1.9.2 Wave Equation

Equation of motion of an object is the equation that gives the position of the object as a function of time. We obtain the entire picture of wave motion only when we consider the harmonic motion of a series of points in the medium. As the oscillations are communicated from point to point, the points in the medium will be in different states of oscillation at different times. The displacement of a particle in the medium is therefore a function of space coordinates as well as a function of time. We denote the displacement by y. Thus,

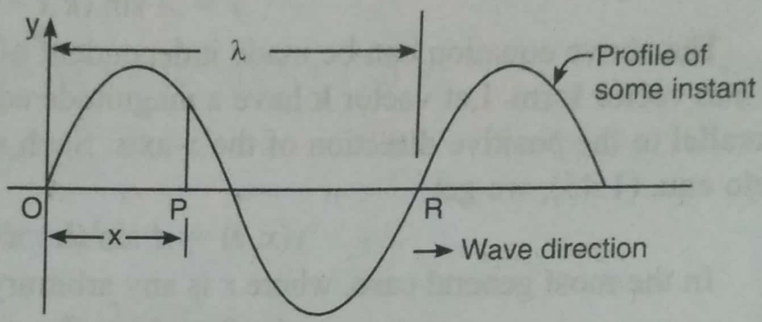

Fig. 1.21

$$y = f(x, t) \tag{1.38}$$

The displacement y is sometimes called the **wave function.**

Let us consider the case of a one-dimensional wave moving along $+\,x$-axis, as in Fig. 1.21.

We first consider the displacement as a *function of time*, at the position $x = 0$. Then,

$$y = f(t)$$

Since the oscillations are sinusoidal, we can describe the displacement y in terms of *time* as

$$y = A \sin \omega t$$

or

$$y = A \sin 2\pi\, \nu\, t \tag{1.39}$$

The wave is travelling forward to the right with a velocity, say υ. Then after time t, the wave has moved through the distance $x = \upsilon\, t$. The displacement at x can be represented by

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$$y = f(x - \upsilon t) \tag{1.40}$$

$$\upsilon = \nu\lambda$$

Also

$$\upsilon = \frac{x}{t}. \quad \text{Therefore, } \nu\lambda = x/t.$$

or

$$\nu = \frac{x}{\lambda t} \tag{1.41}$$

We can rewrite the relation (1.39) using (1.41) as

$$y = A \sin 2\pi\left(\frac{x}{\lambda}\right) \tag{1.42}$$

This describes the displacement in terms of *space*.

Using the equations (1.40) and (1.42), we can describe the displacement of any point on a harmonic wave in terms of both space and time as

$$y = A \sin\left[\frac{2\pi}{\lambda}(x - \upsilon t)\right] \tag{1.43}$$

This equation gives the relationship between the space and time dependence of disturbances in a medium. It is seen from the above that the wave is periodic in both space and time.

The equation (1.43) may be rewritten as

$$y = A \sin k(x - \upsilon t) \tag{1.44}$$

where

$$k = \frac{2\pi}{\lambda}.$$

k is known as **propagation constant or wave number**.

The equation (1.43) may further be rewritten as

$$y = A \sin(kx - \omega t) \tag{1.45}$$

The above equation can be made independent of the system of coordinates by converting it into vector form. Let vector **k** have a magnitude equal to the wave number k and a direction parallel to the positive direction of the x-axis. Such a vector is called a **wave vector**. Using **k** into equ. (1.45), we get

$$y(x, t) = A \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

In the most general case, where r is any arbitrary direction, we replace x by r and write

$$y(r, t) = A \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{1.45a}$$

1.9.3 General Wave Equation

To know how the displacement y varies as a function of space x and time t we have to do partial differentiation of y with respect to x and y in equ. (1.44).

