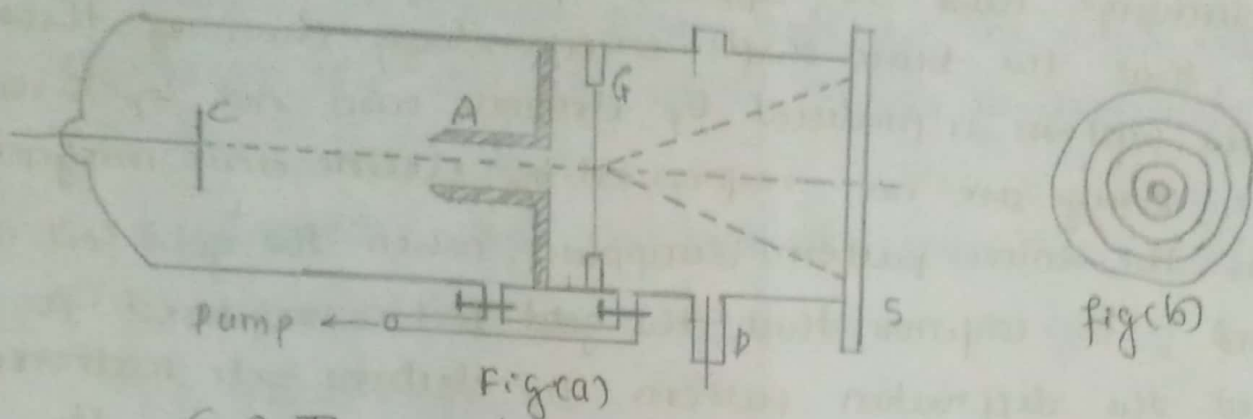


# G. P. Thomson's Experiment:

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G. P. Thomson's experiment is used to study the wave nature of the electron. The experimental arrangement is as shown in figure (a).

The apparatus consists of cathode 'C' from which narrow beam of electron is produced by applying a high potential of 50 kV across the cathode. [A narrow fine beam of electrons is produced from cathode is passing through a fine cylinder 'A'] The fine beam of electrons after coming out from cylinder 'A' are allowed to ~~incide on~~ ~~a photographic plate~~ be incident on a thin gold foil 'G'. The thickness of the foil is about  $10^{-8}$  m. The diffraction of electrons takes place at thin gold foil. These diffracted electrons are either allowed to be incident on a photographic plate 'P' or fluorescent screen 'S'. Both generation of secondary X-rays and diffraction of electrons takes place when electron beam is incident on thin gold foil. A diffraction pattern can be obtained on the screen which verifies the wave nature of electrons. The diffraction pattern obtained on the screen is shown in figure (b). The diffraction pattern consists of concentric rings of increasing diameter due to the periodic arrangement of atoms in

the gold foil. The obtained diffraction pattern is due to electrons and not due to X-rays generated by the electrons in their passage through the gold foil, the cathode rays in the discharge tube are applied by magnetic field. It was observed that the beam shifts correspondingly showing thereby that the pattern is produced by electrons and not by X-rays because X-rays are not responded by electric and magnetic fields. The entire pattern disappears when the gold foil is removed, it confirms that the gold foil is required to obtain the diffraction pattern. The electrons get scattered over a wide range when they strike the gold foil, it confirms the particle nature of electrons. The diffracted electron produces the diffraction pattern.

The G.P. Thomson's experiment demonstrate the wave nature of electrons as diffraction pattern can be obtained by waves only. By knowing the values of the ring diameter and the de Broglie wavelength, the size of the crystal unit cell can be obtained. ~~by same way~~. The size obtained from this method are in good agreement with that obtained from X-ray technique.

## Phase velocity ( $v_p$ )

The phase velocity is the velocity of the individual waves. The velocity with which each phase in a wave moves is called phase velocity. The phase velocity represents the velocity with which a particular phase of the wave moves.

