Vector Integration

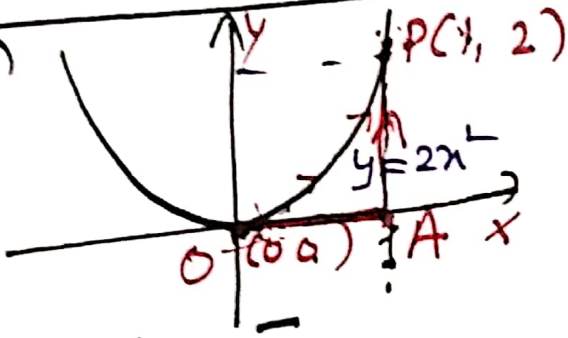
Line Integral.

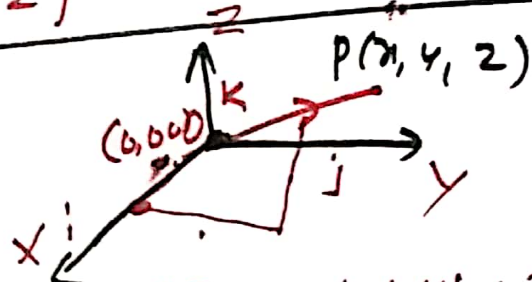
① Compute $\int_C \bar{F}\cdot\bar{dr}$ if $\bar{F} = 3xy\,i - y^2 j$ and C is parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

OR

Find the workdone by the force $F = 3xy\,i - y^2 j$ along parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$

Soln





$\bar{r} = OP = xi + yj + zk$

$d(\bar{r}) = dx\,i + dy\,j + dz\,k$

$$I = \int_C F\cdot\bar{dr}$$

$$= \int_{y=2x^2} (3xy\,i - y^2 j)\cdot(dx\,i + dy\,j + dz\,k)$$

$$= \int_{y=2x^2} 3xy\,dx - y^2\,dy$$

$$= \int_{y=2x^2} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2)$$

$$= \int_{x=0}^{1} 6x^3\,dx - 4x^4\,[2\cdot 2x\,dx]$$

$$= \int_{x=0}^{1} (6x^3 - 16x^5)\,dx$$

$$= \left(6\frac{x^4}{4} - 16\frac{x^6}{6}\right)_{x=0}^{1}$$

$$= \left(\frac{6}{4} - \frac{16}{6}\right) - 0$$

$$= -7/6.$$

VI ②

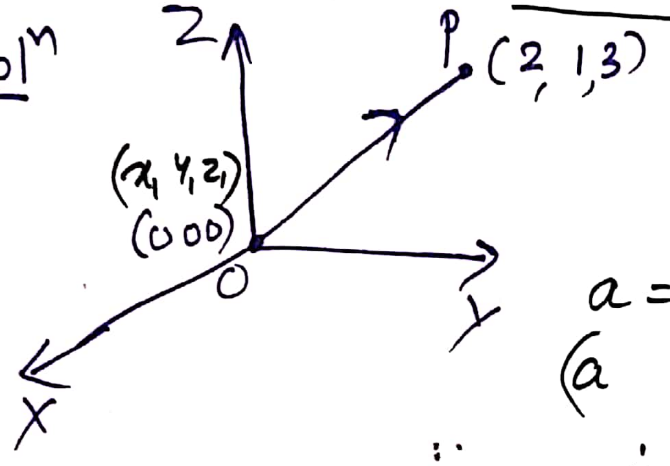
Eg ② Find the workdone in moving a particle in the force field

$\bar{F} = \underset{①}{3x^2} i + (\underset{②}{2zx - y}) j + \underset{③}{z} k$ along

(i) the straight line from (0,0,0) to (2,1,3)

(ii) the curve c, $x^2 = 4y$, $3x^3 = 8z$, $x=0$ to $x=2$.

Sol^n



Equations of st. line $\overline{OP}$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t$$

$a = 2-0 \quad b = 1-0 \quad c = 3-0$

$(a\ b\ c) \equiv (2, 1, 3)$

$$\therefore \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

OR $\boxed{x = 2t \quad y = t \quad z = 3t}$

Limits
$x = 2t$
$x = 0$, $x = 2$; $t = 0$, $t = 1$

$\bar{dr} = d(x) i + d(y) j + d(z) k$

$= d(2t) + d(t) j + d(3t) k$

$\bar{dr} = 2dt\, i + dt\, j + 3dt\, k = (2i + j + 3k) dt$

Work Done (W) $= \int_c F \cdot \bar{dr} = \int_{\overrightarrow{OP}} [① i + ② j + ③ k] \cdot [2i + j + 3k] dt$ (F)

$$w = \int_{t=0}^{1} [3(2t)^2 i + \{2(3t)(2t) - t\} j + (3t) k] \cdot [2i + j + 3k] dt$$

$$w = \int_{t=0}^{1} [3(2t)^2 \times 2 + \{12t^2 - t\} \times 1 + (3t \times 3)] dt$$

$$w = \int_0^1 (36t^2 + 8t) dt = \left(36\frac{t^3}{3} + 8\frac{t^2}{2}\right)_0^1 = \left(\frac{36}{3} + \frac{8}{2}\right)$$

$\boxed{w = 16}$

Eg ② continued.

