UNIT - III. Inverse Laplace Transforms

Contents:

- Inverse Laplace Transforms - problems.
- Convolution thm (only statement) - problems
- Solution of LDE using L.T
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Inverse L.T can as well be regarded as the reverse process of finding the L.T of given fn

Defn: If $f(t)$ is a real valued fn defined for all $t \geq 0$ then the Laplace transform of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\, dt = \bar{f}(s) \text{ provided the integral exists.}$$

Thus $L[f(t)] = \bar{f}(s)$

$\Rightarrow L^{-1}[\bar{f}(s)] = f(t)$. is called inv. L.T

Thus we can say that

$L[f(t)] = \bar{f}(s) \Leftrightarrow L^{-1}[\bar{f}(s)] = f(t)$.

Basic table of inverse Laplace transforms

	Function	Inverse Transform		Function	Inverse Transform
1	$1/s$	1	5	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
2.	$1/s-a$	e^{at}	6	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
3.	s/s^2+a^2	$\cos at$	7	$\frac{1}{s^{n+1}}$ $(n>-1)$	$\frac{t^n}{\Gamma(n+1)}$
4.	s/s^2-a^2	$\cosh at$	8.	$\frac{1}{s^{n+1}}$ $n=1,2,3\cdots$	$\frac{t^n}{n!}$

Find the inv L.T of.

① $\dfrac{3s^2+4}{s^5}$ ② $\dfrac{3(s^2-1)^2}{2s^5}$ ③ $\dfrac{3s+5\sqrt{2}}{s^2+8}$

Ans $3L^{-1}\left(\frac{s^2}{s^5}\right)+4L^{-1}\left(\frac{1}{s^5}\right)$

$3L^{-1}\left(\frac{1}{s^3}\right)+4L^{-1}\left(\frac{1}{s^5}\right)$

$3\cdot\frac{t^2}{2!}+4\,\frac{t^4}{4!}$.

$=\frac{3}{2}t^2+\frac{t^4}{6}$

② $\frac{3}{2}L^{-1}\left[\frac{s^4-2s^2+1}{s^5}\right]$

$=\frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right)-2L^{-1}\left(\frac{1}{s^3}\right)+L^{-1}\left(\frac{1}{s^5}\right)\right]$

$=\frac{3}{2}\left[1-2\,\frac{t^2}{2!}+\frac{t^4}{4!}\right]=\frac{3}{2}\left[1-t^2+\frac{t^4}{24}\right]$.

③ $\dfrac{3s+5\sqrt{2}}{s^2+8}$

$=3L^{-1}\left[\frac{s}{s^2+(\sqrt{8})^2}\right]+5\sqrt{2}\,L^{-1}\left[\frac{1}{s^2+(\sqrt{8})^2}\right]$

$=3.\cos(2\sqrt{2}t)+5\cancel{\sqrt{2}}\cdot\frac{1}{2\cancel{\sqrt{2}}}\sin(2\sqrt{2}t)$

$=3.\cos(2\sqrt{2}t)+5/2\cdot\sin(2\sqrt{2}t)$.

④ $\dfrac{s^3}{s^4-a^4}$

$=s.\frac{s^2}{(s^2+a^2)(s^2-a^2)}$

$=s\left[\frac{1}{2}\left\{\frac{(s^2+a^2)+(s^2-a^2)}{(s^2+a^2)(s^2-a^2)}\right\}\right]$

$=\frac{1}{2}\left\{\frac{s}{s^2-a^2}+\frac{s}{s^2+a^2}\right\}$

$=\frac{1}{2}\{\cosh at+\cos at\}$

⑤ $\dfrac{s^2}{(s^2+4)(s^2+25)}$

Let $\frac{s^2}{(s^2+4)(s^2+25)}=\frac{A(s^2+4)+B(s^2+25)}{}$

$\Rightarrow A=25/21,\ B=-4/21$

$=\frac{25}{21}\cdot\frac{1}{s^2+25}-\frac{4}{21}\,\frac{1}{s^2+4}$.

$=\frac{25}{21}\cdot\frac{1}{5}L^{-1}\left(5\right.$

$$\frac{s-2}{s^2+7s+12} = \frac{s-2}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3}$$

for $s=-4$, $A=6$

for $s=-3$, $B=-5$

Then $\frac{s-2}{s^2+7s+12} = \frac{6}{s+4} - \frac{5}{s+3}$.

$$\therefore L^{-1}\left\{\frac{s-2}{s^2+7s+12}\right\} = 6L^{-1}\left(\frac{1}{s+4}\right) - 5L^{-1}\left(\frac{1}{s+3}\right)$$

$$= 6e^{-4t} - 5e^{-3t}.$$

⑦ $\frac{s^2+s-2}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$

$\Rightarrow A(s-2)(s+3) + Bs(s+3) + Cs(s-2) = s^2+s-2$

for $s=-3$, $C = 4/15$

for $s=2$, $B = 2/5$

for $s=0$, $A = 1/3$.

$$L^{-1}\left\{\frac{s^2+s-2}{s(s-2)(s+3)}\right\} = \frac{1}{3}L^{-1}\left(\frac{1}{s}\right) - \frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right) + \frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{3} + \frac{2}{5}e^{2t} + \frac{4}{15}e^{-3t}.$$

⑧ $\frac{s^2+2s-4}{(s^2+9)(s-5)} = \frac{A}{s-5} + \frac{Bs+C}{s^2+9}$

Then $A(s^2+9) + (Bs+C)(s-5) = s^2+2s-4$

$A = \frac{31}{34}$, $B = \frac{3}{34}$, $C = \frac{83}{34}$

$$\therefore L^{-1}\left\{\frac{s^2+2s-4}{(s^2+9)(s-5)}\right\} = \frac{31}{34}L^{-1}\left(\frac{1}{s-5}\right) + \frac{3}{34}L^{-1}\left(\frac{s}{s^2+9}\right) + \frac{83}{34}L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= \frac{31}{34}e^{5t} + \frac{3}{34}\cos 3t + \frac{83}{34}\cdot\frac{1}{3}\sin 3t.$$

Computation of inverse transform of $e^{-as}\bar{f}(s)$.

