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UNIT IV : Linear Algebra

* Rank of Matrix, by elementary operations.
* Consistency of system of equations
* Gauss - Seidel
* Eighen values & Eighen Vectors (Rayleighs power meth
* Linear transformation.
* Diagonalisation of Square Matrix
* Quadratic forms
* Reduction to connonical form by orthogonal transformation

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Elementary Transformations :

The fallowing operations refer to rows and column, known as elementary transformation
→ The interchange of any two rows or column.
→ The multiplication of any row/column by non zero num
→ The addn of const multiple of any element of row/col to corresponding elements of any row/col.

Rank of Matrix.

Let 'A' be a non zero matrix of order $m \times n$. A positive integer 'r' is said to be rank of A, if fallowing condns are satisfied

→ 'A' has atleast 1 non zero minor of order 'r'
→ Every minor of 'A' who's order is greater than 'r' is equal to 0
→ We denote rank of 'A' by $\boxed{\rho(A)}$

Some consequences :
- Rank of A will be 1 if & only if all minors of A of order greater than or equal to 2 are zero.
- $\rho(A)$ cannot exceed 'm' or 'n' (when A is of order $m \times n$).
- If A is a square matrix of order 'n' then ~~zero~~ $\rho(A) \leq n$, further $\rho(A)$ is equal to n if & only if A is non-singular.

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- If I_n is unit matrix (Identity matrix) of order n, then $\rho(I_n) = n$
- If A is a normal form, i.e if A is of the form $[I_r \;\; 0]$ or $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Then $\rho(A)$ is r.

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Determine rank of the following matrices, reducing it into echelon form

1.) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$\Rightarrow$

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

$R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix}$ (∵ leading ele of non-zero row, must be 1)

$R_3 \rightarrow R_3 - R_2$

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore \rho(A) = 2$

2.) $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{pmatrix}$

$\Rightarrow$ Let $A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{pmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\sim \begin{pmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{pmatrix}$

$R_3 \to 2R_3 - 5R_2$

$\sim \begin{pmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & 0 & 24 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore \underline{\rho(A) = 3}$

3.) $A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix}$

NOTE: In case of non square matrix last element must be made zero, if it is easily possible

$\to$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

$\therefore$ ~~$\rho(A) = 2$~~ $R_3 \to 3R_3 - R_2$

$\sim \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore \rho(A) = \underline{2}$

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4.) $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix}$

$\Rightarrow$

$R_2 \to 2R_2 - 3R_1, \quad R_3 \to R_3 - 2R_2, \quad R_4 \to 2R_4 - 9R_1$

$\sim \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \\ 0 & -7 & -14 & -21 \end{pmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -7 & -14 & -21 \end{pmatrix}$

$R_4 \to R_4/-7$

$\sim \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

$R_4 \to R_4 + R_2$

$\sim \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 & 2 & 5/2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore \quad \rho(A) = \underline{2}$

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5.) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$ (3 = 2)

6.) $\begin{pmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{pmatrix}$

7.) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{pmatrix}$

8.) $\begin{pmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$ (3 = 3)

9.) $\begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix}$ (3 = 4)

5.) $\begin{pmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{pmatrix}$

$\Rightarrow$ $A = \begin{pmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{pmatrix}$ $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 9R_1$, $R_4 \to R_4 - 16R_1$

$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & -20 & -56 & -108 \\ 0 & -39 & -108 & -207 \end{pmatrix}$ ~~$R_3 \leftrightarrow R_2$~~ $R_3 \to 7R_3 - 20R_2$

$-156 +$

$7(-56) + 20(20)$

$\sim \cancel{\begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & 0 & 8 & 24 \\ 0 & -39 & -108 & -207 \end{pmatrix}}$ $R_4 \to R_4/3$

$\sim \cancel{\begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & 0 & 1 & 3 \\ 0 & -13 & -36 & -69 \end{pmatrix}}$ $\sim \cancel{\begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 3 & 69/13 \end{pmatrix}}$

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5.)

$$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & -5 & -14 & -27 \\ 0 & -13 & -36 & -64 \end{pmatrix} \quad \begin{matrix} R_3 \to 7R_3 - 5R_2 \\ R_4 \to 7R_4 - 13R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 8 & -24 \end{pmatrix} \quad R_4 \to R_4 + 4R_3.$$

$$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & 0 & -2 & +6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & 1 & 20/7 & 39/7 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \rho(A) = 3.$

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Determine rank of matrix by reducing it to normal form.

1.) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

$\Rightarrow$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ $\quad R_2 \rightarrow R_2 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & +1 \\ 0 & -1 & -1 \end{bmatrix}$ $\quad C_2 \rightarrow C_2 - 1$
$C_3 \rightarrow C_3 - 2C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$ $\quad R_3 \rightarrow R_3 + R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad C_3 \rightarrow C_3 - C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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2.) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

$\Rightarrow$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -8 & -10 \\ 0 & -5 & -11 & -14 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 2C_1$
$C_3 \rightarrow C_3 - 3C_1$
$C_4 \rightarrow C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -8 & -10 \\ 0 & -5 & -11 & -14 \end{bmatrix}$$

$R_3 \rightarrow 2R_3 - 5R_2$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -8 & -10 \\ 0 & 0 & 18 & 22 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 4C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -10 \\ 0 & 0 & 18 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$C_4 \rightarrow C_4 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 18 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$C_4 \rightarrow C_4/11 - C_3/9$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3$

$[I_3 \ 0]$

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3.) $\begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

$\Rightarrow$ $R_2 \to 2R_2 - R_1$
$R_3 \to 2R_3 - 3R_1$
$R_4 \to R_4 - 3R_1$

$\sim \begin{pmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & -7 & 9 & 1 \\ 0 & -6 & 3 & -4 \end{pmatrix}$

$C_2 \to 2C_2 - 3C_1$
$C_3 \to 2C_3 + C_1$
$C_4 \to 2C_4 + C_1$

$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & -6 & -14 \\ 0 & -14 & 18 & 2 \\ 0 & -12 & 6 & -8 \end{pmatrix}$

$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -5 & -3 & -7 \\ 0 & -7 & 9 & 1 \\ 0 & -6 & 3 & -4 \end{pmatrix}$ $R_3 \to 5R_3 - 7R_2$
$R_4 \to 5R_4 - 6R_2$.

$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -5 & -3 & -7 \\ 0 & 0 & 66 & 54 \\ 0 & 0 & 33 & 22 \end{pmatrix}$ $C_3 \to 5C_3 - 3C_2$
$C_4 \to 5C_4 - 7C_2$

$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 165 & 135 \\ 0 & 0 & 55 & 55 \end{pmatrix}$ $R_4 \to 3R_4 - R_3$

$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 165 & 135 \\ 0 & 0 & 0 & 30 \end{pmatrix}$ $\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 165 & 279 \\ \end{pmatrix}$

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$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -3 & -7 \\ 0 & 0 & 66 & 44 \\ 0 & 0 & 33 & 22 \end{pmatrix}$$

$C_3 \to 5C_3 - 3C_2$

$C_4 \to 5C_4 - 7C_2$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 30 & 20 \\ 0 & 0 & 15 & 10 \end{pmatrix}$$

$R_4 \to 2R_4 - R_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 30 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_4 \to \frac{C_4}{20} - \frac{C_3}{30}$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \rho(A) = 3$

$$\begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

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Consistency of Linear Equations:

→ System of eqn in which all the unknown quantities appear in 1st degree alone, is called a linear system of eqns.

→ A set of 'm' linear eqns in 'n' unknowns is as follows:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

where a_{ij} & b_{ij} are constants.

→ If $b_1, b_2, b_3 \dots$ so on b_m are all '0', the system is said to be $\boxed{\text{homogeneous}}$. The set of values $x_1, x_2, \dots, x_n$ which satisfy all the eqns simultaneously is called $\boxed{\text{soln of system}}$ of eqn.

→ A system of L.E is said to be $\boxed{\text{consistent}}$ if it possess a soln, else it is $\boxed{\text{inconsistent}}$

→ The above eqn can be written in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A \qquad X \qquad = \qquad B \qquad (AX = B)$$

→ $x_1 = x_2 = x_3 = \dots = x_n = 0$, is a soln of homogeneous system of eqn & is called $\boxed{\text{trivial soln}}$

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→ If atleast one x_i $(i = 1, 2 \ldots, n)$ is not equal to '0'. Then it is called non-trivial solⁿ

** Conditions for consistency & types of solⁿˢ

→ The system of eqns represented by the matrix eqn $AX = B$, is consistent if $\rho(A) = \rho(A:B)$

→ Suppose ~~rank~~ of $\rho(A) = \rho(A:B) = r$, then condⁿˢ for various types of solⁿˢ are

- Unique solⁿ - $\rho(A) = \rho(A:B) = r = n$, where 'n' is no. of unknowns
- Infinite solⁿ - $\rho(A) = \rho(A:B) = r < n$, in this case, $(n - r)$ unknowns can take arbitrary values

→ Obviously $\rho(A) \neq \rho(A:B)$, ~~th~~ implies, the system is inconsistent. (Does not posses solⁿ).

1.) Test the consistency of :

$$x + y + z = 6$$
$$x - y + z = 5$$
$$2x + y + z = 10$$

$\Longrightarrow$ Let,

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad [X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad [B] = \begin{bmatrix} 6 \\ 5 \\ 10 \end{bmatrix}$$

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & 1 & 10 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & -1 & -1 & -2 \end{array}\right] \qquad R_3 \to 2R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & -3 \end{array}\right] \qquad \therefore \begin{array}{l} \rho(A) = 3 \\ \rho(A:B) = 3 \end{array}, \; n = 3$$

$\because \rho(A) = \rho(A:B) = n = 3$, it has unique sol^n

$$\therefore x + y + z = 6$$
$$-2y = -1$$
$$-2z = -3$$

$$x + \frac{3}{2} + \frac{1}{2} = 6$$
$$x = 4$$

$$\boxed{\therefore z = \frac{3}{2}, \; y = \frac{1}{2}, \; x = 4.}$$

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2:) $x + 2y + 3z = 0$
$3x + 4y + 4z = 0$
$7x + 10y + 12z = 0$

$\Rightarrow$ $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$

$[A] \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{pmatrix}$ $R_3 \to R_3 - 2R_2$

$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{pmatrix}$

$\rho(A) = 3$
$n = 3$
$\rho(A:B) = 3$

$\because$ Rank of $\rho(A) = 3 = n$, System has trivial soln.

3.) $x + 3y - 2z = 0$
$2x - y + 4z = 0$
$x - 11y + 14z = 0$

$\Rightarrow$

$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix}$ $R_3 \to R_3 - 2R_2$ $\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix}$

$\rho(A) = 2$, $r < n$ $(n = 3)$

$(r<n)$, inifinite $sol^n \rightarrow (n-r)$

$3-2=1$, $\therefore$ 1 unknown can be set for arb soln

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3.) Let $z=K$, be arbitrary sol^n,

$$-7y+8K=0$$
$$8K=7y$$
$$K=\frac{7y}{8} \Rightarrow \frac{8K}{7}=y$$

$$x+3y-2K=0$$
$$x+3y-\frac{7y}{4}=0$$
$$x+\frac{5y}{4}=0$$
$$4x+5y=0$$
$$x=\frac{-5y}{4} \Rightarrow x=\frac{-5}{4}\times\frac{8K}{7}$$

$$\therefore x=\frac{-10K}{7}, \; y=\frac{8K}{7}, \; z=K$$

4.]
$$2x-3y+7z=5$$
$$3x+y-3z=13$$
$$2x+19y-47z=32$$

$\Rightarrow$

$$(A:B)=\begin{pmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{pmatrix}$$

$R_2 \rightarrow 2R_2-3R_1$

$R_3 \rightarrow R_3-R_1$

$$\sim \begin{pmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 22 & -54 & : & 27 \end{pmatrix}$$

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4.] $R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 0 & 0 & : & 5 \end{pmatrix}$$

$\rho(A:B) = 3$

$\rho(A) = 2$ $\quad \therefore \rho(A:B) \neq \rho(A)$

The system of eqns are inconsistent & System has no solⁿ

5.) $x + 2y + z = 3$

$2x + 3y + 2z = 5$

$3x - 5y + 5z = 2$

$\Longrightarrow$

$$(A:B) = \begin{pmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 3 & 2 & : & 5 \\ 3 & -5 & 5 & : & 2 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & -11 & 2 & : & -7 \end{pmatrix} \quad R_3 \to R_3 - 11R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 2 & : & 4 \end{pmatrix} \quad \rho(A) = \rho(A:B) = 3 = n$$

$\therefore 2z = 7$

$z = \frac{7}{2}$, $\quad y = 1$

$\cancel{x + 2(1) + \frac{7}{2} = 3}$

$\cancel{2x + 4 + 7 = 6}$

$\cancel{2x = -5}$

$\cancel{x = -5/2, \ y = 1, \ z = 2.}$

$x + 2 + 2 = 3$

$x = -1$

$y = 1$

$z = 2$

6.] $2x + 6y + 0 = -11$
$6x + 20y - 6z = -3$
$6y - 18z = -1$

$\Rightarrow$ $(A:B) = \begin{pmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{pmatrix}$

$R_2 \to R_2 - 3R_1$ $\sim \begin{pmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{pmatrix}$

$R_3 \to R_3 - 3R_2$ $\sim \begin{pmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{pmatrix}$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\therefore$ System is inconsistent.

7.) $x - 2y + 3t = 2$
$2x + y + z + t = 4$
$4x - 3y + z + 7t = 8$

$\Rightarrow$

$[A:B] = \begin{bmatrix} 1 & -2 & 0 & 3 & : & 2 \\ 2 & 1 & 1 & 1 & : & 4 \\ 4 & -3 & 1 & 7 & : & 8 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$\sim \begin{pmatrix} 1 & -2 & 0 & 3 & : & 2 \\ 0 & 5 & 1 & -5 & : & 0 \\ 0 & 5 & 1 & -5 & : & 0 \end{pmatrix}$ $R_3 \to R_3 - R_2$

7 - 12 = -5

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7.)

$$\sim \left(\begin{array}{cccc:c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\rho(A) = 2$; $\rho(A:B) = 2$, $n = 4$

$r < n$, $(n-r)$ arb sol^n

$4 - 2 = 2$

$t = k_1$, $z = k_2$

Let $t = p$, $z = q$

$\therefore$ $5y + q + 7p = 0$, $x - 2y + 3p = 2$.

$y = \dfrac{-7p - q}{5}$, $x = 2y - 3p + 2$

$= \dfrac{-14p - 2q}{5} - 3p + 2$

$x = \dfrac{-14p - 2q - 15p + 10}{5}$

$\therefore x = \dfrac{-29p - 2q + 10}{5}$

$\therefore x = \dfrac{-29p - 2q + 10}{5}$, $y = \dfrac{-7p - q}{5}$, $t = p$, $z = q$

$5y + k_2 - 5k_1 = 0$, $x = 2y - 3k_1 + 2$

$y = \dfrac{5k_1 - k_2}{5}$

$x = \dfrac{10k_1 - 2k_2 - 15k_1 + 10}{5}$

$y = \dfrac{5k_1 - k_2}{5}$, $t = k_1$, $z = k_2$, $x = \dfrac{-5k_1 - 2k_2 + 10}{5}$.

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8.) $x + 2y + z = 3$. $2x + 3y + 2z = 5$ $3x - 5y + 5z = 2$
$3x + 9y - z = 4$

$\Rightarrow$ $[A:B]$

$$= \begin{pmatrix} 1 & 2 & 1 & : 3 \\ 2 & 3 & 2 & : 5 \\ 3 & -5 & 5 & : 2 \\ 3 & 9 & -1 & : 4 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & : 3 \\ 0 & -1 & 0 & : -1 \\ 0 & -11 & 2 & : -7 \\ 0 & 3 & -4 & : -5 \end{pmatrix} \quad \ldots\ x = -1,\ y = 1,\ z = 2$$

9.) For what value of λ, & μ, the eqns
$x + y + z = 6$
$x + 2y + 3z = 10$ i.) unique sol^n
$x + 2y + \lambda z = \mu$ ii.) ∞ sol^n
iii.) No sol^n

$\Rightarrow$ $$[A:B] = \begin{pmatrix} 1 & 1 & 1 & : 6 \\ 1 & 2 & 3 & : 10 \\ 1 & 2 & \lambda & : \mu \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$(A:B) = \begin{pmatrix} 1 & 1 & 1 & : 6 \\ 0 & 1 & 2 & : 4 \\ 0 & 1 & \lambda - 1 & : \mu - 6 \end{pmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : 6 \\ 0 & 1 & 2 & : 4 \\ 0 & 0 & \lambda - 3 & : \mu - 10 \end{pmatrix}$$

9.] i.) For $\lambda \neq 3$, $\rho(A) = 3$ & $\rho(A:B) = 3$ & given system will have unique solⁿ.

ii.) If $\lambda = 3$, $\rho(A) = 2$.
For $\mu = 10$, $\rho(A:B) = 2$.
i.e for $\lambda = 3$ & $\mu = 10$ the given system will have ∞ no of solutions.

iii.) For $\lambda = 3$, $\rho(A) = 2$
For $\mu \neq 10$, $\rho(A:B) = 3$
$\rho(A) \neq \rho(A:B)$, $\therefore$ System will have no solⁿ.

H.W

i.)
$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = 7$$
$\rho(A) = 2$
$\rho(A:B) = 3$

ii.)
$$x + 2y + 2z = 5$$
$$2x + y + 3z = 6$$
$$3x - y + 2z = 4$$
$$x + y + z = -1$$
$\rho(A) = 3$
$\rho(A:B) = 4$

iii.)
$$x + 2y + 3z = 1$$
$$2x + 3y + 8z = 2$$
$$x + y + z = 3$$
$\rho(A) = \rho(A:B) = 3$. $x = \frac{9}{2}$, $y = -1$, $z = -\frac{1}{2}$

iv.)
$$2x_1 - x_2 + 3x_3 = 1$$
$$-3x_1 + 4x_2 - 5x_3 = 0$$
$$x_1 + 3x_2 - 6x_3 = 0$$
$x_1 = \frac{9}{34}$, $x_2 = \frac{23}{34}$, $x_3 = \frac{13}{34}$

v.)
$$x + y + z = 1$$
$$2x + y + 4z = 1$$
$$4x + y + 10z = 1$$
$x = -3z$
$y = 2z + 1$.

H.W)

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

$x = 1,$
$y = 3k - 2$
$z = k$

Gauss Seidal method :

It is an iterative method for solving a system of linear algebraic eqn which is diagonally dominant and the same is to be understood as follows :

* consider system of 3 independent eqns in 3 unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

* This system of eqn is said to be diagonally dominant if :

$\rightarrow |a_{11}| > |a_{12}| + |a_{13}|$

$\rightarrow |a_{22}| > |a_{21}| + |a_{23}|$

$\rightarrow |a_{33}| > |a_{31}| + |a_{32}|$

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3]$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3]$$

$$x_3 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}x_2]$$

Solve the system of eqn using Gauss Seidal method

i.) $2x + y + 6z = 9$ with $x_0 = y_0 = z_0 = 0$.

$8x + 3y + 2z = 13$

$x + 5y + z = 7$

$\Rightarrow$ Rearranging, because given eqns are not diagonally domi

$8x + 3y + 2z = 13$

$x + 5y + z = 7$

$2x + y + 6z = 9$

$\therefore x = \frac{1}{8}[13 - 3y - 2z]$, $y = \frac{1}{5}[7 - x - z]$, $z = \frac{1}{6}[9 - 2x - y]$

· Given initial values are: $x_0 = y_0 = z_0 = 0$.