(ii) Curve $x^2 = 4y$, $3x^3 = 8z$, $x = 0$ to $x = 2$

OR it can be written as

Curve ≡ C $z = \frac{3}{8}x^3$, $y = \frac{x^2}{4}$ and $x = x$ → (I) (All in terms of x)

Work Done $= W = \int_C \bar{F} \cdot d\bar{r}$

$$= \int_C [3x^2 i + (2zx - y) j + zk] \cdot [d(x) i + d(y) j + d(z) k]$$

Taking Dot product

$$W = \int_C 3x^2 d(x) + \{2zx - y\} d(y) + z\, d(z)$$

Using (I) (to convert all in terms of x)

$$W = \int_{x=0}^{2} 3x^2 dx + \left\{2\left(\frac{3}{8}x^3\right)(x) - \frac{x^2}{4}\right\} d\left(\frac{x^2}{4}\right) + \frac{3}{8}x^3 d\left(\frac{3}{8}x^3\right)$$

$$= \int_{x=0}^{2} 3x^2 dx + \left\{\frac{6}{8}x^4 - \frac{x^2}{4}\right\}\left(\frac{x\,dx}{2}\right) + \frac{3}{8}x^3\left(\frac{9x^2}{8}\right) dx$$

$$= \int_{x=0}^{2} \left[3x^2 + \left(\frac{6}{16}x^5 - \frac{x^3}{8}\right) + \frac{27}{64}x^5\right] dx$$

$$= \left[3\frac{x^3}{3} + \frac{6}{16}\frac{x^6}{6} - \frac{x^4}{8 \times 4} + \frac{27}{64}\frac{x^6}{6}\right]_{x=0}^{2}$$

$$= \left[2^3 + \frac{1}{16}2^6 - \frac{2^4}{8 \times 4} + \frac{27}{64} \times \frac{2^6}{6}\right]$$

$$\boxed{W = 16}$$

$d\left(\frac{x^2}{4}\right) = \frac{2x\,dx}{4} = \left(x\,dx/2\right)$

$d\left(\frac{3}{8}x^3\right) = \frac{3}{8}(3x^2 dx) = \frac{9}{8}x^2 dx$

Green's Theorem (in the plane)

Statement: If $\phi(x,y)$ and $\psi(x,y)$, ϕ_x, ψ_y are continuous in a region E (or R) of the xy-plane bounded by a closed curve C then

$$\int_C \phi\,dx + \psi\,dy = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

Note ① This theorem is a tool to convert integration along a closed curve into double integration.

② Green's Thm is a special case of Stoke's thm.

Example 1. Apply Green's theorem to Evaluate

$$\int_C (x^2+xy)\,dx + \cancel{(x^2+xy)}\,(x^2+y^2)\,dy$$ where C is the square bounded by lines $x=\pm 1$, $y=\pm 1$

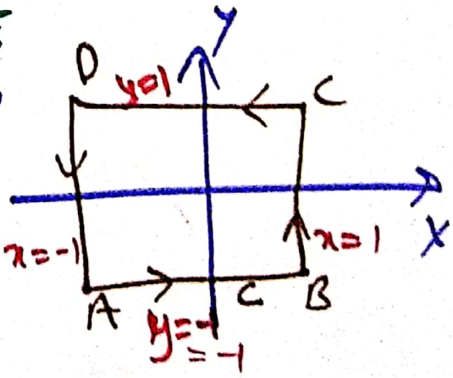
Sol^n $\phi = x^2+xy$, $\psi = x^2+y^2$

$\frac{\partial \phi}{\partial y} = x$, $\frac{\partial \psi}{\partial x} = 2x$

$$\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = 2x - x = x$$

Green's Theorem: $\displaystyle\int_C \phi\,dx + \psi\,dy = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

$$\therefore \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{y=-1}^{1}\int_{x=-1}^{1} x\,dx\,dy$$



$$= \left(\frac{x^2}{2}\right)_{-1}^{1} (y)_{-1}^{1}$$

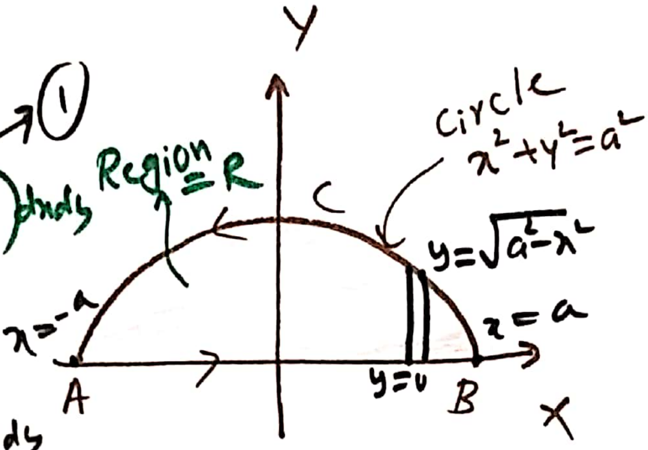
$$= 0 \times 2$$

$$= 0$$

Example 2: Apply Green's Theorem to evaluate $\int_C (2x^2-y^2)dx + (x^2+y^2)dy$ where C is the curve as a boundary of the area enclosed by x-axis and upperhalf of the circle $x^2+y^2=a^2$.

Sol^n Green's Theorem: (1)

$$LHS = \int_C \phi\, dx + \psi\, dy = \iint_R (\psi_x - \phi_y)\, dx\, dy$$



Given $I = \int_C \underbrace{(2x^2-y^2)}_{\phi}dx + \underbrace{(x^2+y^2)}_{\psi}dy$

$C \equiv ABCA$

$$\left.\begin{aligned} &\phi = 2x^2-y^2 \qquad \psi = x^2+y^2 \\ &\phi_y = \frac{\partial \phi}{\partial y} = -2y \qquad \psi_x = \frac{\partial \psi}{\partial x} = 2x \end{aligned}\right\} (2)$$

using (2) in (1)

$$\therefore LHS = \int_C \underbrace{(2x^2-y^2)dx + (x^2+y^2)dy}_{\phi\, dx \;+\; \psi\, dy} = \int_{x=-a}^{a} \int_{y=0}^{y=\sqrt{a^2-x^2}} (2x+2y)\, dx\, dy$$

converting to polar co-ordinates

$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$$LHS = 2\int_{\theta=0}^{\pi} \int_{r=0}^{a} r[\cos\theta + \sin\theta]\, r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi} (\sin\theta + \cos\theta)\, d\theta \times \int_{r=0}^{a} r^2\, dr$$

$$= 2\left[-\cos\theta + \sin\theta\right]_0^{\pi} \times \left(\frac{r^3}{3}\right)_0^{a}$$

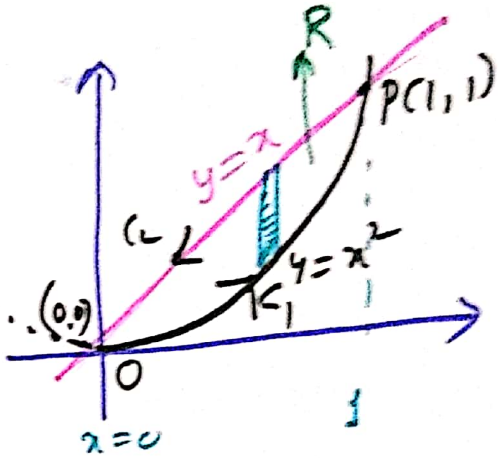
$$= 2[(1+1)+0] \times a^3/3$$

$$= 4a^3/3$$

Example 3: Verify Green's theorem for $\int_C [(xy+y^2)dx + x^2 dy]$ where C is bounded by $y=x$ and $y=x^2$.