W.K.T $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$\therefore L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)u(t-a)$.

Find the inverse L.T of

① $\frac{1+e^{-3s}}{s^2}$

Sol: $L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{e^{-3s}}{s^2}\right)$ we have $L^{-1}\left(\frac{1}{s^2}\right) = t$

Thus $L^{-1}\left[\frac{1+e^{-3s}}{s^2}\right] = t + (t-3)u(t-3)$.

(2) $\dfrac{\cosh 2s}{e^{3s}\, s^2}$

Sol. $\dfrac{e^{-3s}}{s^2}\,\dfrac{(e^{2s}+e^{-2s})}{2} = \frac{1}{2}\left[\dfrac{e^{-s}}{s^2} + \dfrac{e^{-5s}}{s^2}\right]$

$\therefore L^{-1}\left[\dfrac{\cosh 2s}{e^{2s}\, s^2}\right] = \frac{1}{2}\left\{L^{-1}\left(\dfrac{e^{-s}}{s^2}\right) + L^{-1}\left(\dfrac{e^{-5s}}{s^2}\right)\right\}$ but $L^{-1}\left(\frac{1}{s^2}\right) = t$

$= \frac{1}{2}\left[(t-1)u(t-1) + (t-5)u(t-5)\right]$

(3) $\dfrac{e^{-\pi s}}{s^2+1} + \dfrac{s\, e^{-2\pi s}}{s^2+4}$

$L^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right) + L^{-1}\left(e^{-2\pi s}\cdot \dfrac{s}{s^2+4}\right)$

we have $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t.$

$\sin(t-\pi)u(t-\pi) + \cos 2(t-2\pi)u(t-2\pi)$

$L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1} + s\dfrac{e^{-2\pi s}}{s^2+1}\right] = -\sin t\, u(t-\pi) + \cos 2t\, u(t-2\pi).$

Inverse transform by completing the square.

W.K.T if $L[f(t)] = \bar{F}(s)$ then $L[e^{at}f(t)] = \bar{F}(s-a)$

$\Rightarrow L^{-1}[\bar{F}(s-a)] = e^{at} L^{-1}[\bar{F}(s)]$

(1) $\dfrac{1}{s^2+2s+5}$

s^2+2s+5

$s^2+2(s)(1)+1^2-1^2+5$

$(s+1)^2+4$

$= (s+1)^2 + 2^2.$

$L^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\dfrac{1}{(s+1)^2+4}\right\}$

$= e^{-t} L^{-1}\left\{\dfrac{1}{s^2+4}\right\}$

$= \frac{1}{2} e^{-t} \sin 2t.$

$$\frac{s+2}{s^2-4s+13}$$

(4)

$$s^2-4s+13$$
$$(s)^2-2(s)(2)+2^2-2^2+13$$
$$(s-2)^2+3^2$$

$$L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\}=L^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\}+L^{-1}\left\{\frac{4}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{s}{s^2+3^2}\right\}+\frac{4}{3}e^{2t}\sin 3t.$$

$$= e^{2t}\cdot\cos 3t+\frac{4}{3}e^{2t}\sin 3t$$

$$= e^{2t}\{\cos 3t+\tfrac{4}{3}\sin 3t\}.$$

(3) $$\frac{7s+4}{4s^2+4s+9}$$

consider $4s^2+4s+9$

$$=4(s^2+s+9/4)$$
$$=4\{s^2+2(s)(\tfrac{1}{2})+(\tfrac{1}{2})^2-(\tfrac{1}{2})^2+\tfrac{9}{4}\}$$
$$=4\{(s+\tfrac{1}{2})^2+2$$

$$L^{-1}\left\{\frac{7s+4}{4s^2+4s+9}\right\}=\frac{1}{4}L^{-1}\left\{\frac{7(s+\frac{1}{2})+\frac{1}{2}}{(s+\frac{1}{2})^2+2}\right\}$$

$$=\frac{e^{-t/2}}{4}\left\{7L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]\right\}$$

$$=\frac{e^{-t/2}}{4}\cdot\{7\cdot\cos\sqrt{2}t+\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}\sin\sqrt{2}t\}$$

$$=\frac{e^{-t/2}}{4}\{7\cos\sqrt{2}t+\tfrac{1}{2\sqrt{2}}\sin\sqrt{2}t\}.$$

(4) $$\frac{5s+3}{(s+1)(s^2+2s+5)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+2s+5}$$

$$A(s^2+2s+5)+(Bs+C)(s+1)=5s+3.$$

Equating corresponding coeffs. and solving gives $A=1,\ B=-1,\ C=2$

$$\frac{5s+3}{(s+1)(s^2+2s+5)}=\frac{1}{s+1}+\frac{-s+2}{s^2+2s+5}$$

$$s^2+2s+5$$
$$(s)^2+2(s)(1)+1^2-1^2+4$$
$$=(s+1)^2+4$$

$$=\frac{1}{s+1}+\frac{-(s+1)+3}{(s+1)^2+4}$$

$$=\frac{1}{s+1}-\frac{(s+1)}{(s+1)^2+2^2}+\frac{3}{(s+1)^2+4}$$

$$\therefore L^{-1}\left\{\frac{5s+3}{(s+1)(s^2+2s+5)}\right\}=L^{-1}\left(\frac{1}{s+1}\right)-e^{-t}L^{-1}\left(\frac{s}{s^2+2^2}\right)+3e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$=e^{-t}-e^{-t}\cos 2t+\tfrac{3}{2}e^{-t}\sin 2t.$$

