Module - 1

Vector differentiation

Important result :-

1. The position Vector defined on xy plane, yz plane, xz plane is $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and its d can be defined as $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ in other words if x, y, z are in function of t, Then the position Vector $\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ and its derivative is
$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

2. for any $F_1 = F_1(x,y,z)$, $F_2 = F_2(x,y,z)$, $F_3 = \overline{F_3(x,y,z)}$ Then the fun $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ is called The Vector point

3. Suppose the two Vectors point fun $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$. Then $\vec{A}\cdot\vec{B} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\cdot(b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$
$a_1b_1 + a_2b_2 + a_3b_3$ is a Scalar

(ii) $$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\vec{i} - (b_3a_1 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$ is a Vector point function

4. The gradient of a Vector can be denoted by The Symbol ∇ read as del or grad and which will be denoted as
$$\nabla = \frac{\partial\vec{i}}{\partial x} + \frac{\partial\vec{j}}{\partial y} + \frac{\partial\vec{k}}{\partial z}$$
Suppose $\phi(x,y,z) = c$ be a Scalar point function. Then The gradiant of ϕ can be defined as
$$\nabla\phi = \text{grad}\,\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi$$

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}}$$

5. Suppose $\phi_1(x,y,z) = c_1$, $\phi_2(x,y,z) = c_2$ are the two Scalar functions then

(i) $\nabla(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1\nabla\phi_1 + \lambda_2\nabla\phi_2$

(ii) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(iii) The angle between the given two Surfaces can be Evaluated by Using $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

6. Suppose $\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a Vector point function for which F_1, F_2, F_3 are the functions of x,y,z then

(i) $\operatorname{div}\vec{F}$ = can be defined as

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$
$$= \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\cdot\left(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}\right)$$

$\operatorname{div}\vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$ is a Scalar point function

(ii) $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$\operatorname{curl}\vec{F} = \left(\dfrac{\partial F_3}{\partial y}, \dfrac{\partial F_2}{\partial z}\right)\bar{i} - \left(\dfrac{\partial F_3}{\partial x} - \dfrac{\partial F_1}{\partial z}\right)\bar{j} + \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\bar{k}$

is a Vector point function

7. If $\operatorname{div}\vec{F} = 0$ then we say that $\vec{F}$ is a solenoidal and if $\vec{F} = 0\bar{i} + 0\bar{j} + 0\bar{k} = \vec{0}$ then we say that the given $\vec{F}$ is a Irrotational at any point $P(x_0, y_0, z_0)$

I Find the angle between the surface $x^2+y^2+z^2=9$, $x^2+y^2-3=z$ at the point $(2,-1,2)$

$\Rightarrow$

$$\phi_1 = x^2+y^2+z^2-9=0$$

$$\phi_2 = x^2+y^2-3-z=0$$

$$\nabla\phi_1 = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi_1 = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\nabla\phi_1 P(2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} + 2(2)\bar{k}$$

$$\Rightarrow \nabla\phi_1 P(2,-1,2) = 4\bar{i} - 2\bar{j} + 4\bar{k}$$

$$\phi_2 = x^2+y^2-3-z=0$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_2}{\partial z}\bar{k}$$

$$\nabla\phi_2 = 2x\bar{i} + 2y\bar{j} - 1\bar{k}$$

$$\nabla\phi_2 P(2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} - 1\bar{k}$$

$$\Rightarrow \nabla\phi_2 P(2,-1,2) = 4\bar{i} - 2\bar{j} - \bar{k}$$

$$\nabla\phi_1 = 4\bar{i} - 2\bar{j} + 4\bar{k} = \bar{A}$$

$$\nabla\phi_2 = 4\bar{i} - 2\bar{j} - \bar{k} = \bar{B}$$

$$\cos\theta = \frac{\bar{A}\cdot\bar{B}}{|\bar{A}||\bar{B}|}$$

$$\cos\theta = \frac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{(4)^2+(-2)^2+(4)^2}\sqrt{(4)^2+(-2)^2+(-1)^2}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

2] Find the angle between the Surfaces $x^2+y^2-z^2=4$ and $x^2+y^2-13=z$ at the point $(2,1,2)$

$\Rightarrow$

$$\phi_1 = x^2+y^2-z^2-4=0$$

$$\phi_2 = x^2+y^2-13-z=0$$

$$\nabla\phi_1 = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi_1 = 2x\bar{i} + 2y\bar{j} - 2z\bar{k}$$

$$\nabla\phi_1 p(2,1,2) = 2(2)\bar{i} + 2(1)\bar{j} - 2(2)\bar{k}$$

$$\Rightarrow \nabla\phi p(2,1,2) = 4\bar{i} + 2\bar{j} - 4\bar{k}$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_2}{\partial z}\bar{k}$$

$$\nabla\phi_2 = 2x\bar{i} + 2y\bar{j} - 1\bar{k}$$

$$\nabla\phi_2 p(2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} - 1\bar{k}$$

$$\Rightarrow \nabla\phi_2 p(2,-1,2) = 4\bar{i} - 2\bar{j} - \bar{k}$$

$$\nabla\phi_1 = 4\bar{i} + 2\bar{j} - 4\bar{k} \Rightarrow \bar{A}$$

$$\nabla\phi_2 = 4\bar{i} + 2\bar{j} - 1\bar{k} \Rightarrow \bar{B}$$

$$\cos\theta = \frac{\bar{A}\cdot\bar{B}}{|\bar{A}||\bar{B}|}$$

$$\cos\theta = \frac{(4\bar{i}+2\bar{j}-4\bar{k})\cdot(4\bar{i}+2\bar{j}-1\bar{k})}{\sqrt{(4)^2+(2)^2+(4)^2}\sqrt{(4)^2+(2)^2+(-1)^2}}$$

$$\cos\theta = \frac{16+4+4}{6\sqrt{21}}$$

$$\cos\theta = \frac{24}{6\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)}$$

3] Find the values a and b such that surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1,-1,2)$

$\Rightarrow$

$$\phi_1 = ax^2 - byz - (a+2)x = 0 \rightarrow (1)$$

$$\phi_2 = 4x^2y + z^3 - 4 = 0 \rightarrow (2)$$

the point $p(1,-1,2)$ is on both the surfaces

$(1) \Rightarrow a(1)^2 - b(-1)(2) = (a+2)(1)$

$\Rightarrow a + 2b = a + 2$

$\therefore b = 1$

$(1) \Rightarrow \phi_1 = ax^2 - yz - (a+2)x = 0$

$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_1}{\partial z}\bar{k}$

$\nabla\phi_1 = (2ax - a - 2)\bar{i} - z\bar{j} - y\bar{k}$

$\nabla\phi\, p(1,-1,2) = 2a(1) - a - 2)\bar{i} - 2\bar{j} + \bar{k}$

$\nabla\phi_1 = (a-2)\bar{i} - 2\bar{j} + \bar{k}$

$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_2}{\partial z}\bar{k}$

$\nabla\phi_2 = 8xy\bar{i} + 4x^2\bar{j} + 3z^3\bar{k}$

$\nabla\phi_2\, p(1,-1,2) = -8(\bar{i}) + 4\bar{j} + 12\bar{k}$

Given that ϕ_1 and ϕ_2 orthogonally intersect

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$|(a-2)\bar{i} - 2\bar{j} + \bar{k}| \cdot |-8\bar{i} + 4\bar{j} + 12\bar{k}| = 0$

