]$

$E(z) = \mathcal{Z}\left[e(KT)\right]$

since $e(KT) = r(KT) - b(KT)$

$$E(z) = R(z) - B(z)$$

$$E(z) = R(z) - \mathcal{Z}\left[G_{ho}(s) \cdot H(s) \cdot H(s)\right] E(z)$$

$$E(z)\left[1+Z\left[G_{ho}(s)\cdot G(s)\,H(s)\right]\right] = R(z)$$

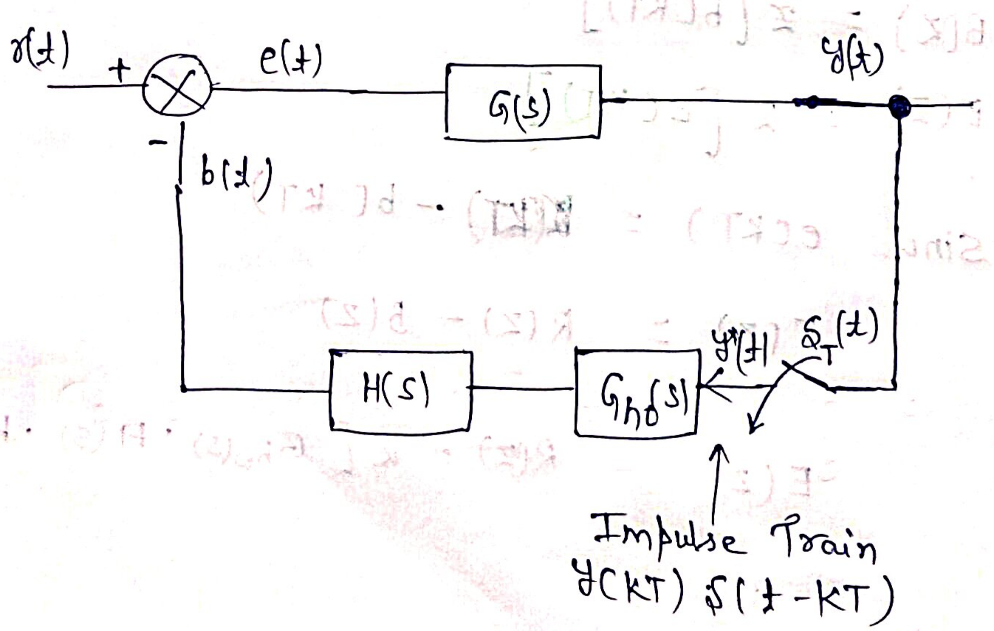
$$\frac{E(z)}{R(s)} = \frac{1}{1+Z\left[G_{ho}(s)\;G(s)\;H(s)\right]}$$

$$Y(z) = Z\left[G_{ho}(s)\cdot G(s)\right]\;E(z)$$

$$Y(z) = Z\left[G_{ho}(s)\;G(s)\right] * \frac{R(s)}{1+Z\left[G_{ho}(s)\;G(s)\,H(s)\right]}$$

$$\boxed{\frac{Y(z)}{R(z)} = \frac{Z\left[G_{ho}(s)\cdot G(s)\right]}{1+Z\left[G_{ho}(s)\cdot G(s)\cdot H(s)\right]}}$$