I stage

$$x^{(1)} = \frac{1}{8}[13 - 3(0) - 2(0)]$$

$$= \frac{13}{8}$$

$$\boxed{= 1.625}$$

$$y^{(1)} = \frac{1}{5}[7 - (1.625) - 0]$$

$$\boxed{= 1.075}$$

$$z^{(1)} = \frac{1}{6}[9 - 2(1.625) - (1.075)]$$

$$z^{(1)} = 0.77916$$

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IInd stage

$$x^{(2)} = \frac{1}{8}[13 - 3(1.075) - 2(0.77916)]$$

$$\boxed{= 1.027085}$$

$$y^{(2)} = \frac{1}{5}[7 - (1.027085) - 0.77916]$$

$$\boxed{= 1.038751}$$

$$z^{(2)} = \frac{1}{6}[9 - 2(1.027085) - (1.038151)]$$

$$\boxed{= 0.98451}$$

IIIrd stage

$$x^{(3)} = \frac{1}{8}[13 - 3(1.038151) - 2(0.98451)]$$

$$\boxed{= 0.98934}$$

$$y^{(3)} = \frac{1}{5}[7 - 0.98934 - 0.98451]$$

$$\boxed{= 1.00523}$$

$$z^{(3)} = \frac{1}{6}[9 - 2(0.98934) - (1.00523)]$$

$$\boxed{= 1.00268}$$

$\underline{\underline{\text{IV}^{th} \text{ stage}}}$

$$x^{(4)} = \frac{1}{8}[13-3(1.00523)-2(1.00268)]$$

$$\boxed{= 0.99736} \approx 1$$

$$y^{(4)} = \frac{1}{5}[7-0.99736-1.00268]$$

$$\boxed{= 0.999992} \approx 1$$

$$z^{(4)} = \frac{1}{6}[9-2(0.99736)-0.999992]$$

$$\boxed{= 1.00088} \approx 1$$

$\therefore$ $x = 1$, $y = 1$, $z = 1$

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Linear Transformation:

* Linear transformation in 2D is represented by

$$y_1 = a_1x_1 + a_2x_2$$
$$y_2 = b_1x_1 + b_2x_2.$$

This can be represented in matrix form as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$Y = A \quad X$$

III^{ly} in 3D, system of eqn will be

$$y_1 = a_1x_1 + a_2x_2 + a_3x_3$$
$$y_2 = b_1x_1 + b_2x_2 + b_3x_3$$
$$y_3 = c_1x_1 + c_2x_2 + c_3x_3$$

This can be represented in matrix form as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where, A, Y, X are associated matrices & A is called transformation matrix.

Further if A is non-singular ($|A| \neq 0$), then $Y = AX$ is called non-singular transformat^n^ or regular transformation.

$\boxed{X = A^{-1}Y}$ is called inverse transformat^n^

NOTE: If $|A| = 0$, transformat^n^ $Y = AX$ is called singular transformat^n^.

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Find inverse transformat?

1.) $y_1 = 5x_1 + 3x_2 - 3x_3$
$y_2 = 3x_1 + 2x_2 - 2x_3$
$y_3 = 2x_1 - x_2 + 2x_3$

$\Rightarrow$ The LT of given sym.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 & 3 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$Y = AX$

$\rightarrow$ Inv transformation $X = A^{-1}Y$.

$$A = \begin{pmatrix} 5 & 3 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & -3 & 0 \\ -10 & 16 & 1 \\ -7 & 11 & 1 \end{pmatrix}$$

$\therefore$ $X = A^{-1}Y$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 0 \\ -10 & 16 & 1 \\ -7 & 11 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$\therefore x_1 = 2y_1 - 3y_2$
$x_2 = -10y_1 + 16y_2 + y_3$
$x_3 = -7y_1 + 11y_2 + y_3$.

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2.) $y_1 = 0.25x_1 - 0.50x_2$
$y_2 = 1.50x_1 - 1x_2$

$\rightarrow$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.50 \\ 1.5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$Y = AX$

$\Rightarrow$ Inv trans $X = A^{-1}Y$.

$$A = \begin{pmatrix} 0.25 & 0.5 \\ 1.5 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} -2 & 1 \\ -3 & 0.5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 0.5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$X = A^{-1}Y$

$\therefore x_1 = -2y_1 + y_2$
$x_2 = -3y_1 + 0.5y_2$

3.) $y_1 = 2x_1 + x_2 + x_3$
$y_2 = x_1 + x_2 + x_3$
$y_3 = x_1 - 2x_3$

$\rightarrow$

$Y = AX$, $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Inv tr $\Rightarrow X = A^{-1}Y$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1.5 & 2.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{pmatrix}$$

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$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1.5 & 2.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$\therefore x_1 = y_1 - y_2$

$x_2 = -1.5y_1 + 2.5y_2 + 0.5y_3$

$x_3 = 0.5y_1 - 0.5y_2 - 0.5y_3$

---

4) $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = 2x_1 + 4x_2 + 11x_3$

$y_3 = -x_2 + 2x_3$.

ST it is regular LT. Hence find inverse of LT

$\Rightarrow$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$Y = AX$

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{pmatrix} \qquad |A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{vmatrix}$$

$|A| = (11-10) + (4-4)$

$|A| = 1$. $\therefore$ it is regular Transformation.

Inverse transform $\rightarrow X = A^{-1}Y$

$$A^{-1} = \begin{pmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$\therefore x_1 = 19y_1 - 9y_2 + 2y_3$

$$X = \begin{pmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_2 = -4y_1 + 2y_2 - y_3$

$x_3 = -2y_1 + y_2$

5.) $\alpha = x\cos\theta - y\sin\theta$; $\beta = x\sin\theta + y\cos\theta$

$A^{-1} = A'$.

write inverse format

$\Rightarrow$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$Y = AX$

$$A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \qquad A^{-1} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$A' = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$X = A^{-1}Y$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$x = \alpha\cos\theta + \beta\sin\theta$

$y = -\alpha\sin\theta + \beta\cos\theta$

Eighen Values & Eighen Vectors of Sq^{re} Matrix :

Given sq. mat A, there exists a scalar λ (real or complex) & a non zero column matrix, such that ~~A X X~~ $\boxed{AX = \lambda X}$ then λ is called Eighen value of A & X is called Eighen vector of A corresponding to Eighen value of λ.

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Find all the eigen values & corresponding Eigen vectors

1.) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow$

Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Character Eqn $\Rightarrow |A - \lambda I| = 0$

$\therefore \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \longrightarrow$ (I)

$(5-\lambda)(2-\lambda) - 4 = 0$

$\therefore \lambda^2 - 7\lambda + 6 = 0$

$\therefore \lambda = 1, 6$

CASE 1: For $\lambda = 1$, (I) becomes

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$\therefore 4x + 4y = 0$

$x + y = 0$

Let arb const $y = k_1$

$\therefore y = k_1$

$\therefore x = -k_1$

$$\therefore X_1 = \begin{pmatrix} -k_1 \\ k_1 \end{pmatrix}$$

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CASE 2: $\lambda = 6$ - (I) becomes

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{aligned} -x + 4y &= 0 \\ x - 4y &= 0 \end{aligned}\right\} \text{Eqns are identical}$$

$\therefore$ we take arb const (θ):

Let $y = K_1$ $\therefore X_2 = \begin{pmatrix} 4K_1 \\ K_1 \end{pmatrix}$

$\therefore x = 4K_1$.

$\therefore \lambda = 1, 6$ are eighen values of

X_1, X_2 " corresponding Eighen vectors of A

2.) $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$

$\rightarrow$

Let $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$

char Eqn $\Rightarrow |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0 \longrightarrow (1)$$

$$\Rightarrow (-1-\lambda)(4-\lambda) + 6 = 0$$

$$\Rightarrow -4 + \lambda - 4\lambda + \lambda^2 + 6 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 2, 1.$$

Mat Eqn is $(A - \lambda I)X = 0$

CASE 1: For $\lambda = 1$, 1 becomes.

$$\begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$-2x + 3y = 0$$
$$-2x + 3y = 0.$$

Let $y = k_1$ $\quad \therefore X_1 = \begin{pmatrix} 3k_1/2 \\ k_1 \end{pmatrix}.$

$\therefore x = \frac{3k_1}{2}$

CASE 2: For $\lambda = 2$ 1 becomes

$$\begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

$$\left.\begin{array}{l} -3x + 3y = 0 \\ -2x + 2y = 0 \end{array}\right\} \text{identical.}$$

$+x - y = 0$.?
$-2x + 2y = 0$
$2x = 0$

Let $y = k_1$
$\therefore x = k_1$

$$\therefore X_2 = \begin{pmatrix} k_1 \\ k_1 \end{pmatrix}.$$

$\therefore \lambda = 1, 2$ are eighen values &
X_1, X_2 are corresponding eighen vectors of A.

3.) $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

$\Rightarrow$

Char Eqn is $|A - \lambda I|$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \longrightarrow (P)$$

$\Rightarrow (8-\lambda)(7-\lambda)(3-\lambda) + 6(-18+6\lambda) + 2(24 - 14 + 2\lambda) = 0$

$\Rightarrow 56 - 15\lambda + \lambda^2 (3-\lambda) - 108 + 36\lambda + 20 + 4\lambda = 0$

$\Rightarrow 168 - 45\lambda + 3\lambda^2 - 56 + 15\lambda - \lambda^3 - 108 + 36\lambda + 20 + 4\lambda = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda = 15, 3, 0$

Mat Eqn is $(A - \lambda I)X = 0$

$$\begin{pmatrix} 8-6-\lambda & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\uparrow$ A(1)

CASE 1: For $\lambda = 0$, A becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$8x - 6y + 2z = 0$

$-6x + 7y - 4z = 0$

$2x - 4y + 3z = 0$

$$\frac{x}{24-14} = \frac{-y}{-32+12} = \frac{z}{56-36}$$

$$\frac{x}{1\cancel{0}} = \frac{-y}{-2\cancel{0}} = \frac{z}{2\cancel{0}}$$

$\lambda = 0$

$x = 1\cancel{0}$

$y = +2\cancel{0}$

$z = 2\cancel{0}$

$\therefore X_1 = (1, 2, 2)' = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

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CASE 2: For $\lambda = 3$.

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y - z = 0$$

$$\frac{x}{24-8} = \frac{-y}{-20+12} = \frac{z}{20-36}.$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad \text{for } \lambda = 3$$

CASE 3: For $\lambda = 15$.

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-7x - 6y + 2z = 0$$
$$-6x - 8y - 4z = 0$$
$$2x - 4y - 12z = 0$$

$$\frac{x}{24+16} = \frac{-y}{28+12} = \frac{z}{56-36}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

$\therefore$ $\lambda = 0, 15, 3$ are Eighen values & X_1, X_2, X_3 are Eighen vectors

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4.) $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

$\Rightarrow$ $A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

Char Eqn $\Rightarrow$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0$$

$(7-\lambda)[30 - 11\lambda + \lambda^2 - 4] + 2[-10 + 2\lambda] = 0$

$182 - 77\lambda + 7\lambda^2 - 30\lambda + 11\lambda^2 - \lambda^3 + 4\lambda - 20 + 4\lambda = 0$

$\Rightarrow 162 - 99\lambda + 18\lambda^2 - \lambda^3 = 0$

$\therefore \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$\lambda = 3, 9, 6$

Matrix Eqn : $(A - \lambda I)X = 0$

$$\begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \rightarrow (1)$$

CASE 1: For $\lambda = 3$, (1) becomes-

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

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4) $4x - 2y + 0z = 0$ $\qquad \dfrac{x}{4} = \dfrac{-y}{-8} = \dfrac{z}{12-4}$

$-2x + 3y - 2z = 0$

$0 - 2y + 2z = 0$

$\therefore x = 1, 2, 2$ (y, z) $\qquad X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

CASE 2: For $\lambda = 6$ ① becomes

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x - 2y + 0 = 0$ $\qquad \dfrac{x}{4} = \dfrac{-y}{-2} = \dfrac{z}{-4}$

$-2x + 0 - 2z = 0$

$0 - 2y - z = 0$

$$\therefore X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

CASE 3: For $\lambda = 9$.

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x - 2y + 0 = 0$ $\qquad \dfrac{x}{4} = \dfrac{-y}{4} = \dfrac{z}{6-4}$

$-2x - 3y - 2z = 0$

$0 - 2y - 4z = 0$

$\therefore X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $\qquad \therefore$ Eigen values $= 3, 6, 9$

Eigen vectors $= X_1, X_2, X_3$

(For ease, take numerically largest value as common)

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Determine largest Eighen value & corresponding Eighen vector of matrices by Ray Leighs Power Method & (POWER METHOD).

1.] $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ with $X_0 = [1, 0]'$

$\Rightarrow$ $Y^{(1)} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0.2 \end{pmatrix}$

$Y' = \lambda' Y_1$

$Y^{(2)} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 5.8 \\ 1.4 \end{pmatrix} = 5.8\begin{pmatrix} 1 \\ 0.24 \end{pmatrix} = \lambda^2 Y_2$

$Y^{(3)} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0.24 \end{pmatrix} = \begin{pmatrix} 5.96 \\ 1.48 \end{pmatrix} = 5.96\begin{pmatrix} 1 \\ 0.248 \end{pmatrix}$

$= \lambda^3 Y_3$

$Y^{(4)} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0.248 \end{pmatrix} = \begin{pmatrix} 5.992 \\ 1.496 \end{pmatrix} = 5.992\begin{pmatrix} 1 \\ 0.249 \end{pmatrix}$

$= \lambda^4 Y_4$

$\Rightarrow$ Largest eighen value $5.992 \approx 6$

Corresponding eighen ~~value~~ vector is $\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$

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2) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ $x_0 = [1 \ 0 \ 0]'$

Carry 5 iterations.

$\Longrightarrow$

$$Y^1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \lambda^1 Y_1$$

$$Y^2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2 \\ +0.5 \end{pmatrix} = 2.5\begin{pmatrix} 1 \\ -0.8 \\ 0.20 \end{pmatrix} = \lambda^2 Y_2$$

$$Y^3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 2.8 \\ -2.8 \\ 1.2 \end{pmatrix} = 2.8\begin{pmatrix} 1 \\ -1 \\ 0.428 \end{pmatrix} = \lambda^3 Y_3$$

$$Y^4 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0.428 \end{pmatrix} = \begin{pmatrix} 3 \\ -3.428 \\ 1.856 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1.1426 \\ 0.6186 \end{pmatrix} = \lambda^4 Y_4$$

$$Y^5 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\cancel{\begin{pmatrix} 0.8574 \\ 0.6666 \\ 0.0946 \end{pmatrix}}\begin{pmatrix} 1 \\ 1.14261 \\ 0.6186 \end{pmatrix} = \begin{pmatrix} 0.8574 \\ 0.6666 \\ 0.0946 \end{pmatrix}$$

$$= 0.8574\begin{pmatrix} 1 \\ 0.7774 \\ 0.1103 \end{pmatrix} = \lambda^5 Y_5.$$

$\therefore$ Largest eigen value $= 0.8574$

" " vector $= \begin{pmatrix} 1 \\ 0.7774 \\ 0.1103 \end{pmatrix}$

If X_0 isn't given, take any 1x3 matrix except 0 matrix.

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3.) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

⇒

Let $X_0 = [1 \ \ 1 \ \ 1]'$

$$Y^1 = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 0 \\ 0.666 \end{pmatrix} = \lambda_1 Y^1$$

$$Y^2 = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ .666 \end{pmatrix} = \begin{pmatrix} 7.2 \\ -2.6 \\ 3.8 \end{pmatrix} = 7.2\begin{pmatrix} 1 \\ -0.3611 \\ 0.527 \end{pmatrix} = \lambda_2 Y^2$$

$$Y^3 = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3611 \\ 0.527 \end{pmatrix} = \begin{pmatrix} 7.7762 \\ -3.6103 \\ 3.9421 \end{pmatrix} = 7.7762\begin{pmatrix} 1 \\ -0.4642 \\ 0.5069 \end{pmatrix} = \lambda^3 Y_3$$

$$Y^4 = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.4642 \\ 0.5069 \end{pmatrix} = \begin{pmatrix} 7.9422 \\ -3.8995 \\ 3.9849 \end{pmatrix} = 7.9422\begin{pmatrix} 1 \\ -0.490 \\ 0.501 \end{pmatrix} = \lambda^4 Y_4$$

$$Y^5 = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.490 \\ 0.501 \end{pmatrix} = \begin{pmatrix} 7.982 \\ -3.971 \\ 3.993 \end{pmatrix} = 7.982\begin{pmatrix} 1 \\ -0.497 \\ 0.500 \end{pmatrix}$$

∴ Highest eighen value $= 7.98 \approx 8$

Highest " vector $= \begin{pmatrix} 1 \\ -0.497 \\ 0.500 \end{pmatrix} \approx \begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$

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4) $\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}$ $\lambda = 6$ $x = [1 \; 1 \; -1]$

5.) $\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ $\lambda = 25.182$ $x = [1 \; 0.045 \; 0.068]$

Diagonalization.

→ Property : If A is □re mat of order n, having n linearly independent eighen vector, then there exists an n^{th} order □re mat P such that $P^{-1}AP$ is a diagonal matrix. i.e $\boxed{P^{-1}AP = D}$ where P is modal matrix or spectral matrix & D is diagonal mat.

→ Computat^n of powers of □re matrix.

- Diagonalisat^n of □re matrix A also helps us to find the powers of A, such as $A^3, A^4, \ldots$ etc.
$A^n = PD^nP^{-1}$

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

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Reduce the follo mat to Diagonal form

1.) $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$ find A^4.

$\Rightarrow$

Charecteristic eqn : $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$-4 - 4\lambda + \lambda + \lambda^2 + 6 = 0$

$\lambda^2 - 3\lambda + 2 = 0.$

$\lambda = 1, 2.$

Matrix Eqn $\Rightarrow (A - \lambda I)X = 0$

$$\begin{pmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \longrightarrow (1)$$

<u>CASE 1</u>. $\lambda = 1$. (1) becomes

$$\begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-2x + 3y = 0$

$-2x + 3y = 0$

$2x = 3y$

$\frac{x}{3} = \frac{y}{2}.$

$\therefore x = 3, \; y = 2$

$X_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

CASE 2 : $\lambda = 2$, (1) becomes.

$$\begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-3x + 3y = 0$

$-2x + 2y = 0$

$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$2x = 2y$

$\frac{x}{2} = \frac{y}{2}$

$x = 1, \; y = 1$

$\therefore \lambda_1 = 1, \quad \lambda_2 = 2$

$X_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The Modal Matrix is : $P = [X_1 \; X_2]$

$\therefore P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$|P| = 1$, since $|P| \neq 0$

$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

$\therefore P^{-1}AP = D$.