$$\frac{\partial y}{\partial x} = \frac{2\pi}{\lambda} A \cos\left[\frac{2\pi}{\lambda}(x - \upsilon t)\right] \tag{1.46a}$$

$$\frac{\partial y}{\partial t} = -\frac{2\pi\nu}{\lambda} A \cos\left[\frac{2\pi}{\lambda}(x - \upsilon t)\right] \tag{1.46b}$$

Combining both these equations and eliminating equal factors, we get

$$\frac{\partial y}{\partial x} = -\frac{1}{\upsilon}\frac{\partial y}{\partial t} \tag{1.47}$$

If we take the second derivatives, it will hold for any sinusoidal wave, independent of the direction of travel, either $-x$ or $+x$.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\upsilon^2}\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = \upsilon^2 \frac{\partial^2 y}{\partial x^2}$$

We replace y by the more general term ξ, which stands for any disturbance.

$$\frac{\partial^2 \xi}{\partial t^2} = \upsilon^2 \frac{\partial^2 \xi}{\partial x^2} \tag{1.48}$$

This is the *one-dimensional wave equation.* It connects the variations in space and time to the velocity of propagation of the wave.

If we are to include waves propagating in any direction, we need to extend the right hand term to the y and z-axes, and replace it by

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}$$

Using the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, we can write the equation as

$$\frac{\partial^2 \xi}{\partial t^2} = \upsilon^2 \Delta^2 \xi \tag{1.49}$$

This is the *general three-dimensional wave equation.*

Example 1.8: *The wave function for a light wave is given by,*

$$E(z, t) = 10^3 \sin \pi(3 \times 10^6 x - 9 \times 10^{14} t).$$

Determine the speed, wavelength and frequency of the wave.

Solution: The given equation resembles the general equation,

$$E(z, t) = E_0 \sin k (x - \upsilon t) \tag{a}$$

The given equation may be written as

$$E(z, t) = 10^3 \sin 3 \times 10^6 \pi(x - 3 \times 10^8 t) \tag{b}$$

Comparing (b) with (a), we find that,

$\upsilon = 3 \times 10^8$ m/s and $k = 3 \times 10^6$ π/m.

As $k = \frac{2\pi}{\lambda}$, $\lambda = \frac{2\pi}{k} = \frac{2\pi}{3\times10^6\pi} = 6666$ Å.

Frequency, $\nu = \frac{\upsilon}{\lambda} = \frac{3\times10^8}{6666\times10^{-10}} = 4.5 \times 10^{14}$ Hz.

Example1.9: *A progressive sinusoidal wave is represented by $y(x, t) = A \sin [(0.2\ m^{-1}) x - (0.4\ s^{-1})t + \pi/6]$ where x and t are in meter and second respectively. Determine the speed of propagation of the wave.* **(B.P.U.T. 2004)**

Solution: Here, we know that,

$$y = A \sin (\omega t - kx + \Phi)$$

$\therefore$ $\omega = 0.4\ s^{-1}$, $k = 0.2\ m^{-1}$ and $\phi = \pi/6$

Now the speed of propagation of the wave is given by, $\upsilon = \frac{\omega}{k}$ $\therefore$ $\upsilon = \frac{0.4}{0.2} = 2$ m/s

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1.12 PRINCIPLE OF SUPERPOSITION

It often happens that two or more waves propagate simultaneously through the same region in the same direction. They pass through one another as if the other wave is not present. When two pebbles are dropped at different points in a pond, the expanding water waves cross each other without either one producing any change in the other. Similarly, sound waves from different instruments in an orchestra propagate in space independent of each other and can be distinguished separately. There occur many such instances in which a number of waves meet and pass through each other without mutual effect. However, the waves act simultaneously on the particles of the medium, in the region in which they are overlapping on each other. In the region of overlap, the waves will simply add to (or subtract from) one another without permanently disrupting each other (see Figs. 1.30 & 1.31). Once having passed through the region, each wave will move out and away without getting affected by the overlap. The resultant displacement of the medium at the location of the overlap will be different from the sum of the displacements caused by the waves individually. The resultant displacement at any point and at any instant of time can be found using the **principle of superposition**. According to this principle the instantaneous displacement of the medium at any point in space or time, is simply the linear sum of the individual displacements that would have occurred for each wave alone. The principle of superposition states that

when a number of waves pass through a medium simultaneously, the instantaneous resultant displacement of the medium at every instant is the algebraic sum of the displacements of the medium due to individual waves in the absence of others.