$\Rightarrow -8(a-2) - 8 + 12 = 0$

$\Rightarrow -8a + 16 - 8 + 12 = 0$

$\Rightarrow 20 - 8a = 0$

$\Rightarrow 20 = 8a$

$\Rightarrow a = \frac{20}{8}$

$$\boxed{\Rightarrow a = \frac{5}{2}}$$

5 Find the angle between the Normals to the Surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

$\Longrightarrow$

$$xy = z^2$$

$$xy - z^2 = 0$$

$$\phi = xy - z^2$$

$$P = (4,1,2) \qquad Q = (3,3,-3)$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$(\nabla\phi)_P = 1\bar{i} + 4\bar{j} - 4\bar{k} = \bar{A}$$

$$(\nabla\phi)_Q = 3\bar{i} + 3\bar{j} + 6\bar{k} = \bar{B}$$

W.K.T $\quad \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$

$$\cos\theta = \frac{(\bar{i} + 4\bar{j} - 4\bar{k})\cdot(3\bar{i} + 3\bar{j} + 6\bar{k})}{\sqrt{(1)^2 + (4)^2 + (4)^2}\sqrt{(3)^2 + (3)^2 + (6)^2}}$$

$$\cos\theta = \frac{3 + 12 - 24}{\sqrt{33}\sqrt{54}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\; 3\sqrt{6}}$$

$$\cos\theta = \frac{-9}{3\sqrt{33 \times 6}}$$

$$\cos\theta = \frac{-3}{\sqrt{198}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-3}{\sqrt{198}}\right)}$$

6] If $\vec{F} = (xy^3z^2)$ (or) $\vec{F} = \text{grad}(xy^3z^2)$ find $\text{div}\vec{F}$ and $\text{curl}\vec{F}$ at the point $(1,-1,1)$

$\Rightarrow$

$$\vec{F} = \nabla(xy^3z^2)$$

$$\vec{F} = \nabla\phi$$

$$\therefore \phi = xy^3z^2$$

$$\frac{\partial\phi}{\partial x} = y^3z^2, \quad \frac{\partial\phi}{\partial y} = 3y^2xz^2, \quad \frac{\partial\phi}{\partial z} = 2xy^3z$$

$$\vec{F} = \nabla\phi$$

$$\vec{F} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\vec{F} = y^3z^2\bar{i} + 3y^2xz^2\bar{j} + 2xy^3z\bar{k}$$

(i) $\text{div}\vec{F} = \nabla\cdot\vec{F}$

$$\text{div}\vec{F} = \left|\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right| \cdot \left|y^3z^2\bar{i} + 3y^2xz^2\bar{j} + 2xy^3z\bar{k}\right|$$

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial(3y^2xz^2)}{\partial y} + \frac{\partial(2xy^3z)}{\partial z}$$

$$\text{div}\vec{F} = 0 + 6xyz^2 + 2xy^3$$

$$\text{div}\vec{F} = 6xyz^2 + 2xy^3$$

$$\text{div}\vec{F}_{P(1,-1,1)} = 6(1)(-1)(1)^2 + 2(1)(-1)^3$$

$$\text{div}\vec{F} = -6-2$$

$$\text{div}\vec{F} = -8$$

(ii) $\text{curl}\vec{F} = \nabla\times\vec{F}$

$$\text{curl}\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$\text{curl}\vec{F} = (6xy^2z - 6xy^2z)\bar{i} - (2y^3z - y^3 2z)\bar{j} + (3y^2z^2 - 3y^2z^2)\bar{k}$

$\text{curl}\vec{F} = \left[\left(6(1)(-1)^2(1) - 6(1)(-1)^2(1)\right)\bar{i}\right] - \left[2(-1)^3(1) - (-1)^3 2(1)\right]\bar{j}$

$+ \left[3(-1)^2(1) - 3(-1)^2(1)\right]$

$\text{curl}\vec{F} = [6-6]\bar{i} - [2-2]\bar{j} + [3-3]\bar{k}$

$\text{curl}\vec{F} = 0\bar{i} - 0\bar{j} + 0\bar{k}$

$\text{curl}\vec{F} = \vec{0}$ i.e irrotational

7. If $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$ find $\text{div}\vec{F}$ and $\text{curl}\vec{F}$

$\Longrightarrow$ $\vec{F} = \nabla(x^3+y^3+z^3-3xyz) \longrightarrow (1)$

$\phi = x^3+y^3+z^3-3xyz$

$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz \Rightarrow 3(x^2 - yz)$

$\frac{\partial\phi}{\partial y} = 3y^2 - 3xz \Rightarrow 3(y^2 - xz)$

$\frac{\partial\phi}{\partial z} = 3z^2 - 3xy \Rightarrow 3(z^2 - xy)$

$(1) \Rightarrow \vec{F} = \nabla\phi$

$\vec{F} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\vec{F} = 3(x^2 - yz)\bar{i} + 3(y^2 - xz)\bar{j} + 3(z^2 - xy)\bar{k}$

(i) $\text{div}\vec{F} = \nabla\cdot\vec{F}$

$\text{div}\vec{F} = \left|\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right| \cdot \left|3(x^2-yz)\bar{i} + 3(y^2-xz)\bar{j} + 3(z^2-xy)\right|$

$\text{div}\vec{F} = 3\frac{\partial}{\partial x}(x^2-yz) + 3\frac{\partial}{\partial y}(y^2-xz) + 3\frac{\partial}{\partial z}(z^2-xy)$

$\text{div}\vec{F} = 6x + 6y + 6z$

$\text{div}\vec{F} = 6(x+y+z)$

(ii) $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy) \end{vmatrix}$$

$$\text{curl}\,\vec{F} = (-3x+3x)\bar{i} - (-3y+3y)\bar{j} + (-3z+3z)\bar{k}$$

$$\text{curl}\,\vec{F} = \vec{0}$$

8] Find 'a' for which $\vec{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is solenoidal

$\Rightarrow$ $\vec{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$

$\vec{F}$ is a solenoidal

$\Rightarrow \text{div}\,\vec{F} = 0$

$\Rightarrow \nabla\cdot\vec{F} = 0$

$\Rightarrow \left[\frac{\partial \bar{i}}{\partial x} + \frac{\partial \bar{j}}{\partial y} + \frac{\partial \bar{k}}{\partial z}\right]\cdot\left[(x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}\right] = 0$

$\Rightarrow 1 + 1 + a = 0$

$a + 2 = 0$

$\boxed{a = -2}$

9] Find the constants a, b and c such that $\vec{F} = (axy + bz^3)\bar{i} + (3x^2 - cz)\bar{j} + (3xz^2 - y)\bar{k}$ is irrotational also find the scalar potential such that $\vec{F} = \nabla\phi$

$\Rightarrow$ $\vec{F} = (axy + bz^3)\bar{i} + (3x^2 - cz)\bar{j} + (3xz^2 - y)\bar{k} \longrightarrow (1)$

Given that $\text{curl}\,\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \vec{0}$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+bz^3) & (3x^2-cz) & (3xz^2-y) \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$\Rightarrow (-1+0)\bar{i} - (3z^2-3bz^2)\bar{j} + (6x-ax)$$

$$\Rightarrow -1\bar{i} - 3z^2(1-b)\bar{j} + x(6-a) = \bar{0}\bar{i} + \bar{0}\bar{j} + \bar{0}\bar{k}$$

