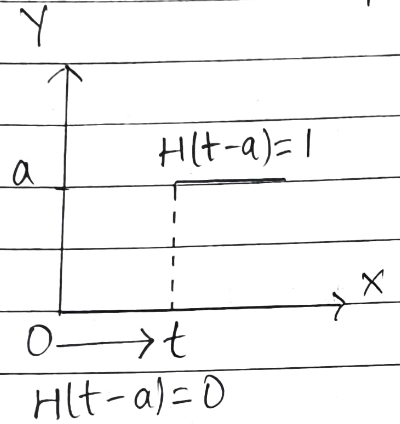
Laplace transform of unit step function & unit impulse function

The unit step function or heaviside function is a discontinuous function denoted by,

$H(t-a)$ or $u(t-a)$ is defined as, ~~$H(t-a)=0$~~
$=1$

$$H(t-a) = 0 \quad , \quad t \le a$$
$$\quad\quad\quad = 1 \quad , \quad t > a$$



Properties:

→ $L[u(t-a)] = \dfrac{e^{-as}}{s}$

→ $L[f(t-a)\,u(t-a)] = e^{-as} F(s) = e^{-as} L[f(t)]$

1] Find, $L[(2t-1)H(t-2)]$

→ $L[(2t-1)H(t-2)]$

here both the brackets should have $(t-2)$

$\Rightarrow \; 2t-1 = 2(t-2+2)-1$
$\quad\quad\quad\;\; = 2(t-2)+4-1$
$\quad\quad\quad\;\; = 2(t-2)+3$

$\Rightarrow L[(2(t-2)+3)H(t-2)] = e^{-as}L[2t+3]$

$= e^{-as}\left[\frac{2}{s^2}+\frac{3}{s}\right]$

$L[(2(t-2)+3)H(t-2)] = e^{-as}\left[\frac{2}{s^2}+\frac{3}{s}\right]$

2] $L[e^{-t}H(t-2)]$.

$\rightarrow$ $e^{-t} = e^{-(t-2+2)}$

$= e^{-(t-2)-2}$

$= e^{-(t-2)}.e^{-2}$

$L[e^{-(t-2)}.e^{-2}H(t-2)] = e^{-2}L[e^{-(t-2)}H(t-2)]$

By, $L[f(t-a)H(t-a)] = e^{-as}f(t)$

$e^{-2}L[e^{-(t-2)}H(t-2)]$

$= e^{-2}\{e^{-2s}L[e^{-t}]\}$

$= e^{-2}.e^{-2s}\frac{1}{s+1}$

$= \frac{e^{-2(s+1)}}{s+1}$

3] $L[(t^2+2t-1)H(t-3)]$

$\rightarrow$ $t^2+2t-1 = (t-3+3)^2+2(t-3+3)-1$

$=(t-3)^2+9+2(3)(t-3)+2(t-3)+6-1$

$= (t-3)^2+8(t-3)+14$

$L[((t-3)^2+8(t-3)+14)H(t-3)] = \cancel{e^{-3s}[t^2]}\ e^{-3s}L[t^2+8t+14]$

$= e^{-3s}\left(\frac{2!}{s^3}+\frac{8}{s^2}+\frac{14}{s}\right)$

4] $L[\sin t \, H(t-\pi)]$

$\rightarrow$ $\sin t = \sin(t-\pi+\pi) = -\sin(t-\pi)$

$L[-\sin(t-\pi)\,H(t-\pi)] = e^{-\pi s} L[-\sin t]$

$= e^{-\pi s} \dfrac{-1}{s^2+1}$

$L[-\sin(t-\pi)H(t-\pi)] = \dfrac{-e^{-\pi s}}{s^2+1}$

5] $L[e^{-t}\sin t \, H(t-\pi)]$

classmate
Date
Page

Note:

Property: If the function $f(t)$ is defined by

$$f(t) = f_1(t) \qquad t \le a$$
$$= f_2(t) \qquad t > a$$

then $f(t) = f_1(t) + [f_2(t) - f_1(t)] H(t-a)$

1] Express the following functions in terms of unit step function & hence find ~~the~~ their Laplace transform

$$f(t) = t^2 \quad \forall \ 0 < t \le 2$$
$$= 4t \quad \forall \ \cancel{2 < t} \ t > 2$$

