LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Definition

A function $f(t)$ is said to be a *periodic function*, if there exists a constant $P\ (>0)$ such that $f(t+P)=f(t)$, for all values of t. Now $f(t+2P)=f(t+P+P)=f(t+P)=f(t)$, for all t. In general, $f(t+nP)=f(t)$, for all t, when n is an integer (positive or negative).

P is called the *period of the function.*

Unlike other functions whose Laplace transforms are expressed in terms of an integral over the semi-infinite interval $0 \le t < \infty$, the Laplace transform of a periodic function $f(t)$ with period P can be expressed in terms of the integral of $e^{-st} f(t)$ over the finite interval $(0, P)$, as established in the following theorem.

Theorem

If $f(t)$ is a piecewise continuous periodic function with period P, then

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \cdot \int_0^P e^{-st} f(t)\,dt.$$

Proof:

By definition, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^P e^{-st} f(t)\,dt + \int_P^\infty e^{-st} f(t)\,dt \qquad (1)$$

In the second integral in (1), put $t = x + P$, $\therefore\ dt = dx$ and the limits for x become 0 and ∞.

$$\therefore \quad \int_P^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(x+P)} f(x+P)\,dx$$

$$= e^{-sP} \cdot \int_0^\infty e^{-sx} f(x)\,dx \qquad [\because f(x+P) = f(x)]$$

$$= e^{-sP} \int_0^\infty e^{-st} f(t)\,dt, \text{ on changing the dummy variable } x \text{ to } t.$$

$$= e^{-sP} \cdot L\{f(t)\} \tag{2}$$

By putting (2) in (1), we have

$$L\{f(t)\} = \int_0^P e^{-st} f(t)\,dt + e^{-sP} \cdot L\{f(t)\}$$

$$\therefore \quad (1 - e^{-Ps})\, L\{f(t)\} = \int_0^P e^{-st} f(t)\,dt$$

$$\therefore \quad L\{f(t)\} = \frac{1}{1 - e^{-Ps}} \int_0^P e^{-st} f(t)\,dt.$$

WORKED EXAMPLE 5(b)

Example 5.1 Find the Laplace transform of the "saw-tooth wave" function $f(t)$ which is periodic with period 1 and defined as $f(t) = kt$, in $0 < t < 1$.

The graph of $f(t)$ is shown in Fig. 5.1 below. If the period of the function $f(t)$ is P, the function will be defined as $f(t) = \dfrac{k}{P}t$ in $0 < t < P$.

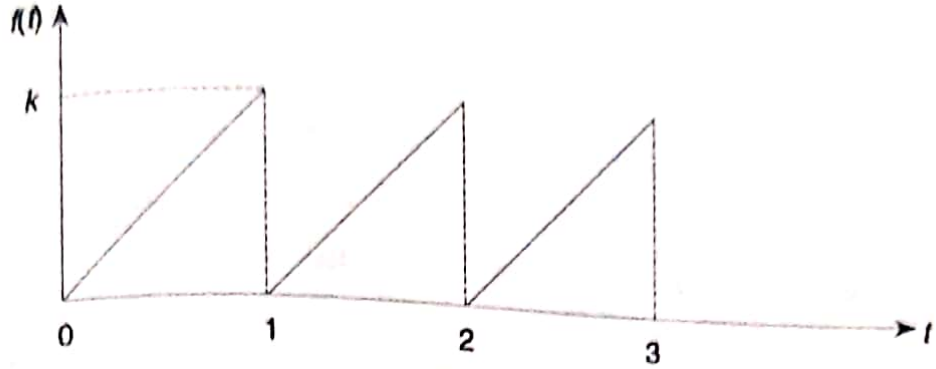

Fig. 5.1

By the formula for the Laplace transform of a periodic function $f(t)$ with period P,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\, dt$$

$\therefore$ For the given function,

$$L\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 kt\, e^{-st}\, dt$$

$$= \frac{k}{1-e^{-s}} \left[t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right]_0^1$$

$$= \frac{k}{1-e^{-s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{k}{1-e^{-s}} \left[\frac{(1-e^{-s})}{s^2} - \frac{e^{-s}}{s} \right]$$

$$= \frac{k}{s^2} - \frac{k e^{-s}}{s(1-e^{-s})}$$

Example 5.2 Find the Laplace transform of the "square wave" function $f(t)$ defined by

$$f(t) = k \text{ in } 0 \le t \le a$$
$$= -k \text{ in } a \le t \le 2a$$

and $\quad f(t+2a) = f(t)$ for all t.

$f(t+2a) = f(t)$ means that $f(t)$ is periodic with period $2a$. The graph of the function is shown in Fig. 5.2.