Soln Green's Thm

$$\int_C \phi\, dx + \psi\, dy = \iint \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$



Note: Verify $\overset{R}{\Rightarrow}$ Both LHS, RHS should be evaluated & LHS ≡ RHS. (unlike previous two examples)

$C = OPO = C_1 + C_2$
O to P along $y=x^2$, P to O along $y=x$

$$LHS = \int_C [\phi\, dx + \psi\, dy] = \int_{C_1} [\ \ ] + \int_{C_2} [\ \ ]$$

Along C_1: $\boxed{y = x^2}$ $\boxed{dy = 2x\, dy}$

$x \rightarrow$ moves from 0 to 1 ①

$\phi = xy + y^2$
$\psi = x^2$

$$\therefore \int_{C_1} [\ \ ] = \int_{C_1} \phi\, dx + \psi\, dy$$

$$= \int_{x=0} [x(y) + (y)^2]\, dx + x^2 d(y) \rightarrow ②$$

① in ②

$$\int_{C_1} = \int_{x=0}^{1} [x(x^2) + (x^2)^2]\, dx + x^2 d(x^2)$$

$$= \int_{x=0}^{1} (3x^3 + x^4)\, dx$$

$$= \left(3\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$\boxed{\int_{C_1} = 19/20}$$

Now along C_2 $y = x$ $dy = dx$
$x \rightarrow 1$ to 0

$$\int_{C_2} \phi\, dx + \psi\, dy = \int_{x=1}^{0} [x(x) + x^2]\, dx + x^2 d(x) \rightarrow$$

$$\int_{C_2} = \int_1^0 3x^2\, dx = \left(3\frac{x^3}{3}\right)_1^0 = -1$$

$$\int_C = \int_{C_1} + \int_{C_2} = \frac{19}{20} - 1 = \boxed{-1/20} = LHS$$

$$RHS = \int_{x=0}^{1} \int_{y=x^2}^{x} \frac{\partial (x^2)}{\partial x} - \frac{\partial}{\partial y}(xy + y^2)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} 2x - [x + 2y]\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} [xy - y^2]_{y=x^2}^{x}\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx = \frac{-1}{20}$$ RHS

LHS = RHS
Greens Thm Verified

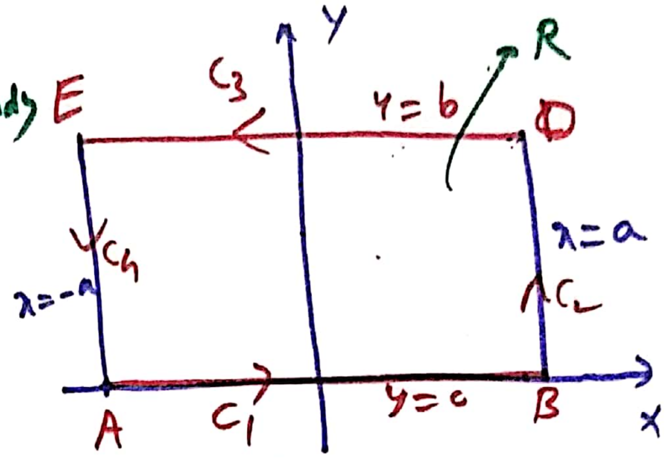
Ex: 4 Verify Green's theorem for

$\bar{F} = (x^2+y^2)i - 2xy\,j$ around the rectangle determined by $x = \pm a$ $y=0$, $y=b$.

Sol^n Green's Theorem

$$\int_C \phi\, dx + \psi\, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

LHS RHS



$C = C_1 + C_2 + C_3 + C_4$

$\text{LHS} = \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

Along $C_1 \equiv AB$, $y=0$ $dy=0$, $x \to -a$ to a (1)

$\phi = x^2+y^2$

$\psi = -2xy$

$$\boxed{\int_{C_1} \phi\,dx + \psi\,dy = \int_{AB} (x^2+y^2)\,dx - 2xy\,dy} \to (2)$$

(1) in (2)

$$\int_{C_1} = \int_{x=-a}^{a} (x^2+0)\,dx - 2x(0)(0)$$

$$\int_{C_1} = \left(x^3/3\right)_{-a}^{a} = \frac{2}{3}a^3$$

BP Along C_2 $x=a$ $dx=0$ $y \to y=0$ to $y=b$ $\to$ (3)

(3) in (2)

$$\int_{C_2} = \int_{y=0}^{b} [(a^2)+y^2](0) - 2(a)y\,dy$$

$$= \int_{y=0}^{b} -2ay\,dy = -2a\left(y^2/2\right)_0^b$$

$$\int_{C_2} = -ab^2$$

DE Along C_3 $y=b$ $dy=0$ $x \to a$ to $-a$

using in (2) as $dy=0$

$$\int_{C_3} = \int_{x=a}^{-a} (x^2+b^2)\,dx + 0$$

$$= \left(x^3/3\right)_a^{-a} + b^2(x)_a^{-a}$$

$$\int_{C_3} = -\frac{2}{3}a^3 - 2ab^2$$

Along $C_4 = EA$ $x=-a$ $dx=0$

$y \to b$ to 0

using in (2)

$$\int_{C_4} = \int_{y=b}^{0} 0 - 2(-a)y\,dy$$

$$= 2a\left(y^2/2\right)_b^0$$

$$\int_{C_4} = -ab^2$$

$$\text{LHS} = \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \frac{2}{3}a^3 - ab^2 - \frac{2}{3}a^3 - 2ab^2 - ab^2$$

