Date
2/9/2021

DATE:

Unit: Multiple Integrals - Double & Triple Integrals.

* Constant limits :→

Ex 1 $\int_0^1 \int_1^2 x^2 y^3 \, dx\, dy = \int_0^1 \left\{ \int_1^2 x^2 dx \right\} y^3 dy$

Solve

$= \int_0^1 \left\{ \frac{x^3}{3} \right\}_1^2 y^3 dy = \int_0^1 \left[\frac{8}{3} - \frac{1}{3} \right] y^3 dy$

$= \int_0^1 \frac{7}{3} y^3 dy = \frac{7}{3} \left\{ \frac{y^4}{4} \right\}_0^1 = \frac{7}{12}$ //

Ex 2. $\int_3^4 \int_1^3 (x^2 y + y^2 x) \, dx\, dy = \int_3^4 \left\{ \frac{x^3 y}{3} + \frac{y^2 x^2}{2} \right\}_1^3 dy$

Solve

$= \int_3^4 \left[\frac{27y}{3} + \frac{9y^2}{2} - \frac{y}{3} - \frac{y^2}{2} \right] dy$

$= \int_3^4 \left(\frac{26y}{3} + 4y^2 \right) dy$

$= \left[\frac{26}{3} \left(\frac{y^2}{2} \right) + \frac{4y^3}{3} \right]_3^4$

$= \left\{ \frac{26}{3} \left[\frac{16}{2} - \frac{9}{2} \right] + 4 \left[\frac{64 - 27}{3} \right] \right\}$

$$= \frac{26}{3}\left(\frac{7}{2}\right) + \frac{148}{3} = \frac{182+296}{6} = \frac{239}{3}$$

Ex 3. Solve $\int_0^1\int_1^2\int_2^3 x y^2 z^3 \, dx\, dy\, dz$

$$= \int_0^1\int_1^2 \left\{ \int_2^3 x y^2 z^3 dx \, dy\, dz \right.$$

$$= \int_0^1\int_1^2 \left\{ \int_2^3 x\, dx \right\} y^2 z^3 dy\, dz$$

$$= \int_0^1\int_1^2 \left[\frac{x^2}{2}\right]_2^3 y^2 z^3 dy\, dz = \int_0^1\int_1^2 \left(\frac{9-4}{2}\right) y^2 z^3 dy\, dz$$

$$= \frac{5}{2}\int_0^1\int_1^2 y^2 z^3 dy\, dz = \frac{5}{2}\int_0^1 \left[\frac{y^3}{3}\right]_1^2 z^3 dz$$

$$= \frac{5}{2}\int_0^1 \left[\frac{8-1}{3}\right] z^3 dz = \frac{35}{6}\int_0^1 z^3 dz$$

$$= \frac{35}{6}\left[\frac{z^4}{4}\right]_0^1 = \frac{35}{6}\left[\frac{1}{4}\right] = \frac{35}{24}$$

Ex 4 Solve. $\int_0^a\int_0^b\int_0^c (x+y+z)\, dx\, dy\, dz$

$$I = \int_0^a\int_0^b \left[\frac{x^2}{2} + xy + xz\right]_0^c dy\, dz$$

$$= \int_0^a\int_0^b \left[\frac{c^2}{2} + cy + cz\right] dy\, dz$$

$$= \int_0^a \left[\frac{c^2 y}{2} + \frac{cy^2}{2} + czy\right]_0^b dz$$

$$= \int_0^a \left[\frac{c^2 b}{2} + \frac{cb^2}{2} + cbz\right] dz$$

$$= \left\{\frac{c^2 bz}{2} + \frac{cb^2 z}{2} + \frac{cbz^2}{2}\right\}_0^a$$

$$= \left\{\frac{c^2 ba}{2} + \frac{cb^2 a}{2} + \frac{cba^2}{2}\right\}$$

$$I = \frac{abc}{2}\left[a+b+c\right]$$

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Ex 7. Solve $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx =$ DATE: Ans $= \frac{16a^2}{3}$

$$= \int_0^{4a} \left[y\right]_{x^2/4a}^{2\sqrt{ax}} = \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx$$

$$= 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)_0^{4a}$$

$$= \frac{8\sqrt{a}}{3}\left(4a\right)^{3/2} - \frac{1}{4a}\left(\frac{(4a)^3}{3}\right)$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}$$