Consider an unclamped wave in motion and the equation for displacement is given by

$$y = A \sin(\omega t - kx)$$

where  $\omega$  is the angular frequency,  $y$  is the displacement along  $y$ -coordinate at the instant  $t$  and  $x$  is the displacement along  $x$ -coordinate at the instant  $t$ .  $k$  is the propagation constant or the wave number.

$$k = \frac{\omega}{v} \text{ is the wave number}$$

The term  $\omega t - kx$  is the phase of the wave

$\therefore$  The phase velocity of the wave is the rate at which the phase of the wave propagates in space

$$\therefore \frac{d}{dt}(\omega t - kx) = 0 \Rightarrow \omega - k \frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$$

$$\Rightarrow \boxed{v_{\text{phase}} = \frac{\omega}{k}}$$

$$v_p = v \lambda$$

$$v = \frac{\omega}{2\pi} \quad \lambda = \frac{2\pi}{k}$$

$$v_p = \frac{\omega}{k}$$

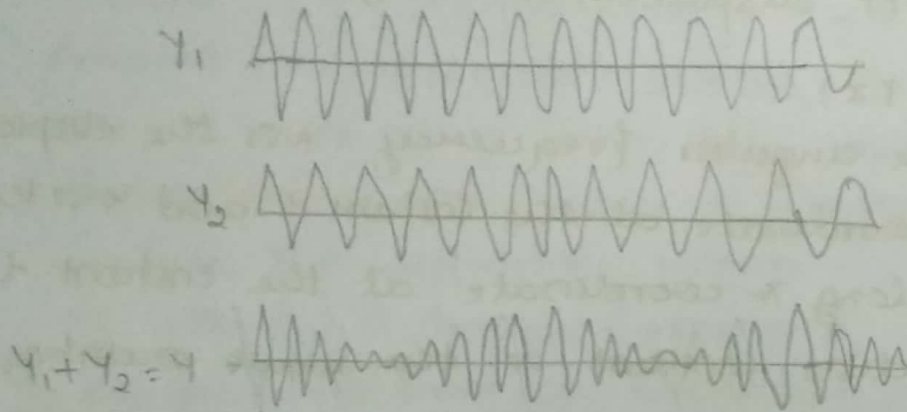
## Group velocity ( $v_g$ )

When a group of two or more waves of different wavelengths and different velocities are superimposed on each other the amplitude of the resultant wave is varied, which forms a wave group or wave packet. The velocity with which the resultant envelope of the group of waves travel is called group velocity and it is denoted by  $v_g$ . In other words the velocity



with which the energy in the wave group is transmitted is called group velocity. The velocity of group wave is different from that of the individual component.

Group velocity is represented by  $\boxed{v_g = \frac{d\omega}{dk}}$



### Relation Between Group Velocity and Phase Velocity

We have the equations for group velocity and phase velocity are as follows

$$v_{\text{group}} = \frac{d\omega}{dk} \quad \text{--- (1)} \quad \text{and} \quad v_{\text{phase}} = \frac{\omega}{k} \quad \text{--- (2)}$$

where  $\omega$  is the angular frequency of the wave and  $k$  is the wave number

$$\text{from eqn (2)} \Rightarrow \omega = k v_{\text{phase}} \rightarrow \text{(3)}$$

Eqn (3) Diff w.r.t  $k$

$$\frac{d\omega}{dk} = v_{\text{phase}} + k \frac{dv_{\text{phase}}}{dk} \rightarrow \text{(4)}$$

$\therefore$  Eqn (1) becomes

$$v_{\text{group}} = v_{\text{phase}} + k \frac{dv_{\text{phase}}}{dk}$$

$$v_{\text{group}} = v_{\text{phase}} + k \frac{dv_{\text{phase}}}{d\lambda} \times \frac{d\lambda}{dk} \rightarrow \text{(5)}$$

We know that  $k = \frac{2\pi}{\lambda}$

Differentiating we get  $\frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2}$

$$\text{or } \frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi}$$

$\therefore$  eqn (5) becomes

$$v_{\text{group}} = v_{\text{phase}} + \left(\frac{2\pi}{\lambda}\right) \left(\frac{dv_{\text{phase}}}{d\lambda}\right) \left(-\frac{\lambda^2}{2\pi}\right)$$

$$v_{\text{group}} = v_{\text{phase}} - \lambda \frac{dv_{\text{phase}}}{d\lambda}$$

This is the relation between group velocity and phase velocity.