$3z^2(1-b)=0$ $\quad$ $x(6-a)=0$ $\quad$ $-1+c=0$

$1-b=0$ $\quad$ $6-a=0$ $\quad$ $c=1$

$b=1$ $\quad$ $6=a$

Scalar potential Equation ①

$$\vec{F} = (6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xz^2-y)\bar{k}$$

$$\vec{F} = \nabla\phi$$

$$(6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xz^2-y)\bar{k} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 6xy+z^3$$

$$\Rightarrow \int \partial\phi = \int (6xy+z^3)\,dx$$

$$\phi = 3x^2y + xz^3 + f_1(y.z) \longrightarrow ②$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z$$

$$\Rightarrow \int \partial\phi = \int 3x^2 - z\,\partial y$$

$$\phi = 3x^2y - yz + f_2(x.z) \longrightarrow ③$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y$$

$$\Rightarrow \int \partial\phi = \int 3xz^2 - y\;dz$$

$$\phi = xz^3 - yz + f_3(x.y) \longrightarrow ④$$

from ② ③ and ④

$$\boxed{\phi = 3x^2y - yz + xz^3 + c}$$

10 Show that $\vec{F} = \frac{x\bar{i} + y\bar{j}}{x^2+y^2}$ is Solenoidal and irrotational

$\Rightarrow$ $\vec{F} = \frac{x\bar{i} + y\bar{j}}{x^2+y^2}$

$\vec{F} = \frac{x\bar{i}}{x^2+y^2} + \frac{y\bar{j}}{x^2+y^2} + 0\bar{k}$

(i) $\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$

$= \left[\frac{\partial \bar{i}}{\partial x} + \frac{\partial \bar{j}}{\partial y} + \frac{\partial \bar{k}}{\partial z}\right] \cdot \left[\frac{x}{x^2+y^2}\bar{i} + \frac{y}{x^2+y^2}\bar{j} + 0\bar{k}\right]$

$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + 0$

$= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$

$= \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$

$= \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2}$

$\operatorname{div}\vec{F} = 0$

$\vec{F}$ is a Solenoidal

(ii) $\operatorname{curl}\vec{F} = \nabla \times \vec{F}$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= (0-0)\bar{i} - (0-0)\bar{j} + \left[\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right]\bar{k}$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$= \vec{0}$$

Curl $\vec{F} = \vec{0}$ it is a irrotational

11. Show that the vector field $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is an irrotational

$\Longrightarrow$ $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix}$$

Curl $\vec{F} = (-1+1)\bar{i} - (-1+1)\bar{j} + (-1+1)\bar{k}$

Curl $\vec{F} = 0\bar{i} + (1-1)\bar{j} + 0\bar{k}$

Curl $\vec{F} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Curl $\vec{F} = \vec{0}$

$\therefore$ $\vec{F}$ is a irrotational

12. Find the constant a, b and c such that $\vec{F} = (x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (x+cy+2z)\bar{k}$ is an irrotational also find the scalar potential ϕ for which $\vec{F} = \nabla\phi$

$\Longrightarrow$ $\vec{F} = (x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (x+cy+2z)\bar{k} \rightarrow (1)$

$\vec{F}$ is an irrotational

$\Rightarrow \text{curl}\,\vec{F} = \vec{0}$

$\Rightarrow \nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y+z & (x+cy+2z) \end{vmatrix} = 0\bar{i}+0\bar{j}+0\bar{k}$$

$\Rightarrow (c+1)\bar{i} - (1-a)\bar{j} + (b-1)\bar{k} = 0\bar{i}+0\bar{j}+0\bar{k}$

$\Rightarrow (c+1)\bar{i} + (a-1)\bar{j} + (b-1)\bar{k} = 0\bar{i}+0\bar{j}+0\bar{k}$

$c=-1 \quad a=1 \quad b=1$

$(a.b.c) \Rightarrow (-1, 1.1)$

By rewritting ① we get

① $\Rightarrow \vec{F} = (x+y+z)\bar{i} + (x+2y-z)\bar{j} + (x-y+2z)\bar{k}$

Given that $\vec{F} = \nabla\phi$

$\Rightarrow (x+y+z)\bar{i} + (x+2y-z)\bar{j} + (x-y+2z)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\frac{\partial\phi}{\partial x} = x+y+z$

$\Rightarrow \int\partial\phi = \int x+y+z \;\partial x$

$\phi = \frac{x^2}{2} + xy + zx + f_1(y.z) \longrightarrow ②$

$\frac{\partial\phi}{\partial y} = x+2y-z$

$\Rightarrow \int\partial\phi = \int x+2y-z \;\partial y$

$\phi = xy + y^2 - zy + f_2(x.z) \longrightarrow ③$

$\frac{\partial\phi}{\partial z} = x-y+2z$

$\Rightarrow \int\partial\phi = \int x-y+2z \;\partial z$

$\phi = xz - yz + z^2 + f_3(x.y) \longrightarrow ④$

from ② ③ and ④

$$\boxed{\phi = \frac{x^2}{2} + y^2 + z^2 + xy - zy + zx + C}$$

13 Find the constant a, b, c so that the vector field $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is an irrotational, also find the scalar potential ϕ for which $\vec{F} = \nabla\phi$

$$\Rightarrow \vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k} \rightarrow ①$$

Given that

$\vec{F}$ is an irrotational

$$\Rightarrow \operatorname{curl}\vec{F} = \vec{0}$$

$$\Rightarrow \nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\Rightarrow (c+1)\vec{i} - (4-a)\vec{j} + (b+2)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$c+1=0$ $\quad$ $4-a=0$ $\quad$ $b+2=0$

$c=-1$ $\quad$ $a=4$ $\quad$ $b=-2$

$\Rightarrow$ ① rewrite as

$$\vec{F} = (x+2y+4z)\vec{i} + (-2x-3y-z)\vec{j} + (4x-y+2z)\vec{k}$$

Given that $\vec{F} = \nabla\phi$

$$\Rightarrow (x+2y+4z)\vec{i} + (-2x-3y-z)\vec{j} + (4x-y+2z)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial \phi}{\partial x} = x - 2y + 4z$$

$$\Longrightarrow \int \partial \phi = \int (x - 2y + 4z)\, dx$$

$$\phi = \frac{x^2}{2} - 2xy + 4xz + f_1(y, z) \longrightarrow (2)$$

$$\frac{\partial \phi}{\partial y} = -2x - 3y - z$$

$$\Longrightarrow \int \partial \phi = \int (-2x - 3y - z)\, \partial y$$

$$\phi = -2yx - \frac{3y^2}{2} - zy + f_2(x, z) \longrightarrow (3)$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z$$

$$\Longrightarrow \int \partial \phi = \int (4x - y + 2z)\, \partial z$$

$$\phi = 4xz - yz + z^2 + f_3(x, y) \longrightarrow (4)$$

from (2) (3) and (4)

$$\boxed{\vec{F} = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 - 2xy + 4xz - zy + C}$$

14) Show that $\vec{F} = (2xy + z^3)\bar{i} + x^2\bar{j} + (3xz^2)\bar{k}$ is irrotational (or) force field (or) conservative force (or) potential field find the scalar potential ϕ such that $\vec{F} = \nabla \phi$

$$\Longrightarrow \text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$= (0-0)\bar{i} - (3z^2 - 3z^2)\bar{j} + (2x - 2x)\bar{k}$

$= 0\bar{i} + 0\bar{j} + 0\bar{k}$

$= \vec{0}$ is a Irrotational

$\vec{F} = \nabla\phi$

$\Rightarrow (2xy + z^3)\bar{i} + x^2\bar{j} + (3xz^2)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\frac{\partial\phi}{\partial x} = 2xy + z^3$