$$\text{LHS} = \boxed{-4ab^2}$$

$$\text{RHS} = \int_{x=-a}^{a}\int_{y=0}^{b} \frac{\partial(-2xy)}{\partial x} - \frac{\partial(x^2+y^2)}{\partial y}\,dx\,dy$$

$$= \int_{-a}^{a}\int_{y=0}^{b} (-2y-2y)\,dy\,dx$$

$$= -4ab^2 = -4ab^2$$

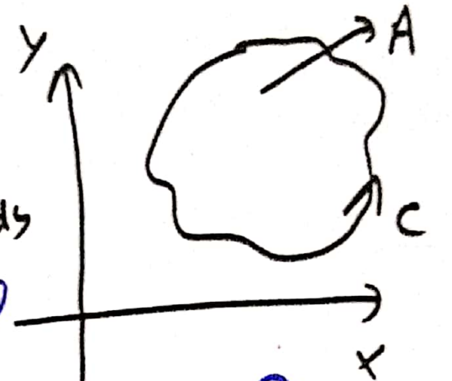
LHS = RHS Verified

Example 6 (An Amazing Fact)

$\equiv$ Area bound by ANY closed curve

If A is the area enclosed by simple closed plane curve C, using Green's thm P.T.

$$A = \frac{1}{2}\int_C x\,dy - y\,dx.$$



Sol^n Green's thm

$$\int_C \phi\,dx + \psi\,dy = \iint_A \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)dx\,dy \quad \text{(1)}$$

Choose $\psi = x$ and $\phi = -y$ and put in (1)

$$\int_C -y\,dx + x\,dy = \iint_A \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\,dx\,dy$$

$$\therefore \int_C x\,dy - y\,dx = \iint_A [1-(-1)]\,dx\,dy$$

OR

$$\iint_A 2\,dx\,dy = \int_C x\,dy - y\,dx$$

OR

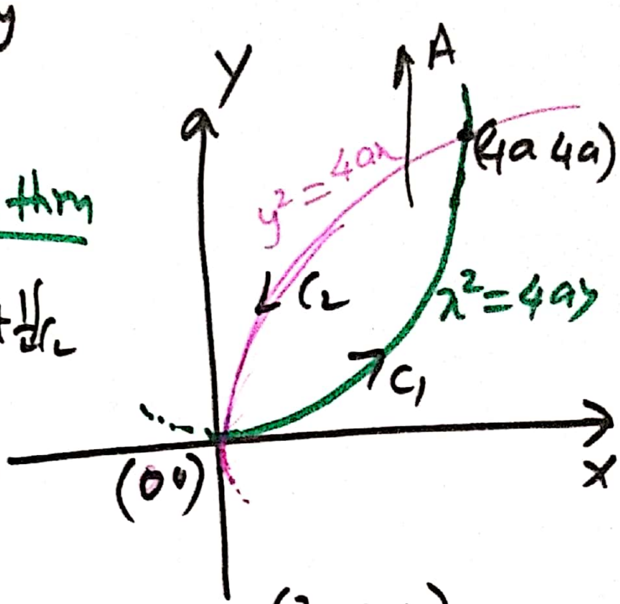
$$2\,\text{Area}(A) = \int_C x\,dy - y\,dx \quad \text{as } \iint dx\,dy = A$$

$$\therefore A = \frac{1}{2}\int_C x\,dy - y\,dx$$

Eg 6 Using Green's thm Find the area of the region bounded by $y^2 = 4ax$ $\quad x^2 = 4ay$

Sol^n

$C = C_1 + C_2$ By Green's thm



$$A = \frac{1}{2}\int_C [x\,dy - y\,dx] = \frac{1}{2}\int_{C_1} + \frac{1}{2}\int_{C_2}$$

Along C_1 $\quad y = x^2/4a \quad dy = \frac{2x\,dx}{4a}$

$x \to 0$ to $4a$

$$\int_{C_1}[\ ] = \int_{x=0}^{4a} x\,\frac{2x\,dx}{4a} - \frac{x^2}{4a}\,dx$$

$$= \frac{1}{4a}\int_0^{4a} x^2\,dx = \frac{16a^2}{3}$$

$C_1 = (x^2 = 4ay)$
$y = x^2/4a$

$C_2 = y^2 = 4ax$
$x = y^2/4a$

Along C_2 $\quad x = y^2/4a \quad dx = \frac{2y\,dy}{4a} \quad y \to 4a$ to 0

$$\int_{C_2} = \int_{y=4a}^{0} \frac{y^2}{4a}(dy) - y\,\frac{2y\,dy}{4a}$$

$$\int_{C_2} = \frac{-1}{4a}\int_{4a}^{0} y^2\,dy = \frac{-16}{3}a^2$$

$$A = \frac{1}{2}\left[\frac{16}{3}a^2 - \left(-\frac{16}{3}a^2\right)\right] = \left(\frac{16}{3}\right)a^2 = \frac{16}{3}a^2$$

PRH ① VI 10

Stoke's Theorem : (Relation Between Line and Surface Integration

Let S be a surface bounded by a simple closed curve 'C' and $\bar{F} = f_1 i + f_2 j + f_3 k$ be continuously differentiable vector point function. Then

$$\oint_C \bar{F} \cdot \bar{dR} = \iint_S \text{Curl} \vec{F} \cdot \vec{N} \, ds$$