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore \cancel{D^4 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}}$

$A^n = P D^n P^{-1}$

$A^4 = P D^4 P^{-1}$

$D^4 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

$\therefore A^4 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$

2.) $A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$ $A^5 = ?$

$\Rightarrow$ $|A - \lambda I| = 0$

$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0$

$(11-\lambda)[12+\lambda+2\lambda+\lambda^2-20] + 4[-56-7\lambda+50] - 7[-28+20+10\lambda]$

$132 + 11\lambda + 22\lambda + 11\lambda^2 - 220 - 12\lambda - \lambda^2 - 2\lambda^2 - \lambda^3 - 20\lambda + 24 + 28\lambda$
$+ 196 - 140 - 70\lambda = 0$

$\cancel{-8 - 41\lambda + 8\lambda^2 - \lambda^3} = 0$

$\cancel{\lambda^3 - 8\lambda^2 + 41\lambda + 8} = 0$

$\lambda = 0, 1, 2$

Mat Eqn = $(A - \lambda I)X = 0$

$$\begin{pmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

CASE 1: $\lambda = 0$

$$\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$11x - 4y - 7z = 0$$
$$7x - 2y - 5z = 0$$
$$10x - 4y - 6z = 0$$

$$\frac{x}{20-14} = \frac{-y}{-55+49} = \frac{z}{-22+28}$$

$\therefore x = 6, \quad y = 6, \quad z = +6$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

CASE 2: $\lambda = 1$.

$$\begin{pmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$10x - 4y - 7z = 0$$
$$7x - 3y - 5z = 0$$
$$10x - 4y - 7z = 0$$

$$\frac{x}{20-21} = \frac{-y}{-50+49} = \frac{z}{-30+28}$$

$$X_2 = \begin{bmatrix} -1 \\ +1 \\ -2 \end{bmatrix}$$

CASE 2: $\lambda = 2$

$$\begin{pmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$9x - 4y - 7z = 0$$
$$7x - 4y - 5z = 0$$
$$10x - 4y - 8z = 0$$

$$\frac{x}{20-28} = \frac{-y}{-45+49} = \frac{z}{-36+28}$$

$$\frac{x}{-8} = \frac{-y}{-4} = \frac{z}{-8}$$

$$X_3 = \begin{bmatrix} +2 \\ +1 \\ +2 \end{bmatrix}$$

$$\therefore P = [X_1 \; X_2 \; X_3]$$

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$$P = \begin{pmatrix} 1 & -1 & +2 \\ 1 & 1 & 1 \\ 1 & -2 & +2 \end{pmatrix} \qquad \cancel{P^{-1} = \begin{pmatrix} 0 & 0.6666 & 0.3333 \\ 1 & 0 & -1 \\ -1 & & \end{pmatrix}}$$

$$|P| = 0 + 1(-2-1) - 2(-2-1)$$
$$= -3 + 6$$
$$|P| = 3 \neq 0.$$

$$P^{-1} = \begin{pmatrix} -4 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & -2 \end{pmatrix}$$

$$P^{-1} A P = D.$$

$$\begin{pmatrix} -4 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{pmatrix} = D$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$D^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \qquad A^5 = P D^5 P^{-1}.$$

$$\therefore A^5 = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} -4 & 2 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 191 & -64 & -127 \\ 97 & -32 & -65 \\ 190 & -64 & -126 \end{pmatrix}$$

3)
$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$\Rightarrow$

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[5-6\lambda+\lambda^2] - (-2-\lambda) + 3(-14+3\lambda) = 0$$
$$5-6\lambda+\lambda^2-5\lambda+6\lambda^2-\lambda^3+2+\lambda-42+9\lambda = 0$$
$$-35-\lambda+7\lambda^2-\lambda^3 = 0$$
$$\lambda^3-7\lambda^2 \cancel{+\lambda} +35 = 0$$

$\lambda = -2, 3, 6$

Mat Eqn

$$(A-\lambda I)x = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

CASE 1: $\lambda = -2$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$3x + y + 3z = 0$$
$$x + 7y + z = 0$$
$$3x + y + 3z = 0$$

$$\frac{x}{1-21} = \frac{-y}{3-3} = \frac{z}{21-1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$x = -1, \ y = 0, \ z = 1$

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CASE 2: $\lambda = 3$.

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-5 \quad -15 = -5.$

$$-2x + y + 3z = 0$$
$$x + 2y + z = 0$$
$$3x + y - 2z = 0$$

$$\frac{x}{1-6} = \frac{-y}{-2-3} = \frac{z}{-4-1}.$$

$$X_2 = \begin{bmatrix} +1 \\ -1 \\ 1 \end{bmatrix}$$

$x = -1, \quad y = 1, \quad z = 1.$

CASE 3, $\lambda = 6$

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-5x + y + 3z = 0$$
$$x - y + z = 0$$
$$3x + y - 5z = 0$$

$$\frac{x}{1+3} = \frac{-y}{-5-3} = \frac{z}{5-1}.$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$= 4 \quad = 8 = 4$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -0.5 & 0 & 0.5 \\ 0.3333 & -0.333 & 0.3333 \\ 0.1666 & 0.3333 & 0.1666 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

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Nothin but Diagonalization

4.) Find spectral matrix & modal matrix.

$\Rightarrow$

1.) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ 2.) $A = \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & 4 & 1 \end{pmatrix}$

3.) $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ 4.) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{pmatrix}$ $A^5 = ?$

$A^4 = ?$

Quadratic form

→ A homogeneous expression of 2^{nd} degree in any number of variables (Q.F)

→ For a given $\square^{re}$ mat of order $\boxed{2\times 2}$ Quad form (QF) will be

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \text{coeff of } x_1x_2 \\ \text{coeff of } x_1x_2 & \text{coeff of } x_2^2 \end{bmatrix}$$

→ $|||^{ly}$ for $\boxed{3\times 3}$

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \frac{1}{2}\text{coeff of } x_1x_2 & \frac{1}{2}\text{coeff of } x_1x_3 \\ \frac{1}{2}\text{coeff } x_1x_2 & \text{coeff of } x_2^2 & \frac{1}{2}\text{coeff of } x_2x_3 \\ \frac{1}{2}\text{coeff } x_1x_3 & \frac{1}{2}\text{coeff of } x_2x_3 & \text{coeff of } x_3^2 \end{bmatrix}$$

NOTE : $P^{-1} = P'$ $(D = P^{-1}AP)$.

Reduction of quadratic form to canonical form

If $B = P'AP$ is diagonal matrix, the transformed QF $= Y'BY$ is sum of square terms known as canonical form.

$$\boxed{d_1y_1^2 + d_2y_2^2 + d_3y_3^2 + d_4y_4^2 + \dots + d_ny_n^2}$$

Nature of quadratic form.

1.) Rank
→ Rank of B or A is called rank of QF.

2.) Index (P)
→ No. of +ve terms in canonical form of QF is known as index.

3.) Signature
→ Diff b/w no. of +ve terms & -ve terms is known as signature

4.) If R is rank, P is index of QF in n variables the nature of QF is identified by follo table
→

Condt^n	Nature of QF	Cannonical form
* $r=n, p=n$	+ve definite	$y_1^2 + y_2^2 + \dots + y_n^2$
* $r=n, p=0$	-ve "	$-y_1^2 - y_2^2 - \dots - y_n^2$
* $r=p, p<n$	+ve semi "	$y_1^2 + \dots + y_r^2$
* $r<n, p=0$	-ve " "	$-y_1^2 - y_2^2 - \dots - y_r^2$

else indefinite

$(a-b)(a^2+b^2-2ab)$ $a^3+ab^2-2a^2b-ba^2-b^3+2ab^2$ $a^3-b^3+3ab^2-3a^2b$ $(a-b)^3$ $\Rightarrow 3C_0a^3-3C_1a^2b$ $a^3-3a^2b+3ab^2-b^3$

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Reduce the following QF into canonical form by orthogonal transformation.

1.) $2x^2+2y^2+2z^2+2xz$

$\rightarrow$ $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ is sym mat A of given Q.F.

Char eqn: $|A-\lambda I| = 0$

$-\lambda^3 + 6\lambda^2 - 11\lambda - 10$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)^3 - 0 + 1(-2+\lambda) = 0$

$4+3+h$

$= \cancel{8+\lambda^3-3(2)(\lambda)(2-\lambda)} = 0$

$\Rightarrow \cancel{8+\lambda^3-12\lambda+6\lambda} = 0$

$\cancel{\lambda^3-6\lambda+8} = 0$

$8-3(4)(\lambda)+3(2)(\lambda^2)-\lambda^3-2+\lambda = 0$

$8-12\lambda+6\lambda^2-\lambda^3-2+\lambda = 0$

$-\lambda^3-11\lambda+6\lambda^2+6 = 0$

$\lambda^3+11\lambda-6\lambda^2-6 = 0$

$2(4)+1(2)$

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$\lambda = 1, 2, 3$

Char Mat $\rightarrow$ $(A-\lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\boxed{\text{Canonical form: } x^2+2y^2+3z^2}$

CASE 1 $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\|X_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

$x + z = 0$

$y = 0$

$x + z = 0$

$x = -z$

$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \therefore X_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

CASE 2: $\lambda = 2$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

CASE 3: $\lambda = 3$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-x + z = 0$

$-y = 0$

$x - z = 0$

$\frac{x}{1} = \frac{z}{1} = \frac{y}{0}$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \therefore X_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

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$$P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$ Modal Matrix.

Reduce the follo QF into sum of sq. of orthogonal transformatn & give the matrix of transformatn

2) $x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 + 6x_1x_3 + 12x_2x_3$

$\Rightarrow$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$$

Char eqn $\rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[4-5\lambda+\lambda^2] - 2(2-2\lambda-18) + 3(12-12+3\lambda) = 0$

$4-5\lambda+\lambda^2 - \cancel{4\lambda} + 5\lambda^2 - \lambda^3 + \cancel{4\lambda} + 32 + 9\lambda = 0$

$36 + 4\lambda + 6\lambda^2 - \lambda^3 = 0$

$\lambda^3 - 6\lambda^2 - 40\lambda = \cancel{36} = 0$

$\lambda = -4, 0, 10.$

Matrix Eqn : $(A - \lambda I)X = 0$

$$\sim \begin{pmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

CASE 1. $\lambda = -4$

$$\begin{cases} 5x + 2y + 3z = 0 \\ 2x + 8y + 6z = 0 \\ 3x + 6y + 5x = 0 \end{cases}$$

$$\frac{x}{12-24} = \frac{-y}{30-6} = \frac{z}{40-4}.$$

$$-12 = 24 = 36.$$

$$x_1 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \qquad ||x_1|| = \sqrt{1+4+9} = \sqrt{14}$$

$$x_1 = \begin{bmatrix} -1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

CASE 2: $\lambda = 0$

$$x + 2y + 3z = 0$$
$$2x + 4y + 6z = 0$$
$$3x + 6y + z = 0$$

$$\frac{x}{4-36} = \frac{-y}{2-18} = \frac{z}{12-12}.$$

$$\Rightarrow -32 = -16 = 0$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad ||x_1|| = \sqrt{4+1+0} = \sqrt{5}$$

$$\therefore \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

CASE 3: $\lambda = 10$

$$-9x + 2y + 3z = 0$$
$$2x - 6y + 6z = 0$$
$$3x + 6y - 9z = 0$$

$$\frac{x}{12+18} = \frac{-y}{-54-6} = \frac{z}{54-4}.$$

$$\frac{x}{30} = \frac{y}{60} = \frac{z}{50}$$

$X_3 = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}$ $\quad \|X_3\| = \sqrt{9+36+25}$ $\quad X_3 = \begin{bmatrix} 3/\sqrt{70} \\ 6/\sqrt{70} \\ 5/\sqrt{70} \end{bmatrix}$

Canonical form: $\boxed{-4y_1^2 + 10y_3^2}$

$$P = \begin{bmatrix} -1/\sqrt{14} & -2\sqrt{5} & 3\sqrt{70} \\ -2\sqrt{14} & 1/\sqrt{5} & 6\sqrt{70} \\ 3\sqrt{14} & 0 & 5\sqrt{70} \end{bmatrix}$$

The corresponding orthogonal transformation

$X = PY$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -1/\sqrt{14} & -2/\sqrt{5} & 3\sqrt{70} \\ -2\sqrt{14} & 1/\sqrt{5} & 6\sqrt{70} \\ 3\sqrt{14} & 0 & 5\sqrt{70} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = -1/\sqrt{14}\,y_1 - 2\sqrt{5}\,y_2 + 3\sqrt{70}\,y_3$$
$$x_2 = -2\sqrt{14}\,y_1 + 1/\sqrt{5}\,y_2 + 6\sqrt{70}\,y_3$$
$$x_3 = 3\sqrt{14}\,y_1 + 5\sqrt{70}\,y_3$$

Discuss nature

3.) $x^2 + 5y^2 + z^2 + 2yz + 6zx + 2xy$

$\Rightarrow$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Ch. Eqn $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[5-6\lambda+\lambda^2] - (1-\lambda-3) + 3(1-15+3\lambda) = 0$

$5 - 6\lambda + \lambda^2 - 1 + \lambda + 3 - 42 + 9\lambda - 5\lambda + 6\lambda^2 - \lambda^3 = 0$

$\cancel{-35 + 4\lambda - \lambda^3 = 0}$

$\cancel{\lambda^3 - 4\lambda + 55 = 0}$

$3\cancel{5 + 7\lambda^2 - \lambda^3 = 0}$ $\qquad \lambda^3 - 7\lambda^2 + 36 = 0.$

$\lambda = -2, 3, 6$

Matrix Eqn $\Rightarrow (A - \lambda I)(X) = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

CASE : 1 $\quad \lambda = -2.$

$3x + y + 3z = 0$

$x + 7y + z = 0$

$3x + y + 3x = 0$

$$\frac{x}{1-7} = \frac{-y}{3-1} = \frac{z}{21-1}$$

$-8 = 2 = 20$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow X_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

CASE 2: $\lambda = 3$

$-2x + y + 3z = 0$
$x + 2y + z = 0$
$3x + y - 2x = 0$

$$\frac{x}{1-6} = \frac{-y}{-2-3} = \frac{z}{-4-1}$$

$X_1 = \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$ $\|X_2\| = \sqrt{1+50}$ $\Rightarrow X_2 =$

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1.) $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$

Nature

2.) ST QF is indefinite : $x^2 + 2y^2 - 7z^2 - 4xy + 8yz$.

3.) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$

4.) $3x^2 - 2y^2 - z^2 + 12yz + 8zx - 4xy$

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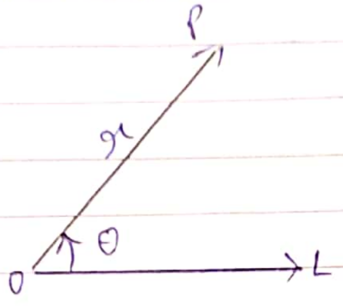
UNIT I : Differential Calculus

* Infinite series.
* Convergence & Divergence of a series & eg.
* Oscillation, +ve term series, alternating series.
* Leibnitz test, Ratio test
* Taylor's thearem.
* Maclawrin's thearem
* Angle b/w polar curves.

Angle b/w polar curves

→ **Polar Coordinates:**



* Initial reference O in the plane is called $\boxed{\text{pole}}$.
* $\overline{OL}$ is called initial line.
* $\angle LOP$ is called vectorial angle.
* The pair r & θ represented by $P = (r, \theta)$ or $p(r, \theta)$ are called polar coordinates of point P.

→ **Relationship b/w cartisian coordinates (x, y) & polar coordinates (r, θ)**

$$\boxed{x = r\cos\theta} \quad \boxed{y = r\sin\theta}$$

$$x^2 + y^2 = r^2$$

$$r = \sqrt{x^2 + y^2}$$

$$\boxed{\frac{y}{x} = \tan\theta}$$

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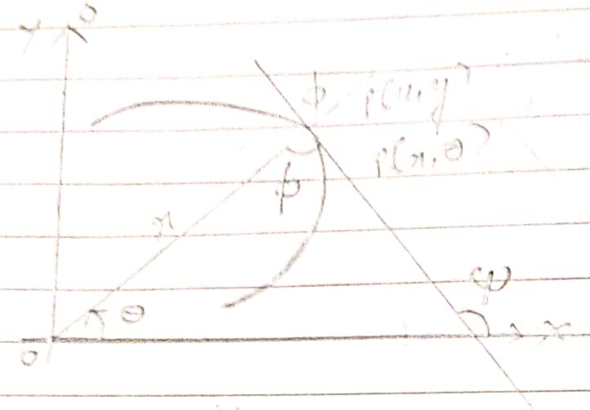
→ Angle b/w radius vector and tangent



Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

$\therefore \angle XOP = \theta$ & $OP = r$

from fig. $\psi = \theta + \phi$

$$\tan\psi = \tan(\phi + \theta)$$
$$= \frac{\tan\phi + \tan\theta}{1 - \tan\phi \cdot \tan\theta} \rightarrow (1)$$

Let (x, y) be the cartesian co-ordinates of 'P', so that

$$x = r\cos\theta, \quad y = r\sin\theta.$$

$\because r$ is funct^n of θ, we can set a parametric eqns interms of θ.

$\tan\psi = \frac{dy}{dx}$ = slope of tangent PL

$\tan\psi = \frac{dy/d\theta}{dx/d\theta}$ $\because$ x & y funct^n of θ

$$\tan\varphi = \frac{d/d\theta\,(r\sin\theta)}{d/d\theta\,(r\cos\theta)} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta} \qquad \left(r' = \frac{dr}{d\theta}\right)$$

Dividing N & D by $r'\cos\theta$.

$$\tan\varphi = \frac{\frac{r}{r'} + \tan\theta}{-\frac{r}{r'}\tan\theta + 1} \longrightarrow (2)$$

Comparing (1) & (2), we observe

$$\tan\phi = \frac{r}{r'} = \frac{r}{\frac{dr}{d\theta}} = \frac{r\,d\theta}{dr}$$

$$\therefore \boxed{\tan\phi = r\cdot\frac{d\theta}{dr}} \qquad \boxed{\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}}$$

Show that following pair of curves intersect each other orthogonally

1.) $r = a(1+\cos\theta)$ & $r = b(1-\cos\theta)$

$\Rightarrow$

$\longrightarrow$ $r = a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$ Now, d. w.r.t θ.

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta}$ (using half angle formulas)

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2}$.

$\cot\phi_1 = -\tan\theta/2$

$\therefore \cot\phi_1 = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\phi_2 = \frac{\pi}{2}+\frac{\theta}{2}$

$\longrightarrow$ $r = b(1-\cos\theta)$

$\log r = \log b + \log(1-\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi_2 = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$

$\cot\phi_2 = \cot(\theta/2)$

$\phi_2 = \frac{\theta}{2}$

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NOTE

1.) $|\phi_1 - \phi_2|$ used to find angle of intersect^n
2.) ($\psi = \phi + \theta$) used to find slope
3.) $\tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - \tan(\phi_2)}{1 + \tan\phi_1 \cdot \tan\phi_2}\right|$

Now, $|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right|$

$= \frac{\pi}{2}$.

$\therefore$ Angle of intersect^n is $\pi/2$.

2.) $r^n = a^n \cos\theta, \quad r^n = b^n \sin(n\theta)$

$\Rightarrow$

$\rightarrow$ $n\log r = n\log a + \log\cos\theta$.

$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{\cos\theta}$

$n\cot\phi = -n\tan\theta$.

$\cancel{n}\cot\phi = \cancel{n}\cot\left(\frac{\pi}{2} + n\theta\right)$

$\phi_1 = \frac{\pi}{2} + n\theta$

$\rightarrow$ $n\log r = n\log b + \log \sin(n\theta)$

$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = 0 + \frac{n\cos(n\theta)}{\sin(n\theta)}$

$\cot\phi = \cot n\phi$

$\phi_2 = n\phi$

$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + n\theta - n\theta\right| = \pi/2$

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3.) $r = ae^{\theta}$, $re^{\theta} = b$

$\Rightarrow$ $\log r = \log a + \theta \log_e e$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1$

$\cot \phi = \frac{\pi}{4}$

$\phi_1 = \frac{\pi}{4}$

$\longrightarrow$ $\log r + \theta \log_e e = \log b$

$\frac{1}{r}\frac{dr}{d\theta} + 1 = 0$

$\cot \phi = -1$

$\phi_2 = \frac{-\pi}{4}$

$|\phi_1 - \phi_2| = \left|\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right|$

$= \frac{\pi}{2}$

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Find the angle of intersection of follo pair of curves.

1) $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$

$\Rightarrow$ $\log r = \log(\sin\theta + \cos\theta)$.

$$\frac{1}{r}\frac{dr}{dt} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1-\tan\theta}{\tan\theta + 1}.$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1-\tan\theta}{1+\tan\theta}.$$

$$\cot\phi_1 = \cot\left(\frac{\pi}{4} + \theta\right) \Rightarrow \phi_1 = \frac{\pi}{4} + \theta.$$

$\Rightarrow$ $r = 2\sin\theta$

$\log r = \log 2 + \log(\sin\theta)$.

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{\sin\theta}$$

$\cot\phi_2 = \cot\theta$

$\phi_2 = \theta$.

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \theta - \theta\right| = \frac{\pi}{4}$$

2.) $r = a(1-\cos\theta)$, $r = 2a\cos\theta$

$\Rightarrow$ $\log r = \log a + \log(1-\cos\theta)$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$$

$$\cot\phi_1 = \frac{2\sin^2\theta/2 \cdot \cos\theta/2}{2\sin^2\theta/2}$$

$$\phi_1 = \frac{\theta}{2}$$

$\log r = \log 2a + \log\cos\theta$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{\cos\theta}$$

$\cot\phi_1 = -\tan\theta$.

$\cot\phi_1 = \cot\left(\frac{\pi}{2} + \theta\right)$.