$\Longrightarrow \int\partial\phi = \int 2xy + z^3$

$\phi = x^2y + xz^3 + f_1(y.z) \longrightarrow (1)$

$\frac{\partial\phi}{\partial y} = x^2$

$\Longrightarrow \int\partial\phi = \int x^2\,\partial y$

$\phi = x^2y + f_2(x,z) \longrightarrow (2)$

$\frac{\partial\phi}{\partial z} = 3xz^2$

$\Longrightarrow \int\partial\phi = \int 3xz^2\,\partial z$

$\phi = xz^3 + f_3(x.y) \longrightarrow (3)$

from (1) (2) and (3)

$$\boxed{\phi = x^2y + xz^3 + c}$$

$\boxed{17}$ Find the constants a & b such that $\vec{F} = (axy+z^3)\bar{i} + (3x^2-z)\bar{j} + (bxz^2-y)\bar{k}$ is an irrotational and also find the scalar potential for which $\vec{F} = \nabla\phi$

$\Rightarrow$ $\vec{F} = (axy+z^3)\bar{i} + (3x^2-z)\bar{j} + (bxz^2-y)\bar{k}$

from given

$\Rightarrow$ curl $\vec{F} = \vec{0}$

$\Rightarrow$ $\nabla\times\vec{F} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+z^3) & (3x^2-z) & (bxz^2-y) \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$\Rightarrow$ $(-1+1)\bar{i} - (bz^2-3z^2)\bar{j} + (6x-ax)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$\Rightarrow$ $0\bar{i} + (3-b)z^2\bar{j} + (6-a)x\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$(3-b)z^2 = 0$ $\quad$ $(6-a)x = 0$

$3-b=0$ $\quad$ $6-a=0$

$3=b$ $\quad$ $a=6$

rewrite equation ①

$\vec{F} = (6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xy^2-y)\bar{k}$

$\vec{F} = \nabla\phi$

$(6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xy^2-y)\bar{k} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 6xy + z^3$

$\Rightarrow \int\partial\phi = \int 6xy + z^3\, \partial x$

$\phi = 3x^2y + xz^3 + f_1(y,z) \longrightarrow$ ②

$\frac{\partial\phi}{\partial y} = 3x^2 - z$

$\Rightarrow \int\partial\phi = \int 3x^2 - z\, \partial y$

$\phi = 3x^2y - yz + f_2(x,z) \longrightarrow$ ③

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y$$

$$\Rightarrow \int \partial\phi = \int 3xz^2 - y \, \partial z$$

$$\phi = xz^3 - yz + f_3(x,y) \rightarrow (4)$$

from (2) (3) and (4)

$$\boxed{\phi = 3x^2y - yz + xz^3 + c}$$

Directional derivatives

16 Find the directional derivatives of $\phi = 4xz^3 - 3x^2y^2z$ and $(2,-1,2)$ along $\vec{e}$ $2\bar{i} - 3\bar{j} + 6\bar{k}$

$\Rightarrow$
$$\phi = 4xz^3 - 3x^2y^2z$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = (4z^3 - 6xy^2z)\bar{i} + (-6x^2yz)\bar{j} + (12xz^3 - 3x^2y^2)$$

$$\nabla\phi_{p(2,-1,2)} = (32-24)\bar{i} + 48\bar{j} + (96-12)\bar{k}$$

$$\nabla\phi = 8\bar{i} + 48\bar{j} + 84\bar{k}$$

and given that $\vec{d} = 2\bar{i} - 3\bar{j} + 6\bar{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{\sqrt{4+9+36}}$$

$$\hat{n} = \frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{7}$$

$$DD = \nabla\phi \cdot \hat{n}$$

$$DD = (8\bar{i} + 48\bar{j} + 84\bar{k}) \cdot \left(\frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{7}\right)$$

$$DD = \frac{1}{7}\left[16 - 144 + 504\right]$$

$$DD = \frac{376}{7}$$

$$\boxed{DD = 53.7143}$$

17 Find the directional derivative of $\phi = x^2 + y^2 + 2z^2$ at $(1,2,3)$ along the direction of line $\overrightarrow{PQ} = 4i - 2j + k$

$\Longrightarrow$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 4z\bar{k}$$

$$\nabla\phi \; P(1,2,3) = 2\bar{i} + 4\bar{j} + 12\bar{k}$$

and given that $\vec{d} = 4\bar{i} - 2\bar{j} + \bar{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{(4)^2 + (-2)^2 + (1)^2}}$$

$$\hat{n} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{16 + 4 + 1}}$$

$$\hat{n} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{12}}$$

$$D.D = \nabla\phi \cdot \hat{n}$$

$$= (2\bar{i} + 4\bar{j} + 12\bar{k}) \cdot \left(\frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{12}}\right)$$

$$= \frac{1}{\sqrt{12}}(8 - 8 + 12)$$

$$\boxed{DD = \frac{12}{\sqrt{12}}}$$

18 Find the DD of $xy^3 + yz^3$ at $(2,-1,1)$ in the direction of the Vector $\bar{i}+2\bar{j}+2\bar{k}$

$\Longrightarrow$

$$\phi = xy^3 + yz^3$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = (y^3)\bar{i} + (3xy^2+z^3)\bar{j} + (3yz^2)\bar{k}$$

$$\nabla\phi_P = (-1)^3\bar{i} + (6+1)\bar{j} + (-3)\bar{k}$$

$$\nabla\phi_P = 1\bar{i} + 7\bar{j} - 3\bar{k}$$

$$\vec{d} = \bar{i}+2\bar{j}+2\bar{k}$$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{(1)^2+(2)^2+(2)^2}}$$

$$\hat{n} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{3}$$

$$DD = \nabla\phi \cdot \hat{n}$$

$$= (-\bar{i}+7\bar{j}-3\bar{k}) \cdot \left(\frac{\bar{i}+2\bar{j}+2\bar{k}}{3}\right)$$

$$= \frac{1}{3}(-1+14-6)$$

$$\boxed{DD = \frac{7}{3}}$$

19 If $\vec{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$ show that $\vec{F} \cdot \text{curl}\,\vec{F} = 0$

$\Longrightarrow$ $\vec{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$\text{curl}\,\vec{F} = (-1-0)\bar{i} - (-1-0)\bar{j} + (0-1)\bar{k}$$

$$\text{curl}\,\vec{F} = (-1)\bar{i} - (-1)\bar{j} - \bar{k}$$

$$\therefore \vec{F} \cdot \text{curl}\,\vec{F} = \left[(x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}\right] \cdot \left(-\bar{i} + \bar{j} - \bar{k}\right)$$

$$= (-x+y+1) + (1+x+y)$$

$$\boxed{\vec{F} \cdot \text{curl}\,\vec{F} = \underline{0}}$$

Vector Integration

Line integral :- Any integral which is to be evaluated along the curve c is called a line integral. Suppose $\vec{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a vector point function in (x, y, z) and $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$ a position vector, then the line integral of c can defined as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c [f_1 dx + f_2 dy + f_3 dz] \cdot [dx\bar{i} + dy\bar{j} + dz\bar{k}]$$

$$\boxed{\int_c \vec{F} \cdot d\vec{r} = \int_c (f_1 dx + f_2 dy + f_3 dz)}$$

19 If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ Evaluate $\int_c \vec{F}\cdot\vec{dr}$ where c is the Clariable of $y = 2x^2$ from point (0,0) to (1,2)

$\Rightarrow$ $\vec{F} = 3xy\vec{i} - y^2\vec{j} \longrightarrow (1)$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\therefore \vec{F}\cdot\vec{dr} = (3xy\vec{i} - y^2\vec{j})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$\Rightarrow \vec{F}\cdot\vec{dr} = 3xy\,dx - y^2dy \longrightarrow (2)$

and given $y = 2x^2$

diff w.r.t 'x'