$$\frac{s}{s^4+4a^4}$$

Sol: $s^4+4a^4 = (s^2+2a^2)^2 - 4a^2s^2$

$= (s^2+2a^2+2as)(s^2+2a^2-2as)$

and $4as = (s^2+2a^2+2as)-(s^2+2a^2-2as)$

or $s = \frac{1}{4a}\left[(s^2+2a^2+2as)-(s^2+2a^2-2as)\right]$

$$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left\{\frac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right\}$$

$$= \frac{1}{4a}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right] - L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$$

$$= \frac{1}{4a}\left\{e^{at}L^{-1}\left(\frac{1}{s^2+a^2}\right) - e^{-at}L^{-1}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{1}{4a}\left\{e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right\}$$

$$= \frac{\sin at}{2a^2}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{\sin at \cdot \sinh at}{2a^2}.$$

H.W.

6 $\frac{1}{s^3(s^3+1)}$

Sol: $\frac{1}{s^3(s^3+1)} = \frac{1}{s^3(s+1)(s^2-s+1)}$. $= \frac{A}{s^3}+\frac{B}{s+1}+\frac{Cs+D}{s^2-s+1}$

⋮

$$= \frac{t^2}{2} - \frac{e^{-t}}{3} + \frac{e^{t/2}}{3}\left\{\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right\}.$$

Inv. transform of logarithmic f!s and inv. f!s.

1) $\log\left(\frac{s+a}{s+b}\right)$ 2) $\log\left(\frac{s^2+1}{s(s+1)}\right)$ 3) $s\log\left(\frac{s+4}{s-4}\right)$

4) $\cot^{-1}\left(\frac{s+a}{b}\right)$ 5) $\tan^{-1}(2/s^2)$ 6) $\tan^{-1}(a/_{s+b})$.

W.K.T $L^{-1}[\bar{f}(s)] = f(t)$, we have the property

$L[t\,f(t)] = -\bar{f}'(s) \Rightarrow L^{-1}[-\bar{f}'(s)] = t\,f(t)$.

1) $\log\left(\frac{s+a}{s+b}\right)$

Ans: Let $\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$.

$$-\bar{f}'(s) = -\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = \frac{1}{s+b} - \frac{1}{s+a}.$$

$$L^{-1}\{-\bar{f}'(s)\} = L^{-1}\left(\frac{1}{s+b}\right) - L^{-1}\left(\frac{1}{s+a}\right) = e^{-bt} - e^{-at}.$$

$$\therefore t\,f(t) = e^{-bt} - e^{-at}.$$

$$\therefore f(t) = \frac{e^{-bt} - e^{-at}}{t}.$$

2) $\log\left[\frac{s^2+1}{s(s+1)}\right]$

Ans: Let $\bar{f}(s) = \log\frac{s^2+1}{s(s+1)} = \log(s^2+1) - \log(s+1) - \log s$.

$$-\bar{f}'(s) = \frac{2s}{s^2+1} - \frac{1}{s+1} - \frac{1}{s}$$

$$L^{-1}\{-\bar{f}'(s)\} = 2L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s}\right)$$

$$f(t) = \left(2\cos t - e^{-t} - 1\right)/t$$

$\mathcal{L}^{-1}\log\left(\frac{s+4}{s-4}\right)$

$\therefore \bar{f}(s) = \mathcal{L}\left[\log(s+4) - \log(s-4)\right]\cdot 1 = tf(t).$

$\bar{f}'(s) = \left(\frac{s}{s+4} - \frac{s}{s-4}\right) + \left[\log(s+4) - \log(s-4)\right]$

diff again

$\bar{f}''(s) = \frac{4}{(s+4)^2} - \frac{(-4)}{(s-4)^2} + \frac{1}{s+4} - \frac{1}{s-4}$

$\mathcal{L}^{-1}\left[\bar{f}''(s)\right] = 4\left[\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\}\right] + \mathcal{L}^{-1}\left(\frac{1}{s+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-4}\right)$

$t^2 f(t) = 4\left\{e^{-4t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + e^{4t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\right\} + e^{-4t} - e^{4t}$

$= 4\left\{e^{-4t}\cdot t + e^{4t}\cdot t\right\} + e^{-4t} - e^{4t}.$

$= 4t(2)\left\{\frac{e^{-4t} + e^{4t}}{2}\right\} \cancel{+} -2\left(\frac{e^{4t} - e^{-4t}}{2}\right)$

$= 8t.\cosh 4t - 2\sinh 4t.$

$f(t) = 2(4t\cosh 4t - \sinh 4t)/t^2$

(4) $\cot^{-1}\left(\frac{s+a}{b}\right).$

Ans: $\bar{f}'(s) = -\frac{1}{1+\left(\frac{s+a}{b}\right)^2}\cdot\frac{1}{b} = -\frac{b^2}{b^2+(s+a)^2}\cdot\frac{1}{b} = \frac{-b}{b^2+(s+a)^2}$

$\mathcal{L}^{-1}\left\{-\bar{f}'(s)\right\} = \mathcal{L}^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = e^{-at}\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}$

$= e^{-at}\cdot b/b\cdot \sin bt$

$tf(t) = e^{-at}\sin bt.$

$f(t) = \frac{e^{-at}\sin bt}{t}.$

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$\tan^{-1}(2/s^2)$ (8)

$$\bar{f}'(s) = \frac{1}{1+(4/s^4)} \cdot \left(-4/s^3\right) = \frac{s^4}{s^4+4}\left(-\frac{4}{s^3}\right) = -\frac{4s}{s^4+4}$$

$$L^{-1}\{-\bar{f}'(s)\} = L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$t f(t) = L^{-1}\left\{\frac{4s}{s^4+4}\right\}.$$

$$s^4+4 = (s^2+2)^2 - 4s^2 = (s^2+2+2s)(s^2+2-2s)$$

and $4s = (s^2+2+2s)-(s^2+2-2s)$.