$\cos 2\theta = 1 + \cos^2\theta/2$

Now $|\phi_1 - \phi_2| = \frac{\pi}{2} + \frac{\theta}{2}$

Now, $\cancel{4}(1-\cos\theta) = 2\cancel{4}\cos\theta$

$1 - \cos\theta = 2\cos\theta$

$\frac{1}{3} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{1}{3}\right)$

$\cancel{\frac{1}{3} = \frac{1+\cos\theta}{2}}$

$\frac{\theta}{2} = \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)$

$\cancel{\frac{1}{3} - 1 = \frac{\cos^2\theta}{2}}$

$\cancel{\frac{-2}{3} = 2\cos^2\theta}$

$\therefore |\phi_1 - \phi_2| = \frac{\pi}{2} + \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)$

$= \frac{\pi + \cos^{-1}(1/3)}{2}$

(ii) $r^n = a^n \sec(n\theta + \alpha)$, $r^n = b^n \sec(n\theta + \beta)$

$\Rightarrow$

$n \log r = n \log a + \sec(n\theta + \alpha)$

$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{\cancel{\sec(n\theta+\alpha)}.\tan(n\theta+\alpha)(n+0)}{\cancel{\sec(n\theta+\alpha)}}$

$\cot\phi_1 = \cancel{\sec(n\theta+\alpha)}.\tan(n\theta+\alpha).$

$\cot\phi_1 = \cot\left(\frac{\pi}{2} - (n\theta + \alpha)\right)$

$\phi_1 = \frac{\pi}{2} - (n\theta + \alpha)$

$1+\tan$

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$\Rightarrow$ IIIly $\phi_2 = \frac{\pi}{2} + n\theta + \beta$

$|\phi_1 - \phi_2| = \alpha - \beta$, $\alpha > \beta$

Q) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

$\Rightarrow$ $\log r = \log a\theta - \log(1+\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \log a + \log\theta - \frac{1}{1+\theta}$

$\cot\phi_1 = 0 + \frac{1}{\theta} - \frac{1}{1+\theta}$ $\theta = \tan\alpha$

$\phi_1 = \frac{1+\theta-\theta}{\theta(1+\theta)}$

$\cot\phi_1 = \frac{1}{\theta+\theta^2} = \cancel{\frac{1}{\tan\alpha + \tan^2\alpha}} = \cancel{\frac{\sec\alpha}{1+\tan\alpha}}$

$\tan\phi_1 = \theta + \theta^2$

$\Rightarrow$ $\cot\phi_2 = \log a - \log(1+\theta^2)$

$= 0 - \frac{2\theta}{1+\theta^2}$

$\cot\phi_2 = \frac{-2\theta}{1+\theta^2}$ $\Rightarrow \tan\phi_2 = -\left(\frac{1+\theta^2}{2\theta}\right)$

$\cancel{\phi_2 = \frac{2\tan\alpha}{1+\tan^2\alpha}}$

$\cancel{\phi_2 = \tan(2\alpha)}$

$\cancel{\phi_2 = \cot\left(\frac{\pi}{2} - 2\alpha\right)}$.

Comparing

$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$

$\theta + \theta^3 = 1 + \theta$

$\theta^3 = 1$

$\theta = 1$.

$|\phi_1 - \phi_2| =$

$\therefore \tan\phi_1 = 2$ $\tan\phi_2 = -1$ $\Rightarrow |\phi_1 - \phi_2| = \tan^{-1}(3)$

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HW: Find $\phi \Rightarrow$

i) $r^m = a^m(\cos m\theta + \sin m\theta)$

$\phi = \pi/4 + m\theta$

ii) $\frac{l}{r} = 1 + e\cos\theta$

$\phi = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$

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5) ST, for the curve, $r = ae^{\theta\cot\alpha}$ (α is const) the radius vector is incline at an constant angle to the tgt at every point.

$\Rightarrow$

diff wrt θ

$$\frac{dr}{d\theta} = ae^{\theta\cot\alpha}(\cot\alpha).$$

$$\frac{dr}{d\theta} = r\cot\alpha$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$$

$$\cot\phi_1 = \cot\alpha$$

$$\phi_1 = \alpha$$

6.) Find angle ϕ for polar curve, $r = a(1+\sin\theta)$ hence determine slope of the curve at $\theta = \pi/2$

$\Rightarrow$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{a\cos\theta}{1+\sin\theta}$$

$$\cot\phi = \frac{\cos\theta}{1+\sin\theta}$$

At, $\theta = \pi/2$

$$\cot\phi = 0$$

$$\phi = \cot^{-1}(0)$$

$$\phi = \frac{\pi}{2}$$

Now Slope $\Rightarrow$ $\psi = \theta + \phi$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\psi = \pi$$

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H/W: Find angle of intersection

i) $r = a(1+\cos\theta)$, $r^2 = a^2\cos 2\theta$ $\Rightarrow \theta = \cos^{-1}$ [illegible]

ii) $r = a\log\theta$, $r = \frac{a}{\log\theta}$ $\Rightarrow \theta = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$

iii) $r = \frac{a}{1+\cos\theta}$, $r = b/1-\cos\theta$

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Infinite Series

→ If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be an infinite sequence of real numbers, then $u_1+u_2+\cdots+u_n+\cdots\infty$ is called infinite series.

→ An ∞ series is denoted by $\sum_1^\infty u_n$ & sum of 1st n terms is denoted by S_n.

Convergence, Divergence, Oscillations

Consider the ∞ series $\sum_1^\infty u_n = u_1+u_2+\cdots+u_n+\cdots\infty$

Let sum of 1st n terms be $S_n = u_1+u_2+\cdots+u_n$.

Clearly, S_n is function of n & n inc indefinitely. 3 possibilities arise

→ If S_n tends finite limit as $n\to\infty$, the series $\sum_1^\infty u_n$ is said to be convergent.

→ If S_n tends to $\pm\infty$, as $n\to\infty$, the series $\sum_1^\infty u_n$ is said to be divergent.

→ If S_n does not tend to unique limit as $n\to\infty$, then the series $\sum_1^\infty u_n$ is said to be oscillatory or (non-convergent)

Positive Term Series

An ∞ series in which all terms after some particular terms are +ve is a +ve term series.

Eg: $-7+-5+-2+2+7+13+\cdots$

* NOTE: A series of +ve terms either converges or diverges to $+\infty$, for the sum of 1st n terms, omitting the -ve terms, tends to either a finite limit or $+\infty$

HW: Orthogonality

i) $r = a\,cosec^2(\theta/2)$

ii) $r = 4\sec^2(\theta/2)$

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Alternating Series

A series in which the terms are alternately +ve / −ve is called alternating series

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Test convergence with help of P-series & limit form

Note: Necessary condn for convergence

If +ve term series $\sum U_n$ = convergent

$\lim_{n\to\infty} U_n = 0$, if $\lim_{n\to\infty} U_n \neq 0$ then series $\sum U_n$ diverge

---

i) $\frac{1}{1\times2\times3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots \infty$

$\Rightarrow$ $1\cdot3\cdot5 \cdots = (2n-1)$

$1\cdot2\cdot3\cdot4 \cdots = n$

$2\cdot3\cdot4 \cdots = n+1$

$3\cdot4\cdot5 \cdots = n+2$

$$\sum U_n = \frac{2n-1}{n(n+1)(n+3)}$$

$$\sum U_n = \frac{n\,(2-1/n)}{n^3(1+1/n)(1+2/n)}$$

$$= \frac{1}{n^2}\frac{(2-1/n)}{(1+1/n)(1+4/n)}$$

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1.) Let $v_n = 1/n^2$

$$\therefore \frac{1}{n^2} \frac{[2-1/n] \cdot n^2}{[1+1/n][1+2/n]}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{(2-1/n)}{(1+1/n)(1+2/n)}$$

$$= 2 > 1$$

$\therefore$ Series is convergent.

2.) $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \dots + \infty$

$\Rightarrow \frac{1}{(n^2)^{2/3}}$ is nth term

$$\therefore \sum_{1} u_n = \frac{1}{n^{4/3}}$$

comparing with std series, $p = 4/3 > 1$

$\therefore$ it is convergent.

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Infinite Series

$u_1 - u_2 + u_3 - u_4 + \cdots$

converges if each term is numerically less than its preceeding term.

$\lim_{n\to\infty} u_n \to 0$,

if $\lim_{n\to\infty} u_n \neq 0$, then series is oscillatory

* p-series test! P-Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \cdots + \frac{1}{n^p} + \cdots \infty$$

* Convergent $p > 1$
* Divergent $p < 1$

3.) $\sum_{n=1}^{\infty} \left[\sqrt{n^2+1} - n\right]$

$\Rightarrow$

$\cancel{n\sqrt{1+\frac{1}{n^2}}} - n$ $\qquad \dfrac{(\sqrt{n^2+1} - n)\times(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n}$

$= \cancel{n}\left[\cancel{\sqrt{1+\frac{1}{n^2}}} - \cancel{1}\right]$ $\qquad = \dfrac{n^2+1-n^2}{n\sqrt{1+n^2}}$

$=$ $\qquad = \dfrac{1}{\sqrt{1+n^2}+n}$

$= \cancel{1}$ $\qquad = \dfrac{1}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]}$

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3.) $v_n = \frac{1}{n}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{n} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} \frac{n}{1}$

$= \frac{1}{\sqrt{1}+1}$

$= \frac{1}{2}$

$= \frac{1}{2} < 1$

$\therefore$ Divergent.

---

4) $\frac{1}{4\cdot 7\cdot 10} + \frac{4}{7\cdot 10\cdot 13} + \frac{9}{10\cdot 13\cdot 16} + \cdots + \infty$

$\Rightarrow$ $1\cdot 4\cdot 9 = n^2$ $\quad$ $a+(n-1)d$

$4\cdot 7\cdot 10 = 3n+1$ $\quad$ $4+(n-1)3 \Rightarrow 4+3n-3$

$7\cdot 10\cdot 13 = 3n+4.$ $\quad$ $7+(n-1)3 = 7+3n-3$

$10\cdot 13\cdot 16 = 3n+7.$ $\quad$ $10+(n-1)3 = 10+3n-3$

$\therefore \sum u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$

$\sum u_n = \frac{n^{\cancel{2}}}{n\left[3+\frac{1}{n}\right]\left[3+\frac{4}{n}\right]\left[3+\frac{7}{n}\right]}$

$v_n = \frac{n}{\cancel{n}}.$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{n}{n\left[3+\frac{1}{n}\right]\left[3+\frac{4}{n}\right]\left[\frac{3}{n}\right]} = \frac{1}{27}.$

$\therefore p = \frac{1}{27} < 1$

$\therefore$ Divergent

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3.) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n \rightarrow$ cgt 4.) $\frac{x}{1\cdot 3} + \frac{x^2}{3\cdot 5} + \frac{x^3}{5\cdot 7} + \cdots$

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1.)

Ratio test (or) (De alembert's test)

In the +ve term series, $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lambda$

Then series converges for $\lambda > 1$.
Diverges for $\lambda < 1$.
Fails for $\lambda = 1$.

1.) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots + \infty$

$\Rightarrow$ $\sum U_n = \frac{n!}{n^n}$ $\sum U_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{n!}{n^n} \frac{(n+1)^{(n+1)}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{\cancel{n!}}{n^n} \frac{(n+1)^{n+1}}{(n+1)\cancel{(n)!}}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \frac{e}{1}$$

$\lambda = 1/e$
$= 1/e < 1$

~~Converge~~ ~~Divergent~~ Convergent

Test convergence:

1.) $\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2} \rightarrow cgt$ 2.) $\sum \frac{n^3+a}{2^n+a}$

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3.) $\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \dots + \infty$

$\Rightarrow$

NOTE: * $x^{n+1} \rightarrow 0$ as $n \rightarrow \infty$ if $x<1$
* $\frac{1}{x^{n+1}} \rightarrow 0$ as $n \rightarrow \infty$ if $x>1$

$\sum U_n = \frac{x^n}{1+x^n}$, $\sum U_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$\lim_{n \to \infty} \frac{\sum U_n}{\sum U_{n+1}} = \lim_{n \to \infty} \left(\frac{x^n}{1+x^n} \cdot \frac{1+x^{n+1}}{x^{n+1}} \right)$

$\Rightarrow \lim_{n \to \infty} \left[\frac{x^n \cdot 1+x^{n+1}}{(1+x^n)\, x^n \cdot x} \right]$

$\Rightarrow \lim_{n \to \infty} \left[\frac{1+x^{n+1}}{x+x^{n+1}} \right]$

$= \frac{1+0}{x+0}$ if $x<1$

$\lambda = \frac{1}{x}$ $\cancel{\text{con}}$

$\therefore$ Series is convergent.

Also, $\lim_{n \to \infty} \frac{U_n}{U_{n+1}} = \frac{1+x^{n+1}}{x+x^{n+1}} = \frac{1/x^{n+1} + 1}{\frac{x}{x^{n+1}} + 1} = 1$

Dvgt
$(x>1)$

For $x=1$, dgt $\because \frac{1}{2} < 1$

$\therefore$ By ratio test ~~Σ~~ ~~ca~~ $\sum$ converges for $x<1$ & diverges for $x \geq 1$

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5) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \dots$

oscillatory

6) $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \dots \rightarrow$

gf

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3) $1 + \frac{x}{2} + \frac{x^2}{5} + \dots + \frac{x^n}{n^2+1} \dots \infty$

$\Rightarrow \quad U_n = \frac{x^n}{n^2+1} \qquad U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

$$\lim_{n\to\infty} \frac{\cancel{x^n}}{n^2+1} \times \frac{[(n+1)^2+1]}{x \cdot \cancel{x^n}}$$

$$\lim_{n\to\infty} \frac{(n^2+2+2n)}{(n^2+1)x}$$

$$\lim_{n\to\infty} \frac{2(1+n)+n^2}{(n^2+1)x}$$

$$\lim_{n\to\infty} \frac{1}{x} \frac{[2(1+n)+n^2]}{[n^2+1]}$$

$\Rightarrow \frac{1}{x}$ if $x < 1$ $\therefore$ converges

$\Rightarrow \frac{1}{x}$ if $x > 1$ diverges

For $x = 1$, $\sum U_n = \frac{x^n}{n^2+1}$

$$\lim_{n\to\infty} \frac{x}{n^2+1} = 0$$

$\therefore$

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7.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ dgt

8.) $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$ .. dgt

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Alternating series.

Leibnitz series test.

1.) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \dots$

$\Rightarrow$ $U_n - U_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} < 0.$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

(The terms of given series are alternatively +ve & -ve,)

{Each term is numerically less than its preceding terms}

~~There~~ Hence by leibnitz test, given series is convergent.

2.) $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \dots$

$\Rightarrow$ (The terms of given seri).

{ }.

$U_n = \frac{2n+3}{2n}$ $\quad U_{n-1} = \frac{2n+1}{2n-2}$

$\therefore U_n - U_{n-1} = \frac{(2n+3)}{2n} - \frac{2n+1}{2n-2}$

$\Rightarrow \frac{4n^2+6n-4n-6-4n^2-2n}{4n^2-4n}$

$U_n - U_{n-1} = \frac{-3}{2(n^2-n)} < 0.$

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2.) for $n>1$

$U_n < U_{n-1}$ for $n>1$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{2n+3}{2n}$$

$$\lim_{n\to\infty} \frac{2+\frac{3}{n}}{2} = \frac{2}{2}$$

$= 1 \ (\neq 0)$ $\quad \sum U_n \neq 0$

Series is oscillatory.

3.) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots$

$\Rightarrow$ ( ) { }

$U_n = \frac{1}{\log(1+n)}$ $\quad U_{n-1} = \frac{1}{\log(n)}$

$$U_n - U_{n-1} = \frac{1}{\log(n+1)} - \frac{1}{\log n}$$

$= -\log(n+1) + \log n$

$= \log(n) - \log(n+1)$

$= \log\left(\frac{n}{n+1}\right)$

Taylors and Maclaurin's series

Taylors and Macllaurians theorem for a function of one variable

→ **Taylors series.**

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \ldots$$

OR

$$y(x) = y(a) + \frac{(x-a)}{1!}y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) \ldots$$

→ **Maclaurins series**

It is particular case of taylors series, i.e in taylors series expansion, if we put $a=0$, the series obtained is Maclaurins series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

OR

$$y(x) = y(0) + x y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots$$

$\log_e x \longrightarrow \ln(x)$

$\log_{10} x \longrightarrow \log_{10}(x)$

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1.) Expand $f(x) = \log_e x$ in series of powers of $(x-1)$ & hence evaluate $\log_e 1.1$ correct to 4 decimal places

$\rightarrow$ We have Taylor's expansion at point $a = 1$.

$$y(x) = y(a) + y'(a)(x-a) + y''(a)\frac{(x-a)^2}{2!} + y'''(a)\frac{(x-a)^3}{3!} + \dots$$

$$① \longrightarrow y(x) = y(1) + y'(1)\frac{(x-1)}{1} + y''(1)\frac{(x-1)^2}{2!} + y'''(1)\frac{(x-1)^3}{3!} + y^{iv}\frac{(x-1)^4}{4!}$$

$y(x) = \log_e x \longrightarrow y(1) = 0$

$y'(x) = \frac{1}{x} \longrightarrow y'(1) = 1$

$y''(x) = -\frac{1}{x^2} \longrightarrow y''(1) = -1$

$y'''(x) = \frac{2}{x^3} \longrightarrow y'''(1) = 2$

$y^{iv}(x) = -\frac{6}{x^4} \longrightarrow y^{iv}(1) = -6$

$$\therefore y(x) = 0 + (x-1) + \frac{(-1)(x-1)^2}{2!} + \frac{(2)(x-1)^3}{3!} + \frac{(-6)(x-1)^4}{4!}$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \longrightarrow ②$$

$$\log_e(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4}$$

$$= 0.1 - \frac{0.01}{2} + \frac{(0.1)^3}{3} - \frac{0.0001}{4}$$

$= \cancel{0.095} = \cancel{0.06}$

$= \underline{0.0953}$

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2) $f(x) = \tan^{-1}(x)$ in power of $(x-1)$ upto term containing $(x-1)^3$

$\Rightarrow$ $f(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!}$

$f(x) = \tan^{-1}(x) \rightarrow f(1) = \pi/4$

$f'(x) = \frac{1}{1+x^2} \rightarrow f'(1) = \frac{1}{2}.$

$f''(x) = (1+x^2)^{-1}$

$= -1(1+x^2)^{-2}$

$= \frac{-2x}{(1+x^2)^2} \rightarrow f''(1) = \frac{-1}{2}$

$f'''(x) = \cancel{-2(1+x^2)^{-2}}$

$= \cancel{4(1+x^2)^{-3}(}$

$f'''(x) = \frac{\cancel{-2x(2x} + 2x[2(1+x^2)(2x)] - (1+x^2)(2)}{[(1+x^2)^2]^2} \rightarrow f'''(1) = \frac{16-4}{16} = \frac{1}{2}$

$\therefore f(x) = \frac{\pi}{4} + \frac{1(x-1)}{2} + \frac{(-1)}{2}\frac{(x-1)^2}{2} + \frac{1}{2}\frac{(x-1)^3}{3!}$

$\tan^{-1}(x) = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12}$

3) $f(x) = \log(\cos x)$, $x = \frac{\pi}{3}$, upto 4^{th} deg

$\Rightarrow$

$f(x) = \log(\cos x) \rightarrow f(\pi/3) = \log(1/2) = 0.3010$

$f'(x) = \frac{1(-\sin x)}{\cos x}$

$= -\tan x$ $\qquad f'(\pi/3) = -\sqrt{3}$

$f''(x) = -\sec^2 x$ $\qquad f''(\pi/3) = -4$

$f'''(x) = -2\sec^2 x \tan x$ $\qquad f'''(\pi/3) = -2(4)(\sqrt{3}) = -13.85$

$f^{iv}(x) = (-2\tan x)(2\sec x \cdot \sec x \tan x) + \sec^2 x(-2\sec^2 x)$

$= -4\sec^2 x \tan^2 x - 2\sec^4 x$

$= \cancel{-2\sec^2 x[2\tan^2 x - 1]} - 2\sec^2 x[2\tan^2 x - \sec^2 x]$

$= -2(4)[2(3) - 4]$

$= -8[6-4]$

$= \cancel{-16}\ -8$

$\therefore f(x) = 0.3010 - 1.732(x-1) - \frac{4(x-1)^2}{2!} - \frac{13.85(x-1)^3}{3!} - \frac{80(x-1)^4}{4!}$