$\Rightarrow \frac{dy}{dx} = 4x$

$\Rightarrow dy = 4x\,dx$

(2) $\Rightarrow \vec{F}\cdot\vec{dr} = 3x(2x^2)dx - (2x^2)^2 4x\,dx$

$= 6x^3dx - 16x^5dx$

$\Rightarrow \vec{F}\cdot\vec{dr} = (6x^3 - 16x^5)dx$

$\therefore \int_c \vec{F}\cdot\vec{dr} = \int_c (6x^3 - 16x^5)\,dx$

$= \int_{x=0}^{1} (6x^3 - 16x^5)\,dx$

$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$

$= \frac{6}{4} - \frac{16}{6}$

$= \frac{3}{2} - \frac{8}{3}$

$= \frac{9-16}{6}$

$$\boxed{\int_c \vec{F}\cdot\vec{dr} = \frac{-7}{6}}$$

Q) If $\vec{F} = x^2\bar{i} + xy\bar{j}$ Evaluate $\int \vec{F}\cdot\vec{dr}$ when from (0.0) TO (1,1) along with the curves i] $y = x$
ii] $y = \sqrt{x}$

$\Rightarrow \vec{F} = x^2\bar{i} + xy\bar{j} \longrightarrow (1)$

Let $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow \vec{dr} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \vec{F}\cdot\vec{dr} = (x^2\bar{i} + xy\bar{j})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$\Rightarrow \vec{F}\cdot\vec{dr} = x^2dx + xydy \longrightarrow (2)$

and given i) Towards the curves $c: y = x$

$\frac{dy}{dx} = 1$

$dy = dx$

$(2) \Rightarrow \vec{F}\cdot\vec{dr} = x^2dx + x.xdx$

$\vec{F}\cdot\vec{dr} = 2x^2dx$

$\int_c \vec{F}\cdot\vec{dr} = \int_c (2x^2)dx$

$= \int_0^1 (2x^2)dx$

$= \frac{2x^3}{3}\Big|_0^1 dx$

$\int_c \vec{F}\cdot\vec{dr} = \frac{2}{3}$

ii) Towards the curve $y = \sqrt{x}$

$\Rightarrow y^2 = x$

$2y = \frac{dx}{dy}$

$\Rightarrow dx = 2y\,dy$

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{3}=t$$

$$x=2t\ ,\quad y=t\ ,\quad z=3t$$

$$dx=2t\ ,\quad dy,dt\ ,dz=3dt$$

$$① \Rightarrow \vec{F}\cdot\vec{dr} = 3(2t)^2 2dt + 2(3t)(2t) - t\,dt + 3t\cdot 3dt$$

$$= 24t^2\,dt + 12t^2\,dt - t\,dt + 3t\,dt$$

$$= (36t^2+8t)\,dt$$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^1 (36t^2+8t)\,dt$$

$$= \left[\frac{36t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$= \frac{36}{6} + \frac{8}{2}$$

$$= 12+4$$

$$\int_C \vec{F}\cdot\vec{dr} = 16$$

(ii) $\vec{F}\cdot\vec{dr} = 3x^2dx + (2zx-y)dy + z\,dz \longrightarrow ①$

$x^2 = 4y \rightarrow ②$

$3x^2 = 8z \longrightarrow ③$

$x=0,\quad x=2$

Let $x=t$

$dx=dt$

$$② \Rightarrow \quad y=\frac{x^2}{4}$$

$$y=\frac{t^2}{4}$$

$$dy=\frac{2t}{4}dt$$

$$dy=\frac{1}{2}t\,dt$$

(2) $\Rightarrow \vec{F}\cdot\vec{dr} = (y^2)^2\, 2y\,dy + y(y^2)\,dy$

$$= 2y^5\,dy + y^3\,dy$$

$$\vec{F}\cdot\vec{dr} = dy(2y^5 + y^3)$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_C (2y^5 + y^3)\,dy$$

$$= \int_0^1 (2y^5 + y^3)\,dy$$

$$= \left[\frac{2y^6}{6} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{2}{6} + \frac{1}{4}$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$\boxed{\int_C \vec{F}\cdot\vec{dr} = \frac{7}{12}}$$

Q] Find the work done is moving a particle the force filled $\vec{F} = 3x^2\vec{i} + (2zx - y)\vec{j} + z\vec{k}$ along

i) A straight line from (0,0,0) to (2,1,3)

ii) The curve defined by $x^2 = 4y$, $3x^2 = 8z$ from $x=0$ to $x=2$

$\Rightarrow$

$$\vec{F} = 3x^2\vec{i} + (2zx - y)\vec{j} + z\vec{k}$$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow \vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot\vec{dr} = [3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}]\cdot[dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$\vec{F}\cdot\vec{dr} = 3x^2dx + (2zx - y)dy + z\,dz \longrightarrow (1)$$

i] Given the curve along the straight line passing through (0.0.0) to (1,2,3)

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$x = 2t$, $y = t$, $z = 3t$

$dx = 2t$, dy, dt , $dz = 3dt$

① $\Rightarrow \vec{F}\cdot\vec{dr} = 3(2t)^2 2\,dt + 2(3t)(2t) - t\,dt + 3t \cdot 3dt$

$= 24t^2\,dt + 12t^2\,dt - t\,dt + 3t\,9t$

$= (36t^2 + 8t)\,dt$

$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^1 (36t^2 + 8t)\,dt$

$= \left[\frac{36t^3}{3} + 8\frac{t^2}{2}\right]_0^1$

$= \frac{36}{6} + \frac{8}{2}$

$= 12 + 4$

$\int_C \vec{F}\cdot\vec{dr} = 16$

(ii) $\vec{F}\cdot\vec{dr} = 3x^2 dx + (2zx - y)dy + z\,dz \longrightarrow$ ①

$x^2 = 4y \to$ ②

$3x^2 = 8z \longrightarrow$ ③

$x = 0, \; x = 2$

Let $x = t$

$dx = dt$

② $\Rightarrow$ $y = \frac{x^2}{4}$

$y = \frac{t^2}{4}$

$dy = \frac{2t}{4}\,dt$

$dy = \frac{1}{2}t\,dt$

$$③ \Rightarrow z = \frac{3x^2}{8}$$

$$z = \frac{3t^2}{8}$$

$$dz = \frac{9t^2}{8} dt$$

$$① \Rightarrow \vec{F} \cdot \vec{dr} = (3t^2)(dt) - \left[2 \times \frac{3t^2}{8} \times t - \frac{t^2}{4}\right]\left[\frac{t^2}{2} dt\right] + \frac{3t^3}{8} \times \frac{9t^2}{8} dt$$

$$= \left[3t^2 + \left(\frac{3t^4}{4} - \frac{t^2}{4}\right)\frac{t}{2} + \frac{27t^5}{64}\right] dt$$

$$= \left[3t^2 + \frac{3t^5}{8} - \frac{t^3}{8} + \frac{27t^5}{64}\right] dt$$

$$= \left[3t^2 - \frac{t^3}{8} + \frac{51t^5}{64}\right] dt$$

$$\int_c \vec{F} \cdot \vec{dr} = \int_0^2 \left[3t^2 - \frac{t^3}{8} + \frac{51}{64} t^5\right] dt$$

$$= \left[t^3 - \frac{t^4}{32} + \frac{51}{64} \frac{t^6}{6}\right]_0^2$$

$$= 8 - \frac{16}{32} + \frac{51}{384}(64)$$

$$= 8 - \frac{1}{2} + \frac{51}{6}$$

$$= \frac{48 - 3 + 51}{6}$$

$$= \frac{96}{6}$$

$$\boxed{\int_c \vec{F} \cdot \vec{dr} = 16}$$

22) If $\vec{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ Evaluate $\int_c \vec{F}\cdot\vec{dr}$ from (0.0.0) to (1.1.1) along the curve given by $x=t$. $y=t^2$, $z=t^3$