Now $$\frac{4s}{s^4+4} = \frac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)(s^2+2-2s)}$$

$$= \frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}.$$

$$L^{-1}\left[\frac{4s}{s^4+4}\right] = L^{-1}\left[\frac{1}{s^2+2-2s}\right] - L^{-1}\left[\frac{1}{s^2+2+2s}\right]$$

$$t f(t) = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{t} L^{-1}\left(\frac{1}{s^2+1}\right) - e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= e^{t}.\sin t - e^{-t}\sin t = \sin t\,(e^{t}-e^{-t})$$

$$\therefore f(t) = \frac{\sin t . 2\sinh t}{t}.$$

(6) $\tan^{-1}\left(\frac{a}{s+b}\right)$

$$\bar{f}'(s) = \frac{1}{1+\left(\frac{a}{s+b}\right)^2}\left\{-\frac{a}{(s+b)^2}\right\} = \frac{\cancel{(s+b)^2}}{(s+b)^2+a^2}\left\{-\frac{a}{\cancel{(s+b)^2}}\right\} = -\frac{a}{(s+b)^2+a^2}$$

$$L^{-1}\{-\bar{f}'(s)\} = L^{-1}\left\{\frac{a}{(s+b)^2+a^2}\right\} = e^{-bt} L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = e^{-bt}.\sin at.$$

$$t f(t) = e^{-bt}\sin at$$

$$f(t) = \frac{e^{-bt}\sin at}{t}.$$

$L^{-1}[-\bar{f}'(s)] = t\,f(t)$.

$\frac{s}{(s^2+a^2)^2}$ ② $\frac{s+2}{(s^2+4s+5)^2}$ ③ $\frac{1}{(s^2+a^2)^2}$

① $\frac{s}{(s^2+a^2)^2}$

Sol: we have $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$.

by using the property $L^{-1}[-\bar{f}'(s)] = t.f(t)$

$$L^{-1}\left[-\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right] = t.\frac{1}{a}\sin at.$$

$$L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{a}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}.$$

② $\frac{s+2}{(s^2+4s+5)^2}$

Sol: $L^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{-2t}.\sin t.$

$$\therefore L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\} = L^{-1}\{\bar{f}'(s)\} = -t\,e^{-2t}\sin t.$$

$$L^{-1}\left\{-\frac{(2s+4)}{(s^2+4s+5)^2}\right\} = -t\,e^{-2t}\sin t$$

or $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{t}{2}e^{-2t}\sin t.$

③ $\frac{1}{(s^2+a^2)^2}$

Sol: Let $\bar{f}(s) = \frac{1}{(s^2+a^2)^2}$

$\Rightarrow f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

using $L^{-1}\{\bar{f}'(s)\} = -t\,f(t)$

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\} = -t\cdot\left(\frac{1}{a}\sin at\right)$$

$$L^{-1}\left\{-\frac{2s}{(s^2+a^2)^2}\right\} = -\frac{t}{a}\sin at.$$

or $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at = f(t)$ say

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s}\cdot F(s)\right\} = \int_0^t f(t)\,dt = \frac{1}{2a}\int_0^t t\sin at\,dt.$$

$$= \frac{1}{2a}\left\{\left[-\frac{t}{a}\cos at\right]_0^t - \int_0^t \frac{\cos at}{-a}\,dt\right\}$$

$$= \frac{1}{2a}\left[-\frac{t}{a}\cos at + \frac{1}{a^2}\sin at\right] = \frac{1}{2a^3}\left[\sin at - at\cos at\right].$$

Computation of inverse of $\frac{1}{s}F(s)$

W.K.T, if $L[f(t)] = F(s)$ then

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \Rightarrow L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt.$$

① $\frac{1}{s(s+2)^3}$ ② $\frac{1}{s(s^2+a^2)}$ ③ $\frac{1}{s(s^2-1)(s^2+1)}$

① $\frac{1}{s(s+2)^3}$

Soln, Let $F(s) = \frac{1}{(s+2)^3}$ and $f(t) = L^{-1}[F(s)]$

$$\therefore f(t) = L^{-1}\left[\frac{1}{(s+2)^3}\right] = e^{-2t}L^{-1}\left(\frac{1}{s^3}\right) = e^{-2t}\,t^2/2$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$$

$$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t e^{-2t}\frac{t^2}{2}\,dt = \frac{1}{2}\int_0^t t^2 e^{-2t}\,dt.$$

$$= \frac{1}{2}\left\{t^2\left(\frac{e^{-2t}}{-2}\right) - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right\}_{t=0}^{t}$$

$$= \frac{1}{2}\left[-\frac{1}{2}t^2e^{-2t} - \frac{1}{2}te^{-2t} - \frac{1}{4}(e^{-2t}-1)\right]$$

$$= -\frac{1}{4}\left[t^2e^{-2t} + te^{-2t} + \frac{1}{2}(e^{-2t}-1)\right]$$

② $\frac{1}{s(s^2+a^2)}$, $F(s) = \frac{1}{s^2+a^2}$ and $f(t) = L^{-1}[F(s)]$.

$$\therefore f(t) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t f(t)\,dt = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a}\left[\frac{-\cos at}{a}\right]_0^t$$

$$= \frac{1}{a^2}(1-\cos at)$$

$$\frac{1}{s(s^2-1)(s^2+1)}$$

$\therefore$ Let $\bar{f}(s) = \frac{1}{(s^2-1)(s^2+1)}$ and $f(t) = L^{-1}[\bar{f}(s)]$

$\cancel{f(t)} = L^{-1}\left\{\frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right)\right\}$ $\quad\Big|\quad \because \frac{1}{2}\left\{\frac{(s^2+1)-(s^2-1)}{(s^2-1)(s^2+1)}\right\}$

$= \frac{1}{2}\left[L^{-1}\left(\frac{1}{s^2-1}\right) - L^{-1}\left(\frac{1}{s^2+1}\right)\right]$

$= \frac{1}{2}(\sinh t - \sin t)$

Convolution thm (only statement.)