$\log(\cos \pi/3) = \underline{\underline{0.3010}}$

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NOTE: Taylors theorem can also be written as

$$\left(f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots \right)$$

1.) Expand $\sin\left(\frac{\pi}{4}+h\right)$ in ascending powers of h upto h^4 and hence deduce an approximate value of $\sin 50^\circ$

$\Rightarrow$ Taylors thm can also be written as

$$\left(\qquad \right)$$

Given $a = \pi/4$.

$$f\left(\frac{\pi}{4}+h\right) = \sin\left(\frac{\pi}{4}+h\right)$$

$f(x) = \sin x$

$y = \sin x \longrightarrow f(\pi/4) = 0.7071$

$y' = \cos x \longrightarrow f'(\pi/4) = 0.7071$

$y'' = -\sin x \longrightarrow f''(\pi/4) = -0.7071$

$y''' = -\cos x \longrightarrow f'''(\pi/4) = -0.7071$

$y^{iv} = \sin x. \longrightarrow f^{iv}(\pi/4) = +0.7071$

$$\rightarrow \therefore f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f^{iv}(a)$$

$$f\left(\frac{\pi}{4}+h\right) = 0.7071 + h(0.7071) - \frac{h^2(0.7071)}{2} - \frac{h^3(0.7071)}{6} + \frac{h^4}{24} 0.7071$$

$$f\left(\frac{\pi}{4}+\frac{\pi}{36}\right) = \cancel{0.7071\left[1+5-\frac{25}{2}-\frac{125}{6}+\frac{625}{4}\right]}$$

$$= 0.7071\left[1 + (0.087) - \frac{(0.087)^2}{2} - \frac{(0.087)^3}{6} + \frac{(0.087)}{24}\right]$$

$$\sin 50^\circ = 0.765999 \approx 0.766$$

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$\frac{u}{v} = \frac{u'v - vu'}{v^2}$

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Maclaurins

1.) Expand a^x by using Maclaurian's series upto 4th deg term

$\Rightarrow$

$f(x) = a^x \cancel{\log a}$ $\rightarrow f(0) = \cancel{\log a} \; 1$

$f'(x) = \cancel{\log a}[a^x \log a]$ $\rightarrow f'(0) = (\log a)$

$f''(x) = (\log a)[a^x \log a]$ $\rightarrow f''(0) = (\log a)^2$

$f'''(x) = (\log a)^2[a^x \log a]$ $\rightarrow f'''(0) = (\log a)^3$

$f^{iv}(x) = (\log a)^3[a^x \log a]$ $\rightarrow f^{iv}(0) = (\log a)^4$

$y(x) = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \frac{x^4}{4!}(\log a)^5$

$y(0) = 1$

2.) $\tan^{-1}(x)$, 5th deg & hence ST $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} + \dots\right)$

$\Rightarrow$

$f(x) = \tan^{-1} x$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\qquad f''(0) = 0$

$f'''(x) = \frac{-2[(1+x^2)^2] - [2(1+x^2)(2x)](-2x)}{(1+x^2)^2}$

$= \frac{-2(1+x^2)^2 + [8x^2(1+x^2)]}{(1+x^2)^2}$

$= -2(\cancel{1+x^2})[(1+x^2) + 8x^2] / (1+x^2)^4$

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$f'''(x) = \frac{-2(1+9x^2)}{(1+x^2)}$ $f'''(0) = -2$

$f^{iv}(x) \Rightarrow \cancel{-2(0+18x) = (0+2x)(}$
$\frac{-2(0+18x)(1+x^2) + (0+2x)2(1+9x^2)}{(1+x^2)^2} \Rightarrow f^{iv}(0) = 0$
$= \frac{-36x(1+x^2) + 4x(1+9x^2)}{(1+x^2)^2} \Rightarrow \frac{-36x - \cancel{36x^3} + 4x + \cancel{36x^3}}{(1+x^2)^2}$

$f^{v}(x) \Rightarrow \frac{-32(1+x^2)^2 - [2(1+x^2)(2x)(-32x)]}{(1+x^4)^2}$

$f^{(v)}(0)$ $f^{v}(0) = +24$

$\tan^{-1}x = 1 + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0)$

$= \text{d}\pi +$

Given: $\pi = 4(1 - 1/3 + 1/5 + \dots)$
$\frac{\pi}{4} = (1 - 1/3 + 1/5 + \dots)$
$\tan^{-1}(1) = [1 - 1/3 + 1/5 + \dots]$

$\therefore \tan^{-1}(1) = \frac{\pi}{2}$

$\tan^{-1}(1) = 0 + 1 + 0 - \frac{2}{6} + 0 - \frac{24}{120}$

$\frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} + \dots\right)$

$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} + \dots\right)$

$\frac{vu' - u'v}{v^2}$

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3.) $e^{\tan^{-1}x}$, 5 deg

$\Rightarrow$

$\rightarrow$ $y(x) = e^{\tan^{-1}x}$ $\qquad y(0) = 1$

$\rightarrow$ $y'(x) = e^{\tan^{-1}x}\frac{1}{1+x^2} \Rightarrow \frac{y(x)}{1+x^2}$ $\qquad y'(0) = 1$

$\rightarrow$ $y''(x) = \frac{y(x)}{(1+x^2)}$

$= \frac{y(x)(2x) - y'(x)(1+x^2)}{(1+x^2)^2}$ $\qquad y''(0) = +1$

$2(1+x^2)\, y'''(x)(0+2x) = y'(x)(2) + (2x)y'(x) - y''(x)(1+x^2) - y'(x)(1+x^2)$

$y'''(x) = \frac{2y(x) + 2x(y'(x)) - y''(x)(1+x^2) - y'(x)(1+x^2)}{y''(x)\,2(1+x^2)(2x) +}$

$\rightarrow$

$y''(x)(1+x^2) = 2x(y(x)) - y'(x)(1+x^2)$

$y''(x)(0+2x) - y'''(x)(1+x^2) = 2y(x) + y'(x)(2x) - y''(x)(1+x^2) - 2x(y'(x))$

$-y'''(x) = \frac{2y(x) + y'(x)(2x)}{y''(x)} - \frac{y''(x)(1+x^2) - 2x\,y'(x)}{(2x)}$

$y'''(0) = \frac{2 + 0 - 1 - 0}{}$

$y'''(0) = -1$

$y^{iv}(0) = -7$

$y^{v}(0) = 5$

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$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \frac{x^5}{5!} y^{v}(0)$$

$$= 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \frac{5x^5}{5!}$$

4.) $\frac{x}{e^{x-1}}$ 4^{th} deg

$\Rightarrow$

$\longrightarrow$ $y(x) = \frac{x}{e^{x-1}}$ $y(0) = 0$

$\longrightarrow$ $y(x) \cdot e^{x-1} = x$

$y(x) \cdot e^{x-1} + y'(x) \cdot e^{x-1} = 1.$

$y(0) e^{0-1} + y'(0) \cdot e^{0-1} = 1$

$0 + \frac{y'(0)}{e} = 1$ $y'(0) = e$

$\longrightarrow$ $y'(x) e^{x-1} + e^{x-1} y(x) + y''(x) \cdot e^{x-1} + y'(x) e^{x-1} = 1.$

$1 + 0 + \frac{y''(0)}{e} + 0 = 1$

$y''(0) = -2e$

$\longrightarrow$ $y'(x) e^{x-1} + y''(x) e^{x-1} + e^{x-1} y'(x) + e^{x-1} y''(x) + y'(x) \cdot e^{x-1} + y'''(x) e^{x-1} + y'(x) e^{x-1} + y''(x) e^{x-1} = 1.$

$y'(x) + y''(x) + y'(x) + y''(x) + y''(x) + y'''(x) + y'(x) + y''(x) = \frac{1}{e^{x-1}}$

$\cancel{e} + \cancel{-e} + \cancel{e} - \cancel{e} - e + y'''(0) + \cancel{e} - e = e$

$y'''(0) = 3e$, $y^{iv}(0) = -4e$

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$\int \tan x\,dx = -\log|\cos x|$ $= \frac{1}{x} + \frac{\sin x}{\cos x} = \frac{1}{x}$

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$y(x) = 0 + ex - \frac{2ex^2}{2!} + \frac{3ex^3}{3!} - \frac{4ex^4}{4!}$

HW) Upto 4^{th} dy

T.S

1.) P.T $\tan^{-1}x = \tan^{-1}\left(\frac{\pi}{4}\right) + \frac{(x - \pi/4)}{1 + (\pi^2/16)} - \frac{\pi (x - \pi/4)^2}{4(1 + \pi^2/16)} + \ldots$

M.S

1.) $y = e^{x\cos x}$ 2.) $\log(1+\sin x)$ 3.) $\log(1+e^x)$

4.) $\tan x$ 5.) $e^{\sin x}$ 6.) e^{-ax}

7.) $\sin^{-1}x$ 8.) $\cos^{-1}x$

UNIT 2 : Partial Differentiation

* Definition and simple problems.
* Total differentiation.
* Partial differentiation of composite function

Partial Differentiation

⟹ Applications:

→ Heat conduction, electrostatics involving 2+ variables

⟹ Defn:

→ Differentiation of a function of many independent variables is partial diff.

Ex: Area of rectangle = $l \times b$
ie " depends on l & b

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[or] $[f_x]$ [or] $[f_y]$

Obtain Z_x & Z_y

1) $Z = \sin(x^2y^3)$

$\Rightarrow$

$Z_x = \cos(x^2y^3).(2x.y^3)$

$Z_y = \cos(x^2y^3)(3y^2x^2)$

2) $Z = \cos(x^3y^2)$

$\Rightarrow$

$Z_x = -\sin(x^3y^2)\, 3x^2y^2$

$Z_y = -\sin(x^3y^2)\, 2yx^3$

3) $Z = \tan(2x+3y) + \sec(3x+2y)$

$\Rightarrow$

$Z_x = \sec^2(2x+3y)(2) + \sec(3x+2y).\tan(3x+2y)(3)$

$Z_y = \sec^2(2x+3y)(3) + \sec(3x+2y).\tan(3x+2y)(2)$

Obtain Z_x & Z_y

1.) $Z = \phi(x^2y^3)$

$\Rightarrow$

$Z_x = \phi'(x^2y^3)(2xy^3)$

$Z_y = \phi'(x^2y^3)(3y^2x^2)$

2.) $Z = \varphi(x^3y^2)$

$\Rightarrow$

$Z_x = \varphi'(x^3y^2)\, 3x^2y^2$

$Z_y = \varphi'(x^3y^2)\, 2yx^3$

3.) $Z = f(2x+3y) + g(3x+2y)$

$\Rightarrow$

$Z_x = f'(2x+3y)2 + g'(3x+2y)3$

$Z_y = f'(2x+3y)3 + g'(3x+2y)2$

$ax^2 + 2gh x + 2fy + c$ $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

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4.) $z = \phi(2x+3y) + \psi(3x+3y)$

$\Rightarrow$

$z_x = \phi'(2x+3y)2 + \psi'(3x+3y)3$

$z_y = \phi'(2x+3y)3 + \psi'(3x+3y)3$

1.) $z = f(x+ct) + \phi(x-ct)$

PT $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$

$\Rightarrow$ $\dfrac{\partial z}{\partial t} = f'(x+ct)c + \phi'(x-ct)(-c)$

$\dfrac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + \phi'' c^2 (x-ct)$ ~~$+ \phi'(x-ct)t$~~

$\dfrac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$

$\dfrac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$

$\therefore \dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$

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H.W

1.) $z = \sin(x+ct) + \cos(x-ct)$

Verify $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

Symmetric function.

→ A function $f(x,y)$ is symmetric :

if $f(x,y) = f(y,x)$

→ |||ly $f(x,y,z)$ is symmetric :

if $f(x,y,z) = f(y,z,x) = f(z,x,y)$.

→ In general we say that the function of several variable is symm if funct remains unchanged, when the variables are cyclically rotated.

---

*.) Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

i.) $z = x^2 y - x\sin(xy)$

⇒ $\frac{\partial z}{\partial x} = 2xy - yx\cos(xy) - \sin(xy)$

$\frac{\partial z}{\partial y} = x^2 - x^2\cos(xy)$

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ii.) $z = \log(x^2+y^2)$

$\Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$$

Since ~~f(x,y)~~ z is symmetric

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

iii.) $\log_e z = x+y+z$

$z = e^{x+y+z}$

$$\frac{\partial z}{\partial x} = e^{x+y+z} \cancel{(y+z)} \left(1+\partial z/\partial x\right)$$

$$\frac{\partial z}{\partial y} = e^{x+y+z} \cancel{(x+\partial z/\partial x)} \left(1+\partial z/\partial y\right)$$

$\cancel{\partial z}$

iv.)

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*.) $\cancel{z = e^{ax+by}}$

*.) $z = e^{ax+by} \cdot f(ax - by)$

P.T $b \cdot \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$.

$\Rightarrow$ $\frac{\partial z}{\partial x} = e^{ax+by} \cdot f'(ax - by)(a) + e^{ax+by} a \cdot f(ax - by)$. $\to$ (1)

$\frac{\partial z}{\partial y} = e^{ax+by} \cdot f'(ax - by)(-b) + e^{ax+by} b \cdot f(ax - by)$. $\to$ (2)

Now $b \times$ (1) $+$ a (2) $\Rightarrow b\partial z/\partial x + a\partial z/\partial y$

$abe^{ax+by} f'(ax - by) + abe^{ax+by} f(ax - by) + (-ab)e^{ax+by} f'(ax-by)$
$+ ab\, e^{ax+by} f(ax - by)$

$= 2abe^{ax+by} f(ax - by)$

$= 2abz$

$=$ RHS.

Find z_x & z_y.

1.) $z = x^2y - x\cos(xy)$

2.) $z = \log(x^3 + y^3)$

3.) $z(x+y) = x^2 + y^2$.

ST. $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

13.) If $z = x^6 + y^6 + 4x^2y^2$ ST

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$\Rightarrow$ $\frac{\partial z}{\partial x} = 6x^5 + 2xy^2$

$\frac{\partial^2 z}{\partial y \partial x} = 0 + 4xy \rightarrow (1)$

$\Rightarrow$ $\frac{\partial z}{\partial y} = 6y^5 + 2x^2y$

$\frac{\partial^2 z}{\partial x \partial y} = 0 + 4xy \rightarrow (2)$

$(1) = (2)$ $\therefore$ LHS = RHS Hence proved.

14.) $u = \frac{y}{z} + \frac{z}{x}$ PT.

$x \cdot u_x + y u_y + z u_z = 0$

$\Rightarrow$ $u_x = 0 - \frac{z}{x^2}$ | $u_y = \frac{1}{z}$.

$x u_x = \frac{-z}{x} \rightarrow (1)$ | $y u_y = \frac{y}{z} \rightarrow (2)$

$u_z = \frac{-y}{z^2} + \frac{1}{x}$

$z u_z = -\frac{y}{z} + \frac{z}{x} \rightarrow (3)$

$\therefore (1) + (2) + (3)$

$\frac{-z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$

$= 0$

$=$ RHS

15.) If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$

$\Rightarrow$ Find $\dfrac{\partial^2 u}{\partial x \partial y}$

$\Rightarrow$ $\cancel{\dfrac{\partial u}{\partial y} = x^2 \cdot \dfrac{1}{\left(1+\frac{y^2}{x^2}\right)} \cdot \dfrac{1}{x} - y^2 \cdot \dfrac{1}{\left(1+\frac{x^2}{y^2}\right)} \cdot \left(\dfrac{-x}{y^2}\right)}$

$\cancel{= \dfrac{x^3}{x^2+y^2} + \dfrac{xy^2}{x^2+y^2}}$

$$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 2y \tan^{-1}(x/y) - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}(x/y) + \frac{xy^2}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^3(2x) - 3x^2(x^2+y^2)}{(x^2+y^2)^2} - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right) + \frac{xy^2(2x) - \cancel{(2x)(xy^2)}(x^2+y^2)(y^2)}{(x^2+y^2)^2}$$

$$= \frac{2x^4 - 3x^4 - 3x^2y^2 + 2x^2y^2 - y^3 - x^3}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)}$$

$$= \frac{-x^4 - x^2y^2 - x^3 - y^3 - 2x^2y^2 - 2y^4}{(x^2+y^2)^2}$$

$$= \frac{-(x^4 + 3x^2y^2 + x^3 + y^3 + 2y^4)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

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1)(a) HW: i) $z = \tan(y+ax) + (y-ax)^{3/2}$

PT $= \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

2) $u = e^{xyz}$. Find $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

$\rightarrow e^{xyz}(x^2y^2z^2 + 3yxz + 1)$.

16.) If $V = r^m$ & $r^2 = x^2+y^2+z^2$.

PT. $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$

$\Rightarrow$ Given $V = r^m$

$V_x = m r^{(m-1)} \frac{\partial r}{\partial x}$.

Consider. $r^2 = x^2+y^2+z^2$

Now

$2r \frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $|||^{ly}$ $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$\therefore V_x = m r^{(m-1)} \cdot \frac{x}{r}$

$\therefore V_x = x m r^{(m-2)}$

$\therefore V_{xx} = m r^{(m-2)} + x^2 m(m-2) r^{m-3} \rightarrow ①$

$|||^{ly}$ $V_{yy} = m r^{(m-2)} + r\, y^2/r \cdot m(m-2) r^{m-3} \rightarrow ②$

$V_{zz} = m r^{(m-2)} + z^2/r \cdot m(m-2) r^{m-3} \rightarrow ③$

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HW 1) $u = f(r)$. $r = \sqrt{x^2+y^2+z^2}$

ST. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$

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(1)+(2)+(3)

$3mr^{m-2} + \frac{x^2+y^2+z^2}{r} \cdot m(m-2)r^{m-3}$

$3mr^{m-2} + m(m-2)r^{m-2}$.

$3mr^{m-2} + m^2r^{m-2} - 2mr^{m-2}$

$= mr^{m-2} + m^2r^{m-2}$

$= m(1+m)r^{m-2}$

17.) Given $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$

P.T $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

$\Rightarrow$ $\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x}$

$1 = \frac{\partial r}{\partial x} \cdot \cos\theta$.

$\frac{\partial r}{\partial x} = \frac{1}{\cos\theta}$.

$\therefore \frac{\partial u}{\partial x} = \frac{f'(r)}{\cos\theta}$.

$\frac{\partial^2 u}{\partial x^2} = \frac{f''(r)}{\cos\theta^2}$.

$\frac{\partial u}{\partial y} = f'(r) \frac{\partial r}{\partial y}$

$1 = \frac{\partial r}{\partial y} \sin\theta$

$\frac{\partial u}{\partial y} = \frac{f'(r)}{\sin\theta}$

$\frac{\partial^2 \varphi}{\partial y^2} = \frac{f''(r)}{\sin^2\theta}$.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f''(r)}{\cos^2\theta} + \frac{f''(r)}{\sin^2\theta} \rightarrow r^2 \frac{f''(r)}{x^2} + \frac{f''(r)}{y^2} r^2$

$= (f''(r))$

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2.) $x = r\cos\theta$, $y = r\sin\theta$.

PT) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$.