If y_1, y_2, y_3,... are the displacement vectors due to waves 1, 2, 3, ... acting separately, then the resultant displacement y is given by

$$y = y_1 + y_2 + y_3 + \ldots \tag{1.51}$$

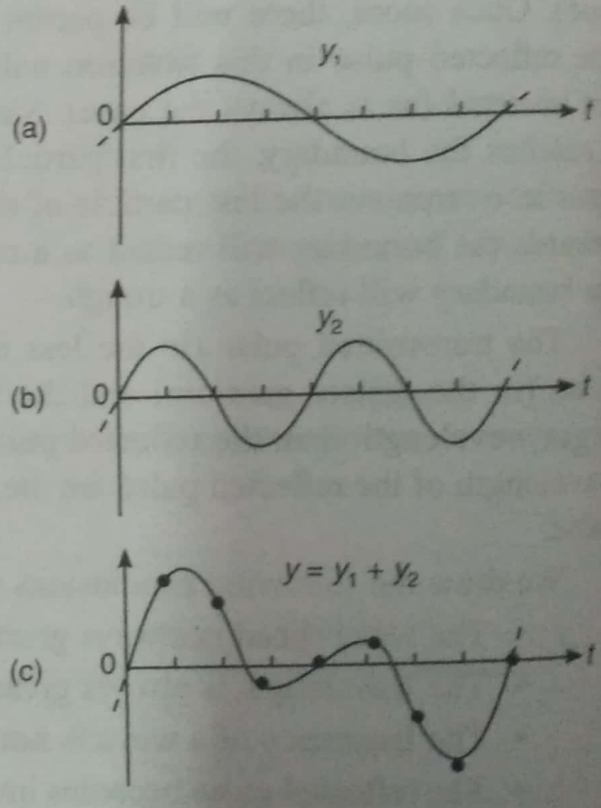


Fig. 1.29

As an example we consider two waves travelling simultaneously along the same path. Let y_1 (x, t) and y_2 (x, t) be the displacements that the medium would experience if each wave acted alone. When both the waves act, then the displacement of the medium is

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

the sum being an algebraic sum (see Fig. 1.29). The superposition principle holds as long as the amplitudes of the waves are not very large. This principle is applicable to all kind of waves and is very useful in the study of sound waves, electromagnetic waves and quantum physics also. This is true of waves which are finite in length (wave pulses) or which are continuous sine waves.

The superposition of the waves may result in the following cases;

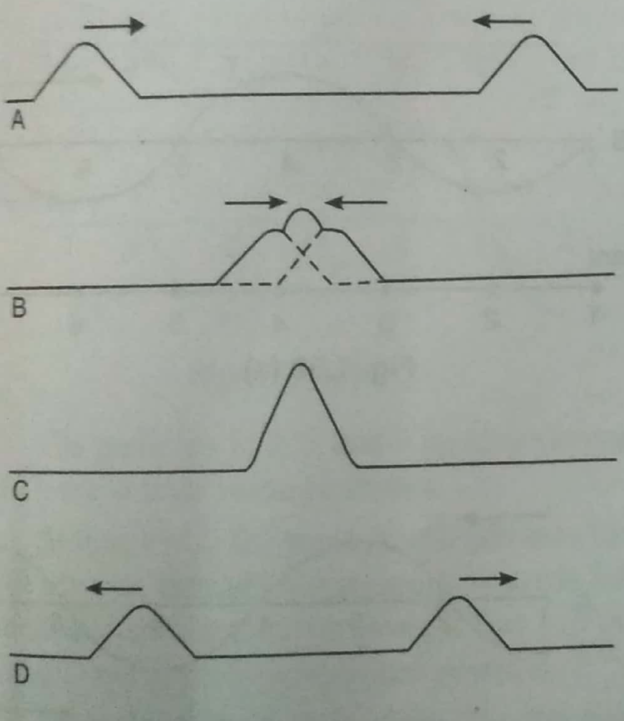
(*i*) The superposition of two waves of the same frequency moving in the same direction leads to **interference**.