Defn: The convolution of two fns $f(t)$ and $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows.

$$f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du.$$

Note: property $f(t) * g(t) = g(t) * f(t)$.

That is to say that the convolution operation $*$ is Commutative.

Verify convolution thm for the following fns.

① $f(t) = \sin t,\ g(t) = e^{-t}$ ② $f(t) = \cos at,\ g(t) = \cos bt$.

③ $f(t) = t,\ g(t) = t\,e^{-t}$.

① $f(t) = \sin t,\ g(t) = e^{-t}$

Sol: $\bar{f}(s) = L(\sin t) = \frac{1}{s^2+1}$, $\bar{g}(s) = L(e^{-t}) = \frac{1}{s+1}$

$f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du = \int_{u=0}^{t} \sin u \cdot e^{-(t-u)}\, du$

$= e^{-t} \int_{u=0}^{t} e^{u} \sin u\, du$

$\frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$

$= e^{-t}\left[\frac{e^{u}}{1+1}(\sin u - \cos u)\right]_{u=0}^{t}$

$$= e^{-t}\left[\frac{e^{u}}{1+1}(\sin u - \cos u)\right]_{u=0}^{t}$$

$$= \frac{e^{-t}}{2}\left\{e^{t}(\sin t - \cos t) + 1\right\} = \frac{\sin t - \cos t}{2} + \frac{e^{-t}}{2}$$

$$\therefore L[f(t) * g(t)] = \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s+1}\right]$$

$$= \frac{1}{2}\left\{\frac{(s+1) - s(s+1) + (s^2+1)}{(s+1)(s^2+1)}\right\} = \frac{1}{(s+1)(s^2+1)}$$

$$\bar{f}(s)\cdot\bar{g}(s) = \frac{1}{(s^2+1)(s+1)}$$

$$L[f(t) * g(t)] = \bar{f}(s)\cdot\bar{g}(s).$$

(2) $f(t) = \cos at$, $g(t) = \cos bt$.

Ans. $\bar{f}(s) = \cos at$, $L[\bar{f}(s)] = L(\cos at) = s/(s^2+a^2)$

$\bar{g}(s) = \cos bt$, $L[\bar{g}(s)] = L(\cos bt) = s/(s^2+b^2)$

$$f(t) * g(t) = \int_{u=0}^{t} f(u)\cdot g(t-u)\,du$$

$$= \int_{u=0}^{t} \cos au \cdot \cos b(t-u)\,du$$

$$= \int_{u=0}^{t} \cos au \cdot \cos(bt - bu)\,du.$$

$$\cos A\cdot\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{2}\int_{u=0}^{t} \cos(au + bt - bu) + \cos(au - bt + bu)\,du$$

$$= \frac{1}{2}\left\{\frac{\sin(au+bt-bu)}{a-b} + \frac{\sin(au-bt+bu)}{a+b}\right\}_{u=0}^{t}$$

$$= \frac{1}{2}\left\{\frac{1}{a-b}[\sin at - \sin bt] + \frac{1}{a+b}[\sin at + \sin bt]\right\}$$

$$= \frac{1}{2}\left\{\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right\}$$

$$= \frac{1}{2}\left[\sin at\,\frac{2a}{a^2-b^2} + \sin bt\cdot\left(\frac{-2b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt), \quad a \neq 0$$

$$[f(t) * g(t)] = \frac{1}{a^2-b^2}\left[a \cdot \frac{a}{s^2+a^2} - b \cdot \frac{b}{a^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{a^2(s^2+b^2) - b^2(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{\cancel{a^2-b^2}} \cdot \frac{s^2(a^2-b^2)}{(s^2+a^2)(s^2+b^2)} = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

$$\bar{f}(s) \cdot \bar{g}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

$$L[f(t) * g(t)] = \bar{f}(s) \cdot \bar{g}(s)$$

(3) $f(t) = t, \quad g(t) = t\, e^{-t}$

Sol: $\bar{f}(s) = L(t) = 1/s^2, \quad \bar{g}(s) = L[e^{-t} \cdot t] = \frac{1}{(s+1)^2}$

$$f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du = \int_{u=0}^{t} u\, e^{-(t-u)} (t-u)\, du$$

$$= e^{-t} \int_{u=0}^{t} (tu - u^2)\, e^{u}\, du$$

$$= e^{-t}\left\{(tu - u^2)e^{u} - (t-2u)e^{u} + (-2)e^{u}\right\}_{u=0}^{t}$$

$$= e^{-t}\left\{(0-0) - (-t\,e^{t} - t) - 2(e^{t} - 1)\right\}$$

$$= t + t\,e^{-t} - 2 + 2e^{-t}$$

$$\therefore L[f(t) * g(t)] = \frac{1}{s^2} + \frac{1}{(s+1)^2} - \frac{2}{s} + \frac{2}{s+1}$$

$$= \left[(s+1)^2 + s^2 - 2s(s+1)^2 + 2s^2(s+1)\right] \Big/ s^2(s+1)^2$$

$$= \frac{1}{s^2(s+1)^2} \;//.$$

Thus $L[f(t) * g(t)] = \bar{f}(s) \cdot \bar{g}(s)$. Thm verified.

Sing convolution thm obtain Inv L.T of the following f's:

1) $\frac{1}{s(s^2+a^2)}$

Sol: Let $\bar{f}(s) = \frac{1}{s}$; $\bar{g}(s) = \frac{1}{s^2+a^2}$

Taking inv.