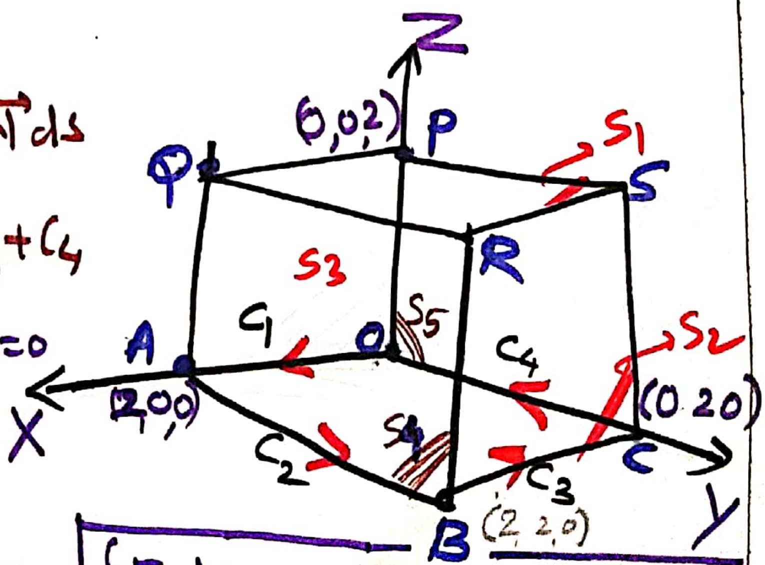
where $\vec{N}$ is unit outward normal to 'S'

Example 1. Verify stoke's theorem for the vector field $\boxed{\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - 4zx\bar{k}}$ where 'S' is the surface of the cube bounded by planes $X=0, Y=0, Z=0, X=2, Y=2, Z=2$ above the xy-plane.

Solⁿ stokes thm

$$\oint_C F \cdot \bar{dR} = \iint_S \text{Curl} F \cdot \vec{N} \, ds$$

$C = OABCO = C_1 + C_2 + C_3 + C_4$



LHS

Along C_1 $y=0$ $z=0$ $dy=0$ $dz=0$

$x \to 0$ to 2

$$\int_{C_1} \bar{F} \cdot \bar{dR} = \int_{x=0}^{2} 2\,dx = 4$$

$C_2 \to x=2$ $z=0$ $dx=0$ $dz=0$

$y \to 0$ to 2

$$\int_{C_2} F \cdot \bar{dR} = \int_{y=0}^{2} 4\,dy = 8$$

$$\int_{C_3} F \cdot dR = \int_2^0 4\,dx = -8$$

$\int_{C_4} = -8$ $\boxed{LHS = 4+8-8-8 = -4}$

PRH (2)
V I
(11)

$$\text{Curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= i\left[0-y\right] - j\left[-z-(-1)\right] + k(0-1)$$

$$\boxed{\text{Curl} F = -iy + (z-1)j - k}$$

$S = PQRS + BCSR + OAPQ + ABRQ + OCSP$

$\quad\; S_1 + S_2 + S_3 + S_4 + S_5$

On S_1 $\quad ds = dx\,dy \quad \vec{N} = k$

$$\iint_{S_1} \text{Curl} F \cdot N\,ds = \int_{x=0}^{2}\int_{y=0}^{2}(-1)\,dx\,dy = -4$$

on S_2 $\quad ds = dx\,dz \quad \boxed{y=2} \quad \vec{N} = j$

$$\iint_{S_2} \text{Curl} F \cdot N\,ds = \int_{x=0}^{2}\int_{z=0}^{2}(z-1)\,dz\,dx = \left(\frac{z^2}{2}-z\right)_0^2 (x)_0^2 = 0$$

on S_3 $\quad ds = dx\,dz \quad \boxed{y=0} \quad \vec{N} = -j, \quad \iint_{S_3} -(z-1)\,dx\,dz = 0$

S_4 $\quad ds = dy\,dz \quad \boxed{x=2} \quad \vec{N} = i \quad \int_{z=0}^{2}\int_{y=0}^{2} -y\,dy\,dz = \left(-\frac{y^2}{2}\right)_0^2 (z)_0^2$

S_5 $\quad ds = dy\,dz \quad \boxed{x=0} \quad \vec{N} = -i \quad \int_{z=0}^{2}\int_{y=0}^{2} y\,dy\,dz = 4 \qquad = -4$

$$\iint_S \text{Curl} F \cdot N\,ds = \iint_{S_1+S_2+S_3+S_4+S_5} \text{Curl} F \cdot N\,ds = -4+0+0-4+4 = \boxed{-4}$$

RHS

$\boxed{LHS = RHS = -4}$ Stokes thm Verified

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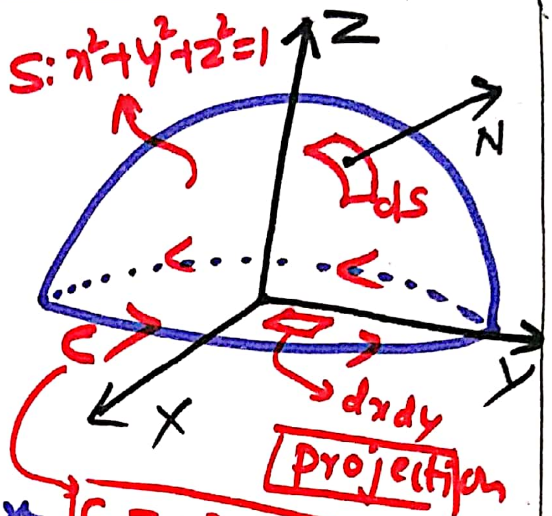
Example 2: Verify Stokes theorem for $\bar{F} = (2x-y)i - yz^2 j - y^2 z k$ for upper half of the sphere $x^2+y^2+z^2=1$.

Sol^n Given $\bar{F} = (2x-y)i - yz^2 j - y^2 z k$



Stokes Theorem

$$\boxed{\int_C F\cdot dR = \iint_S \text{Curl}F\cdot N\,ds}$$

$C \equiv \boxed{x^2+y^2=1}$ Circle in XY-plane $\quad S = \boxed{x^2+y^2+z^2=1}$ upper half of sphere