(*ii*) The superposition of two waves of slightly different frequencies moving in the same direction leads to **beats**.

(*iii*) The superposition of two waves of the same frequency moving in the opposite direction leads to **stationary waves**.

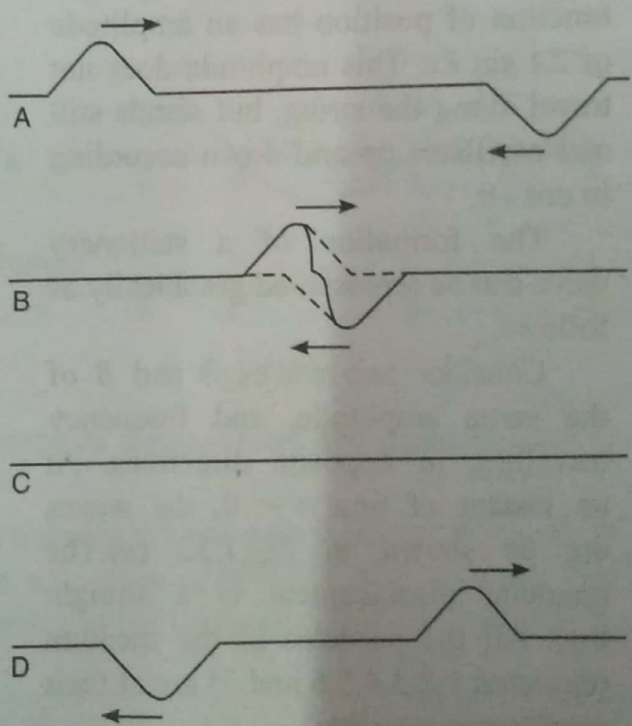
1.12.1 Interference

The superposition of two or more pulses (waves) in a given region may give rise to interference. When the two interfering wave pulses have a displacement in the same direction, the resultant displacement is greater than the displacement of either wave. This type of interference is called **constructive interference**. Constructive interference is observed when a crest meets a crest; and when a trough meets a trough as shown in Fig. 1.30.



(a) Constructive interference

Fig. 1.30



(b) Destructive interference

Fig. 1.31

On the other hand, when the pulses have displacements in opposite directions, the resultant displacement is smaller than that of either pulse. This type of interference is called **destructive interference**.

In Fig. 1.31, the interfering pulses have the same maximum displacement but in opposite directions. They completely destroy each other when they have completely overlapped. At the instant of complete overlap, there is no resulting disturbance in the medium. This "destruction" is *not a permanent condition*. Destructive interference leads to only a momentary condition in which the displacement of the medium is zero. At the point of total destructive interference, when the net wave shape and hence potential energy are zero, the wave energy is stored in the medium completely in the form of kinetic energy.

1.13 STATIONARY WAVES

Standing waves are produced whenever two waves of equal frequency amplitude interfere with one another while travelling in **opposite directions** along the same medium.

Let the two component waves be represented by

$$y_1(x, t) = A \sin(kx - \omega t) \tag{1.51a}$$

and

$$y_2(x, t) = A \sin(kx + \omega t) \tag{1.51b}$$

Using the principle of superposition, the resulting string displacement may be written as:

$$y(x, t) = y_1(x, t) + y_2(x, t) = A \sin(kx - \omega t) + A \sin(kx + \omega t) \tag{1.52}$$

By using the identity $\sin A + \sin B = 2 \sin[(A + B)/2] \cos[(A - B)/2]$, we simplify the above equation to obtain

$$y(x, t) = 2A \cos(\omega t) \sin kx \tag{1.53}$$

This wave is no longer a travelling wave because the position and time dependence have been separated. The displacement of the string as a function of position has an amplitude of $2A \sin kx$. This amplitude does not travel along the string, but stands still and oscillates up and down according to $\cos \omega t$.