Sampled data system with sampler and **ZOH** in feedback path



$$E(s) = R(s) - G_{ho}(s)\, H(s)\, Y^*(s)$$

$$Y(s) = [R(s) - B(s)]\, G(s) = [R(s) - G_{ho}(s)\, H(s)\, Y^*(s)]\, G(s)$$

$$Y(s) = -\, G_{ho}(s)\, G(s)\, H(s)\, Y^*(s) + G(s)\, R(s)$$

Taking Z Transform :-

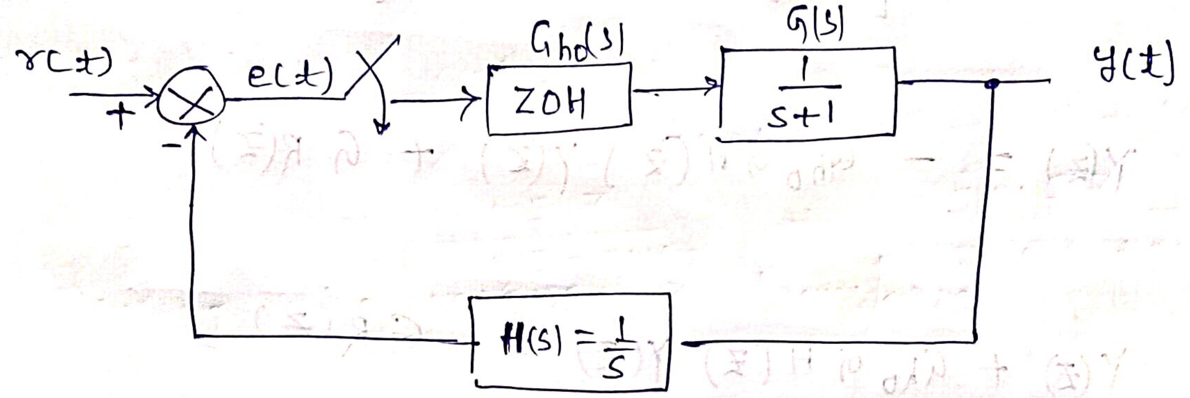
$$\therefore Y(z) = Z\left[-G_{ho}(s)\, G(s)\, H(s)\, Y^*(s)\right] + Z\left[G(s) \cdot R(s)\right]$$

$$Y(z) = -\, G_{ho}GH(z)\, Y(z) + GR(z)$$

$$Y(z) + G_{ho}GH(z)\, Y(z) = GR(z)$$

$$\boxed{Y(z) = \frac{GR(z)}{1 + G_{ho}GH(z)}}$$

Q: For the sampled data system shown below find the output $y(k)$ for $r(t)$ = unit step. $T = 1$ second.



$$Z\left[G_{ho}(s) \cdot G(s)\right] = Z\left[\frac{1-e^{-Ts}}{s} * \frac{1}{s+1}\right]$$

$$= (1-z^{-1})\, Z\left[\frac{1}{s(s+1)}\right]$$

$$= [1-z^{-1}]\left[\frac{z(1-e^{-1})}{(z-1)(z-e^{-1})}\right]$$

$$= \frac{(1-e^{-1})}{(z-e^{-1})}$$

$$Z\left[G_{ho}(s) \cdot G(s)\, H(s)\right] = Z\left[\frac{1-e^{-Ts}}{s} * \frac{1}{s+1} * \frac{1}{s}\right]$$

$$= (1-z^{-1}) \; Z\left[\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1}) \left[\frac{z}{(z-1)^2} - \frac{z(1-e^{-1})}{(z-1)(z-e^{-1})}\right]$$

$$= \left[\frac{1}{z-1} - \frac{1-e^{-1}}{z-e^{-1}}\right]$$

$$= \frac{z-e^{-1} - (z-1)(1-e^{-1})}{(z-1)(z-e^{-1})}$$

$$= \frac{e^{-1}z - 2e^{-1}+1}{(z-1)(z-e^{-1})}$$

$$\frac{Y(z)}{R(z)} = \frac{Z\left[G_{ho}(s)\; G(s)\right]}{1+Z\left[G_{ho}(s)\, G(s)\, H(s)\right]}$$

$$R(z) = \frac{z}{(z-1)}$$

$$\frac{Y(z)}{R(z)} = \frac{(1-e^{-1})/(z-e^{-1})}{1+\left[\frac{e^{-1}z-2e^{-1}+1}{(z-1)(z-e^{-1})}\right]}$$

$$\frac{Y(z)}{R(z)} = \frac{(1-e^{-1})(z-1)}{z^2-z+1-e^{-1}}$$

$$Y(z) = \frac{(1-e^{-1})\,(z-1)}{z^2 - z + (1-e^{-1})} * \frac{z}{(z-1)}$$

$$= \frac{(1-e^{-1})\,z}{z^2 - z + 1 - e^{-1}}$$

$$= \frac{0.632z}{(z-0.5-j0.62)(z-0.5+j0.62)}$$

$$= \frac{-j0.51z}{z-0.5-j0.62} + \frac{j0.51z}{z-0.5+j0.62}$$

Taking the inverse z-transform:

$$y(k) = -j0.51(0.5+j0.62)^k + j0.51(0.5-j0.62)^k$$

Ans

Jury Stability Test

$$\Delta(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1$$

Necessary conditions for stability are:

$\Delta(1) = 2+7+10+4+1 = 24 > 0$ Satisfied

$(-1)^4 \Delta(-1) = 2-7+10-4+1 = 2 > 0$ Satisfied

To check the requirement of sufficiency, we construct the jury table:

Row	z^0	z^1	z^2	z^3	z^4
1	1 α_4	4 α_3	10 α_2	7 α_1	2 α_0
2	2 α_0	7 α_1	10 α_2	4 α_3	1 α_4
3	-3 b_3	-10 b_2	-10 b_1	-1 b_0	
4	-1 b_0	-10 b_1	-10 b_2	-3 b_3	
5	8 q_2	20 q_1	20 q_0		

(i) $|1| < |2|$ Satisfied $[|\alpha_4| < |\alpha_0|]$

(ii) $|-3| > |-1|$ Satisfied $[|b_3| > |b_0|]$

(iii) $|8| > |20|$ Not satisfied $(|q_2| > |q_0|)$

Q Therefore system is unstable.

$$b_0 = \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} = 7-8 = -1$$

$$b_1 = \begin{vmatrix} 1 & 10 \\ 2 & 10 \end{vmatrix} = 10-20 = -10$$

$$b_2 = \begin{vmatrix} 1 & 7 \\ 2 & 4 \end{vmatrix} = 4-14 = -10$$

$$b_3 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3$$

Thus necessary conditions for stability are:

$$\Delta(1) > 0$$

$$(-1)^n \Delta(-1) > 0$$

Sufficient conditions for stability are:

$$\left.\begin{array}{l} |\alpha_n| < |\alpha_0| \\ |b_{n-1}| > |b_0| \\ |c_{n-2}| > |c_0| \\ \vdots \\ |q_2| > |q_0| \end{array}\right\} (n-1) \text{ constraints}$$

* The Jury table is continued until $(2n-3)^{rd}$ row is reached which will exactly three elements.

let $n = 4$

$$(2n-3) = 2\times4-3 = 8-3 = 5$$

Total number of row in jury table will be 5. And in the last row of the jury table there will be exactly three elements.

Q: Consider the characteristics polynomial:

$$\Delta(z) = z^3 + 3.3z^2 + 4z + 0.8$$

using jury table, determine whether system is stable or not (show all steps).

Soln

$$\Delta(z) = z^3 + 3.3z^2 + 4z + 0.8$$

$$\Delta(1) = 1 + 3.3 + 4 + 0.8 = 9.1$$

$$(-1)^3 \Delta(-1) = (-1)(-1 + 3.3 - 4 + 0.8)$$

$$= (-1)(-0.9) = +0.9$$

$\Delta(1) > 0$; Satisfied

$(-1)^3 \, \Delta(-1) > 0$; Satisfied

ROW	z^0	z^1	z^2	z^3
1	0.8 α_3	4 α_2	3.3 α_1	1 α_0
2	1 α_0	3.3 α_1	4 α_2	0.8 α_3
3	-0.36 q_2	-0.1 q_1	-1.36 q_0	

$$q_0 = \begin{vmatrix} 0.8 & 4 \\ 1 & 3.3 \end{vmatrix} = 0.8 \times 3.3 - 1 \times 4$$
$$= 2.64 - 4$$
$$= -1.36$$

$$q_1 = \begin{vmatrix} 0.8 & 3.3 \\ 1 & 4 \end{vmatrix} = 0.8 \times 4 - 1 \times 3.3$$
$$= 3.2 - 3.3 = -0.1$$

$$q_2 = \begin{vmatrix} 0.8 & 1 \\ 1 & 0.8 \end{vmatrix} = (0.8)(0.8) - 1 \times 1$$
$$= .64 - 1$$
$$= -.36$$

(ii) $|0.8| < |1|$; $|\alpha_3| < |\alpha_0|$ satisfied

(iii) $|-0.36| \not> |-1.36|$; $|a_2| > a_0$

This condition is not satisfied

Therefore system is <u>unstable</u>.

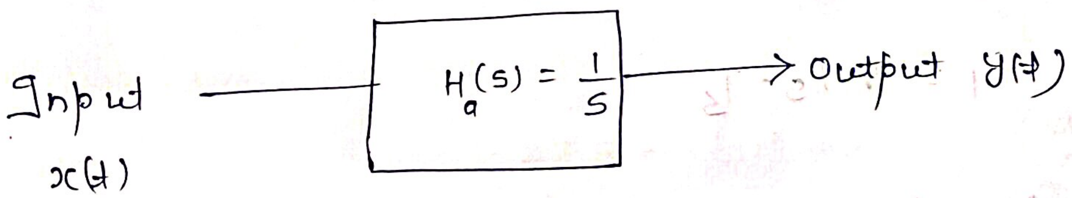
Bilinear Transformation

* This method is used to convert analog filter / anlog transfer function to digital filter / discrete transfer function.