$\boxed{C = x^2+y^2=1}$

$\boxed{LHS}$

$F\cdot dR = (2x-y)dx - yz^2 dy - y^2 z\,dz$

Along C, $z=0$ $dz=0$ $\quad \bar{F}\cdot d\bar{R} = (2x-y)dx$

$LHS = \int_C \bar{F}\cdot d\bar{R} = \int_C (2x-y)dx$

$x = \cos\theta \quad y = \sin\theta \quad \boxed{\theta \to 0 \text{ to } 2\pi}$ $\quad dx = -\sin\theta\,d\theta$

$$LHS = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta)$$

$$ds = \frac{dx\,dy}{|N\cdot k|}$$

$$= \int_{\theta=0}^{2\pi} [-2\sin\theta\cos\theta + \sin^2\theta]\,d\theta$$

$$= -\int_{\theta=0}^{2\pi} \sin(2\theta)\,d\theta + \int_0^{2\pi} \sin^2\theta\,d\theta$$

$$= +\left[\frac{\cos(2\theta)}{2}\right]_0^{2\pi} + 4\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \frac{1-1}{2} + 4\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] \Rightarrow \boxed{LHS = \pi}$$

$$\text{Curl}F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= i[-2yz+2yz] - j[0-0] + k[0-(-1)]$$

$$\boxed{\text{Curl}\bar{F} = k}$$

$$RHS = \iint_S \text{Curl}F \cdot N\, ds$$

$$= \iint_{\substack{\text{Projection} \\ \text{of } S \text{ on } xy\text{-plane}}} k \cdot N \left(\frac{dx\,dy}{|N \cdot k|}\right)$$

$$RHS = \iint_{x^2+y^2 \le 1} dx\,dy$$

$= \text{Area of } \bigcirc^{le}\ x^2+y^2=1$

$$\boxed{RHS = \pi}$$

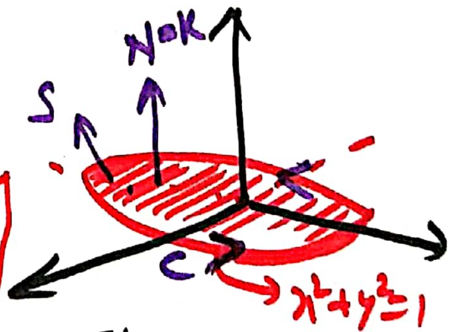
$LHS = RHS = \pi$ Stokes thm verified

Eg 3 Verify Stoke's thm for $f = -y^3 i + x^3 j$ where 's' is the circular disc $x^2+y^2 \le 1$, $z=0$

Soln $\bar{f} = -y^3 i + x^3 j$

S : Circular Disc $x^2+y^2 \le 1$

C : Circle $x^2+y^2=1$



Stokes thm $\boxed{\int_C F \cdot d\bar{R} = \iint_S \text{Curl}F \cdot N\, ds}$

$$LHS = \int_C -y^3 dx + x^3 dy$$

$x = \cos\theta \quad y = \sin\theta$

$$LHS = \int_{\theta=0}^{2\pi} -\sin^3\theta(-\sin\theta)\, d\theta + \cos^3\theta \cos\theta\, d\theta$$

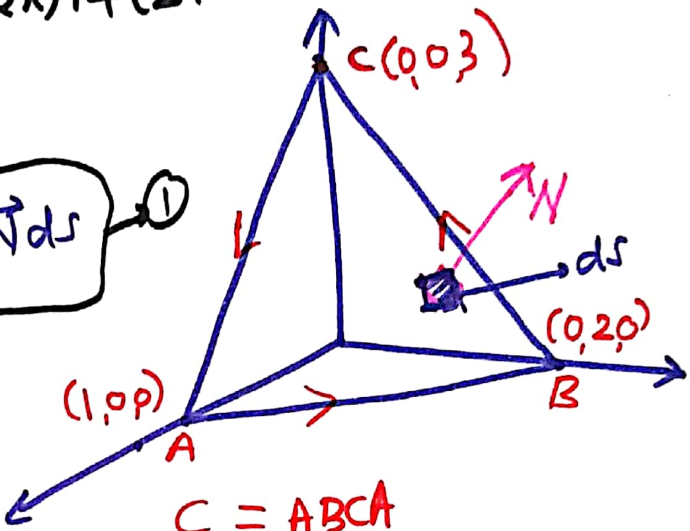
$$LHS = 4\int_0^{\pi/2} (\sin^4\theta + \cos^4\theta)\, d\theta$$

$$= 4\left[\frac{3 \cdot 1}{4 \cdot 2}\frac{\pi}{2} + \frac{3 \cdot 1}{4 \cdot 2}\frac{\pi}{2}\right]$$

$$LHS = 3\pi/2$$

Eg 4 Employ Stokes Theorem to Evaluate $\int_C F \cdot d\bar{r}$ where C is triangular boundary with $(1,0,0)$ $(0,2,0)$ $(0,0,3)$ as vertices For $\bar{F} = (y+z-2x)i + (z+x-2y)j + (x+y-2z)k$

Sol^n $C \equiv ABCA$



$C \equiv ABCA$

$ABC =$ surface S

Stokes theorem

$$\boxed{\int_C F \cdot d\bar{r} = \iint_S \text{Curl} F \cdot \vec{N} \, ds} \quad (1)$$

Eq^n of the plane (intercept form) passing through points A, B, C is

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1 = ax + by + cz$$

$(a, b, c) \equiv (1, \tfrac{1}{2}, \tfrac{1}{3})$ are direction ratios of $\vec{N}$, Normal (perpendicular to plane AB)

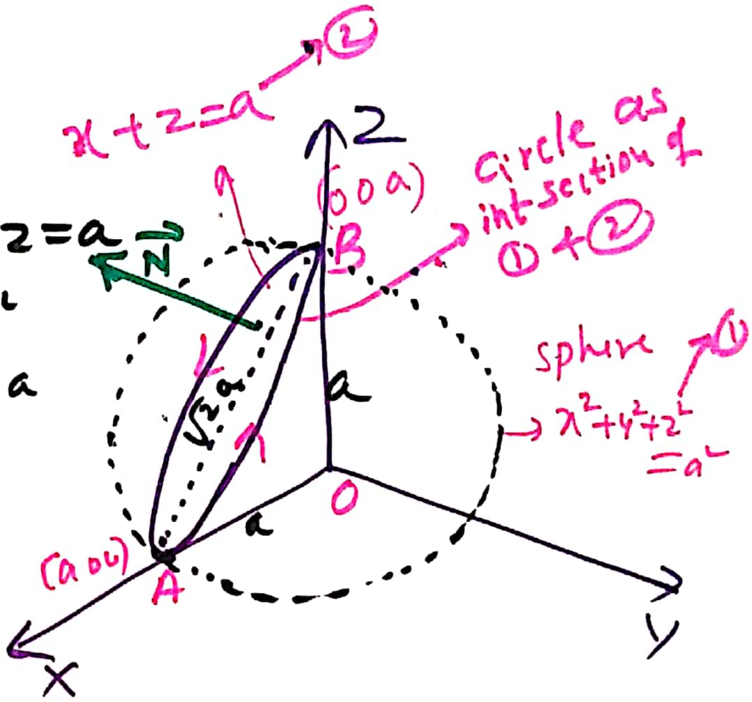
$$\vec{N} = \frac{1i + \tfrac{1}{2}j + \tfrac{1}{3}k}{\sqrt{1 + \tfrac{1}{4} + \tfrac{1}{9}}} = \frac{1}{7}(6i + 3j + 2k)$$