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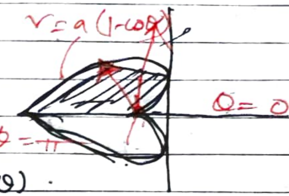
Ex 8. Solve $\iint r\sin\theta\,dr\,d\theta$ over $r=a(1-\cos\theta)$ DATE:

$\Rightarrow$ Consider a radial strip $r=0$ to $r=a(1-\cos\theta)$. In the region above the initial line $\theta=0$ to $\theta=\pi$



Then

$$\iint r\sin\theta\,dr\,d\theta = \int_0^{\pi}\int_0^{r(1-\cos\theta)} r\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= \int_0^{\pi} \frac{\sin\theta}{2}\left[a^2(1-\cos\theta)^2 - 0\right] d\theta$$

$$= \int_0^{\pi} \frac{a^2\sin\theta}{2}\left[1-2\cos\theta+\cos^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left[\sin\theta - 2\sin\theta\cos\theta + \sin\theta\cos^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left[\sin\theta - \sin 2\theta - \cos^2\theta(-\sin\theta)\right] d\theta$$

$$= \frac{a^2}{2}\left\{\int_0^{\pi}\sin\theta\,d\theta - \int_0^{\pi}\sin 2\theta\,d\theta - \int_0^{\pi}\cos^2\theta(-\sin\theta)\,d\theta\right.$$

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$$= \frac{a^2}{2}\left\{\left[-\cos\theta\right]_0^{\pi} + \left[\frac{\cos 2\theta}{2}\right]_0^{\pi} - \left(\frac{\cos^3\theta}{3}\right)_0^{\pi}\right\}$$

$$= \frac{a^2}{2}\left\{(1+1) + \left(\frac{-1-1}{3}\right) + 0\right\} = \frac{a^2}{2}\left\{2 - \frac{2}{3}\right\}$$

$$\boxed{I = \frac{4a^2}{3}}$$

Ex 9. Solve. $\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}(x^2+y^2+z^2)\,dz\,dy\,dx.$

Solution:

$$I = \int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c}(x^2+y^2+z^2)\,dz\,dy\,dx.$$

$$= \int_{-a}^{a}\int_{-b}^{b}\left\{x^2z + y^2z + \frac{z^3}{3}\right\}_{-c}^{c} dy\,dx$$

$$= \int_{-a}^{a}\int_{-b}^{b}\left\{\left(cx^2 + cy^2 + \frac{c^3}{3}\right) - \left(-cx^2 - cy^2 - \frac{c^3}{3}\right)\right\} dy\,dx$$

$$= 2\int_{-a}^{a}\int_{-b}^{b}\left\{cx^2 + cy^2 + c^3/3\right\} dy\,dx$$

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$$= 2\int_{-a}^{a}\left\{cx^2y + \frac{cy^3}{3} + \frac{c^3y}{3}\right\}_{-b}^{b} dx$$

$$= 2\int_{-a}^{a}\left\{\left(cx^2b + \frac{cb^3}{3} + \frac{c^3b}{3}\right) - \left(-cx^2b - \frac{cb^3}{3} - \frac{c^3b}{3}\right)\right\} dx$$

$$= 4\int_{-a}^{a}\left(cx^2b + \frac{cb^3}{3} + \frac{c^3b}{3}\right) dx$$

$$= 4\left\{\frac{cx^3b}{3} + \frac{cb^3x}{3} + \frac{c^3bx}{3}\right\}_{-a}^{a}$$

$$= 4\left\{\left(\frac{ca^3b}{3} + \frac{cb^3a}{3} + \frac{c^3ba}{3}\right) - \left(-\frac{ca^3b}{3} - \frac{cb^3a}{3} - \frac{c^3ba}{3}\right)\right\}$$

$$I = 8\left\{\frac{ca^3b}{3} + \frac{cb^3a}{3} + \frac{c^3ba}{3}\right\} = \frac{8abc}{3}(a+b+c)$$