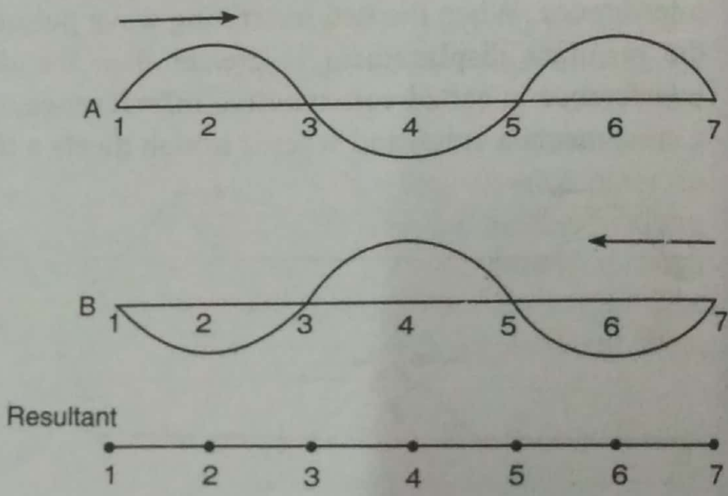


Fig. 1.32 (a)

The formation of a stationary wave can be represented graphically as follows.

Consider two waves A and B of the same amplitude, and frequency travelling in opposite directions. At an instant of time $t = 0$, the waves are as shown in Fig.1.32 (*a*).The resultant displacement is a straight line. All the particles of the medium (depicted 1,2,3,4,5,6 and 7) are at their equilibrium position.

Consider the waves after a time $t = T/4$. During this time, the wave A will advance through a distance $\lambda/4$ towards right, and the wave B will advance through a distance $\lambda/4$ towards the left. The resultant displacement pattern is shown in Fig. 1.32 (*b*).

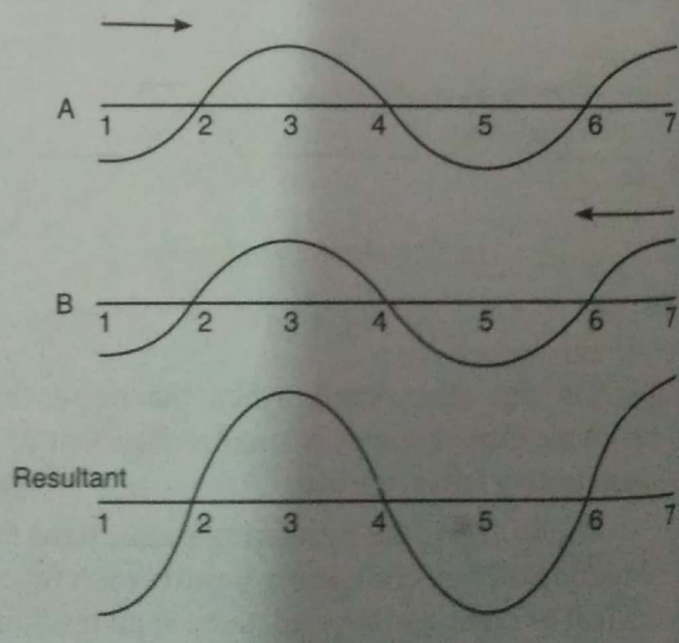


Fig. 1.32 (b)

The particles at 1, 3, 5, and 7 undergo maximum displacement and the particles at 2, 4, and 6 are at their equilibrium positions.