$\Longrightarrow$ $\vec{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$

$\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\vec{dr} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\vec{F}\cdot\vec{dr} = [(3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}] \cdot [dx\bar{i} + dy\bar{j} + dz\bar{k}]$

$\vec{F}\cdot\vec{dr} = (3x^2+6y)dx - 14yzdy + 20xz^2dz$

$\vec{F}\cdot\vec{dr} = 3x^2 + 6ydx - 14yzdy + 20xz^2dz \rightarrow (1)$

$x = t \rightarrow (1)$ $\quad z = t^3$

$(2) \Rightarrow dx = dt$ $\quad (4) \Rightarrow dz = 3t^2dt$

$y = t^2 \rightarrow (2)$

$(3) \Rightarrow dy = 2tdt$

$(1) \Rightarrow \vec{F}\cdot\vec{dr} = (3t^2+6t^2)dt - 14t^2t^3\,2tdt + 20t\,t^6\,3t^2dt$

$\int_c \vec{F}\cdot\vec{dr} = \int_c (9t^2 - 28t^6 + 60t^9)\,dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$

$= \left[3t^3 - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$

$= 3 - \frac{28}{7} + 6$

$= \frac{21 - 28 + 42}{7}$

$= \frac{35}{7}$

$\int_c \vec{F}\cdot\vec{dr} = 5$

Green's theorem :-

Statement :- If $M(x,y)$ & $N(x,y)$ be the two continous functions of x & y having continous partial Integratives $\frac{\partial m}{\partial y}$ & $\frac{\partial n}{\partial x}$ in the region R of the xy - plane bounded by a closed Curve

$$\int_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial m}{\partial x} - \frac{\partial n}{\partial y}\right) dx\,dy$$

23\. Evaluate $\oint_C (3x^2-8y^2)dx + (4y-6xy)$ where c is the boundary of the region Enclosed by $y=\sqrt{x}$ & $y=x^2$

$\Longrightarrow$

$$\oint_C M dx + \oint_C N dy = \oint_C (3x^2-8y^2)dx + (4y-6xy)dy$$

$M = 3x^2-8y^2 \qquad N = 4y-6xy$

$\frac{\partial m}{\partial y} = -16y \qquad \frac{\partial n}{\partial x} = -6y$

$$\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = -6y+16y$$

$$\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 10y$$

The curve bounded by $y=\sqrt{x}$ and $y=x^2$

$\therefore$ By the Green's theoram W.K.T

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial m}{\partial x} - \frac{\partial n}{\partial y}\right) dx\,dy$$

$$= \iint_R (10y)\,dx\,dy$$

$$= 10\int_{x=0}^{1}\int_{y=\sqrt{x}}^{x^2} y\,dy\,dx$$

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_{\sqrt{x}}^{x^2} dx$$

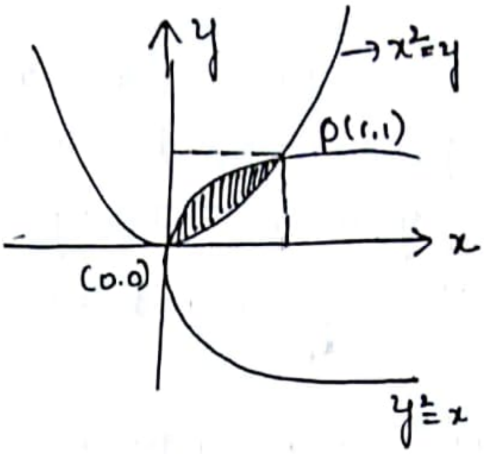
$$= 5\int_0^1 (x^4 - x)\,dx$$

$$= 5\left[\frac{x^5}{5} - \frac{x^2}{2}\right]_0^1$$

$$= 5\left[\frac{-3}{10}\right]$$

$$= \left|\frac{-3}{2}\right|$$

$$= \frac{3}{2}$$



$\boxed{24}$ Evaluate $\oint_c (xy+y^2)dx + x^2dy$, where c is the closed curve of the region bounded by $y=x$ and $y=x^2$ Using Green's theoram

$\Longrightarrow$

$$\oint_c M dx + \oint_c N dy = \oint (xy+y^2)\,dx + x^2 dy$$

$M = xy + y^2$ $\qquad$ $N = x^2$

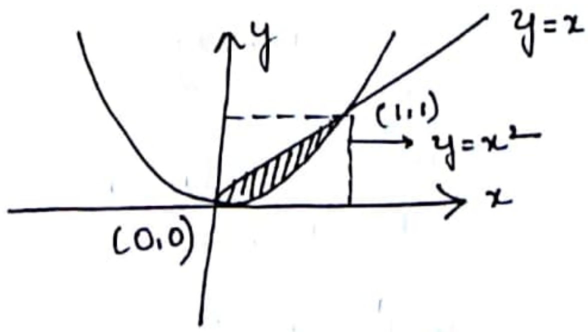
$\frac{\partial m}{\partial y} = x + 2y$ $\qquad$ $\frac{\partial n}{\partial x} = 2x$

$$\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 2x - x - 2y$$

$$\frac{\partial m}{\partial x} - \frac{\partial m}{\partial y} = x - 2y$$

The curve c bounded by $y=x$ & $y=x^2$

$\therefore$ By the Green's theoram. W.K.T



$$\oint_C M\,dx + N\,dy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow I = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[(x^2 - x^2) - (x^3 - x^4)\right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \frac{x^5}{5} - \frac{x^4}{4}\Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$I = \frac{-1}{20}$$

25. Use Green's theorem to Evaluate $\oint (x^2+y^2)\,dx + 3x^2y\,dy$ where c is the circle $x^2+y^2=4$ traced in the positive Signs upper half of the circle

$$\Rightarrow \oint_C M\,dx + \oint_C N\,dy = \oint (x^2+y^2)\,dx + 3x^2y\,dy$$

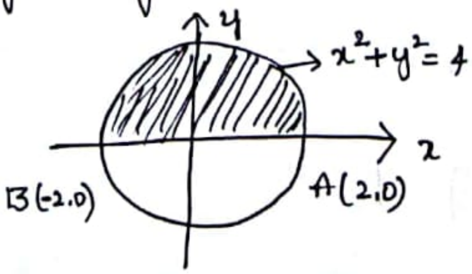
$M = x^2+y^2 \qquad N = 3x^2y$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 6xy$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6xy - 2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2y(3x-1)$$

$\therefore$ By the green's theoram



$$\oint_C M\,dx + N\,dy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$I = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} 2 4 (3x-1)\,dy\,dx$$

$$= 2\int_{x=-2}^{2} (3x-1)(4-x^2)\,dx$$

$$= \int_{x=-2}^{2} (12x - 3x^3 - 4 + x^2)\,dx$$

$$= \int_{-2}^{2} (-3x^3 + x^2 + 12x - 4)\,dx$$

$$= \left[\frac{-3}{4}x^4 + \frac{x^3}{3} + 6x^2 - 4x \right]_{-2}^{2}$$

$$= \left[-12 + \frac{8}{3} + 24 - 8 \right] - \left[-12 - \frac{8}{3} + 24 + 8 \right]$$

$$= \cancel{-12} + \frac{8}{3} + \cancel{24} - 8 + \cancel{12} + \frac{8}{3} - \cancel{24} - 8$$

$$I = \frac{-32}{3}$$

$\boxed{26}$ Find the Area between the parabolas $y^2 = 4x$ & $x^2 = 4y$ by Using Green's theoram