### Relation Between Group Velocity and Particle Velocity

We have the equation for group velocity

$$v_{\text{group}} = \frac{d\omega}{dk} \quad \text{--- (1)}$$

$$\text{But } \omega = 2\pi\nu = 2\pi \frac{E}{h} \quad \because E = h\nu \quad \rightarrow \text{--- (2)}$$

$$\therefore d\omega = \left(\frac{2\pi}{h}\right) dE \quad \rightarrow \text{--- (3)}$$

$$\text{Also we have } k = \frac{2\pi}{\lambda} = 2\pi \frac{p}{h} \quad \because \lambda = \frac{h}{p} \quad \rightarrow \text{--- (4)}$$

$$dk = \left(\frac{2\pi}{h}\right) dp \quad \rightarrow \text{--- (5)}$$

Dividing eqn (3) by eqn (5) we get

$$\frac{d\omega}{dk} = \frac{dE}{dp} \quad \rightarrow \text{--- (6)}$$

We have  $E = \frac{p^2}{2m}$ , where  $p$  is the momentum of the particle

$$\therefore dE = \frac{2p dp}{2m} = \frac{p dp}{m}$$

$$\therefore \frac{dE}{dp} = \frac{p}{m}$$

$\therefore$  eqn ⑥ becomes

$$\frac{d\omega}{dk} = \frac{p}{m}$$

But  $p = mv_{\text{particle}}$

where  $v_{\text{particle}}$  is the velocity of the particle

$$\frac{d\omega}{dk} = \frac{mv_{\text{particle}}}{m} = v_{\text{particle}} \rightarrow \textcircled{7}$$

$\therefore$  from eqn ⑥ and eqn ⑦  
we get  $\boxed{v_{\text{group}} = v_{\text{particle}}}$

The velocity of the de Broglie wave group associated with the particle is equal to the velocity of the particle, itself.

Relation Between Phase velocity, Group velocity and **Velocity of light**

$$\boxed{v_{\text{phase}} \cdot v_{\text{group}} = c^2}$$



Q: Explain Heisenberg's Uncertainty Principle; Give its physical significance.

Statement: "It states that it is impossible to determine precisely and simultaneously the exact position and exact momentum of a particle at the same time. The product of the corresponding uncertainties inherently present in the measurement is equal to or greater than  $(h/4\pi)$ ".

$$\text{i.e. } \Delta x \Delta p_x \geq \frac{h}{4\pi}$$

Where  $\Delta x$  and  $\Delta p_x$  are the uncertainties in the measured values of the position and momentum of the particle.

Similarly Energy and time is  $\Delta E \Delta t \geq \frac{h}{4\pi}$

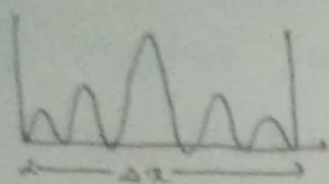
Where  $\Delta E$  and  $\Delta t$  are the uncertainties in the measured values of the energy and time ~~of the~~ respectively.

and angular displacement ( $\theta$ ) and angular momentum

$$(L) : \Delta \theta \Delta L \geq \frac{h}{4\pi}$$

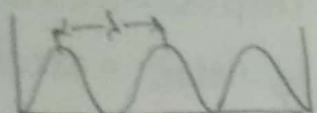
Where  $\Delta \theta$  and  $\Delta L$  are the uncertainties involved in the displacement and angular momentum.

Explanation: In classical mechanics the position and momentum of a particle can be measured accurately at any instant of time. In quantum mechanics a moving particle is associated with wave (wave packet) its probability of finding is proportional to amplitude of the wave at that instant.



Small wave packet

→ For small wave packet, the amplitude of the wave is much large, position is found but the momentum becomes uncertain.



Wider wave packet

→ For wider wave packet the momentum of the particle can be measured accurately but position becomes uncertain.

## Physical Significance of Heisenberg's Uncertainty

### Principle :

It is physically impossible to know simultaneously the exact position and the exact momentum of a particle or to determine accurately the simultaneous values of position and momentum of a particle at any time.

It is possible to know the probability of finding the particle at a certain position or of the probable value for the momentum of the particle.

The estimation of such probability are made by means of certain mathematical functions named probability density function in quantum mechanics similarly for  $\Delta E$  and  $\Delta t$ .



Q: What is wave function? Give its physical significance and properties.

The quantity whose periodic variation make up the matter waves (de-Broglie waves) is called wave function denoted by  $\Psi$ .

It contains all possible information about the system. Hence it is also called state function.

The wave function varies with both position coordinates and time of physical system it is called total wave function represented by  $\Psi$ .

The total wave function can be represented as  
 $\Psi = A e^{-i(\omega t - kx)}$  or  $\Psi = A e^{i(kx - \omega t)}$

$A$  - is the amplitude of the wave  
 $\omega$  - angular frequency.