Ex 10. Solve. $\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta = \frac{5\pi a^3}{64}$ Ans

Solution $I = \int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta.$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} r\,dr\,d\theta\,\left[z\right]_0^{a^2-r^2/a}$$

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$$= \int_0^{\pi/2}\int_0^{a\sin\theta} \left(\frac{a^2-r^2}{a}\right) r\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\int_0^{a\sin\theta} (a^2r - r^3)\,dr\,d\theta.$$

$$= \frac{1}{a}\int_0^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta.$$

$$= \frac{1}{a}\int_0^{\pi/2}\left\{\frac{a^4\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right\} d\theta.$$

$$= \frac{a^4}{a}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{2} - \frac{\sin^4\theta}{4}\right) d\theta.$$

$$= a^3\left\{\frac{1}{2}\int_0^{\pi/2}\sin^2\theta\,d\theta - \frac{1}{4}\int_0^{\pi/2}\sin^4\theta\,d\theta\right\}$$

$$= a^3\left\{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{4}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\}$$

$$= \frac{\pi a^3}{2}\left\{\frac{1}{4} - \frac{3}{8}\right\} = \frac{5\pi a^3}{64}$$

Cont.

Ex 11. Solve $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$.

Solution: $$I=\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx.$$

x= y= z=

$$=\int_0^a\int_0^x\int_0^{x+y} e^x.e^y.e^z\,dz\,dy\,dx.$$

$$=\int_0^a\int_0^x e^x.e^y\left\{e^z\right\}_0^{x+y}dy\,dx$$

$$=\int_0^a\int_0^x e^x.e^y\left\{e^{x+y}-1\right\}dy\,dx.$$

$$=\int_0^a\int_0^x\left\{e^{2x}.e^{2y}-e^xe^y\right\}dy\,dx.$$

$$=\int_0^a\left[e^{2x}.\frac{e^{2y}}{2}-e^x.e^y\right]_0^x dx$$

$$=\int_0^a\left\{\left[e^{2x}.\frac{e^{2x}}{2}-e^{2x}\left(\frac{1}{2}\right)\right]-\left[e^x.e^x-e^x(1)\right]\right\}dx$$

$$=\int_0^a\left[\frac{e^{4x}}{2}-\frac{e^{2x}}{2}-e^{2x}+e^x\right]dx.$$

$$=\int_0^a\left[\frac{e^{4x}}{2}-\frac{3}{2}e^{2x}+e^x\right]dx$$

$$=\left[\frac{e^{4x}}{8}-\frac{3}{2}\cdot\frac{e^{2x}}{2}+e^x\right]_0^a$$

$$=\left\{\left(\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a\right)-\left(\frac{1}{8}-\frac{3}{4}+1\right)\right\}$$

$$=\left\{\left(\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a\right)-\left(3/8\right)\right\}$$

$$I=\left(\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a\right)-3/8$$

$$\boxed{I=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a-3/8}$$

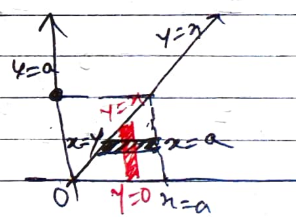
* Change of Order :

Ex 12. Solve.. $\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$.

Soln. $$I=\int_0^a\int_y^a\frac{x\,dx\,dy}{x^2+y^2} \quad \text{—— (I)}$$

limits : Given

$x=y$	$x=a$
$y=0$	$y=a$



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~~Change the Order of Integration~~

~~Suppose~~ $\cancel{x^2+y^2=r^2}$ — (1) | $\cancel{x = r\cos\theta}$ — (2)

~~Put~~ $\cancel{x=0 \quad y=a}$ | ~~Put~~ $\cancel{x=0}$ &

$\cancel{Eq^n (1) \Rightarrow a^2 = r^2}$ | $\cancel{Eq^n (2) \quad 0 = a\cos\theta \quad \theta =}$