$f(t) = L^{-1}\left(\frac{1}{s}\right) = 1$; $g(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

we have convolution thm

$$L^{-1}[\bar{f}(s).\bar{g}(s)] = \int_{u=0}^{t} f(u).g(t-u)du.$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_{u=0}^{t} 1.\frac{\sin(at-au)}{a}du$$

$$= \left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t} = \frac{1}{a^2}(1-\cos at)$$

(2) $\frac{1}{(s^2+4s+13)^2}$

Sol: $\bar{f}(s) = \frac{1}{s^2+4s+13} = \bar{g}(s)$

$$f(t) = L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] = g(t)$$

$$f(t) = e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{e^{-2t}\sin 3t}{3} = g(t).$$

Now applying convolution thm

$$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \int_{u=0}^{t} \frac{e^{-2u}\sin 3u}{3}.\frac{e^{-2(t-u)}\sin(3t-3u)}{3}du$$

$$= \frac{e^{-2t}}{9}\int_{u=0}^{t}\sin 3u.\sin(3t-3u)du.$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t}\cos(3u-3t+3u)-\cos(3u+3t-3u)du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t}[\cos(6u-3t)-\cos 3t]du.$$

→

$$= \frac{e^{-2t}}{18}\left\{\left[\frac{\sin(6u-3t)}{6}\right]_0^t - \cos 3t\,(u)_0^t\right\}$$

$$= \frac{e^{-2t}}{18}\left\{\frac{\sin 3t + \sin 3t}{6} - \cos 3t \cdot t\right\} = \frac{e^{-2t}}{54}(\sin 3t - 3t\cos 3t)$$

(3) $\dfrac{4s+5}{(s-1)^2(s+2)}$.

Sol: Let $\bar{f}(s) = \dfrac{1}{s+2}$ $\qquad \bar{g}(s) = \dfrac{4s+5}{(s-1)^2}$

$$\Rightarrow f(t) = L^{-1}[\bar{f}(s)] = e^{-2t}$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{4s+5}{(s-1)^2}\right] = L^{-1}\left[\frac{4(s-1)+9}{(s-1)^2}\right]$$

$$g(t) = e^{t} L^{-1}\left[\frac{4s+9}{s^2}\right] = e^{t}(4+9t)$$

By applying convolution thm

$$L^{-1}\left[\frac{1}{s+2}\,\frac{4s+5}{(s-1)^2}\right] = \int_{u=0}^{t} e^{-2u}\cdot e^{(t-u)}[4+9(t-u)]\,du$$

$$= e^{t}\int_{u=0}^{t} e^{-3u}(4+9t-9u)\,du$$

$$= e^{t}\int_{u=0}^{t}(4+9t-9u)\,e^{-3u}\,du$$

$$= e^{t}\left\{(4+9t-9u)\frac{e^{-3u}}{-3} - (-9)\left(\frac{e^{-3u}}{9}\right)\right\}_{u=0}^{t}$$

$$= e^{t}\left\{4\cdot\frac{e^{-3t}}{-3} - \frac{(4+9t)}{-3} + e^{-3t} - 1\right\}$$

$$= e^{t}\left\{\frac{1}{3} - \frac{1}{3}e^{-3t} + 3t\right\} = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} + 3t\,e^{t}$$

(4) $\dfrac{s}{(s^2+a^2)^2}$

Sol. Let $\bar{f}(s) = \dfrac{1}{s^2+a^2}$; $\bar{g}(s) = \dfrac{s}{s^2+a^2}$

$$\Rightarrow f(t) = L^{-1}[\bar{f}(s)] = \frac{\sin at}{a}, \quad g(t) = L^{-1}[\bar{g}(s)] = \cos at$$

By convolution thm

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)\cdot g(t-u)\,du$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \cos(at-au)\, du$$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$= \frac{1}{2a}\int_{u=0}^{t} [\sin(au+at-au) + \sin(au-at+au)]\, du$$

$$= \frac{1}{2a}\int_{u=0}^{t} [\sin at + \sin(2au-at)]\, du$$

$$= \frac{1}{2a}\left\{ [\sin at\,(u)]_{u=0}^{t} - \left[\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t} \right\}$$

$$= \frac{1}{2a}\left\{ \sin at\,(t-0) - \frac{1}{2a}(\cos at - \cos at)\right\} = \frac{t\sin at}{2a}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}.$$

(5) $\int_0^t (t-u)\sin 2u\, du$

Sol: To find $L\left[\int_0^t (t-u)\sin 2u\, du\right]$ we use the result

$$L\left[\int_0^t f(t-u).g(u)\, du\right] = \bar{f}(s).\bar{g}(s). \rightarrow (1)$$

Note

$\Rightarrow f(t-u) = (t-u)$ and $g(u) = \sin 2u$.

evidently $f(t) = t$ and $g(t) = \sin 2t$

$\therefore \bar{f}(s) = L(t) = L(t) = 1/s^2$, $\bar{g}(s) = L(\sin 2t) = \frac{2}{s^2+4}$

$\therefore$ from (1)

$$L\left[\int_0^t (t-u)\sin 2u\, du\right] = \frac{1}{s^2}\cdot\frac{2}{s^2+4} = \frac{2}{s^2(s^2+4)}$$

(6) $\int_0^t e^{-u}\sin(t-u)\, du$

Sol: To find $L\left[\int_0^t e^{-u}\sin(t-u)\, du\right]$, we use the result

$$L\left[\int_0^t f(u)\, g(t-u)\, du\right] = \bar{f}(s).\bar{g}(s) \rightarrow (1).$$

$\Rightarrow f(u) = e^{-u}$, $g(t-u) = \sin(t-u)$

evidently $f(t) = e^{-t}$, $g(t) = \sin t$

$\therefore \bar{f}(s). = L(e^{-t}) = 1/s+1$, $\bar{g}(s) = L(\sin t) = 1/s^2+1$

from (1) $L\left[\int_0^t e^{-u}\sin(t-u)\,du\right] = \frac{1}{(s+1)(s^2+1)}$ //.