If the wave function has variation only with position & not with time. it is represented as  $\Psi = A e^{ikx}$   
Then separating space and time dependent parts

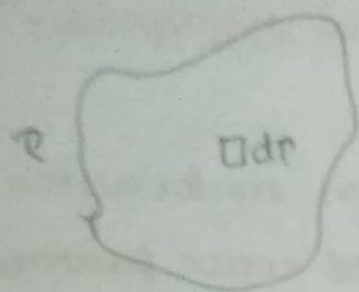
$$\Psi = A e^{ikx} e^{-i\omega t}$$

$$\therefore \Psi = \psi e^{-i\omega t}$$

The wave function in quantum mechanics accounts for the wave-like properties of a particle.

Physical Significance of wave function:  
probability density:

If we consider a single particle and  $\Psi$  is the wave function associated with the particle, then  $|\Psi|^2$  is the probability per unit volume that the particle, will be found at the given region.



Let  $V$  be a volume inside which a particle is known to be present, but where exactly the particle is situated inside  $V$  is not known.

Then the probability of finding the particle in a certain element of volume  $dV$  of  $V$  is equal to  $|\psi|^2 dV$ .

According to Max Born  $|\psi|^2$  gives the probability density. It is also called probability function.

The wave function is either real or complex. The product of  $\psi$  and its complex conjugate  $\psi^*$  is always real and positive.

$\therefore \psi \psi^* = |\psi|^2$  is called probability density.

### Normalization:

If  $\psi$  is the wave function associated with a particle, then we have  $|\psi|^2 dV$  as the probability of finding the particle in a volume  $dV$ . If it is certain that particle is definitely present in a particular region of volume  $V$  then

$$\int_V |\psi|^2 dV = 1 \quad \text{--- (1)}$$

or for some where in space then eqn (1) becomes

$$\int_{-\infty}^{\infty} |\psi|^2 dV = 1$$

In most cases, the result of evaluating of  $\int_{-\infty}^{\infty} |\psi|^2 dV$  will not be unity but involves a constant.

that existed in the equation for  $\psi$ . However, the actual result is equated to unity and the value of the constant is determined.

The wave equation satisfies the above condition, (eqn) is called Normalized wave equation.

## Properties of wave function:

- (i) wave function ( $\psi$ ) must be finite for all values of  $x, y, z$ .
- (ii) wave function ( $\psi$ ) must be single-valued i.e. for each set of values of  $x, y, z$ ,  $\psi$  must have one value only.
- (iii) wave function ( $\psi$ ) must be continuous in all regions, except in those where the potential energy  $V(x, y, z) = \infty$ .
- (iv) wave function ( $\psi$ ) and its first derivatives with its variable are continuous.

## Q7) What are Eigen values and Eigen functions?

"Eigen functions are those wave functions of quantum mechanics which possess the properties that they are single valued & finite everywhere, and also their first derivatives with respect to their variables are continuous everywhere"

In general the eigen value equation can be written as

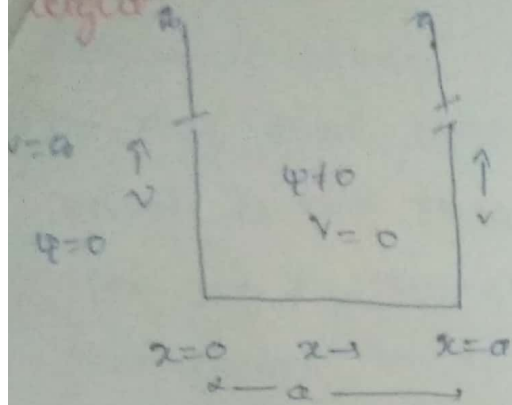
$$F_{op} \psi = f \psi \quad \text{or} \quad \hat{A} \psi = \lambda \psi$$

where  $\hat{A}$  is the operator for the physical quantity, and  $\psi$  is eigen function.

The eigen function should be such that, the operator operating on it produces back the wave function multiplied by a constant ( $\lambda$ ). Such values obtained for a physical observable are called eigen values.



Find the eigen value and eigen functions for an electron in one dimensional potential well of infinite depth.



at  $x=0$  and  $x=a$   
 $\psi=0$   $\psi=0$

A free particle of mass  $m$  is to move in the  $x$ -direction only in the region from  $x=0$  and  $x=a$ , where potential  $V$  will be zero.