$\cancel{\boxed{r=a}}$

New limits

$x=0$	$x=a$
$y=0$	$y=x$

$$=\int_{x=0}^{a}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dx\,dy = \int_0^a x\,dx\int_{y=0}^{x}\frac{1}{x^2+y^2}\,dy$$

$$=\int_0^a x\,dx \times\left\{\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right\}_0^x$$

$$\int_0^a x\,dx \times\left\{\frac{1}{x}\tan^{-1}(1)-\frac{1}{x}\tan^{-1}(0)\right\}$$

$$=\int_0^a x\,dx \times \frac{1}{x}\left(\pi/4 - 0\right) = \pi/4\int_0^a dx$$

$$=\pi/4\,\left[x\right]_0^a = \frac{\pi a}{4}\ //$$

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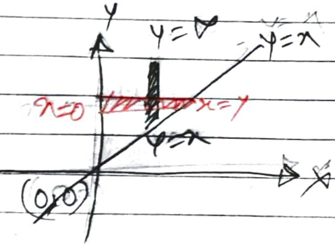
Ex 13. Evaluate

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$$

$\Longrightarrow$ Given limits

$x=0$	$x=\infty$
$y=x$	$y=\infty$



New limits

$y=0$	$y=\infty$
$x=0$	$x=y$

$$I = \int_0^\infty\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy$$

Integrate wrt x, y = con

$$=\int_0^\infty \frac{e^{-y}}{y}\,(x)_0^y\,dy$$

$$=\int_0^\infty \frac{e^{-y}}{y}\,(y-0)\,dy$$

$$=\int_0^\infty e^{-y}\,dy$$

$$=\left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$=\frac{0-1}{-1}$$

$$\boxed{Ans = 1}$$

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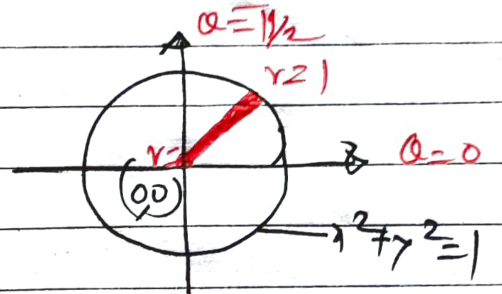
Change into Polar form

Ex 1. Evaluate by Changing into Polar form $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

$\Rightarrow$

$$I = \int_0^1 \int_{x=0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$$



Given limits

$y=0$	$y=1$
$x=0$	$x=\sqrt{1-y^2}$

$x^2+y^2=1.$

$\longrightarrow$ $x^2+y^2=r^2$

$dx\,dy = r\,dr\,d\theta$

New limits into Polar form

$\theta=0$	$\theta=\pi/2$
$r=0$	$r=1$

Polar form $x^2+y^2=r^2$

$$I = \int_0^1\int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2 (r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3\,dr\,d\theta =$$