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$\Rightarrow$ $x^2 + y^2 = r^2$

$$\frac{\partial v}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \text{III}^{ly} \ \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial v}{\partial x} = f'(r)\cdot\frac{x}{r}.$$

$$\frac{\partial^2 v}{\partial x^2} = f'(r)\frac{[r - x\,\partial r/\partial x]}{r^2} + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = f'(r)\left(\frac{r^2 - x^2}{r^3}\right) + \frac{x^2}{r^2}f''(r) \longrightarrow (1)$$

Since funct^n is symmetric

$$\frac{\partial^2 v}{\partial y^2} = f'(r)\left(\frac{r^2 - y^2}{r^3}\right) + \frac{y^2}{r^2}f''(r) \longrightarrow (2)$$

$$(1) + (2) \Rightarrow f'(r)\left[\frac{r^2 - x^2 + r^2 - y^2}{r^3}\right] + f''(r)\left[\frac{x^2 + y^2}{r^2}\right]$$

$$\Rightarrow \frac{f'(r)}{r} + f''(r)$$

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HW: 2.) $U = f(2x-3y, 3y-4z, 4z-2x)$

PT $\Rightarrow \frac{U_x}{2} + \frac{U_y}{3} = -\frac{U_z}{4}$

3.) $U = f(r,s,t)$, $r = x/y$, $s = y/z$, $t = z/x$

PT: $xU_x + yU_y + zU_z = 0$

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Differentiation of Composite funct^n

→ If $u = f(x,y)$; $x = x(r,s)$; $y = y(r,s)$, then

u is composite funct^n of 2 independent variables r & s.

→ Here we should be able to differentiate u wrt r & also wrt s partially, i.e

$$u \to (x,y) \to (r,s) \Rightarrow u \to (r,s) \begin{cases} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{cases}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \end{aligned}}$$

Q.) 1.) $u = x^2 + y^2$, $x = r^2 + s$, $y = s^2 + r$. Find $\frac{\partial u}{\partial r}$ & $\frac{\partial u}{\partial s}$ in terms of r & s.

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= 2x(2r) + (2y)(1)$$

$$= 4xr + 2y \Rightarrow 4(r^2+s)x + 2y$$

$$= 4r^3 + 4sr + 2s^2 + 2r$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2x(1) + 2y(2s)$$

$$= 2x + 4ys$$

$$= 2(r^2+s) + 4(s^2+r)s$$

$$= 2r^2 + 2s + 4s^3 + 4rs$$

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4) $x = u+v+w$, $y = uv+wu+wv$, $z = uvw$ & F is fun^n of x, y, z, then PT.

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z}$$

34.) $u = \sin x + \cos y$; $x = r+s$, $y = r-s$. $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}$.

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos x\,(1) + (-\sin y)(1)$$

$$= \cos x - \sin y$$

$$= \cos(r+s) - \sin(r-s)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \cos x(1) + (-\sin y)(-1)$$

$$= \cos x + \sin y$$

$$= \cos(r+s) + \sin(r-s)$$

35.) $u = x^2 + y^2 + z^2$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r$.

obtain $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$= 2x(\cos\theta) + 2y(\sin\theta) + 2z$$

$$= 2x\cos\theta + 2y\sin\theta + 2z.$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= 2x(r(-\sin\theta)) + 2y(r\cos\theta) + z(0)$$

$$= -2xr\sin\theta + 2yr\cos\theta$$

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36.) $u = f(r,s)$, $r = x + at$, $s = y + bt$ ST.

$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$ where x, y, t are independent

$\rightarrow$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t}$$

$$= \cancel{f'(r,s)\cdot(a) + f'(r,s)(b)}$$

$$\cancel{\frac{\partial u}{\partial t} = a f'(r,s) + b f'(r,s)}$$

$$\cancel{= a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}}$$

$u \rightarrow (r,s) \rightarrow (x,y,t) \Rightarrow u \rightarrow (x,y,t)$

* $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} \rightarrow (1)$$

$$= \frac{\partial u}{\partial r} + 0$$

* $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= 0 + \frac{\partial u}{\partial s} \rightarrow (2)$$

* $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial t}$$

$$= a\frac{\partial u}{\partial r} + b\frac{\partial u}{\partial s}$$

By (1) & (2) $\Rightarrow$ $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$

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11) If $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$

PT: $(u_x)^2 + (u_y)^2 = (u_r)^2 + \frac{(u_\theta)^2}{r^2}$

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57) $u = f(x-y, y-z, z-x)$. PT. $u_x + u_y + u_z = 0$

$\Rightarrow$ $\frac{\partial u}{\partial x}$ // Let $p = x - y$

$q = y - z$

$r = z - x$

$\therefore u = f(p, q, r)$

$u \to f(p, q, r) \to (x, y, z) \Rightarrow u \to (x, y, z)$

* $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$

$= \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$

$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \to (1)$

* $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$

$= \frac{\partial u}{\partial p}(-1) + 0 + \frac{\partial u}{\partial q} \to (2)$

IMP HW

Derivation

* Since they are symm. $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \to (3)$

$\therefore (1) + (2) + (3)$

$\frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$

$= 0$

$=$ RHS

Hence proved

Change the Laplacian eqn in cartesian, Change the Laplacian eqn in cartesian coordinates $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, into polar coordinates

Ans:

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

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21) If $z = u^2 + v^2$, $u = at^2$, $v = 2at$ $\frac{dz}{dt} = ?$

$\Rightarrow$ $z \to (u, v) \to (t)$,

$\Rightarrow$ $\therefore$ $\cancel{z = (at^2)^2 + (2at)^2}$, $z \to (t)$.

$\cancel{z = a^2t^4 + 4a^2t^2}$

$\cancel{\frac{dz}{dt} = 4a^2t^3 + 8a^2t}$.

$$\frac{\partial z}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

$$= 2u(2at) + 2v(2a)$$

$$= 4(at^2)at + 2(2at)2a$$

$$= 4a^2t^3 + 8a^2t.$$

---

22) If $u = x^3 + y^3$, $x = t^2$, $y = t^3$ $\frac{du}{dt}$

$\Rightarrow$ $u \to (x, y) \to (t)$

$u \to (t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{du}{dy}$$

$$= 3x^2(2t) + 3y^2(3t^2)$$

$$= 6t(t^2)^2 + 9t^2(t^3)^2$$

$$= 6t^5 + 9t^8$$

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23.) $u = \sin x + \cos y + \tan z$ $\quad x = e^t, \; y = \log t, \; z = t^4$

$\frac{du}{dt} = ?$

$\Rightarrow$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= \cos x\,(e^t) + \left(\frac{-\sin y}{t}\right) + \sec^2 z\,(4t^3)$$

$$= e^t \cos(e^t) - \frac{\sin(\log t)}{t} + 4\sec^2(t^4)\,t^3$$

24.) If $u = \tan^{-1}\left(\frac{y}{x}\right)$; $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

$\frac{du}{dt}$

$\Rightarrow$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)(e^t - e^{-t}) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)(e^t + e^{-t})$$

$$= \frac{-x^2 y}{x^2+y^2}\frac{[e^t - e^{-t}]}{x^2} + \frac{x^2}{x^2+y^2}\cdot\frac{e^t + e^{-t}}{(x)}$$

$$= \frac{-x^2 y[e^t - e^{-t}] + x[e^t + e^{-t}]}{x^2+y^2}$$

$$= \frac{-(e^t + e^{-t})(e^t - e^{-t})(e^t - e^{-t})(e^t - e^{-t}) + (e^t - e^{-t})(e^t + e^{-t})}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$$

$$= (e^t - e^{-t})^2 [(e^t + e^{-t})(e^t - e^{-t}) + \ldots]$$

$$-\frac{(e^{t}-e^{-t})^2\,[e^{2t}-e^{-2t}+1]}{2(e^{2t}-e^{-2t})}$$

$$= -\frac{[e^{2t}-e^{-2t}+2e^{t}\cdot e^{-t}][e^{2t}-e^{-2t}+1]}{2(e^{2t}-e^{-4t})}$$

$$= -\frac{[e^{2t}-e^{-4t}+2][e^{2t}-e^{-4t}+1]}{2(e^{2t}-e^{-4t})}$$

$$= -\frac{(e^{2t}-e^{-2t})[(e^{t}-e^{-t})^2+1]}{2(e^{2t}-e^{-2t})}$$

$$= -\frac{[e^{2t}-e^{-2t}+3]}{2}$$

$$= \frac{e^{-2t}-e^{2t}+3}{2}$$

$$= \frac{-y[e^{t}-e^{-t}]+x[e^{t}+e^{-t}]}{x^2+y^2}$$

$$= \frac{(e^{t}-e^{-t})(e^{t}-e^{-t})-(e^{t}+e^{-t})(e^{t}+e^{-t})}{(e^{t}-e^{-t})^2+(e^{t}+e^{-t})^2}$$

$$= \frac{e^{2t}+e^{-2t}-2-[e^{2t}-e^{-2t}+2]}{e^{2t}+e^{-4t}-2+e^{2t}+e^{-4t}+2}$$

$$= \frac{-4}{2(e^{4t}+e^{-4t})}$$

$$= \frac{-2}{(e^{2t}+e^{-4t})}$$

25) If $u = x^2 + y^2 + z^2$; $x = e^{2t}$, $y = e^{2t}\cos(3t)$, $z = e^{2t}\sin(2t)$
$du/dt = ?$. Verify by direct substitution

$\Rightarrow$ $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x(e^{2t})(2) + 2y[e^{2t}\cos(3t) + e^{2t}(-\sin(3t))3] + 2z[e^{2t}\sin(2t) + e^{2t}(\cos(2t)2)]$$

$$= 4xe^{2t} + 2ye^{2t}\cos(3t) - 3e^{2t}\sin(3t)2y + 2ze^{2t}\sin 2t + 4ze^{2t}\cos(2t)$$

$$= 4e^{4t} + 2e^{4t}\cos^2(3t) - 3e^{4t}\sin(3t)\cos(3t) + 2e^{4t}\sin^2 2t + 4e^{4t}\sin 2t\cdot\cos 2t \quad (1)$$

Consider

$\Rightarrow$ $u = (e^{2t})^2 + (e^{2t}\cdot\cos 3t)^2 + (e^{2t}\cdot\sin(2t))^2$, direct

$$\frac{du}{dt} = \frac{d}{dt}[e^{4t} + e^{4t}\cos^2 3t + e^{4t}\cdot\sin^2 3t]$$

$\Rightarrow 4e^{4t} + e^{4t}\cos^2 3t + \cancel{2e^{4t}\cos(3t)}\ 2e^{4t}\cos(3t)(-\sin 3t)(3) + 4e^{4t}\sin^2 3t + 2e^{4t}\sin^2 3t\cdot\cos 3t(3) \quad (2)$

$(1) = (2)$

$\therefore$ Verified ✔

Hw] i) $z = xy^2 + x^2y$; $x = at^2$, $y = 2at$ $\Rightarrow$ ($dz/dt = ?$) & Verify

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26.) Find $\frac{du}{dt}$, if $u = y^2 - 4ax$, $x = at^2$, $y = 2at$.

$\Rightarrow$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= -4a(2at) + 2y(2a)$$

$$= -8a^2t + 4a^2t$$

$$= \underline{-4a^2t}$$

27.) At a given instant, the sides of rectangle are 4 ft & 3 ft respectively, & increasing at rate of 1.5 ft/s & 0.5 ft respectively. Apply Total derivative to find the rate at which area is increasing at that instant.

$\Rightarrow$

Area of $\square^{le}$ $(A) = x \times y$

Given:

$x = 4ft$, $y = 3ft$

$\frac{dx}{dt} = 1.5 ft/s$, $\frac{dy}{dt} = 0.5 ft/s$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial A}{\partial y}\cdot\frac{dy}{dt} \Rightarrow y(1.5) + x(0.5)$$

$$3(1.5) + 4(0.5)$$

$$= \underline{6.5 \text{ sq. ft}}$$

$$\cancel{= y\left[y + \frac{dy}{dt}\right] + x\left[x + \frac{dx}{dt}\right]}$$

$$\cancel{= 3[3 + 0.5] + 4[4 + 1.5]}$$

$$\cancel{= 3(3.5) + 4(5.5)}$$

$=$

Implicit Funct^{ns}

$$\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

28.) If $x^6 + y^6 + 5x^2y^2 = 8$. Find dy/dx using implicit funct^n

$\rightarrow$

Let $f = x^6 + y^6 + 5x^2y^2 - 8$

$$\frac{\partial f}{\partial x} = 6x^5 + 10xy^2$$

$$\frac{\partial f}{\partial y} = 6y^5 + 10x^2y$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{6x^5 + 10xy^2}{6y^5 + 10yx^2}$$

$$= -\frac{3x^5 + 5xy^2}{3y^5 + 5yx^2}$$

29) If $\sin^{-1}x + \sin^{-1}y = 100$, Find $\frac{dy}{dx}$

$\Rightarrow$ $f = \sin^{-1}x + \sin^{-1}y - 100$

$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-x^2}}$

$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-y^2}}$ $\quad \frac{dy}{dx} = -\left(\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right)$

30) If $z = \sin x + \cos y$, $x^2+y^2 = 4$
Find: $\cancel{\frac{dy}{dx}}$ $\frac{dz}{dx} = ?$

$\Rightarrow$ $\frac{\partial z}{\partial x} = \cos x$, $\frac{\partial z}{\partial y} = -\sin x$.

$\frac{dy}{dx} = \cancel{\left(\frac{\cos y}{\sin x}\right)}$ $\quad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$
$= \cancel{\cot x}$

$\frac{dz}{dx} = \cos x - \sin x \cdot \frac{dy}{dx}$.

$\cancel{\cos x + \sin y \frac{dy}{dx} = \cos x - \sin x \frac{dy}{dx}}$.

$\cancel{2\sin y \frac{dy}{dx} = 2\cos x}$.

$\cancel{\frac{dy}{dx} = 2\cot x}$

Consider $x^2 + y^2 = 4$

d.w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\therefore \frac{dz}{dx} = \cos x + \sin y\left(\frac{x}{y}\right)$$

2.) If $u = x\log(xy)$ where $x^4 + y^4 + 4xy = 1$

Find $\frac{du}{dx}$

$\Rightarrow$ $u = x[\log x + \log y]$

$u = x\log x + x\log y$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cancel{+} \frac{dy}{dx}$$

$$= \left(\frac{x}{x} + \log xy\right) + \left(\frac{x}{y}\right) + \frac{dy}{dx}$$

$$= 1 + \log xy + \frac{x}{y} + \frac{dy}{dx}$$

Consider: $x^4 + y^4 + 4xy - 1 = 0$ d.w.r.t x

$$\frac{dy}{dx} : 4x^3 + 4y^3\frac{dy}{dx} + 4y + 4x\frac{dy}{dx} = 0$$

$\frac{dy}{dx} = \cancel{4x^3} + \cancel{(4y)} \quad x^3 + y + \frac{dy}{dx}(y^3+x) = 0$

$\frac{dy}{dx} = -\frac{(x^3+y)}{(y^3+x)}$

$\therefore \frac{du}{dx} = 1 + \log xy + \frac{x}{y}\left(-\frac{x^3+y}{y^3+x}\right)$

HW

32.) If x inc at rate of 2cm/s, at the instant when $x = 3$cm & $y = 1$cm, at what rate must y be changing in order that the funct^n $2x - 3xy^2$ shall be neither inc or dec. ANS: $\frac{32}{21}$ cm/s.

$\Rightarrow$

$\frac{dx}{dt} \quad \frac{dy}{dt}$

33.) Same sum as 31.)... $x^3 + y^3 - 3axy = 1$. Find $\frac{du}{dx}$

ANS: $1 + \log xy + \frac{x[x^2 - ay]}{y[ax - y]}$

UNIT : 3

Applications of Partial Diffrentiation

* Jacobians.
  > Def^n.
  > Problems.

* Properties & problems

* Tangent plane and Normal line

* Extreme values of functions of 2 variables.

* Lagranges method of undetermined multipliers.

Jacobians :

→ The application of Jacobians is significant in evaluat^n of double integrals of the form: $\iint f(x,y)\,dx\,dy$ & triple integrals of form $\iiint f(x,y,z)\,dx\,dy\,dz$.

→ If u & v are functions of 2 independent variables, x & y, then

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$ is called Jacobian of u & v, wrt x & y.

and is written as $\dfrac{\partial(u,v)}{\partial(x,y)}$ or $J\dfrac{(u,v)}{(x,y)}$

→ III^{ly} Jacobian of u, v, w wrt x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

→ **Properties of Jacobians :**

i) If $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ & $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ then $\boxed{J \cdot J' = 1}$

ii) If u & v are functions of r & s, & r & s are funct^{ns} of x & y. Then, $\boxed{\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \cdot \dfrac{\partial(r,s)}{\partial(x,y)}}$ is called

~~Chain~~ of chain rule of Jacobians

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iii.) If $J=0$, then we say that the given funct^n u & v are not functionally independent

---

1.] If $u = x(1-y)$, $v = xy$ Find Jacobian $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$ Also verify $J \cdot J' = 1$.

$\Rightarrow$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = (1-y), \quad \frac{\partial u}{\partial y} = -x, \quad \frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$$

$$J = x - xy + xy$$
$$J = x$$

$\Rightarrow$ Now, for J', we have to obtain x & y in terms of u & v.

Consider $u = x(1-y)$, $v = xy$

$$u + v = x - xy + xy$$
$$\therefore x = u + v.$$

$$\& \quad y = \frac{v}{u+v}$$

$\frac{x}{\cos(\tan^{-1}(\ldots))}$

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$\therefore J' = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = \frac{-v}{(u+v)^2}, \quad \frac{\partial y}{\partial v} = \frac{u}{(v+u)^2}$

$\therefore J' = \begin{vmatrix} 1 & 1 \\ -v/(u+v)^2 & u/(u+v)^2 \end{vmatrix}$

$= \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$

$= \frac{1}{(u+v)}$

$= \frac{1}{x}$

$\therefore J \cdot J' = x \cdot \frac{1}{x}$ Verified.

$\therefore J J' = 1$

2.) If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $J = J\frac{(x,y)}{(r,\theta)}$, $J' = J\frac{(r,\theta)}{(x,y)}$

$\Rightarrow$ $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$J = r$

$\Rightarrow$ $r = \frac{x}{\cos\theta}, \quad r = \frac{y}{\sin\theta}$.

$\frac{x}{\cos\theta} = \frac{y}{\sin\theta}$.

$\tan\theta = \frac{y}{x} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad r = \sqrt{x^2+y^2}$

$r = \sqrt{x^2+y^2}$

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$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \partial r/\partial x & \partial r/y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} \qquad \frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$J' = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$J' = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$J' = \frac{1}{(x^2+y^2)^{1/2}}$$

$$J' = \frac{1}{r}$$

$$\therefore J' \cdot J = 1$$

Verified

$vu' - uv'$

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3.) $u = \tan^{-1}x + \tan^{-1}y$, $v = \frac{x+y}{1-xy}$ PT u & v are not functionally independent

$\Rightarrow$ $$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$\frac{\partial v}{\partial x} = \frac{(1-xy) - (x+y)(-x)}{(1-xy)^2} = \frac{1-xy+x^2+xy}{1-xy} = \frac{1+x^2}{(1-xy)^2}$$

$$= \cancel{\frac{1}{1+x^2}} \quad \cancel{\frac{y}{1+y^2}}$$

$$\frac{\partial v}{\partial y} = \frac{(1-xy) - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$J = \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix} \Rightarrow \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$\cancel{\frac{1+y^2 - 1 - x^2}{(1-xy)^2}}$$

$$\cancel{J = \frac{y^2 - x^2}{(1-xy)^2}} \quad \therefore J = 0$$

Hence proved

Since $J = 0$, it follows that u & v are not functionally independent.

Also, $u = \tan^{-1}\frac{x+y}{1-xy} = \tan u$, $\tan^{-1}v = u$.