For a periodic function $f(t)$ with period P,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\, dt$$

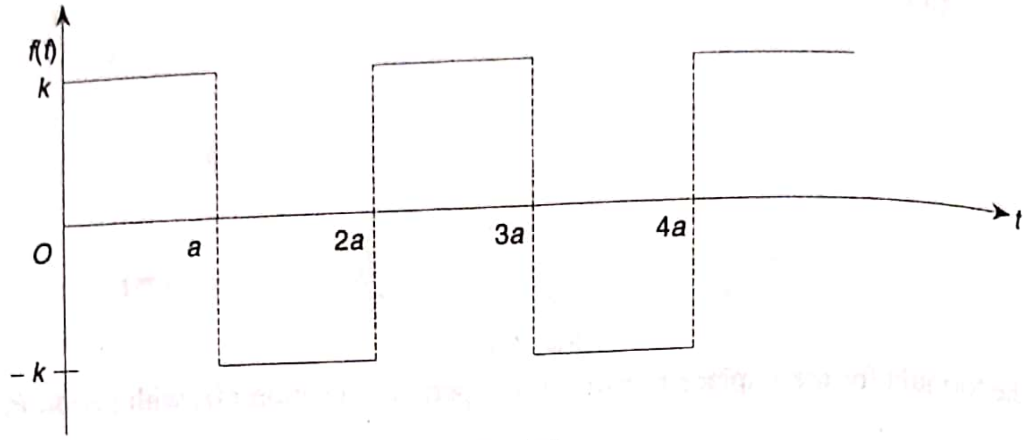

Fig. 5.2

$\therefore$ For the given function;

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left[\int_0^a k\,e^{-st}\,dt + \int_a^{2a}(-k)\,e^{-st}\,dt\right]$$

$$= \frac{k}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{k}{s(1-e^{-2as})}\left[1-e^{-as}-e^{-as}+e^{-2as}\right]$$

$$= \frac{k\,(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{k}{s}\frac{(1-e^{-as})}{(1+e^{-as})} = \frac{k}{s}\frac{\left(e^{as/2}-e^{-as/2}\right)}{\left(e^{as/2}+e^{-as/2}\right)}$$

$$= \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

Example 5.3 Find the Laplace transform of "triangular wave function $f(t)$ whose graph is given below in Fig. 5.3.

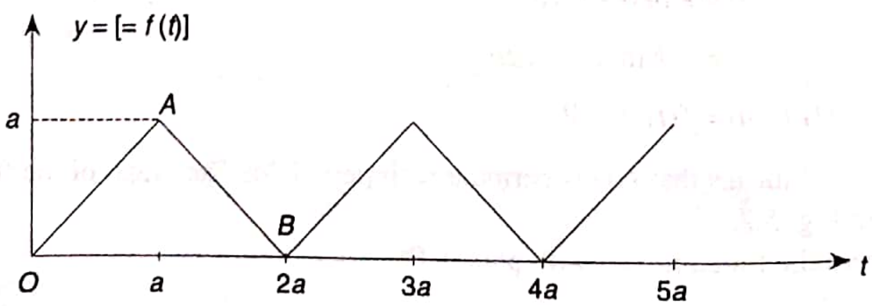

Fig. 5.3

From the graph it is obvious that $f(t)$ is periodic with period $2a$.

Let us find the value of $f(t)$ in $0 \le t \le 2a$, by finding the equations of the lines OA and AB.

OA passes through the origin and has a slope 1.

$\therefore$ Equation of OA is $y = t$, in $0 \le t \le a$

AB passes through the point $B(2a, 0)$ and has a slope -1.

$\therefore$ Equation of AB is $y - 0 = (-1)(t - 2a)$

or $$y = 2a - t \text{ in } a \le t \le 2a.$$

Thus the definition of $f(t)$ [$= y$] can be taken as

$$f(t) = t, \text{ in } 0 \le t \le a$$
$$= 2a - t, \text{ in } a \le t \le 2a$$

and $$f(t + 2a) = f(a).$$

Now $$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st}\, dt + \int_a^{2a} (2a-t) e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left\{ t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right\}_0^a + \left\{ (2a-t)\left(\frac{e^{-st}}{-s}\right) + 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right\}_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{a}{s} e^{-as} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a}{s} e^{-as} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1 - 2e^{-as} + e^{-2as}}{s^2(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})}{(1+e^{-as})} = \frac{1}{s^2}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right)$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

Example 5.4 Find the Laplace transform of the "half-sine wave rectifier" function $f(t)$ whose graph is given in Fig. 5.4.