$$= \int_0^{\pi/2} d\theta \left[\frac{r^4}{4}\right]_0^1$$

$$= \int_0^{\pi/2} d\theta \left[1/4 - 0\right] = 1/4 \int_0^{\pi/2} d\theta.$$

$$\boxed{I = \pi/8}$$

(16) **Ex 2** Evaluate $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

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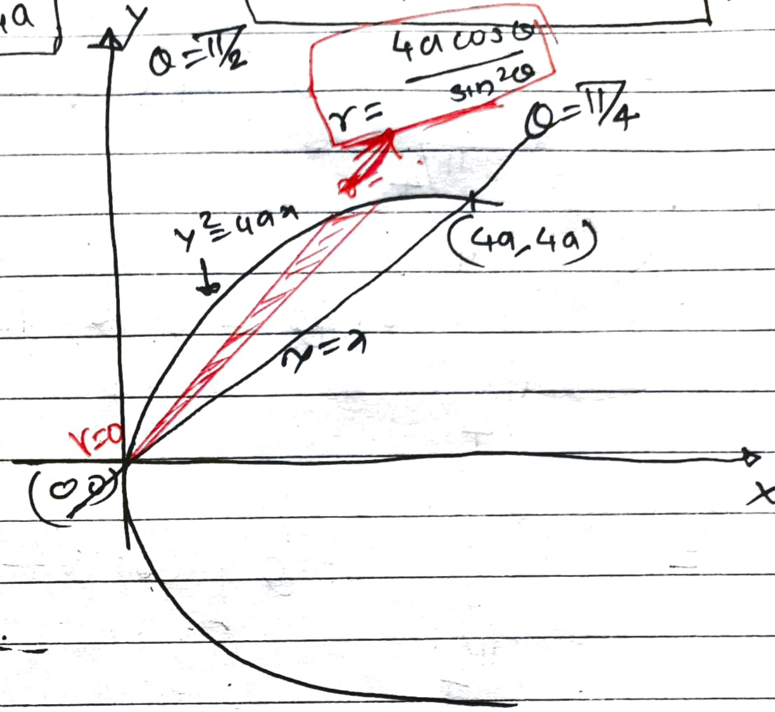
$\Rightarrow$ Given limits $\boxed{\begin{array}{c} y=0 \\ \hline y=4a \end{array}}$ and $\boxed{x = y^2/4a \ \& \ x = y}$.

Point of Intersection



$y = x \ , \ y^2 = 4ax$

$\Rightarrow x^2 = 4ax$

$\Rightarrow x(x-4a) = 0$

$\Rightarrow x = 0 \ , \ x = 4a$

$\Rightarrow x = 0 \ , \ x = 4a$

$\Rightarrow (0,0) \ , \ (4a, 4a)$

For Polar Coordinates.

$x = r\cos\theta \ , \ y = r\sin\theta$ and replace

$dx\,dy$ by $r\,dr\,d\theta$.

$(x^2+y^2 = r^2)$

New limits

$\theta = \pi/4$	$\theta = \pi/2$
$r = 0$	$r = \frac{4a\cos\theta}{\sin^2\theta}$

$$I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\,d\theta \,.\, r\,dr$$

$$= \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\,d\theta \left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}}.$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta\,(\cot^2\theta - 1)\,d\theta.$$

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$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \,(\cot^2\theta - 1)\, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\, d\theta.$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \left[\cot^2\theta - 1\right] d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \left[\operatorname{cosec}^2\theta - 1 - 1\right] d\theta.$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \operatorname{cosec}^2\theta \, d\theta - 16a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \, d\theta$$

$$= 8a^2 \left[-\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} - 16a^2 \int_{\pi/4}^{\pi/2} \left[\operatorname{cosec}^2\theta - 1\right] d\theta$$

$$8a^2 \left[\frac{-\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} - 16a^2 \left\{ \left[-\cot\theta\right]_{\pi/4}^{\pi/2} - \left[\theta\right]_{\pi/4}^{\pi/2} \right\}$$

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$$= \frac{8a^2}{3}\left[0+1\right] + 16a^2\left[(0 - 1) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= \frac{8a^2}{3} + 16a^2\left[-1 + \frac{\pi}{4}\right] = \frac{8a^2}{3} + 16a^2\left[\frac{\pi}{4} - 1\right]$$

$$= \frac{8a^2}{3} - 16a^2 + 4a^2\pi$$

$$= -\frac{40a^2}{3} + 4a^2\pi = 4a^2\left[\pi - \frac{10}{3}\right]$$

$$\text{OR} = 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

Home Work Examples:

(19) DATE:

* Using Double/Tri integration find Area & Volume

1) Find the area of ellipse in the first quadrant and find full area.
2) Find area between $x^2 = 4ay$ & $y^2 = 4ax$
3) Find Area inside cardioid $r = a(1+\cos\theta)$ outside circle $r = a$

* find volume by double / Triple Integration

Ex1. find volume bounded by cylinder $x^2+y^2=4$ & plane $z=0$, $y+z=4$.