* s domain $\longrightarrow$ z domain

* Analog filter $\longrightarrow$ Digital filter

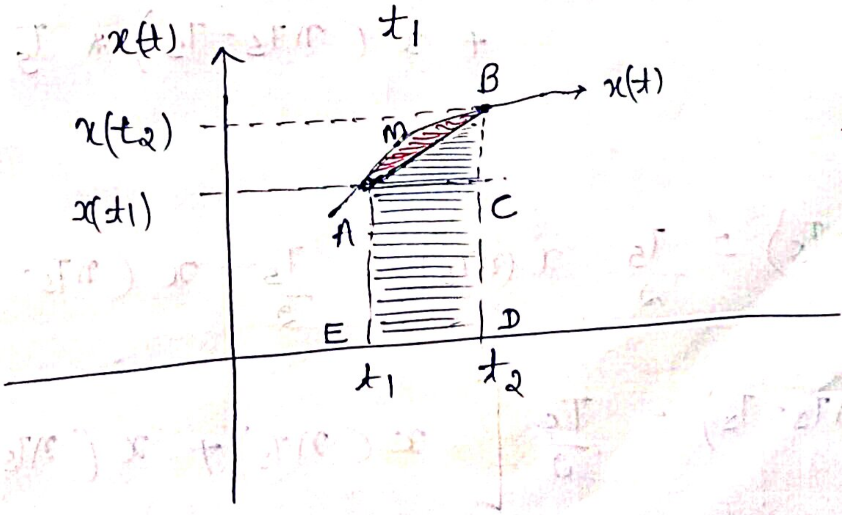
Consider an analog integrator:



For the time period T, the difference in the output is given by

$y(t) = \int x(t)\,dt$

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(t)\,dt$$



* area ABm is very small so it can be neglected.

* We have assumed two input positions $x(t_1)$ and $x(t_2)$. Corresponding output is denoted by $y(t_1)$ and $y(t_2)$.

* Area under the curve = Area of triangle ABC + area of rectangle ACDE + Area ABM

→ neglecting

$$\simeq \text{area of } \Delta ABC + \text{area of rectangle } ACDE$$

$$= \frac{1}{2}(t_2 - t_1) * [x(t_2) - x(t_1)] + x(t_1)(t_2 - t_1)$$

Let $t_2 = nT_s$

$t_1 = nT_s - T_s$

$t_2 - t_1 = nT_s - (nT_s - T_s)$

$t_2 - t_1 = T_s$

$$y(nT_s) - y(nT_s - T_s) = \frac{1}{2} T_s [x(nT_s) - x(nT_s - T_s)] + x(nT_s - T_s) * T_s$$

$$y(nT_s) - y(nT_s - T_s) = \frac{T_s}{2} x(nT_s) + \frac{T_s}{2} x(nT_s - T_s)$$

$$y(nT_s) - y(nT_s - T_s) = \frac{T_s}{2} \left[x(nT_s) + x(nT_s - T_s) \right]$$