Solution of LDE using Laplace transforms.

Note: $L[y'(t)] = sLy(t) - y(0)$

$L[y''(t)] = s^2 Ly(t) - sy(0) - y'(0)$

$L[y'''(t)] = s^3 Ly(t) - s^2 y(0) - sy'(0) - y''(0)$

① solve by using L.T $\frac{d^2y}{dt^2} + k^2 y = 0$ given that $y(0) = 2$, $y'(0) = 0$

Sol. The given eqⁿ. is $y''(t) + k^2 y(t) = 0$

Taking L.T on both sides

$L[y''(t)] + k^2 Ly(t) = L(0)$

$\{s^2 L[y(t)] - sy(0) - y'(0)\} + k^2 L[y(t)] = 0$

using initial condⁿs

$(s^2+k^2) L[y(t)] - 2s = 0$ or $L[y(t)] = \frac{2s}{s^2+k^2}$

$\therefore y(t) = L^{-1}\left[\frac{2s}{s^2+k^2}\right] = 2\cos kt$.

$y(t) = 2\cos kt$.

② solve $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$ by using L.T

Sol: $L[y'''(t)] + 2L[y''(t)] - L[y'(t)] - 2L[y(t)] = L(0)$

$\{s^3 L[y(t)] - s^2 y(0) - sy'(0) - y''(0)\} + 2\{s^2 L[y(t)] - sy(0) - y'(0)\}$

$- \{sL[y(t)] - y(0)\} - 2L[y(t)] = 0$

using given initial condⁿ.

$L[y(t)]\{s^3 + 2s^2 - s - 2\} - 6 = 0$

$L[y(t)]\{s^2(s+2) - (s+2)\} = 6$

$L[y(t)]\{(s+2)(s^2-1)\} = 6.$

$L[y(t)] = \frac{6}{(s+2)(s-1)(s+1)}$

$$L[y(t)] = \frac{6}{(s+2)(s-1)(s+1)}$$

$$y(t) = L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1}$$

for $s=-2$, $A=2$

for $s=1$, $B=1$, for $s=-1$, $C=-3$

$$\therefore L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\} = 2L^{-1}\left(\frac{1}{s+2}\right) + L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+1}\right)$$

$$y(t) = 2e^{-2t} + e^{t} - 3e^{-t}$$

③ Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin t$ given that $y(0) = 0 = y'(0)$.

Sol: $L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = 5L(\sin t)$

$$\{s^2L[y(t)] - sy(0) - y'(0)\} + 2\{sL[y(t)] - y(0)\} + 2L[y(t)] = \frac{5}{s^2+1}$$

using initial cond^{n}s

$$L[y(t)]\{s^2+2s+2\} = \frac{5}{s^2+1} \text{ or } L[y(t)] = \frac{5}{(s^2+1)(s^2+2s+2)}$$

$$y(t) = L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

$$\frac{5}{(s^2+1)(s^2+2s+2)} = L^{-1}\left[\frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+2s+2)}\right]$$

$$5 = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$5 = (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing the coeff on both side

$A+C=0$, $2A+B+D=0$, $2A+2B+C=0$, $2B+D=5$.

Solving simultaneously we obtain

$A=-2, B=1, C=2, D=3$

Hence $$\frac{5}{(s^2+1)(s^2+2s+2)} = \frac{-2s+1}{s^2+1} + \frac{2s+3}{s^2+2s+2}$$

→

$$L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

$$y(t) = -2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{2s+3}{s^2+2s+2}\right)$$

$$y(t) = -2\cos t + \sin t + L^{-1}\left\{\frac{2(s+1)+1}{(s+1)^2+1}\right\}$$

$$= -2\cos t + \sin t + e^{-t}L^{-1}\left\{\frac{2s+1}{s^2+1}\right\}$$

$$= -2\cos t + \sin t + e^{-t}\left[2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -2\cos t + \sin t + e^{-t}\left[2\cdot\cos t + \sin t\right]$$

(4) Solve $y'' + 6y' + 9y = 12t^2e^{-3t}$, given that $y(0) = 0 = y'(0)$ by using L.T

Sol: $y''(t) + 6y'(t) + 9y(t) = 12t^2e^{-3t}$

$$L[y''(t)] + 6L[y'(t)] + 9L[y(t)] = 12L(e^{-3t}t^2)$$

$$\{s^2L[y(t)] - sy(0) - y'(0)\} + 6\{sL[y(t)] - y(0)\} + 9L[y(t)] = \frac{12.2}{(s+3)^3}$$

$$(s^2+6s+9)\,L[y(t)] = \frac{24}{(s+3)^3}$$

$$\text{or } L[y(t)] = \frac{24}{(s+3)^5}$$

$$y(t) = L^{-1}\left[\frac{24}{(s+3)^5}\right] = 24.\,e^{-3t}L^{-1}\left(\frac{1}{s^5}\right) = \cancel{24}.e^{-3t}.\frac{t^4}{\cancel{4!}}$$

$$\therefore y(t) = e^{-3t}t^4.$$

(5) Solve $y'' - 3y' + 2y = 1 - e^{2t}$, given that $y(0) = 1,\ y'(0) = 0$ by using L.T.