Ex2. Find volume bounded by $x=0$, $y=0$, $z=0$, $x+y+z=a$

Ex 3. Find volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

find volume of Sphere $= \frac{4\pi a^3}{3}$

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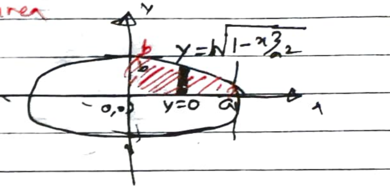
Ex 1. find the Area of Ellipse in the first quadrant. and find full area

$\Rightarrow$ Eqn of Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Formula Area $= \iint_A dx\,dy$.

limits :

$x=0$	& $x=a$
$y=0$	$y=b\sqrt{1-x^2/a^2}$

full Area of Ellipse $= 4\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}} dx\,dy$

$$=4\int_0^a dx\left\{y\right\}_0^{b\sqrt{1-x^2/a^2}}$$

$$=4\int_0^a dx\left\{b\sqrt{1-x^2/a^2}-0\right\}dx$$

$$=4\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

Formula $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}($

$$=\frac{4b}{a}\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}(x/a)\right\}_0^a$$

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$$Area = \frac{4b}{a}\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}(x/a)\right\}_0^a$$

$$= \frac{4b}{a}\left\{\left(\frac{a}{2}\sqrt{a^2-a^2}\right)-\left(\frac{0}{2}\sqrt{a^2-0}\right)+\left(\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right)-0\right\}$$

$$=\frac{4b}{a}\left\{0-0+\frac{a^2}{2}\sin^{-1}(1)\right\} = \frac{b}{a}\times\frac{a^2}{2}\sin^{-1}(1)$$

Formula $\left\{\because \sin^{-1}(1)=\pi/2\right\}$

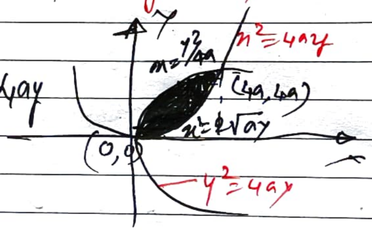
$$= 4b/a \times a^2/2 \times \pi/2$$

$\boxed{A = \frac{\cancel{4}\pi ab}{\cancel{4}}}$ OR $\boxed{Area = \pi ab}$

Ex 2. Find the area between $x^2=4ay$ & $y^2=4ax$ using double integration

⇒ Solution

$y^2=4ax$ & $x^2=4ay$



To find point of intersection

$$\left(\frac{y^2}{4a}\right)^2 = 4ay.$$

$$\frac{y^4}{16a^2}=4ay. \Rightarrow \frac{y^{4}}{y}^{3} = 4a\times16a^2$$

$$y^3=(4a)^3$$

(22) DATE:

$\therefore \boxed{y=4a}$ $\therefore \boxed{x=4a}$

from fig Limit:

$y=0$	$y=4a$
$x=2\sqrt{ay}$	$x=y^2/4a$

$$\therefore Area = \iint_A dy\,dx$$

$$= \int_{y=0}^{y=4a}\int_{x=2\sqrt{ay}}^{y^2/4a} dx\,dy.$$

$$= \int_{y=0}^{y=4a} dy.\left\{x\right\}_{2\sqrt{ay}}^{y^2/4a}$$

$$= \int_0^{4a} dy\left\{y^2/4a - 2\sqrt{ay}\right\}$$

$$= \frac{1}{4a}\int_0^{4a} y^2 dy - 2\sqrt{a}\int_0^{4a}\sqrt{y}\,dy$$

$$= \frac{1}{4a}\left\{\frac{y^3}{3}\right\}_0^{4a} - 2\sqrt{a}\left\{\frac{y^{3/2}}{3/2}\right\}_0^{4a}$$

$$= \frac{1}{4a}\left\{\frac{(4a)^3}{3}\right\} - \frac{4\sqrt{a}}{3}\left\{(4a)^{3/2}\right\}$$

$$= \left(\frac{\cancel{64}^{16}\, a^{\cancel{3}\,2}}{\cancel{12a}_{3}}\right) - \left(\frac{4a^{1/2}}{3}\right)\left(2^3 \times a^{3/2}\right)$$

$$= \frac{16a^2}{3} - \frac{32}{3}a^{1/2+3/2} = \frac{16a^2}{3} - \frac{32a^2}{3}$$

$$A = \frac{-16a^2}{3} \quad \text{or} \quad \text{Area} = \frac{16a^2}{3}$$

Alternative Method

We can take different limits.