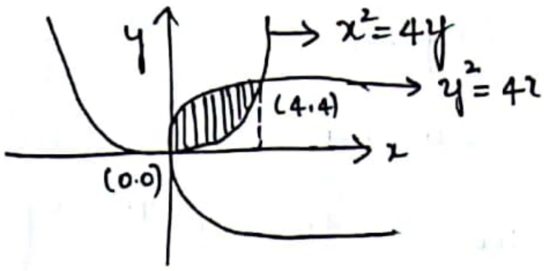
$\Longrightarrow$
$$\int_C M\,dx + N\,dy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy \longrightarrow (1)$$

$$\int_C M\,dx + N\,dy = \frac{1}{2} \int (-y\,dx + x\,dy)$$

$$M = \frac{-y}{2}, \quad N = \frac{x}{2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2} \qquad \frac{\partial N}{\partial x} = \frac{1}{2}$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$



$$① \Rightarrow \int M\,dx + N\,dy = \iint_{R_1} 1\,dx\,dy$$

$$= \int_{x=0}^{4} \int_{y=x^2/4}^{2\sqrt{x}} (1)\,dy\,dx$$

$$= \int_{x=0}^{4} [y]_{x^2/4}^{2\sqrt{x}}$$

$$= \int_{x=0}^{4} \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= 2\int_0^4 x^{1/2} - \frac{1}{4}\int_0^4 x^2\,dx$$

$$= \left[\frac{2x^{3/2}}{\frac{3}{2}}\right]_0^4 - \frac{1}{4}\left[\left[\frac{x^3}{3}\right]\right]_0^4$$

$$= \frac{4}{3}(4^{3/2}) - \frac{1}{12}(4^3)$$

$$= \frac{4\times 8}{3} - \frac{64}{12}$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3} \text{ sq. units}$$

27] Evaluate $\int_C (x^2+y^2)dx + 3x^2y\,dy$. where c is the circle $x^2+y^2=4$ traced in the positive sigus

$\Rightarrow$ $$\int M\,dx + N\,dy = \int_C (x^2+y^2)dx + 3x^2y\,dy$$

$M = x^2+y^2 \qquad N = 3x^2y$

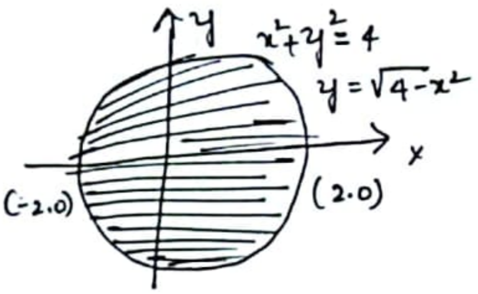
$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 6xy$

$$\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=6xy-2y$$

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=2y(3x-1)$$

And given that the circle having the radius 2 and $x^2+y^2=4$ in the +ve sense

$\therefore$ By green's theorem



$$\int_C M\,dx+N\,dy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$\Rightarrow \int_C (x^2+y^2)dx+3x^2y\,dy=\int_{x=-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y(3x-1)\,dy\,dx$$

$$=\int_{-2}^{2} 2(3x-1)\int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx$$

$$=\int_{-2}^{2} 2(3x-1)\left[\frac{y^2}{2}\right]_{\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$=\int_{-2}^{2}(3x-1)\left[(4-x^2)-(4-x^2)\right]dx$$

$$=\underline{\underline{0}}$$

Stoke's theoram

Suppose S be an open surface bounded by a closed curve c & if $\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be any vector point function having the continuous fun then $\oint_c \vec{F} \cdot \vec{dr} = \iint_S \text{curl} \vec{F} \cdot \hat{n}\, ds$, where $\hat{n}$ is the draw unit Normal vector to the surface S (or)

$$\hat{n}\, ds = dy\,dz\,\bar{i} + dz\,dx\,\bar{i} + dx\,dy\,\bar{k}$$

$\boxed{28}$ Using Stoke's theoram for $\vec{F} = y\bar{i} + z\bar{j} + x\bar{k}$. where S is a upper half of the sphere $x^2+y^2+z^2=1$ and c is its boundary

$\Longrightarrow$ $\vec{F} = y\bar{i} + z\bar{j} + x\bar{k}$

$$\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= (0-1)\bar{i} - (1-0)\bar{j} + (0-1)\bar{k}$$

$$\text{Curl} \vec{F} = -\bar{i} - \bar{j} - \bar{k}$$

Given that surface 'S' is upper half of the sphere $x^2+y^2+z^2=1$ its Normal $\hat{n} = \bar{k}$

$$\therefore \text{curl} \vec{F} \cdot \hat{n} = (-\bar{i} - \bar{j} - \bar{k}) \cdot \bar{k} = -1$$

By the Stoke's Theoram W.R.T

$$\oint \vec{F} \cdot \vec{dr} = \iint_S \text{curl} \vec{F} \cdot \hat{n}\, ds$$

$$= \iint -1\, ds$$

$$= -\iint 1\, ds$$

$$= -A$$

where A is the Area of the circle $x^2+y^2=1$ for which $z=0$

$$A = \pi r^2$$
$$= \pi \times 1$$
$$A = \pi$$
$$\therefore \int_c \vec{F}\cdot d\vec{r} = -\pi$$

Gauss divergence theoram :-

If $\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is a Vector point function Having Continous function in the region V bounded by a Closed Surface S, then $\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{N}\,ds$ where $\hat{n}$ is the Unit Normal Vector to the Surface S

29 Evaluate $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ Taken Over the rectangular parallel piped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ find $\iiint_V \nabla\cdot\vec{F}\cdot dv$

$\Longrightarrow$ $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$\therefore \operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right]\cdot\left[(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy\bar{k})\right]$$

$$= 2x + 2y + 2z$$

$$\operatorname{div}\vec{F} = 2(x+y+z)$$

also given that given Surface is parallello piped bounded b/w $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

$$\iiint_V \operatorname{div}\vec{F} = \iiint_V 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x+y+z)\,dz\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xz+yz+\frac{z^2}{2}\right]_0^c dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xc+cy+\frac{c^2}{2}\right]dy\,dx$$

$$= 2\int_{x=0}^{a}\left[xy+cy^2+y\frac{c^2}{2}\right]_0^b dx$$

$$= 2\int_0^a\left[bc\frac{x^2}{2}+\frac{b^2cx}{2}+\frac{bc^2x}{2}\right]_0^a$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$= abc\,[a+b+c]$$

$\boxed{30}$ By using divergence theorem. Evaluate $\iint \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ & S is the surface enclosing the region for which $x^2+y^2 \le 4$, $0 \le z \le 3$,

$\Longrightarrow$ $\vec{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$

$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$

$$\operatorname{div}\vec{F} = \left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}+\frac{\partial}{\partial z}\bar{k}\right)\cdot\left(4x\bar{i}-2y^2\bar{j}+z^2\bar{k}\right)$$

$\Longrightarrow$ $\operatorname{div}\vec{F} = 4 - 4y + 2z$

z varies from 0 to 3

by using $x^2+y^2=4$

$\Longrightarrow$ $y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

and x varies from -2 to 2

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\cdot dv$$

$$\Rightarrow I = \iiint_V (4-4y+2z)\,dv$$

$$I = 2\int_{x=-2}^{2}\int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(2-2y+z)\,dz\,dy\,dx$$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[2z-2yz+\frac{z^2}{2}\right]\Big|_0^3\,dy\,dx$$

$$= 2\int_{x=-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(6-6y+9/2\right)dy\,dx$$

$$= 2\int_{x=-2}^{2}\left(6y-3y^2+\frac{9}{2}y\right)\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx$$

$$= \int_{x=-2}^{2}\left[21y-6y^2\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\,dx$$

$$= \int_{x=-2}^{2}\left[21\sqrt{4-x^2}-6(4-x^2)\right]-\left[-21\sqrt{4-x^2}-6(4-x^2)\right]$$

$$= 42\int_{x=-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 42*2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84*2*\frac{\pi}{4}$$

$$= \underline{84\pi}$$

31) $\vec{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$. Evaluate $\iiint_V \nabla\cdot\vec{F}\,dv$ where V is the region bounded by the planes $x=0, y=0, z=0$ $2x+2y+z=4$