Taking Z transform of both the sides

$$Y(z) - z^{-1} Y(z) = \frac{T_s}{2}\left[X(z) + z^{-1} X(z)\right]$$

$$Y(z)\left[1 - z^{-1}\right] = \frac{T_s}{2} X(z)\left(1+z^{-1}\right)$$

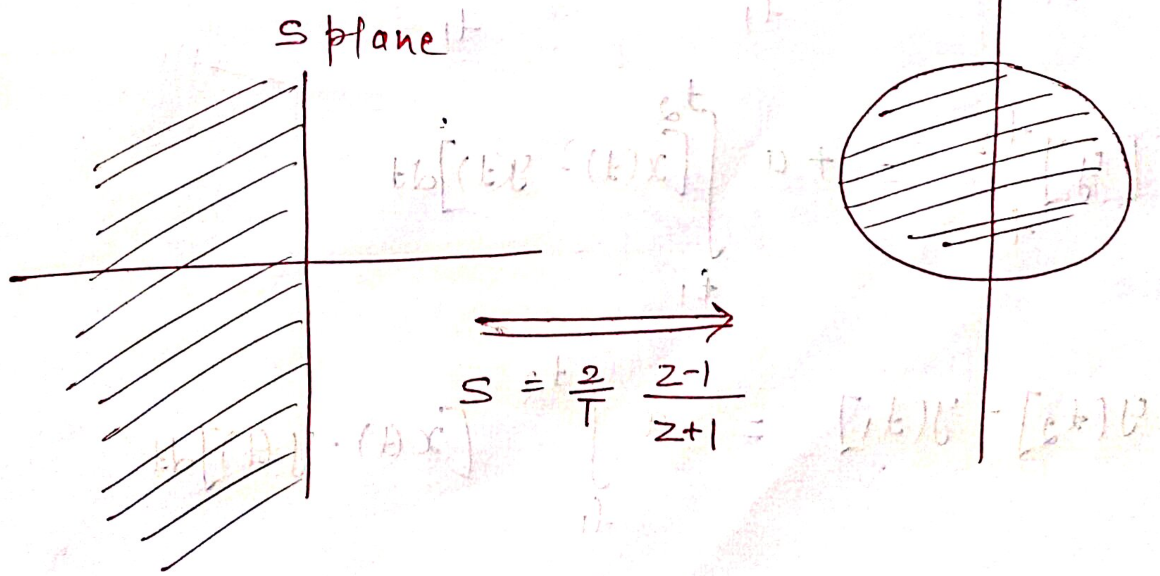
$$\frac{Y(z)}{X(z)} = \frac{T_s}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right] = H(z) = \frac{1}{\frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

We have $\dfrac{Y(s)}{X(s)} = H_a(s) = \dfrac{1}{s}$

$$\therefore \frac{1}{s} = \frac{T_s}{2}\,\frac{1+z^{-1}}{1-z^{-1}}$$

$$s = \frac{2}{T_s}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$\boxed{s = \frac{2}{T_s}\,\frac{z-1}{z+1}}$$



Note: Let transfer function of analog filter is $\frac{a}{s+a}$.

$$X(s) \longrightarrow \boxed{\frac{a}{s+a}} \longrightarrow Y(s)$$

$$\frac{Y(s)}{X(s)} = H_a(s) = \frac{a}{(s+a)}$$

$$sY(s) + aY(s) = aX(s)$$

$$\therefore \quad \frac{dy}{dt} + a\,y(t) = a\,x(t)$$

$$\frac{dy}{dt} = -a\,y(t) + a\,x(t)$$

$$dy = -a\,y(t)\,dt + a\,x(t)\,dt$$

$$\int_{t_1}^{t_2} dy = -a\int_{t_1}^{t_2} y(t)\,dt + a\int_{t_1}^{t_2} x(t)\,dt$$

$$[y]_{t_1}^{t_2} = +a\int_{t_1}^{t_2} [x(t) - y(t)]\,dt$$

$$y(t_2] - y(t_1] = a\int_{t_1}^{t_2} [x(t) - y(t)]\,dt$$

$$y(nT_s) - y(nT_s - T_s) = a\left[\frac{x(nT_s) + x(nT_s - T_s)}{1} * \frac{T_s}{2} - \left\{\frac{y(nT_s) + y(nT_s - T_s)}{1} * \frac{T_s}{2}\right\}\right]$$

$$y(nT_s) - y(nT_s - T_s) = \frac{aT}{2} x(nT_s) + \frac{aT}{2} x(nT_s - T_s) - \frac{aT_s}{2} y(nT_s) - \frac{aT_s}{2} y(nT_s - T_s)$$

$$\left(1 + \frac{aT_s}{2}\right) y(nT_s) - \left(1 - \frac{aT_s}{2}\right) y(nT_s - T_s) = \frac{aT_s}{2} x(nT_s) + \frac{aT_s}{2} x(nT_s - T_s)$$

$$\left(1 + \frac{aT_s}{2}\right) Y(z) - \left(1 - \frac{aT_s}{2}\right) z^{-1} Y(z) = \frac{aT_s}{2} X(z) + \frac{aT_s}{2} z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{aT_s}{2}(1 + z^{-1})}{\left(1 + \frac{aT_s}{2}\right) - \left(1 - \frac{aT_s}{2}\right) z^{-1}}$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{aT_s}{2}(1 + z^{-1})}{(1 - z^{-1}) + \frac{aT_s}{2}(1 + z^{-1})}$$

$$\frac{Y(z)}{X(z)} = \frac{a}{\frac{2}{T_s}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a}$$

$$\frac{Y(s)}{X(s)} = \frac{a}{s + a}$$

$$\therefore \quad s = \frac{2}{T_s} * \frac{1-z^{-1}}{1+z^{-1}}$$

$$\boxed{s = \frac{2}{T_s} \frac{z-1}{z+1}}$$