$$\text{Curl}\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z-2x & z+x-2y & x+y-2z \end{vmatrix} = (1-1)i + (1-1)j + k(1-1) = \bar{0}$$

using in (1) $\int_C F \cdot d\bar{r} = \iint_S 0 \cdot \vec{N} \, ds = 0$ Ans

Ex5 Apply Stokes theorem to evaluate $\int_c (y\,dx + z\,dy + x\,dz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$

SolM Curve C is circle intersection of plane $x+z=a$ and sphere $x^2+y^2+z^2=a^2$
AB is diametre & $AB=\sqrt{2}a$



$F = yi + zj + xk$

$\text{curl} F = [-(i+j+k)]$

$\vec{N} = \text{grad}(f)$

$= \nabla(x+z)$

$= i\dfrac{\partial(x+z)}{\partial x} + j\dfrac{\partial(x+z)}{\partial y} + k\dfrac{\partial(x+z)}{\partial z}$

~~unit~~ outward normal $= \vec{N} = i+k$

unit " " $= \hat{N} = \frac{1}{\sqrt{2}}(i+k)$

Stokes thm $\int_c F\cdot d\bar{r} = \iint_S \text{curl} F \cdot \hat{N}\, ds$

$= \iint_S \left[\dfrac{i+k}{\sqrt{2}}\right]\cdot[-(i+j+k)]\,ds$ → Dot product

$= -\frac{1}{\sqrt{2}}\iint_S (1+0+1)\,ds$

$= -\frac{2}{\sqrt{2}}\iint_S ds$

$= -\frac{2}{\sqrt{2}}$ (area of Ol)

$= \frac{-2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2$

$= -\frac{\pi a^2}{\sqrt{2}}$ // Ans

Gauss's Divergence theorem: (or Divergence theorem)

(Relation Between Surface and volume integration)

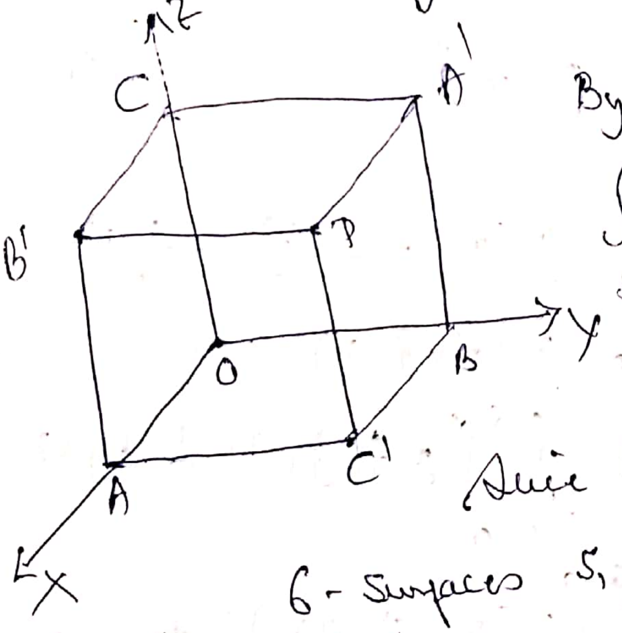
If V is the volume bounded by a closed surface S and $\vec{f}$ is a vector point function with continuous derivatives in V

then $$\iint_S \vec{f}\cdot\hat{n}\,ds = \iiint_V (\operatorname{div}\vec{f})\,dv$$

Where $\hat{n}$ is the outward unit normal vector to S.

Ex: ① Verify divergence theorem for $\vec{f} = x^3 i + y^3 j + z^3 k$ for the cube bounded by $x=0$, $y=0$, $z=0$, $x=a$, $y=a$ & $z=a$.

Sol^n:



By Divergence theorem

$$\iint_S \vec{f}\cdot\hat{n}\,ds = \iiint_V (\operatorname{div}\vec{f})\,dv.$$

Since the surface S bounded by 6 - surfaces $S_1, S_2, \ldots S_6$

ie $$\iint_S \vec{f}\cdot\hat{n}\,ds = \int_{S_1} + \int_{S_2} + \cdots + \int_{S_6}$$

Along S_1 (OAC'B) $z=0$ then $dz=0$ $\hat{n}=-k$ $ds=dx\,dy$

$$\vec{f}\cdot\hat{n}\,ds = -z^3\,dx\,dy = 0$$

$$\boxed{\int_{S_1} \vec{f}\cdot\hat{n}\,ds = 0}$$

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Along S_2, $(PA'CB')$ $z = a$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$\vec{f} \cdot \hat{n}\, ds = z^3\, dx\,dy = a^3\, dx\,dy$

$$\int_{S_2} \vec{f} \cdot \hat{n}\, ds = \int_0^a \int_0^a a^3\, dx\,dy = a^3 \int_0^a (x)_0^a\, dy$$

$$= a^4 (y)_0^a = a^5$$

Along S_3 $(OB A' C)$ $x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

& $\vec{f} \cdot \hat{n}\, ds = -x^3\, dy\,dz = 0$

$$\int_{S_3} \vec{f} \cdot \hat{n}\, ds = 0$$

Illy

$$\int_{S_4} \vec{f} \cdot \hat{n}\, ds = a^5, \quad \int_{S_5} \vec{f} \cdot \hat{n}\, ds = 0 \quad \& \int_{S_6} \vec{f} \cdot \hat{n}\, ds = a^5$$

$$\therefore \iint_S \vec{f} \cdot \hat{n}\, ds = a^5 + a^5 + a^5 = 3a^5 \longrightarrow (1)$$