$\Longrightarrow$ $\vec{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$

$$\operatorname{div}\vec{F} = \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right]\cdot\left[(2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}\right]$$

$$= 4x - 2x + 0$$

$$\operatorname{div}\vec{F} = 2x$$

The planes $x=0, y=0, z=0$ and $2x+2y+z=4$

$\therefore$ By the Gauss theorem we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\cdot dV$$

$$\Longrightarrow \bar{I} = \iiint_V 2x\,dv$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} x\,[z]_0^{4-2x-2y}\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} (4x - 2x^2 - 2xy)\,dy\,dx$$

$$= 2\int_{x=0}^{2} (4xy - 2x^2y - xy^2)_0^{2-x}\,dx$$

$$= 2\int_{x=0}^{2} 4x(2-x) - 2x^2(2-x) - x^2(2-x)^2\,dx$$

$$= 2\int_{x=0}^{2} \left(8x - 4x^2 - 4x^2 + 2x^3 - 4x - x^4 + 4x^3\right)dx$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\left[\frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2\right]_0^2$$

$$= 2\left[4 - \frac{32}{3} + 8\right]$$

$$= 2\left[12 - \frac{32}{3}\right]$$

$$= 2\left[\frac{4}{3}\right]$$

$$I = \frac{8}{3}$$

$\boxed{32}$ Use divergence Theoram to Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ Over the Entire Surface of the region above xy-plane bounded by the Cone $z^2 = x^2 + y^2$ and the plane $z = 4$,

$\vec{F} = 4xz\bar{i} + xyz^2\bar{j} + 3z\bar{k}$

$\Longrightarrow \vec{F} = 4xz\bar{i} + xyz^2\bar{j} + 3z\bar{k}$

$\therefore \operatorname{div}\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\cdot\left(4xz\bar{i} + xyz^2\bar{j} + 3z\bar{k}\right)$$

$\operatorname{div}\vec{F} = 4z + xz^2 + 3$

$x^2 + y^2 = z^2 \longrightarrow (1)$ The plane $z = 4$

$\therefore z$ Varies from 0 to 4

$(1) \Rightarrow x^2 + y^2 = 4^2$

$x^2 + y^2 = 16 \longrightarrow (2)$

when $y = 0$

$(2) \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

$x = -1, \; x = 4$

and $y^2 = 16 - x^2$

$y = \pm\sqrt{16-x^2}$

$y = +\sqrt{16-x^2} \qquad y = -\sqrt{16-x^2}$

W.K.T = $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\cdot dv$

$$\Longrightarrow \iiint_V \operatorname{div}\vec{F}\cdot dv = \iiint_V (4z + xz^3 + 3)\,dv$$

$$= \int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{z=0}^{4} (4z + xz^2 + 3)\,dz\,dy\,dx$$

$$= \int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 2z^2 + 3\frac{z^3}{3} + 3z\Big|_0^4\,dy\,dx$$

$$= \int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(32 + \frac{64}{3}x + 12\right)dy\,dx$$

$$= \int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(44 + \frac{64x}{3} + \right)dy\,dx$$

$$= \frac{1}{3}\int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (132 + 64x)\,dy\,dx$$

$$= \frac{4}{3}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (33 + 16x)\,dy\,dx$$

$$= \frac{4}{3}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (33 + 16x)\,dy\,dx$$

$$= \frac{4}{3}\int_{x=-4}^{4} (33+16x)\,[y]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\, dx$$

$$= \frac{4}{3}\int_{x=-4}^{4} (33+16x)\left(\sqrt{16-x^2}+\sqrt{16-x^2}\right) dx$$

$$= \frac{8}{3}\int_{x=-4}^{4} (16x+33)\sqrt{16-x^2}\, dx$$

$$= \frac{8}{3} * 16 \int_{x=-4}^{4} x\sqrt{16-x^2}\, dx + \frac{88}{3} \times 33 \int_{x=-4}^{4} \sqrt{16-x^2}\, dx$$

$$= \frac{128}{3}(0) + 88 \int_{x=-4}^{4} \sqrt{16-x^2}\, dx$$

$$= 176 \int_{0}^{4} \sqrt{16-x^2}\, dx$$

Put $x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$\Longrightarrow \sin\theta = \frac{x}{4} \Longrightarrow \theta = \sin^{-1}(x/4)$

$x = 4 \Rightarrow \theta = \pi/2$

$x = 0 \Rightarrow \theta = 0$

$$I = 176 \int_{0}^{\pi/2} \sqrt{16 - 16\sin^2\theta}\,.\,(4\cos\theta)\, d\theta$$

$$= 176 \int_{0}^{\pi/2} 4^2 \cos^2\theta\, d\theta$$

$$= 2816 \int_{0}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2816 * \frac{2-1}{2} * \pi/2$$

$$\boxed{I = 704\pi}$$

33] If $\vec{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ is its boundary. Use Stokes theorem to find $\int_C \vec{F}\cdot\vec{dr}$

$\Longrightarrow$ $\vec{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

$x^2+y^2+z^2=1$

$\therefore \text{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & -y^2z \end{vmatrix}$

$= (-2yz+2yz)\bar{i} - (0-0)\bar{j} + (0+1)\bar{k}$

$\text{curl}\vec{F} = \bar{k}$

W.K.T

$\hat{n}ds = dydz\bar{i} + dxdz\bar{j} + dxdy\bar{k}$

$\therefore \text{curl}\vec{F}\cdot\hat{n}ds = [\bar{k}]\cdot[dydz\bar{i} + dxdz\bar{j} + dxdy\bar{k}]$

$= dxdy$

Area of a circle $x^2+y^2=1$

$= \pi r^2$

$= \pi(1)$

$= \pi$

34] A Vector field is given by $\vec{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$ Evaluate the line integral over a circular path given by $x^2+y^2=a^2$, $z=0$

$\Longrightarrow$ $\vec{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$

$\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\vec{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\vec{F}\cdot\vec{dr} = [\sin y\, i + x(1+\cos y)\, j + 0k]\cdot[dx\, i + dy\, j + dz\, k]$$

$$= \sin y\, dx + x(1+\cos y)dy$$

$$= \sin y\, dx + x\, dy + x\cos y\, dy$$

$$= [\sin y\, dx + x\cos y\, dy] + x\, dy$$

$$\Longrightarrow \vec{F}\cdot\vec{dr} = d[x\sin y] + x\, dy \longrightarrow ①$$

and given the curve $x^2+y^2=a^2 \longrightarrow ②$

Let $x = a\cos\theta$ $\quad y = a\sin\theta$

$\Rightarrow dx = -a\sin\theta \quad dy = a\cos\theta$

θ Varies form 0 to 2π

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_C [x\sin y) + x\,dy]$$

$$= \int_0^{2\pi} d[a\cos\theta\,.\sin(a\sin\theta)] + \int_0^{2\pi} a\cos\theta\,.\,a\cos\theta\, d\theta$$

$$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= (0-0) + a^2\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= a^2\int_0^{2\pi}\left[\frac{1+\cos 2\theta}{2}\right]_0^{2\pi} d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_2^{2\pi}$$

$$= \frac{a^2}{2}\times 2\pi$$

$$\boxed{\int_C \vec{F}\cdot\vec{dr} = \pi a^2}$$