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At time $t = T/2$, the wave A will advance through a distance $\lambda/2$ towards right while the wave B advances through a distance $\lambda/2$ towards the left. The resultant displacement pattern is shown in Fig. 1.32 (c).

All the particles of the medium are at their equilibrium positions.

At time $t = 3T/4$, the wave A will advance through a distance $3\lambda/4$ towards right and the wave B will advance through a distance $3\lambda/4$ towards left. The resultant displacement pattern is shown in Fig. 1.32(d).

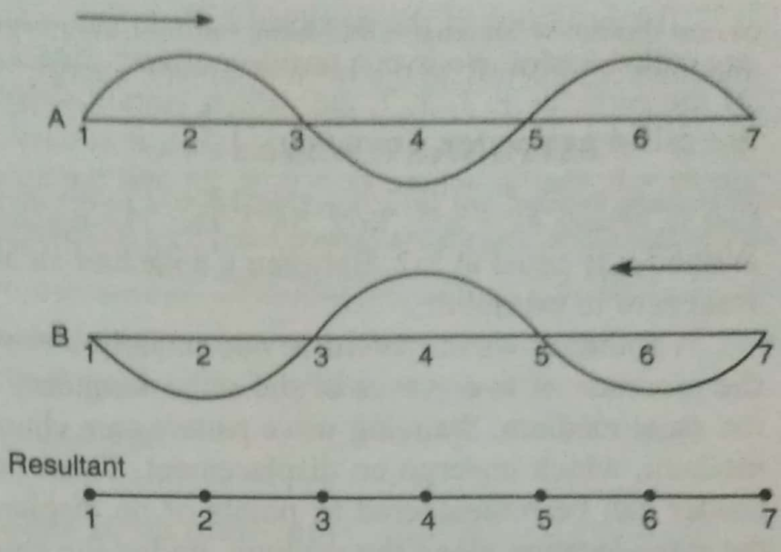

Fig. 1.32 (c)

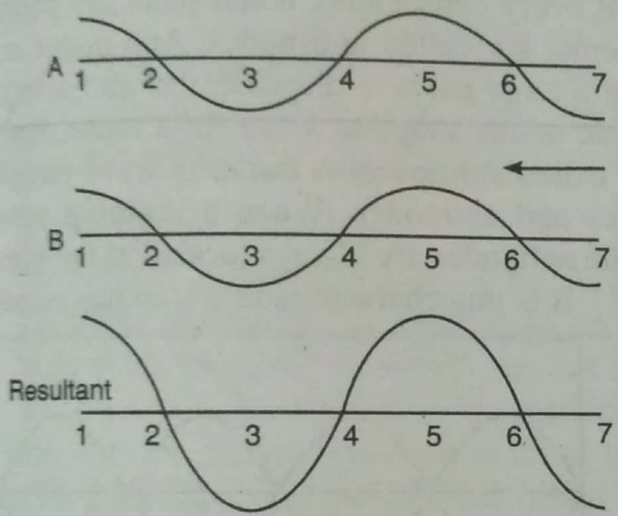

Fig. 1.32 (d)

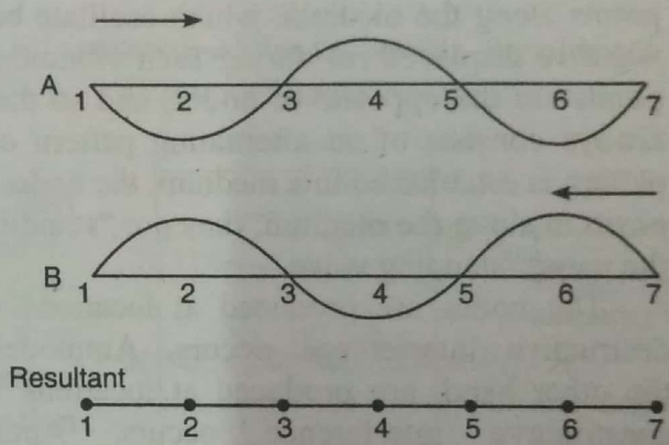

Fig. 1.32 (e)

The particles 1, 3, 5 and 7 have undergone maximum displacement and particles at 2, 4, and 6 are at their mean positions.