Let the potential  $V$  outside this region be taken as infinite. Such a configuration is infinite potential well.

Consider time-independent schrodinger wave equation,

$$\text{i.e. } \frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h^2} (E - V)\psi = 0 \quad \text{--- (1)}$$

The particle inside the well, the schrodinger equation is given by

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h^2} (E - 0)\psi = 0 \quad (\text{since } V=0) \quad \text{--- (2)}$$

$$\text{Let } k^2 = \frac{8\pi^2mE}{h^2} \quad \text{--- (3)}$$

$\therefore$  Eqn (2) becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

This is the second order differential equation the solution of this equation is,

$$\psi = C \cos kx + D \sin kx \quad \text{--- (4)}$$

But as per the condition

(i)  $\psi = 0$  at  $x = 0$

$\therefore$  Eqn (4) becomes

$$0 = C \cos 0 + D \sin 0$$

$$C = 0 \quad \text{--- (5)}$$

(ii)  $\psi = 0$  at  $x = a$

$\therefore$  Eqn (4) becomes

$$0 = C \cos ka + D \sin ka$$

from eqn (5) we have  $C = 0$ , Hence the above eqn becomes

$$0 = D \sin ka$$

Here,  $D$  need not be zero (i.e.  $D \neq 0$ ).

$$\therefore \sin ka = 0$$

$$ka = n\pi$$

where  $n = 0, 1, 2, 3, \dots$

$n$  is called quantum number which is either zero or a positive integer.

$$k = \frac{n\pi}{a} \quad \text{--- (6)}$$

Substituting eqn (5) and eqn (6) in eqn (4) we get

$$\psi_n = D \sin \frac{n\pi}{a} x \quad \text{--- (7)}$$

From eqn (3)  $E = \frac{k^2 h^2}{8\pi^2 m}$

Sub eqn (6) in above eqn

$$E = \frac{n^2 h^2}{8ma^2} \quad \text{--- (8)}$$

As mentioned earlier 'n' can have only integer values i.e. 1, 2, 3, .... The corresponding values that  $E$  takes are the energy eigen values.

In eqn (8), the value zero for  $n$  is not acceptable because when  $n=0$ ,  $\Psi_0=0$  by eqn (4), which means to say that electron is not present inside the box which is not true. Hence the lowest possible values for  $n$  is 1. Consequently the lowest allowed energy corresponds to  $n=1$  which is called the zero-point energy. The zero point energy of an electron in an infinite potential well is given by

$$E_{\text{zero-point}} = E_1 = \frac{h^2}{8ma^2} \quad \text{or } E_1 - E_0 = \frac{h^2}{8ma^2}$$

which represent permitted solutions.

The lowest permitted state of energy is referred to as ground state energy. The energy state corresponding to  $n > 1$  are called excited states.

For  $n=2$  from eqn (8)

$$E_2 = \frac{4h^2}{8ma^2} = 4E_0$$

For  $n=3$  from eqn (8)

$$E_3 = \frac{9h^2}{8ma^2} = 9E_0$$

$\therefore$  From  $E_2$ , the energy in first excited state is 4 times the zero point energy. This is the energy eigen value in first excited state.

$\therefore$  From  $E_3$ , the energy in second excited state is 9 times the zero point energy. This is the energy eigen value in second excited state.



Normalization:

To evaluate  $D$  in eqn (7) we have to perform normalization of the wavefunction.

$\therefore$  The integral of the square of the wave function over the entire space in the well must be equal to unity because, there is only one particle and at any time, it is present somewhere inside the well only.

$$\therefore \int_0^a |\psi_n|^2 dx = 1$$

Sub eqn (7) for  $\psi_n$  in above eqn

$$\int_0^a D^2 \sin^2 \frac{n\pi x}{a} dx = 1$$

~~we know that~~  $\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)$

$$D^2 \left[ \frac{1}{2} \int_0^a dx - \frac{1}{2} \int_0^a \cos \frac{2n\pi x}{a} dx \right] = 1$$

$$\frac{D^2}{2} \left[ x - \frac{a}{2n\pi} \sin \left( \frac{2n\pi x}{a} \right) \right]_0^a = 1$$

$$\frac{D^2}{2} \left[ a - \frac{a}{2n\pi} \sin(2n\pi) \right] = 1$$

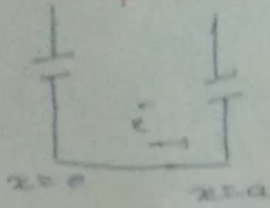
$$\frac{D^2 a}{2} = 1$$

$$\therefore D = \sqrt{2/a}$$

Thus the normalized wavefunction of a particle in a one-dimensional infinite potential well is given by

$$\boxed{\psi_n = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right)} \quad \text{--- (9)}$$

Discuss the <sup>allowed</sup> energy eigenvalues and eigen functions for an electron in one dimensional potential well of infinite height.