Also $\operatorname{div} \vec{f} = \nabla \cdot \vec{f} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$

$$= 3(x^2 + y^2 + z^2)$$

$$\text{R.H.S} = \iiint_V (\operatorname{div} \vec{f})\, dv = \int_0^a \int_0^a \int_0^a 3(x^2 + y^2 + z^2)\, dx\,dy\,dz$$

$$= \int_0^a \int_0^a 3\left[\frac{x^3}{3} + (y^2 + z^2)x\right]_0^a dy\,dz$$

$$= 3 \int_0^a \int_0^a \left[\frac{a^3}{3} + a(y^2 + z^2)\right] dy\,dz$$

$$= 3 \int_0^a \left[\frac{a^3}{3} y + a\left(\frac{y^3}{3} + z^2 y\right)\right]_0^a dz$$

$$= 3 \int_0^a \left[\frac{a^4}{3} + a\left(\frac{a^3}{3} + z^2 a\right)\right] dz$$

$$= 3 \left[\frac{2a^4}{3} z + a^2 \frac{z^3}{3}\right]_0^a = 3a^5 \rightarrow (2)$$

From (1) & (2)
D.T. is verified

(2) Use divergence theorem to evaluate $\iint_S \vec{f}\cdot\hat{n}\,ds$

where $\vec{f} = 4xz\,i - y^2 j + yz\,k$ & S is surface bounded by

$x=0,\ x=1,\ y=0,\ y=1,\ z=0$ & $z=1$.

Sol^n Given $\vec{f} = 4xz\,i - y^2 j + yz\,k$

Consider $\text{div}\,\vec{f} = \nabla\cdot\vec{f} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$

$$= 4z - 2y + y = 4z - y.$$

By D.T. we have

$$\iint_S \vec{f}\cdot\hat{n}\,ds = \iiint_V (\text{div}\,\vec{f})\,dv$$

$$= \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (4z-y)(x)_0^1\,dy\,dz.$$

$$= \int_0^1 \left[\frac{(4z-y)^2}{-2}\right]_0^1 dz$$

$$= -\frac{1}{2}\int_0^1 \left[(4z-1)^2 - (4z)^2\right] dz$$

$$= -\frac{1}{2}\int_0^1 \left(\cancel{16z^2} - 8z + 1 - \cancel{16z^2}\right) dz$$

$$= -\frac{1}{2}\left(z - 4z^2\right)_0^1 = \frac{3}{2}$$

(3) If S is the surface of the sphere $x^2+y^2+z^2=a^2$ (VI 19)

P.T. $\int_S (x^3 dy dz + y^3 dz dx + z^3 dx dz) = \frac{12}{5} \pi a^5$.

Given $\vec{f} \cdot \hat{n} \, ds = \vec{f} \cdot \vec{ds} = x^3 i + y^3 j + z^3 k$.

By D.T. $\iint_S \vec{f} \cdot \hat{n} \, ds = \iiint_V (\operatorname{div} \vec{f}) \, dv$

Consider

$\operatorname{div} \vec{f} = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$

Since $x^2+y^2+z^2=a^2$ is symmetrical $\therefore$ volume ~~is~~ 8 times volume in 1^{st} octant.

Change into spherical polar co-ordinates (r, θ, ϕ)

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ & $z = r\cos\theta$ then

$dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also $0 \le r \le a$, $0 \le \theta \le \pi$ & $0 \le \phi \le 2\pi$

$$\therefore \iint_S \vec{f}\cdot\hat{n}\, ds = 3\int_0^a\int_0^\pi\int_0^{2\pi} r^2\, r^2 \sin\theta\, d\phi\, d\theta\, dr$$

$$= 3\left(\frac{r^5}{5}\right)_0^a \int_0^\pi\int_0^{2\pi} \sin\theta\, d\phi\, d\theta$$

$$= \frac{3a^5}{5}\left(-\cos\theta\right)_0^\pi \left(\phi\right)_0^{2\pi}$$

$$= -\frac{3a^5}{5}\left(\cos\pi - \cos 0\right)(2\pi - 0)$$

$$= \frac{12\pi a^5}{5}$$

④ Using divergence theorem evaluate $\int_S \vec{f}\cdot d\vec{s}$.

where $\vec{f} = 4x\,i - 2y^2\,j + z^2\,k$ & S is the surface bounded the region $x^2+y^2=4$, $z=0$ & $z=3$

Sol^n By D.T $\iint_S \vec{f}\cdot\hat{n}\, ds = \iiint (\mathrm{div}\,\vec{f})\, dv$

Consider $\mathrm{div}\,\vec{f} = \nabla\cdot\vec{f}$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z$$

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Here x varies from $-a$ to a

y " " $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$.

OR 2 limits region in 1st octant.

$$\int_S \vec{F}\cdot d\vec{s} = 4\int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} (4-4y+2z)\, dy\, dx\, dz.$$

$$\cancel{= 4\int_0^3\int_0^2 \left[(4+2z)y - 2y^2\right]}$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} \left[(4-4y)z + z^2\right]_0^3 dy\, dx$$

$$= 4\int_0^2 (12-12y+27)\, dy\, dx$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} (39-12y)\, dy\, dx.$$

$$= 4\int_0^2 \left[39y - 6y^2\right]_0^{\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left[39\sqrt{4-x^2} - 6(4-x^2)\right] dx$$

$$= 4\left[39\left[\cancel{\frac{x}{2}\sqrt{4-x^2}}^{\,0} + \frac{4}{2}\sqrt{4-x^2}\right]_0^2 - 6\left(4x-\frac{x^3}{3}\right)_0^2\right]$$

$$= 4\left[39(4) - 6\left(8-\frac{8}{3}\right)\right]$$

$$= 4\left[136 - \cancel{6}^{\,2}\times\frac{16}{\cancel{3}}\right] = 4\left[136-38\right]$$

$$= 4(98) = \underline{\underline{392}}$$