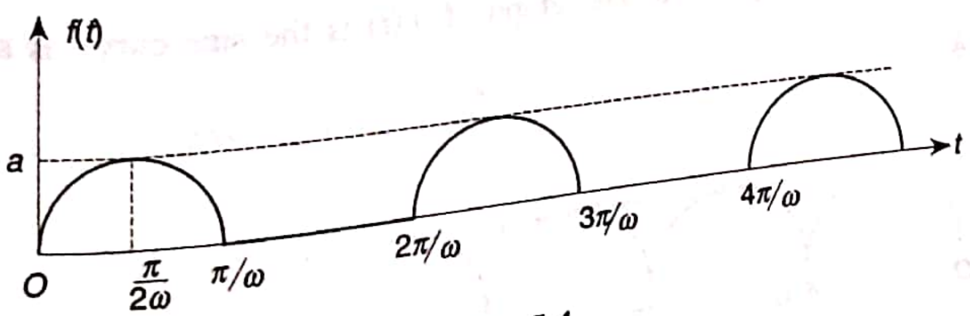

Fig. 5.4

From the graph, it is obvious that $f(t)$ is a periodic function with period $2\pi/\omega$. The graph of $f(t)$ in $0 \le t \le \pi/\omega$ is a sine curve that passes through $(0, 0)$, $\left(\frac{\pi}{2\omega}, a\right)$ and $\left(\frac{\pi}{\omega}, 0\right)$

$\therefore$ The definition of $f(t)$ is given by

$$f(t) = a \sin \omega t, \text{ in } 0 \le t \le \pi/\omega$$
$$= 0, \text{ in } \pi/\omega \le t \le 2\pi/\omega$$

and $$f\left(t + \frac{2\pi}{\omega}\right) = f(t).$$

Now $$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$= \frac{a}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt$$

$$= \frac{a}{1-e^{-2\pi s/\omega}} \cdot \left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t\right]_0^{\pi/\omega}$$

$$= \frac{a}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[\omega e^{-\pi s/\omega} + \omega\right]$$

$$= \frac{\omega a\left(1+e^{-\pi s/\omega}\right)}{(s^2+\omega^2)\left(1-e^{-2\pi s/\omega}\right)} = \frac{\omega a}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)}$$

Example 5.5 Find the Laplace transform of the "full-sine wave rectifier" function $f(t)$, defined as

$$f(t) = |\sin \omega t|, t \ge 0$$

We note that $$f(t + \pi/\omega) = |\sin \omega (t + \pi/\omega)|$$
$$= |\sin \omega t|$$
$$= f(t)$$

$\therefore f(t)$ is periodic with period π/ω.

Also $f(t)$ is always positive. The graph of $f(t)$ is the sine curve as shown in Fig. 5.5.

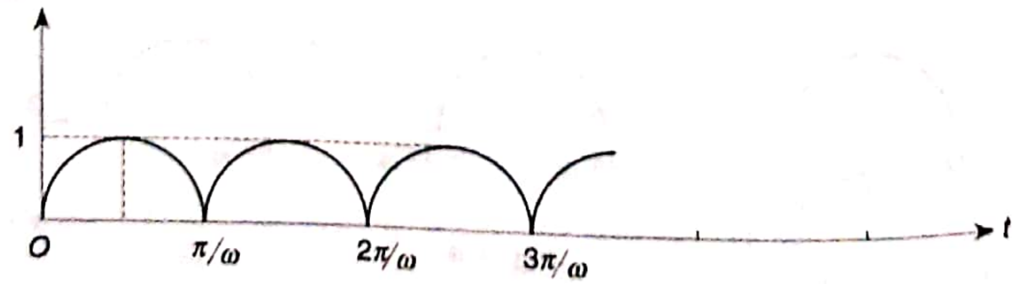

Fig. 5.5

Now $L\{f(t)\} = \dfrac{1}{1-e^{-\pi s/\omega}} \displaystyle\int_0^{\pi/\omega} e^{-st} |\sin \omega t| \, dt$

$$= \frac{1}{1-e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt \; [\because \sin \omega t > 0 \text{ in } 0 \le t \le \pi/\omega]$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{(s^2+\omega^2)(1-e^{-\pi s/\omega})} \left(\omega e^{-\pi s/\omega} + \omega\right) = \frac{\omega}{s^2+\omega^2} \left(\frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}} \right)$$

$$= \frac{\omega}{s^2+\omega^2} \left(\frac{e^{\pi s/2\omega} + e^{-\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/2\omega}} \right), \text{ on integration and simplification}$$

$$= \frac{\omega}{s^2+\omega^2} \coth \left(\frac{\pi s}{2\omega} \right)$$