4) If $x = a\cosh(\alpha)\cdot\cos\beta$, $y = a\sinh(\alpha)\sin\beta$
Find $\frac{J(\alpha,\beta)}{(x,y)}$

$\Rightarrow$

NOTE (i) $\cosh^2\alpha + \sinh^2\alpha = \cosh 2\alpha$
(ii) $\cosh^2\alpha - \sinh^2\alpha = 1$

$\cancel{\frac{J(x,\beta)}{(x,\beta)} = \begin{vmatrix} \partial\alpha/\partial x & \partial\alpha/\partial y \\ \partial\beta/\partial x & \partial\beta/\partial y \end{vmatrix}}$

$$\frac{J(x,y)}{(\alpha,\beta)} = \begin{vmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{vmatrix}$$

$\Rightarrow \frac{\partial x}{\partial\alpha} = a\sinh(\alpha)\cos\beta$

$\frac{\partial x}{\partial\beta} = -a\cosh(\alpha)\sin\beta$

$\frac{\partial y}{\partial\alpha} = a\cosh(\alpha)\cdot\sin\beta$

$\frac{\partial y}{\partial\beta} = a\sinh(\alpha)\cos\beta$

$$\therefore J = \begin{vmatrix} a\sinh(\alpha)\cos\beta & -a\cosh(\alpha)\sin\beta \\ a\cosh(\alpha)\sin\beta & a\sinh(\alpha)\cos\beta \end{vmatrix}$$

$= a^2\sinh^2(\alpha)\cdot\cos^2\beta + a^2\cosh^2(\alpha)\cdot\sin^2\beta$

$= \cancel{a^2[\sinh^2(\alpha)\cos^2\beta - a^2\cosh^2(\alpha)\cos^2\beta + a^2\cosh^2(\alpha)}$

$= \cancel{a^2[-1 + a^2\cosh^2(\alpha)]}$

$= a^2[\sinh^2\alpha(1-\sin^2\beta) + a^2\cosh^2(\alpha)\cdot\sin^2\beta)]$

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$= a^2[\sinh^2\alpha - \sinh^2\alpha\cdot\sin^2\beta + \cosh^2\alpha\cdot\sin^2\beta]$

$= \cancel{a^2[\cosh 2\alpha}$

$= \cancel{a^2[\sinh^2\alpha\ [\sin^2\beta}(\cosh^2 - \sinh^2)$.

$= a^2\cosh 2\alpha - \cos^2\beta/2$

$\therefore J\frac{(\alpha,\beta)}{(x,y)} = \frac{2}{a^2\cosh 2\alpha - \cos 2\beta}$

8.) If $x+y+z=u$, $y+z=uv$, $z=uvw$. $\partial\left(\frac{xyz}{uvw}\right)$

$\Rightarrow$ $\partial\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} \longrightarrow (1)$.

$x+y+z=u \longrightarrow (2)$ $\quad y+z=uv \longrightarrow (3)$ $\quad z=uvw \longrightarrow (4)$

using (3) in (2)

$x+uv=u$

$x=u-uv$; $y=uv-uwv$; $z=uvw$

$$\begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$(1-v)(u^2v - u^2w + u^2vw) + u(uvw - uvw^2 - uvw + uw^2v)$

$= \underline{u^2v}$.

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HW

1) S.T. f^n; $u = x+y-z$, $v = x-y+z$, $w = x^2+y^2+z^2-2yz$ are not independent, find the relatⁿ b/w them.

2) $x = a(u+v)$, $y = b(u-v)$ & $u = r^2\cos 2\theta$, $v = r^2\sin 2\theta$
S.T. $\dfrac{\partial(x,y)}{\partial(r,\theta)} = -8abr^3$

3) If $x = R\cos\phi = r\sin\theta\cos\phi$
$y = R\sin\phi = r\sin\theta\sin\phi$
$z = \cancel{R\sin\phi}\ r\cos\theta$.
} spherical coordinates

Evaluate : i) $\partial\left(\dfrac{x\,y\,z}{r,\theta,\phi}\right)$ ii) $\partial\left(\dfrac{x\,y\,z}{R,\phi,z}\right)$

4) Find weather f^n $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ are functionally independent.

5) If $x = u(1-v)$, $y = uv$, P.T. $JJ' = 1$

6) $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$. } cylindrical coordinates
P.T $J = \rho$

7) $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$.

P.T $J\left(\dfrac{x\,y\,z}{u\,v\,w}\right) = \dfrac{1}{2(x-y)(y-z)(z-x)}$

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Tangent Plane and Normal line

→ **NOTE**: Eqn of tgt plane to $F(x,y,z)=0$ at point P is:

$$\boxed{\frac{\partial F}{\partial x}(X-x)+\frac{\partial F}{\partial y}(Y-y)+\frac{\partial F}{\partial z}(Z-z)}$$

where X, Y, Z are current coordinates at any point of tgt plane.

→ Eqn of normal to the surface at point P is:

$$\boxed{\frac{X-x}{\frac{\partial F}{\partial x}}=\frac{Y-y}{\frac{\partial F}{\partial y}}=\frac{Z-z}{\frac{\partial F}{\partial z}}}$$

1.) Find the eqn of tgt plane and normal to the surface $z^2=4(1+x^2+y^2)$ at $(2,2,6)$.

2.) $x^3+y^3+3xyz=3$ at $(1,2,-1)$

3.) $2x^2+y^2+2z-3=0$ at $(2,1,-3)$.

1.) $z^2=4+4x^2+4y^2$ at $(2,2,6)$

⇒ $F(x,y,z)=4x^2+4y^2-z^2+4$.

$\frac{\partial F}{\partial x}=8x$, $\frac{\partial F}{\partial y}=8y$, $\frac{\partial F}{\partial z}=-2z$

$\left(\frac{\partial F}{\partial x}\right)_{2,2,6}=16$, $\left(\frac{\partial F}{\partial y}\right)=16$, $\left(\frac{\partial F}{\partial z}\right)=-12$.

1.)

→ Tgt Eqn: $\frac{\partial F}{\partial x}(X-x) + \frac{\partial F}{\partial y}(Y-y) + \frac{\partial F}{\partial z}(Z-z)$

$\Rightarrow 16(X-2) + 16(Y-2) - 12(Z-6) = 0$

$\Rightarrow 16X + 16Y - 12Z = 8$

$4x + 4y - 3z = 2$

→ Normal Eqn at $(2,2,6) \Rightarrow \frac{X-x}{\partial F/\partial x} = \frac{Y-y}{\partial F/\partial y} = \frac{Z-z}{\partial F/\partial z}$

$\frac{X-2}{4} = \frac{Y-2}{4} = \frac{Z-6}{-3}$

2.) $F(x,y,z) = x^3 + y^3 + 3xyz - 3$ at $(1,2,-1)$

→ $\cancel{\frac{\partial F}{\partial x} = \frac{3x^2}{3}}$, $\cancel{\frac{\partial F}{\partial y} = \frac{3y^2}{12}}$, $\cancel{\partial F}$

$\frac{\partial F}{\partial x} = 3x^2 + 3yz \Rightarrow 3 - 3(2)(-1)$
$\Rightarrow -3$

$\frac{\partial F}{\partial y} = 3y^2 + 3xz \Rightarrow 12 + 3(1)(-1)$
$\Rightarrow 9$

$\frac{\partial F}{\partial z} = 3xy \Rightarrow 3(1)(2)$
$= 6$

$\Rightarrow$ Tgt Eqn:

$-6(X-1) + 9(Y-2) + 6(Z-1) = 0$

$-6x + 6 + 9y - 18 + 6z - 6 = 0$

$-3x - 9y - 6z = -18$

$x - 3y - 2z = -3$

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⇒ Normal Eqn

$$\frac{x-1}{-6} = \frac{y-2}{9} = \frac{z+1}{6}$$

3.) $2x^2 + y^2 + 2z - 3 = 0$ at $(2,1,-3)$

→ $\frac{\partial F}{\partial x} = 4x = 8$, $\frac{\partial F}{\partial y} = 2y = 2$, $\frac{\partial F}{\partial z} = 2$.

Tgt : $8(x-2) + 2(y-1) + 2(z+3) = 0$

$8x + 2y + 2z = 12$

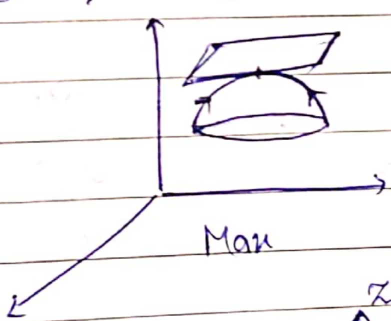
$4x + y + z = 3$

Normal : $\frac{x-2}{4} = \frac{y-1}{1} = \frac{z+3}{1}$

Maxima & Minima of funct^n of 2 variables

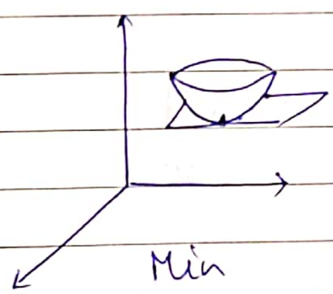
→ $f(a,b)$ is said to be an extreme value of $f(x,y)$, if it is a max value or min value.

i.)



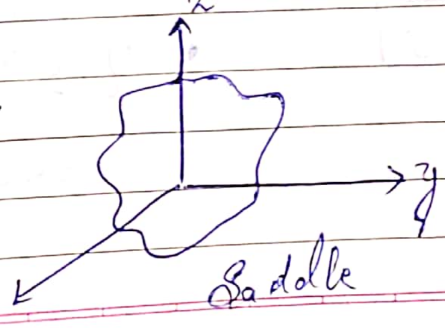
Max

ii.)



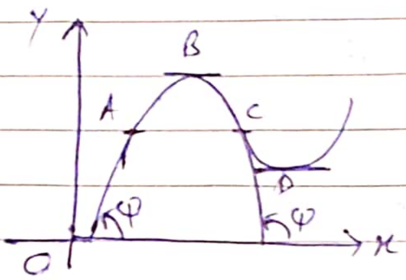
Min

iii.)



Saddle

Maxima & Minima



→ The funct^n $y = f(x)$, if y inc as x inc (as at A), it is called increasing function of x

→ If y dec, even though x increases, (as at C), it is called decreasing funct^n

NOTE:

→ If the derivate is zero (as at B or D), then y is neither increasing nor decreasing, then we say that the funct^n is stationary.

→ Notations

* $p = \dfrac{\partial F}{\partial x}$, $q = \dfrac{\partial F}{\partial y}$

* $r = \dfrac{\partial^2 F}{\partial x^2}$, $s = \dfrac{\partial^2 F}{\partial x \partial y}$, $t = \dfrac{\partial^2 F}{\partial y^2}$

* $\boxed{\delta = rt - s^2}$

* if $\delta > 0$ – max, min exists
  $\delta = 0$: need more data
  $\delta < 0$ – no max, min

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* If $r<0$ & $\delta>0$ - Maxima
(at an extremum point (a,b), then we conclud (a,b) is)

* If $r>0$ & $\delta>0$ - , at ( ) - Minima.

* If $\delta<0$, then $f(a,b)$ is not an extreme value & $f(x,y)$ is neither max nor min at point (a,b) & the point is called saddle.

* If $\delta=0$ at (a,b) then the case is doubtfull, needs further investigatⁿ

---

1) Examine the functⁿ $f(x,y) = x^4+y^4-2(x-y)^2$ far extreme values

$\Rightarrow$

$$f(x,y) = x^4+y^4-2x^2-2y^2+4xy$$

$R = \frac{\partial^2 F}{\partial x^2} \Rightarrow \frac{\partial F}{\partial x} = 4x^3-4x+4y$. Set $F_x=0$

$$\frac{\partial^2 F}{\partial x^2} = 12x^2-4$$

$S = \frac{\partial^2 F}{\partial y^2} \Rightarrow \frac{\partial F}{\partial y} = 12y^2-4$. Set $F_y=0$

$t = \frac{\partial F}{\partial y} = 4y^3-4y+4x$

$$\frac{\partial F}{\partial x \partial y} = 4.$$

$$\delta = rt-s^2$$

$$\delta = 4(12x^2$$

1.) $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

$F_x = 4x^3 - 4x + 4y$
$= x^3 - x + y$
Set $F_x = 0$
$x^3 - (x - y) = 0 \longrightarrow (1)$

$F_y = 4y^3 - 4y + 4x$
$= y^3 - y + x.$
Set $F_y = 0$
$y^3 - (-x + y) = 0 \longrightarrow (2)$

$x^3 + y^3 = 0 \Rightarrow (x+y)(x^2 - xy + y^2) = 0$
$x = -y \quad , \quad x^2 - xy + y^2 = 0$

Put $y = -x$ in F_x
$x^3 - x - x = 0$
$x^3 = 2x \quad x^3 - 2x = 0$
$x^2 = 2 \quad x = 0, \sqrt{2}, -\sqrt{2}$
$x = \pm\sqrt{2}$

Gives $y = 0, -\sqrt{2}, \sqrt{2}$

Hence set of stationary points $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

	$r = f_{xx} = 12x^2 - 4$	$s = 4$	$t = 12y^2 - 4$	$\delta = rt - s^2$
1.) $(0,0)$	$\Rightarrow -4$	4	$\Rightarrow -4$	0
2.) $(\sqrt{2}, -\sqrt{2})$	$12(2) - 4 \Rightarrow 20$	4	$\Rightarrow$ ~~20~~ 20	384 $\therefore$ Min
3.) $(-\sqrt{2}, \sqrt{2})$	20	4	20	384 $\therefore$ Min

Conclusion: 1.) Needs more investigation
2.), 3.) $r > 0$ & $\delta > 0$ $\therefore$ They are minima
$\therefore F(x,y)$ is minimum at $(\sqrt{2}, -\sqrt{2})$ $\therefore$ Min value of
$f(x,y) = 4 + 4 - 4 - 4 - 8 \Rightarrow \therefore f(x,y) = -8$ |||ly for $(-\sqrt{2}, \sqrt{2})$

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2) Extreme values $f(x,y) = x^3y^2(1-x-y)$

$\Rightarrow$

$F_x = 3x^2y^2(1-x-y) + x^3y^2(0-1)$

$= 3x^2y^2 - 3x^3y^2 - 3x^2y^3 - x^3y^2$

$\Rightarrow 2x^2y^2 - 3x^2y^2(x-y) = 0$

$\cancel{2x^2y^2 = 3x^2y^2(x-y)}$

$\cancel{\frac{2}{3} = x-y}$

$x^2y^2(3-4x-3y) = 0$

$F_y = 2x^3y(1-x-y) + x^3y^2(0-1)$

$\Rightarrow 2x^3y - 2x^4y - 2x^3y^2 - x^3y^2 = 0$

$\Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$

$x^3y(2-2x-3y) = 0$

$F_x = 0$

$x^2y^2(3-4x-3y) = 0$

$\rightarrow (x=0, y=0)$

$4x+3y = 3$

$\rightarrow \left(x=\frac{1}{2}, y=\frac{1}{3}\right)$

$\rightarrow$ For $x=0$,

$3y=3$

$y=1$

$(0,1)$

$\rightarrow$ For $y=0$

$x = \frac{3}{4}$

$(4/3, 0)$

$F_y = 0$

$x^3y(2-2x-3y) = 0$

$2x+3y = 2$

$\rightarrow x=0$

$y = \frac{2}{3}$ $\left(0, \frac{2}{3}\right)$

$\rightarrow y=0$

$x=1$

$(1,0)$

$\therefore (0,0)\ (0,1)\ (1,0)\ \left(\frac{3}{4}, 0\right)\ \left(0, \frac{2}{3}\right)\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

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$\longrightarrow$ $r = f_{xx}$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_{xx} = 6xy^2 - 12y^2x^2 - 6xy^3$

$* \; r = 6xy^2 - 12x^2y^2 - 6xy^3$

$\longrightarrow$ $s = f_{xy}$

$f_y = 2x^3y - 2x^4y - 3x^3y^2$

$* \; f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$\longrightarrow$ $t = f_{yy}$

$* \; t = 2x^3 - 2x^4 - 6x^3y$

	r $6xy^2 - 12x^2y^2 - 6xy^3$	s $6x^2y - 8x^3y - 9x^2y^2$	t $2x^3 - 2x^4 - 6x^3y$	z $rt - s^2$
i.) $(0,0)$	$0 - 0 - 0$ $r = 0$	$s = 0$	$t = 0$	$z = 0$
ii.) $(0,1)$	$r = 0$	$s = 0$	$t = 0$	$z = 0$
iii.) $(1,0)$	$r = 0$	$s = 0$	$t = 0$	$z = 0$
iv.) $(\frac{3}{4}, 0)$	$r = 0$	$s = 0$	$2\left(\frac{3}{4}\right)^3 - 2\left(\frac{3}{4}\right)^4$ $= 27/128$	$z = 0$
v.) $(0, \frac{2}{3})$	$r = 0$	$s = 0$	$t = 0$	$z = 0$
vi.) $(\frac{1}{2}, \frac{1}{3})$	$r = -\frac{1}{9}$	$s = -\frac{1}{12}$	$t = -\frac{1}{8}$	$z = \frac{1}{144}$

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vi.) $r < 0$ & $t > 0$

$\therefore f^n$ is maxima.

Max value of $f(x,y)$ at $\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$.

3.) In a plane Δ^{le} find the max value of $\cos A \cdot \cos B \cdot \cos C$

$\Rightarrow$

WKT $\quad A + B + C = \pi$

$C = \pi - (A+B)$

$\cos C = \cos(\pi - (A+B))$

$\cos A \cdot \cos B \cdot \cos(\pi - (A+B))$

$- \cos A \cdot \cos B \cdot \cos(A+B)$

$= f(A,B)$

$\rightarrow$ $f_A = \sin A \cdot \cos B \cdot \cos(A+B) + \cos A \cdot \cos B \sin(A+B)(\cancel{B})$

$= \cos B\,[\sin A \cos(A+B) + \cos A \cdot \sin(A+B).\cancel{B}]$

$= \cos B\,[\sin(A + A + B)$

$f_A = \sin(2A+B) \cdot \cos B$

$\rightarrow$ $f_B = \cos A \sin B \cos(A+B) \cancel{+} - \cos A \cdot \cos B \sin(A+B)$

$= \cos A\,[\sin(2A+B)]$

$f_B = \sin(\cancel{2}A + 2B) \cdot \cos A$

$\rightarrow$ $f_{AA} = 2\cos B \cdot \cos(2A+B)$

$\rightarrow$ $f_{AB} = \sin A \cdot \sin(A+2B) \cancel{+} \cos A \cdot \cos(A+2B)$

$= \cos(A + A \cancel{+} 2B)$

$= \cos(2B + 2A)$

$\rightarrow$ $f_{BB} = 2\cos A \cdot \cos(A+2B)$

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3) $f_A = 0$ $\qquad$ $f_B = 0$

$\cos B \cdot \sin(2A+B) = 0$ $\qquad$ $\cos A \cdot \sin(A+2B) = 0$

$\therefore \quad A = \frac{\pi}{3} = B$

$r = f_{AA} = 2\cos B \cdot \cos(2A+B)$

$t = f_{BB} = 2\cos A \cdot \cos(A+2B)$

$s = f_{AB} = \cos(2A+2B)$

	r	s	t	$z = (rt - s^2)$
$A = B = \frac{\pi}{3}$	-1	$-1/2$	-1	$1 - \frac{1}{4} = \frac{3}{4}$

$\therefore z > 0 \quad \& \quad r < 0$

It implies that given function is maximum.

$\& \; C = \pi - (\pi/3 + \pi/3)$

$= \frac{\pi}{3}$

$\therefore \; A = B = C = \frac{\pi}{3}$

$\therefore$ All 3 angles are equal and is equilateral

Hence $\cos A \cdot \cos B \cdot \cos C$ is Max when each angle is $\pi/3$.
i.e triangle is equi

& Max value $= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

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1.) Find I & II derivate

Lagranges Method of undetermined multipliers.

Sometimes it is required to find the stationary values of functions of several variables, which are not all independent but are connected by some given relations. Here we try to convert the given funct to the one which will have least no. of independent variables with the help of given relat

Working rule:

i.) write $F = f(x,y,z) + \lambda \phi(x,y,z)$

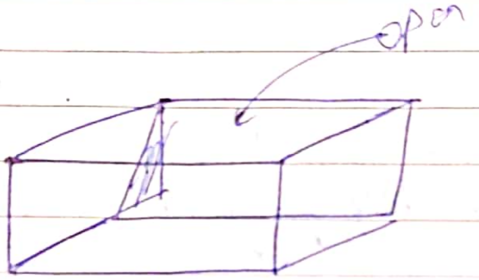
ii.) obtain, $F_x = 0$, $F_y = 0$, $F_z = 0$

iii.) Solve above eqns together with $\phi(x,y,z) = 0$. The values of x, y, z so obtained, will give stationary values of x, y, z.