$x=0$	$x=4a$
$y=\frac{x^2}{4a}$	$x=2\sqrt{ay}$

and solve it. Ans $= \frac{16a^2}{3}$ //.

Ex 3. Find Area inside Cardiode $r=a(1+\cos\theta)$ out side circle $r=a$.

$\implies$ Given Cardiode $r=a(1+\cos\theta)$ & $r=a$ circle

$$\therefore \quad a = a(1+\cos\theta)$$

$$1 = (1+\cos\theta) \implies \boxed{\cos\theta = 0}$$

Limits. $\therefore \boxed{\theta = \pi/2}$

$\theta=0$	$\theta=\pi/2$
$r=a$	$r=a(1+\cos\theta)$

$$\text{Area} = \iint_A r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{r=a}^{a(1+\cos\theta)} d\theta = \int_0^{\pi/2}\left[\frac{a^2(1+\cos\theta)^2}{2} - \frac{a^2}{2}\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left\{\cancel{1} + 2\cos\theta + \cos^2\theta - \cancel{1}\right\} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[2\cos\theta + \cos^2\theta\right] d\theta$$

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$$= \frac{a^2}{2} \int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{a^2}{2} \left\{ 2\int_0^{\pi/2} \cos\theta\, d\theta + \int_0^{\pi/2} \cos^2\theta\, d\theta \right\}$$

$$= a^2 \Big[\sin\theta\Big]_0^{\pi/2} + \frac{a^2}{2}\left\{ \frac{1}{2} \cdot \frac{\pi}{2} \right\}$$

$$= a^2[1-0] + \frac{a^2\pi}{8} = a^2 + \frac{a^2\pi}{8}$$

$$\boxed{A = a^2\left[\frac{\pi}{8} + 1\right]}$$

OR $A = \frac{a^2}{8}[\pi + 8]$

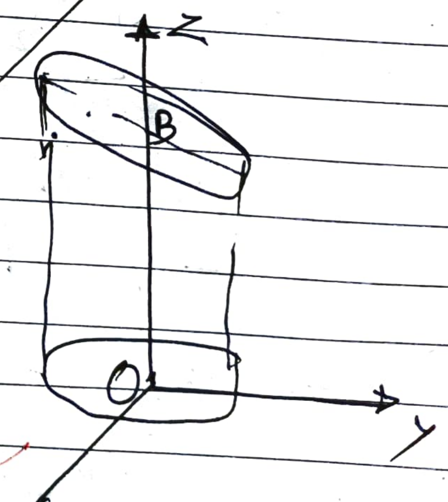
* To find Volume using Triple Integration.

Ex 1. Find volume bounded by cylinder $x^2+y^2=4$ & plane $z=0$, $y+z=4$.

⇒ Given cylinder $x^2+y^2=4$

& plane portion is $z=0$, $y+z=4$.

(i.e $z=4-y$ bounded by the cylinder $x^2+y^2=4$



The projection of this plane area on the xy-plane is area A.

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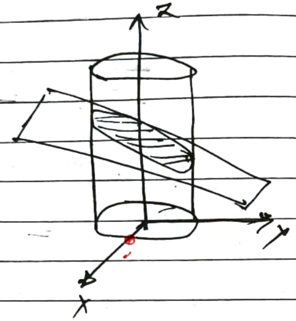
Ex 1. Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$ & $z=0$

Solution:

Given cylinder $x^2+y^2=4$
and plane portion
$y+z=4$ & $z=0$



∴ Region bounded by

Limits

x-limits.	-2	to 2
y limits.	$-\sqrt{4-x^2}$	to $\sqrt{4-x^2}$
z limits	0	to $(4-y)$

Required Volume

$$V = \iiint_V dx\,dy\,dz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{(4-y)} dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{(4-y)}\,dy\,dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