Electron in potential well of infinite height  
consider an electron bound in potential well  
in the region  $x=0$  to  $x=a$

consider equation  $\Psi_n = D \sin\left(\frac{n\pi}{a} x\right)$ .

By using this equation we can write the  
eigen functions  $\Psi_1, \Psi_2, \Psi_3, \dots$  for the electron  
in well by putting  $n=1, 2, 3, \dots$

Similarly by using equation  $E = \frac{n^2 h^2}{8ma^2}$  we can

write the energy eigen values for the electron i.e.

$E_1, E_2, E_3, \dots$

Let us consider the first cases.

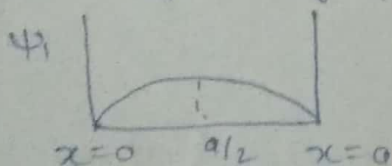
case (i)  $n=1$  : This is the ground state and the particle  
is normally found in this state

For  $n=1$ , the eigen function is  $\Psi_1 = D \sin\left(\frac{\pi}{a} x\right)$

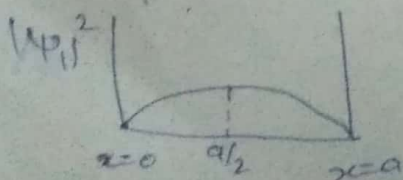
$\Psi_1 = 0$  for both  $x=0$  and  $x=a$

But  $\Psi_1$  is maximum at  $x=a/2$

Thus a plot of  $\Psi_1$  versus  $x$  will be as follows.



The plot of  $|\Psi_1|^2$ , the probability density versus  $x$  is  
as follows



From the figure  $|\Psi_1|^2$  is 0 both at  $x=0$  &  $x=a$   
and maximum at  $x=a/2$  which means probability  
of finding particle at wall is zero and  
maximum at centre

The energy eigen value in ground state is given by putting  $n=1$

$$E_1 = \frac{h^2}{8ma^2} = E_0$$

The energy in the ground state  $E_1$  is same as the zero point energy  $E_0$ .

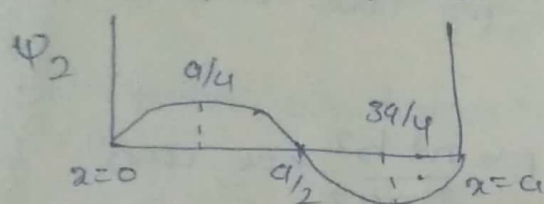
case ii :  $n=2$  : This is the first excited state. The eigen function for this state is

$$\Psi_2 = D \sin\left(\frac{2\pi}{a}\right)x$$

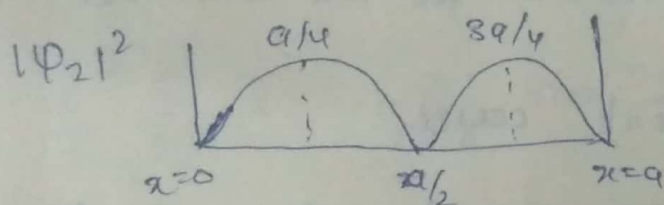
$\Psi_2 = 0$  for  $x=0, a/2$  and  $a$

$\Psi_2$  is maximum at for  $x=a/4$  and  $3a/4$

The plot of  $\Psi_2$  versus  $x$  is as follows



The plot of  $|\Psi_2|^2$  versus  $x$  is as follows.



From the figure  $|\Psi_2|^2 = 0$  at  $x=0, a/2$  and  $a$ , which means in first excited state, particle cannot be observed either at  $x=0$  or at  $x=a$ .

The energy eigen value in first excited state is given by putting  $n=2$

$$\therefore E_2 = 4 \left[ \frac{h^2}{8ma^2} \right] = 4E_0$$

Thus the energy in the first excited state is 4 times the ground state energy.