→ let $f_1(t) = t^2$
$f_2(t) = 4t$

wkt, $f(t) = t^2 + [(4t - t^2] H(t-2)]$

taking laplace transform,

$L[f(t)] = L[t^2] + L[(4t - t^2) H(t-2)]$

$= \frac{2}{s^3} + L[4(t-2) + 8 - [\{(t-2)^2 + 4 + 2(t-2)2\} H(t-2)]]$

$= \frac{2}{s^3} + L[4(t-2) + 8 - (t-2)^2 - 4$

$= \frac{2}{s^3} + e^{-2s}[L[4 - (t-2)^2]$

$= \frac{2}{s^3} + e^{-2s}\left[\frac{4}{s} - \frac{2}{s^3}\right]$

2] Express in the form of heaviside function,

$f(t) = 2t \quad 0 < t \leq \pi$

$\quad = 1 \quad t > \pi$

→ let $f_1(t) = 2t$

$f_2(t) = 1$

wkt, $f(t) = 2t + [1 - 2t\, H(t-\pi)]$

$L[f(t)] = L[2t] + L[1 - 2t\, H(t-\pi)]$

$= \frac{2}{s^2} + L[1 - 2(t-\pi) - 2\pi\, H(t-\pi)]$

$= \frac{2}{s^2} + L[1 - 2(t-\pi) - 2\pi\, H(t-\pi)]$

$= \frac{2}{s^2} + e^{-\pi s}[1 - 2t - 2\pi]$

$= \frac{2}{s^2} + e^{-\pi s}\left[\frac{1}{s} - \frac{2\pi}{s} - \frac{2}{s^2}\right]$

$L[f(t)] = \frac{2}{s^2} + e^{-\pi s}\left[\frac{(1-2\pi)}{s} - \frac{2}{s^2}\right]$

3] $f(t) = e^{-t} \quad 0 < t \leq 3$

$\quad = 0 \quad t > 3$

→ let $f_1(t) = e^{-t}$

$f_2(t) = 0$

wkt, $f(t) = e^{-t} + [(0 - e^{-t}) H(t-3)]$

$L[f(t)] = L[e^{-t}] + L[(0 - e^{-t}) H(t-3)]$ — ①

$\cancel{L[f(t)]} =$

consider, $L[e^{-t}H(t-3)] = L[e^{-(t-3+3)}H(t-3)]$
$= L[e^{-(t-3)}e^{-3}.H(t-3)]$
$= e^{-3}L[e^{-(t-3)}.H(t-3)]$

By heaviside function, $a=-3$ $f(t)=-t$ $t-a\to t$

$= e^{-3}.e^{-3s}L[e^{-t}]$

$= e^{-3}.e^{-3s}\dfrac{1}{s+1}$ or $\dfrac{e^{-3(s+1)}}{s+1}$

substitute in (1)

$L[f(t)] = \cancel{L[e^{-t}]}\dfrac{1}{s+1} + \left[-e^{-3(s+1)}\cdot\dfrac{1}{(s+1)}\right]$

$L[f(t)] = \dfrac{1-e^{-3(s+1)}}{s+1}$

4] $f(t) = 1$ $0<t<1$
$= t$ $1<t<3$
$= t^2$ $t>3$

→ let $f_1(t)=1$
$f_2(t)=t$

wkt, $f(t) = 1+[(t-1)H(t-1)]$

$L[f(t)] = L[1]+L[(t-1)H(t-1)]$

PTO →

$L[f(t)] = \dfrac{2}{s}$

Now, $f_1(t)=t$ & $f_2(t)=t^2$

Property

$$f(t) = f_1(t) \quad 0<t<a$$
$$= f_2(t) \quad a<t<b$$
$$= f_3(t) \quad t>b$$

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]H(t-a) + [f_3(t) - f_2(t)]H(t-b)$$

4) $f(t) = 1 \quad 0<t<1$
$= t \quad 1<t<3$
$= t^2 \quad t>3$

→ $f_1(t) = 1$
$f_2(t) = t$
$f_3(t) = t^2$

$$f(t) = 1 + [(t-1)]H(t-1) + (t^2 - t)H(t-3)$$

$$L[f(t)] = L[1] + L[(t-1)H(t-1)] + L[(t^2-t)H(t-3)]$$

$t \to t-3$

$$t^2 - t = (t-3+3)^2 - (t-3+3)$$
$$= (t-3)^2 + 2(3)(t-3) + 9 - (t-3) - 3$$
$$= (t-3)^2 + 6(t-3) - (t-3) + 6$$

$$L[f(t)] = L[1] + L[(t-1)H(t-1)] + L[(t-3)^2 + 6(t-3) - (t-3) + 6\, H(t-3)]$$

$t-3 \to t \qquad a=3$

$$L[f(t)] = \frac{1}{s} \; e^{-s}L[t] + e^{-3s}L[t^2 + 6t - t + 6]$$

$$L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} - \frac{1}{s^2} + \frac{6}{s}\right]$$

$$L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-3s}\left[\frac{2}{s^3} + \frac{5}{s^2} + \frac{6}{s}\right]$$

classmate

Date
Page

5) $f(t) = \sin t \qquad 0 < t < \pi$
$\quad = \sin 2t \qquad \pi < t < 2\pi$
$\quad = \sin 3t \qquad t > 2\pi$

$\rightarrow f_1(t) = \sin t \quad f_2(t) = \sin 2t \quad f_3(t) = \sin 3t$

~~$f(t) = \sin t + [(\sin 2t - \sin t)] + [$~~

$f(t) = \sin t + [(\sin 2t - \sin t) H(t-\pi)] + [(\sin 3t - \sin 2t) H(t-2\pi)]$

$L[f(t)] = L[\sin t] + L[(\sin 2t - \sin t) H(t-\pi)] + L[(\sin 3t - \sin 2t) H(t-2\pi)]$

$\sin 2t - \sin t \rightarrow t - \pi$