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$$= \int_{-2}^{2} \left\{4y - \frac{y^2}{2}\right\}_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left\{\left[4\sqrt{4-x^2} + 4\sqrt{4-x^2}\right] - \left[\cancel{\frac{(4-x^2)}{2}} - \cancel{\frac{(4-x^2)}{2}}\right]\right\} dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx. \quad = 8\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

formula $\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}(x/a) + \frac{x\sqrt{a^2-x^2}}{2}$

$$= 8\left\{\frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2}\right\}_{-2}^{2}$$

$$= 8\left\{\left(\frac{4}{2}\sin^{-1}(1) + \frac{2\sqrt{4-4}}{2}\right) - \left(\frac{4}{2}\sin^{-1}(-1) + \frac{-2\sqrt{4-4}}{2}\right)\right\}$$

$$= 8\left\{(2\times\pi/2 + 0) - (2\times(-\pi/2) - 0)\right\}$$

$$= 8\left\{\pi + \pi\right\} = 16\pi \text{ c.c} //$$

$\boxed{\text{Volume} = 16\pi \text{ cc}}$

Ex 2 Find the volume bounded by the surface $x=0$, $y=0$, $x+y+z=a$ & $z=0$

Sol^n. $x=0, y=0$ & $z=0$ & $x+y+z=a$

Limits for:

x limits $x=0$ to a

y limits $y=0$ to $a-x$.

z limits $z=0$ to $a-x-y$.

$$\text{volume} = \iiint_V dx\,dy\,dz$$

$$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dx\,dy\,dz$$

$$= \int_0^a \int_0^{a-x} dx\,dy \left\{ z \right\}_0^{a-x-y}$$

$$= \int_0^a \int_0^{a-x} dx\,dy \left\{ a-x-y \right\}$$

$$= \int_0^a dx \left\{ ay - xy - y^2/2 \right\}_0^{a-x}$$

$$= \int_0^a dx \left\{ a(a-x) - x(a-x) - \frac{(a-x)^2}{2} \right\}$$

$$= \int_0^a \left\{ a^2 - \cancel{ax} - ax + x^2 - \frac{a^2}{2} + \frac{\cancel{2ax}}{2} - \frac{x^2}{2} \right\} dx$$

$$= \int_0^a \left\{ \frac{a^2}{2} - ax + \frac{x^2}{2} \right\} dx$$

$$= \left\{ \frac{a^2 x}{2} - \frac{ax^2}{2} + \frac{x^3}{6} \right\}_0^a$$

$$= \left\{ \cancel{\frac{a^3}{2}} - \cancel{\frac{a^3}{2}} + \frac{a^3}{6} \right\} = \frac{a^3}{6}$$

$$\boxed{\text{Volume} = a^3/6}$$

Ex 3 Find volume of Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

find volume of sphere $\boxed{\text{Ans} = \frac{4\pi a^3}{3}}$

$\Rightarrow$ ~~Vol~~ Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\text{volume} = V = 8 \iiint_V dx\,dy\,dz$$

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limits for

z Varies from 0 to $c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$

y Varies from 0 to $\left(\frac{b}{a}\right)\sqrt{a^2-x^2}$

x Varies from 0 to a

$$\text{Volume} = 8\int_0^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx$$

$$= 8c\int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{1}{b}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\; dy\,dx$$

Take $k^2 = b^2\left\{1-\frac{x^2}{a^2}\right\} = \frac{b^2(a^2-x^2)}{a^2}$

Using standard integration formula

$$\boxed{\int\sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]}$$

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Next we have

$$V = \frac{8c}{b}\int_0^a \int_0^k \sqrt{k^2-y^2}\; dy\,dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{k^2-y^2}}{2} + \frac{k^2}{2}\sin^{-1}\left(\frac{y}{k}\right)\right]_0^k dx$$

$$= \frac{8c}{b}\int_0^a \left[0+\frac{k^2}{2}\left\{\sin^{-1}(1)-\sin^{-1}(0)\right\}\right] dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{\pi}{2}\right)\cdot\frac{1}{2}\cdot\frac{b^2}{a}(a^2-x^2)\,dx$$

$$V = \frac{2bc\pi}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{2bc\pi}{a^2}\times\frac{2a^3}{3}$$

$$\boxed{V = \frac{4\pi abc}{3}}$$ Cubic units.