1.) A rectangular box open @ the top is to have volume of $32 ft^3$. Find the dimensions of the box, requiring least material for its constructⁿ

$\longrightarrow$

Let x, y, z be the dimensions of given box.



Volume $V = xyz$.

SA is given (S) = $xy + 2yz + 2zx$ (no 2xy, because open)

Given $V = 32$

$32 = xyz$

$\therefore z = \frac{32}{xy} \longrightarrow (1)$

$\therefore S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x$

$S = xy + \frac{64}{x} + \frac{64}{y}$

$S = f(x, y)$ (say) $\longrightarrow (2)$

$f_x = y - \frac{64}{x^2}$ $\qquad$ $f_y = x - \frac{64}{y^2}$

$f_x = 0$ $\qquad$ $f_y = 0$

$x^2 y = 64$ $\qquad$ $xy^2 = 64$

$x^2 y = xy^2$

$xy = y^2$

$\therefore x = y$ or $\boxed{x - y = 0}$

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1) consider
$x^4y = 64$
Let $y = x$
$\therefore x^5 = 64$
$x = 4$
Since $x = 4$, $y = 4 = x$

$\therefore (4,4)$ is extreme point for $f(x,y)$

$r = f_{xx}$ $\qquad$ $t = f_{yy}$
$= 0 + \frac{64(2x)}{x^4}$ $\qquad$ $= \frac{-128}{y^3}$
$= + \frac{128}{x^3}$

$s = f_{xy} = 1$

$\delta = rt - s^2$
$= \left(\frac{-128}{x^3}\right)\left(\frac{-128}{y^5}\right) - 1$
$\delta = 3$

$r > 0$, $\delta > 0$
$\therefore f$ is min

$z = \frac{32}{xy} = 2$.

$\therefore$ Dimensions of x, y, z are 4, 4, 2 req for constructing $\boxed{}^{lr}$ box.

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2.) Find the max & min distances of point (3,4,12) from sphere $x^2+y^2+z^2=4$

⇒ Let $p(x,y,z)$ be any point on the sphere

$A(3,4,12)$ so that

$$AP^2 = (x-3)^2+(y-4)^2+(z-12)^2$$
$$= f(x,y,z) \longrightarrow (1)$$

Now, we have to find max & min value of $f(x,y,z)$ subject the condⁿ

$$\phi(x,y,z) = x^2+y^2+z^2-4=0$$

$$F = f(x,y,z)+\lambda\,\phi(x,y,z).$$

$$F = (x-3)^2+(y-4)^2+(z-12)^2+\lambda(x^2+y^2+z^2-4)$$

* $\dfrac{\partial F}{\partial x} = 2x-6+2\lambda x$

* $\dfrac{\partial F}{\partial y} = 2y-8+2\lambda y$

* $\dfrac{\partial F}{\partial z} = 2z-24+2\lambda z$

$F_x=0$	$F_y=0$	$F_z=0$
$x-3+\lambda x=0$	$y-4+\lambda y=0$	$z-12+\lambda z=0$
$x(1+\lambda)=3$	$y(1+\lambda)=4$	$z(1+\lambda)=12$
$\lambda=\dfrac{3-x}{x}$	$\lambda=\dfrac{4-y}{y}$	$\lambda=\dfrac{12-z}{z}$

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$$\lambda = \frac{\pm\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\lambda = \frac{\pm\sqrt{F}}{\sqrt{4}}$$

$$\lambda = \frac{\pm\sqrt{F}}{2}$$

Substitute for λ in (3)

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \frac{\sqrt{F}}{2}}$$

$$y = \frac{4}{1+\lambda} = \frac{4}{1 \pm \frac{\sqrt{F}}{2}}$$

$$z = \frac{12}{1+\lambda} = \frac{12}{1 \pm \frac{\sqrt{F}}{2}}$$

$$x^2+y^2+z^2 = \frac{4\left[(3)^2+(4)^2+(12)^2\right]}{(2\pm\sqrt{F})^2}$$

$$4 = \frac{4\,[169]}{(2\pm\sqrt{F})^2}$$

$$2 \pm \sqrt{F} = \mp 13$$

$$\sqrt{F} = 11 \ \& \ 15$$

$\therefore$ Max dist $= 15$.

min dist $= 11$.

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3.) $V = 8xyz$

$x^2+y^2+z^2 = R^2$

$\longrightarrow$ $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$= 8xyz + \lambda(x^2+y^2+z^2-R^2)$.

- $\frac{\partial F}{\partial x} = 0$, $8yz + 2\lambda x = 0$
  $2\lambda x = -8yz$
  $2\lambda x^2 = -8xyz$.

- $\frac{\partial F}{\partial y} = 0$, $8xz + 2\lambda y = 0$
  $2\lambda y = -8xz$
  $2\lambda y^2 = -8xyz$

- $\frac{\partial F}{\partial z} = 0$, $8xy + 2\lambda z = 0$
  $2\lambda z = -8xy$
  $2\lambda z^2 = -8xyz$.

$2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$

$2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2 \Rightarrow x^2 = y^2 = z^2 \Rightarrow x = y = z$.

Thus for max val. $x = y = z$ Rectangular solid is cube.
$x^1y^1z^1$.

Ex: H.W

i) $x^4 + 2x^2y - x^2 - y^2$

ii) $x^3 + y^3 - 3axy$.

Unit 4: Vector Calculus

* Scalar & Vector point function.
* Gradient, Divergence & Curl
* Solenoidal.
* Irrotational vector field
* Applications.
* Vector identities.

→ Vector is a quantity having both magnitude & direction.

→ Vector quantities like force, velocity & acceleration have a lot of reference in physical & engineering prob.

→ Posi^n vector at point P(x, y, z) in space is : $\vec{r} = xi + yj + zk$. If x, y, z are functions of single parameter t, then $\vec{r} = x(t)i + y(t)j + z(t)k$, is called vector equation of curve.

→ $\frac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at P.

Scalar & Vector point Funct^n

→ If to every point (x, y, z) of region R in space, there corresponds a scalar $\phi(x, y, z)$ the ϕ is called scalar point function defined in r.

Ex: ① $\phi = x^3 + 2y^2 + z^3$

② $\phi = xy^3 z$

IIIly if to every point (x, y, z) of region R in space, there corresponds vector $\vec{A}(x, y, z)$, the vector A is vector point function.

$\vec{A} = x^3 i + 2y^2 j + z^3 k$.

→ Operators :

1) The vector differential operator is read as 'nabla' or 'delta' ∇ & is defined as :

$$\boxed{\nabla = \frac{\partial i}{\partial x} + \frac{\partial j}{\partial y} + \frac{\partial k}{\partial z} = \sum \frac{\partial i}{\partial x}}$$

- Gradient divergence & Curl.

* Gradient of scalar field

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = \sum \frac{\partial \phi}{\partial x} i$$

* Divergence of vector field

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial i}{\partial x} + \frac{\partial j}{\partial y} + \frac{\partial k}{\partial z} \right)(a_1 i + a_2 j + a_3 k)$$

$$\text{div } \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}.$$

$$\text{Curl } \vec{A} = \nabla \times \vec{A}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

1) If $f = x^3 + y^3 + z^3 - 3xyz$. Find grad f, at $(1, 2, 3)$.

→ Let $f = x^3 + y^3 + z^3 - 3xyz$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= (3x^2 - 3yz) i + (3y^2 - 3xz) j + (3z^2 - 3xy) k$$

$$\boxed{\nabla f}_{(1,2,3)} = -15i + 3j + 21k.$$

2.) If $\phi = \log(x^2+y^2+z^2)$. Find $\nabla\phi$ at $(1,1,1)$.

→

NOTE: If θ angle b/w 2 surfaces at the given point then angle b/w surfaces is given by

$$\boxed{\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}}.$$

3.) Find angle b/w surfaces $x^2+y^2+z^2=9$, & $z = x^2+y^2-z$ at $(2,-1,2)$

→

Let $\phi_1 = x^2+y^2+z^2$ & $\phi_2 = x^2+y^2-z$

* $\nabla\phi_1 = \frac{\partial(x^2)}{\partial x}i + \frac{\partial(y^2)}{\partial x}j + \frac{\partial(z^2)}{\partial x}k$

$\nabla\phi_1 = 2x + 2y + 2z$

* $\nabla\phi_1(2,-1,2) = 4i - 2j + 4k$

=

* $\nabla\phi_2 = \frac{\partial(x^2)}{\partial x}i + \frac{\partial(y^2)}{\partial y}j - \frac{\partial(z)}{\partial z}k$

$= 2xi + 2yj - 1k$

* $\nabla\phi_2 = 4i + 2j - k$

* $|\nabla\phi_1| = \sqrt{16+4+16}$

$= \sqrt{36} = 6$

* $|\nabla\phi_2| = \sqrt{16+4+1}$

$= \sqrt{21}$

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$\rightarrow \quad \therefore \cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$

$= \dfrac{(4i-2j+4k)\cdot(4i-2j-k)}{\cancel{5\sqrt{84}}\; 6 \cdot \sqrt{21}}$

$= \dfrac{16+4-4}{6\sqrt{24}}$

$\cos\theta = \dfrac{8}{3\sqrt{21}} \qquad \therefore \theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$

$\theta =$

4.) $\phi_1 = x\log z - y^2 \qquad \phi_2 = x^2y - x + z \quad$ at $(1,1,1)$

$\rightarrow \quad \nabla\phi_1 = (\log z)i - (2y)j + \left(\dfrac{x}{z}\right)k$

$(\nabla\phi_1)_{111} = 0i - 2j + k$

$\Rightarrow \quad \nabla\phi_2 = (2xy)i + (1)k + (x^2)j$

$(\nabla\phi_2)_{111} = 2i + k + j$

$\therefore |\nabla\phi_1| = \sqrt{4+1} = \sqrt{5} \qquad |\nabla\phi_2| = \sqrt{4+1+1} = \sqrt{6}$

$\cos\theta = \dfrac{(-2j+k)\cdot(2i+k+j)}{\sqrt{5}\sqrt{6}} = \dfrac{0-2+1}{\sqrt{5}\sqrt{6}}$

$= \dfrac{-1}{\sqrt{30}}$

$= \cancel{0}\ \cancel{0}\ 0 + 0 +$

3.) Find value of a & b (constant) such that the surfaces $ax^2 - byz = (a+2)x$ & $4x^2y + z^3 = 4$ are orthogonal at point P(1, -1, 2)

$\Longrightarrow$

NOTE: The condit^n for 2 surfaces to be orthogonal is

$$\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$$

First we have to ensure that points lie on given surf

$\phi_1 = ax^2 - byz - (a+2)x \longrightarrow (1)$

Put P

$\phi_1 = 0$

$a + 2b - a - 2 = 0$

$b = 1$

$\phi_2 = 4x^2y + z^3 \longrightarrow (2)$

Consider $4x^2y + z^3 = 4$

Put P

$4(-1) + (2)^3 = 4$

LHS = RHS.

When $b = 1$, given point lie on surfaces.

The orthogonal condit^n is $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$\nabla\phi_1 = (2ax - (a+2))i + (-bz)j + (-by)k$

$[\nabla\phi_1]_{1,-1,2} = (a-2)i - 2bj + bk$

$\nabla\phi_2 = (8xy)i + (4x^2)j + (3z^2)k$

$[\nabla\phi_2]_{1,-1,2} = -8i + 4j + 12k$

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$\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$-8(a-2) - 8 + 12 = 0$

$a = \frac{20}{8}$

$a = \frac{5}{9}$

i.e. $a = \frac{5}{2}$ & $b = 1$

1.) In what direct^n from $(3, 1, -2)$, $\phi = x^2y^2z^4$ has max. Find mag & direct^n

→

$\phi = x^2y^2z^4$

$\nabla\phi = 2xy^2z^4 + 2yx^2z^4 + 4z^3x^2y^2$

$= 2(3)(1)(16) + 2(9)(16) + 4(-8)(9)(4)$

$= 80 + 288 + 1152$

$|\nabla\phi| = \sqrt{1520}$

---

2) If $T = \frac{z}{x^2+y^2}$ $P((0, 0, 2))$. Find direct^n & magnit.

→

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Solenoidal or Zero diversion

A vector $\vec{F}$ is said to solenoidal / Zero diversion, if $div\ \vec{F} = 0$.

Verify the following for zero divergence

1) $F = (-x^2 + yz)i + (4y - z^2x)j + (2xz - 4z)k$

$\nabla \cdot \vec{F} = div\ \vec{F}$

$= \left(\frac{\partial i}{\partial x} + \frac{\partial j}{\partial y} + \frac{\partial k}{\partial z}\right) \cdot \left[(-x^2+yz)i + (4y - z^2x)j + (2xz - 4z)k\right]$

$= (-2xi + 4j + 2x - 4]$

$= 0$

$\therefore \vec{F}$ is solenoidal

2.) $F = (3y^5z^3)i + (\sin x \cos z)j + 3x^2y^2k$

$\Rightarrow \nabla \cdot \vec{F} = 0 + 0 + 0$

$\nabla \cdot \vec{F} = 0$

$\vec{F}$ is solenoidal

3.) Find a, if $\vec{F} = (ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$ is solenoid.

$\Rightarrow \nabla F = 0$

$\Rightarrow \therefore \nabla \vec{F} \Rightarrow 2axy + 2xy + 2xy = 0$

$2axy = -4xy$

$a = -2$

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a.) P.T $div.(r^n \vec{R}) = (n+3)r^n$ ~~other~~ hence, ST $\frac{R}{r^3}$ is solenoidal.

$\Rightarrow$

$R = xi + yj + zk$ & $r = \sqrt{x^2+y^2+z^2}$

$div(r^n \vec{R}) = \nabla \cdot (x^2+y^2+z^2)^{n/2}(xi + yj + zk)$

$= \left(\frac{\partial i}{\partial x} + \frac{\partial j}{\partial y} + \frac{\partial k}{\partial z}\right) \cdot \left[(x^2+y^2+z^2)^{n/2} \cdot (xi + yj + zk)\right]$

$= \frac{\partial}{\partial x}[x(x^2+y^2+z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2+y^2+z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2+y^2+z^2)^{n/2}]$

$= \sum \frac{\partial}{\partial x}[x \cdot (x^2+y^2+z^2)^{n/2}]$

$= \sum 1(x^2+y^2+z^2)^{n/2} + x \cdot \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2x$

$= \sum r^n + \cancel{x^2 n (x^2+y^2+z^2)^{(n-2)/2}}$

$= \sum r^n + n\sum x^2 (x^2+y^2+z^2)^{(n-2)/2}$

||| for others

$= 3r^n + nr^2(x^2+y^2+z^2)^{(n-2)/2}$

$= 3r^n + nr^2(r^{n-2})$

$= 3r^n + nr^{2+n-2}$

$= (n+3)r^n$

$\therefore$ for f^n to be solenoid, $n = -3$, $div(R/r^3) = 0$

Then $\left(\frac{R}{r^3}\right)$ is solenoid.

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Find curl of the following

1) $\vec{F} = x^2y\,i + y^2z\,j + z^2x\,k$ at $(1,1,1)$

$\rightarrow$

$$\text{Curl } \vec{F} \; (\nabla \times \vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$\nabla \times \vec{F} = i\left[\frac{\partial(z^2x)}{\partial y} - \frac{\partial(y^2z)}{\partial z}\right] - j\left[\frac{\partial(z^2x)}{\partial x} - \frac{\partial(x^2y)}{\partial z}\right]$$
$$+ k\left[\frac{\partial(y^2z)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right]$$

$$= (0 - y^2)i - j(z^2) + k(0 - x^2)$$

$$= -y^2 i - z^2 j - x^2 k$$

$$\nabla \times \vec{F}_{(1,1,1)} = -(i + j + k)$$

2) $\vec{F} = x^2 i + y^2 j + z^2 k$ at $(1,1,1)$

$\rightarrow$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= i\left[\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z}\right] - j\left[\frac{\partial z^2}{\partial x} - \frac{\partial x^2}{\partial z}\right] + k\left[\frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y}\right]$$

$$= 0 - 0 + 0$$

$$\nabla \times \vec{F}_{(1,1,1)} = \vec{0}$$

3.) $\vec{F} = (x^2yz)i + (xy^2z)j + (xyz)k$ Find curl & Div F at $(1,2,3)$

→

→ $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz \end{vmatrix}$

$\nabla \times \vec{F} = i\left[\frac{\partial(xyz)}{\partial y} - \frac{\partial(xy^2z)}{\partial z}\right] - j\left[\frac{\partial xyz}{\partial x} - \frac{\partial x^2yz}{\partial z}\right] + k\left[\frac{\partial xy^2z}{\partial x} - \frac{\partial x^2yz}{\partial y}\right]$

$= (xz - xy^2)i - j(yz - x^2y) + k(2xyz - x^2z)$

$= (3-4)i - j(6-2)j + k(12-3)$

$= -i - 4j + 9k$

→ $\nabla \cdot F = \frac{\partial (x^2yz)}{\partial x} + \frac{\partial (xy^2z)}{\partial y} + \frac{\partial (xyz)}{\partial z}$

$= 2xyz\,i + 2xyz\,j + xy\,k$

$= 12i + 12j + 2k$

$= \underline{26}$

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4] Evaluate $\operatorname{div}\vec{F}$ & $\operatorname{Curl}\vec{F}$ at $(1,2,3)$

$\vec{F} = \operatorname{grad}(x^2y + y^3z + z^3x - x^2y^2z^2)$

Sol^n $\vec{F} = \operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$$\nabla\phi = \underbrace{(3x^2y + 3z^3 - 2xy^2z^2)}_{a}i + \underbrace{(x^3 + 3y^2z - 2x^2yz^2)}_{b}j + \underbrace{(y^3 + 3z^2x - 2x^2y^2z)}_{c}$$

$\rightarrow$ $\operatorname{div}\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left((a)i + (b)j + (c)k\right)$

$\operatorname{div}\vec{F} = 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2$

$\operatorname{div}\vec{F}_{(1,2,3)} = 6(1)(2) - 2(4)(9) + 6(2)(3) - 2(1)(9) + 6(3) - 2(1)(4)$

$\operatorname{div}\vec{F} = -32$

$\rightarrow$ $\operatorname{Curl}\vec{F} = \nabla\times\vec{F}$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (a) & (b) & (c) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y^3 + 3z^2x - 2x^2y^2z) - \frac{\partial}{\partial z}(x^3 + 3y^2z - 2x^2yz^2)\right]$$
$$-j\left[\frac{\partial}{\partial x}(y^3 + 3z^2x - 2x^2y^2z) - \frac{\partial}{\partial z}(3x^2y + z^3 - 2xy^2z^2)\right]$$
$$+k\left[\frac{\partial}{\partial x}(x^3 + 3y^2z - 2x^2yz^2) - \frac{\partial}{\partial y}(3x^2y + z^3 - 2xy^2z^2)\right]$$

$= \sum i\{3y^2 - 4x^2yz - 3y^2 + 4x^2yz\}$

$\nabla\times\vec{F}_{(1,2,3)} = 0$

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5) $\vec{F} = (x+y+1)i + j - (x+y)k$

ST $\vec{F} \cdot \text{curl}\,\vec{F} = 0$

$\rightarrow$ $\text{Curl}\,F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$

$= i\left(\frac{\partial(-x-y)}{\partial y} - \frac{\partial(1)}{\partial z}\right) - j\left(\frac{\partial(-x-y)}{\partial x} - \frac{\partial(x+y+1)}{\partial z}\right)$

$+ k\left(\frac{\partial(1)}{\partial x} - \frac{\partial(x+y+1)}{\partial y}\right)$

$= i[-1] - j[-1] + k(0-1)$

$\vec{\nabla \times F} = -i + j - k$

$\therefore \vec{F} \cdot \text{Curl}\,\vec{F} = -x-y-1+1+x+y$

$\vec{F} \cdot (\nabla \times \vec{F}) = 0$ $\therefore$ Hence proved.

Vector F is $\perp^r$ to curl F in this (question only)

$\therefore \vec{F} \cdot (\nabla \times \vec{F}) = 0$