Quantification of noise

Process noise (W_k) :

- It refers to uncertainity in the system model. In real world system it is impossible to perfectly model system's evolution, so we assume there is a random disturbance affecting the state.

- W_k is assumed to be Gaussian white noise[uncorrelated].
- We assume that W_k is wide sense stationary.

$$E[W_k] = 0$$

$$E[W_k \cdot W_k^T] = Q_k$$

Q_k is the process noise covariance matrix.

- Process noise is disturbing the state variables (X_k).

- Q_k is positive semidefinite matrix.

Measurement noise (V_K) :

It accounts for errors in the observation due to sensor inaccuracies or environmental factors.

- V_K is assumed to be Gaussian white noise. [uncorrelated]
- We assume that V_K is widesense stationary.

$$E[V_K] = 0$$

$$E[V_K \cdot V_K^T] = R_K$$

R_K is the measurement noise covariance matrix.

- Measurement noise is disturbing the measurement output (y_K).

- R_K is positive definite matrix.

Independence of Process noise and measurement noise :-

- W_K and V_K are assumed to be independent of each other.

$$E[W_t \cdot V_s^T] = 0 \qquad \text{For all } t \text{ and } s$$

t and s are the time instant.

Autocorrelation : It is the correlation between two time slices of same random variable.

$$E[W_{t,i} \cdot W_{s,i}^T] = 0$$

$$E[V_{t,j} \cdot V_{s,j}^T] = 0$$

$\Big\}$ For all time t and s, $t \neq s$

Note here i and j represents the noise component which you are selecting [It is associate with respective sensor output].

Cross correlation : The correlation between two different componentas of same noise vector or the one between measurement noise and process noise.

$$E[V_{t,i} \cdot V^T_{s,j}] = 0$$

$$E[W_{t,i} \cdot W^T_{s,j}] = 0$$

For all t and s, $t \neq s$

and

for all i and j $i \neq j$

$$E[W_{t\,i} \cdot V^T_{s\,j}] = 0$$

For all t and s

and

For all i and j

MSE = Variance + Bias^2

Proof:

True value $= x$

Estimated value $= \hat{x}$

$$MSE = \mathbf{E}\left[(\hat{x}-x)^2\right]$$

$$Bias = E[\hat{x}] - x$$

$$Variance = E\left[(\hat{x} - E[\hat{x}])^2\right]$$

$$\because MSE = E\left[(\hat{x}-x)^2\right]$$

$$\hat{x} - x = \hat{x} - E[\hat{x}] + E[\hat{x}] - x$$

$$\therefore (\hat{x}-x)^2 = \left(\hat{x} - E[\hat{x}] + E[\hat{x}] - x\right)^2$$

$$= \left(\hat{x} - E[\hat{x}]\right)^2 + 2\left(\hat{x} - E[\hat{x}]\right)\left(E[\hat{x}] - x\right) + \left(E[\hat{x}] - x\right)^2$$

Taking expection operator $\mathbf{E}$ on both sides:

$$E[(\hat{x}-x)^2] = E\left[(\hat{x} - E[\hat{x}])^2\right] + 2E\left[(\hat{x} - E[\hat{x}]) \cdot (E[\hat{x}] - x)\right] + E\left[(E[\hat{x}] - x)^2\right]$$

$$= E\left[(\hat{x} - E[\hat{x}])^2\right] + 2\left(E[\hat{x}] - x\right) \cdot \mathbf{E}\left[(\hat{x} - E[\hat{x}])\right] + E\left[(E[\hat{x}] - x)^2\right]$$

This is constant term so it will come outside.

This is aconstant term

Note $E[(\hat{X} - E[\hat{X}])] = E[\hat{X}] - E\{E[\hat{X}]\}$

$$= E[\hat{X}] - E[\hat{X}]$$ Constant Value

$$= 0$$

$\therefore$ from above :

$$E[(\hat{X}-X)^2] = E[(\hat{X}-E[\hat{X}])^2] + 2(E[\hat{X}]-X)\cdot 0 + (E[\hat{X}]-X)^2$$

$$E[(\hat{X}-X)^2] = E[(\hat{X}-E[\hat{X}])^2] + (E[\hat{X}]-X)^2$$

$$\boxed{\text{MSE} = \text{Variance} + \text{Bias}^2}$$

Note $E[\text{Constant}] = \text{constant}$

UnBiased Estimator

* When Bias = 0 then estimator is called unbiased estimator.

* When bias = 0

$$MSE = Variance + Bias^2$$

$$MSE = Variance + 0$$

$$\boxed{MSE = Variance}$$

* $E[error] = Bias$

$$E[error] = E[\hat{X} - X] = E[\hat{X}] - X = Bias$$

* When Bias = 0

$$Bias = E[\hat{X}] - X = 0$$

$$E[\hat{X}] = X$$

* Expectation of error $= E[error]$

$= E[\text{expected value} - \text{True value}]$

$= E[\hat{X} - X]$

$= E[\hat{X}] - E[X]$

$= E[\hat{X}] - X$

This X is true value and this is a constant
E[constant] = constant

$= X - X$

$E[error] = 0$

* $E[error] = 0$
* $Bias = 0$