Sol: $L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(1) - L(e^{2t})$

$$\{s^2L[y(t)] - sy(0) - y'(0)\} - 3\{sL[y(t)] - y(0)\} + 9L[y(t)] = \frac{1}{s} - \frac{1}{s-2}$$

By using initial condns.

$$(s^2 - 3s + 2)\,Ly(t) = s - 3 + \frac{1}{s} - \frac{1}{s-2}$$

$$\text{or } Ly(t) = \frac{s-3}{(s-1)(s-2)} + \frac{1}{s}\cdot\frac{1}{(s-1)(s-2)} - \frac{1}{(s-1)(s-2)^2}$$

$$= \left(\frac{2}{s-1} - \frac{1}{s-2}\right) + \left(\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right) - \left(\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{(s-2)^2}\right)$$

using partial fractions

$$= 1/2\left[\frac{1}{s} + 1/{s-2}\right] - 1/(s-2)^2$$

$$\therefore y(t) = L^{-1}\left\{\frac{1}{2}\left(\frac{1}{s} + \frac{1}{s-2}\right)\right\} - L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= \frac{1}{2}(1 + e^{2t}) - e^{2t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$y(t) = \frac{1}{2}(1 + e^{2t}) - t e^{2t}.$$

Applications: L-C ckt.

Q.1. The current i and charge q in a series ckt containing an inductance L, capacitance C, e.m.f E satisfy the D.E $L\frac{di}{dt} + \frac{q}{C} = E$; $i = \frac{dq}{dt}$. express i and q in terms of t given that L, C, E are constants and the value of i, q are both zero initially.

Soln: since $i = \frac{dq}{dt}$, the DE becomes

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \quad \text{or} \quad \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L}$$

ie $q''(t) + \lambda^2 q(t) = \mu$; $\lambda^2 = 1/LC$ and $\mu = E/L$

Taking (L_T) on both sides

$$L_T[q''(t)] + \lambda^2 L_T[q(t)] = L_T(\mu)$$

i.e. $\{s^2 L_T[q(t)] - sq(0) - q'(0)\} + \lambda^2 L_T[q(t)] = \frac{\mu}{s}$

But $i = 0, q = 0$ at $t = 0$, by data

ie $q(0) = 0$, $q'(0) = 0$

hence $(s^2 + \lambda^2) L_T[q(t)] = \frac{\mu}{s}$ or $L_T[q(t)] = \frac{\mu}{s(s^2+\lambda^2)}$

$$\therefore q(t) = L_T^{-1}\left[\frac{\mu}{s(s^2+\lambda^2)}\right] \rightarrow (1)$$

Now $\frac{1}{s(s^2+\lambda^2)} = \frac{1}{\lambda^2}\left(\frac{1}{s} - \frac{s}{s^2+\lambda^2}\right)$ by partial fraction

$$L_T^{-1}\left[\frac{\mu}{s(s^2+\lambda^2)}\right] = \frac{\mu}{\lambda^2} L_T^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\lambda^2}\right]$$

$q(t) = \frac{\mu}{\lambda^2}(1 - \cos\lambda t)$; $\lambda^2 = 1/Lc$ and $\mu = E/L$

Thus $q(t) = EC\{1 - \cos(\sqrt{1/LC}\, t)\}$.

(21)

An alternating e.m.f $E\sin\omega t$ is applied to an inductance L and capacitance C in series. show by transform method, that the current in the ckt is $\frac{E\omega}{(p^2-\omega^2)L}(\cos\omega t - \cos pt)$ where $p^2 = 1/LC$

Sol: If 'i' be the current and 'q' be the charge at time 't' in the ckt the D.E is

$$L\frac{di}{dt} + \frac{q}{C} = E\sin\omega t \quad | \because R=0$$

$$L\{L_T[i(t)]\} + \frac{1}{C}L_T\{q\} = E.\frac{\omega}{s^2+\omega^2}$$

$$L\{s\,i(0) - i(0)\} + \frac{1}{C}L_T(q) = E.\frac{\omega}{s^2+\omega^2}$$

$\because i=0$ and $q=0$ at $t=0$

$$L\{s\,L_T[i(t)]\} + 1/C\,L_T[q] = \frac{E\omega}{s^2+\omega^2} \quad \text{--- (1)}$$

Laplace of $i = \frac{dq}{dt}$, we get [$\because$ rate of change of i

$$L_T\{i(t)\} = L_T\left(\frac{dq}{dt}\right) = s\,L_T(q) - q(0)$$

$$L_T(q) = L_T\left[\frac{i(t)}{s}\right]$$

$$\{L[s\,L_T[i(t)]\} + 1/C\,L_T\left[\frac{i(t)}{s}\right] = \frac{E\omega}{s^2+\omega^2}$$

(1) becomes

$$\left[Ls + \frac{1}{Cs}\right]L_T[i(t)] = \frac{E\omega}{s^2+\omega^2}$$

$$L_T[i(t)] = \frac{E\omega}{\left[Ls+\frac{1}{Cs}\right](s^2+\omega^2)} = \frac{E\omega}{\left[\frac{Ls^2C+1}{Cs}\right](s^2+\omega^2)}$$

$$= \frac{E\omega s}{\left[\frac{Ls^2C+1}{C}\right](s^2+\omega^2)} = \frac{E\omega s}{L\left[\frac{s^2\cancel{C}}{\cancel{C}} + \frac{1}{LC}\right](s^2+\omega^2)} = \frac{E\omega s}{L\left[s^2+\frac{1}{LC}\right](s^2+\omega^2)}$$

$\because p = 1/LC$

$$= \frac{E\omega}{L}\,\frac{s}{(s^2+p^2)(s^2+\omega^2)}$$

$$L_T\{i(t)\} = \frac{E\omega}{L(p^2-\omega^2)}\left\{\frac{s}{s^2+\omega^2} - \frac{s}{s^2+p^2}\right\}$$

$$\therefore i(t) = \frac{E\omega}{L(p^2-\omega^2)}\,L^{-1}\left\{\frac{s}{s^2+\omega^2} - \frac{s}{s^2+p^2}\right\}$$

$$= \frac{E\omega}{L(p^2-\omega^2)}\{\cos\omega t - \cos pt\}.$$

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