At time $t = T$, the wave A will advance through a distance λ towards right and the wave B will advance through a distance λ towards left. The waves are as shown in Fig. 1.32 (e). The resultant displacement is a straight line. All the particles of the medium (depicted 1, 2, 3, 4, 5, 6 and 7) are at their equilibrium position.

From the figures, it is clear that the particles of the medium such as 2, 4, and 6 etc. always remain at their equilibrium positions. The particles such as 1, 3, 5, 7 etc. continue to vibrate simple harmonically about their equilibrium positions with double the amplitude of each wave, as shown in Fig. 1.32 (f). It appears as though the wave pattern is stationary in space.

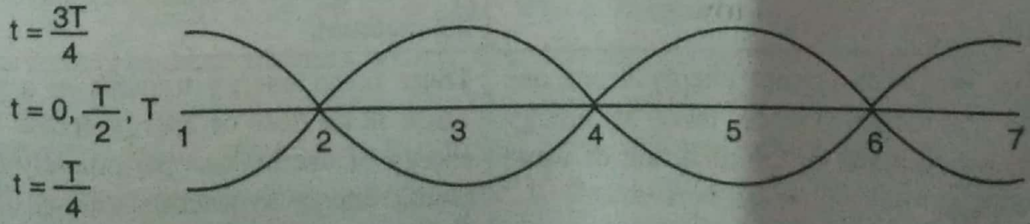

Fig. 1.32 (f)

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The positions of the particles 2, 4, 6, etc. which always remain at their mean positions, are called **nodes**. Node is a position of zero displacement and maximum strain. The positions of the particles 1, 3, 5, 7, etc. which vibrate simple harmonically with maximum amplitude are called **antinodes**. From equ. (1.59), it is easy to see that the nodes are produced where $\sin kx = 0$, that is, where $kx = 0, \pi, 2\pi$ and the antinodes occur at points where $\sin kx = \pm 1$, that is, where $kx = \pi/2, 3\pi/2, 5\pi/2$, etc. The distance between any two consecutive nodes or antinodes is equal to $\lambda/2$. Between a node and an antinode, the amplitude gradually increases from zero to maximum.

A standing wave pattern is not actually a wave; rather it is the pattern resulting from the presence of two waves of the same frequency with different directions of travel within the same medium. Standing wave patterns are characterized by certain fixed points along the medium, which undergo no displacement. These points of no displacement are called *nodes* (nodes can be remembered as points of no displacement). The nodes are always located at the same location along the medium, giving the entire pattern an appearance of standing still. There are other points along the medium, which undergo vibrations between a large *positive* and large *negative* displacement. Midway between every consecutive nodal point are points which undergo maximum displacement. These points are called anti-nodes. Anti-nodes are points along the medium, which oscillate between a large *positive* displacement and a large *negative* displacement during each vibrational cycle of the standing wave. In a sense, these points are the opposite of nodes, and so they are called antinodes. A standing wave pattern always consists of an alternating pattern of nodes and antinodes. When a standing wave pattern is established in a medium, the nodes and the antinodes are always located at the same position along the medium; they are "standing still." It is this characteristic which has earned the name "standing wave."

The nodes are produced at locations where destructive interference occurs. Antinodes, on the other hand, are produced at locations where constructive interference occurs. Antinodes are always vibrating back and forth between these points of large positive and large negative displacement; this is because during a complete cycle of vibration, a crest will meet a crest; and then one-half cycle later, a trough will meet a trough. Because antinodes are vibrating back and forth between positive and negative displacements, a diagram of a standing wave is sometimes depicted by drawing the shape of the medium at an instant in time and at an instant one-half vibrational cycle later. This is shown in Fig. 1.33.

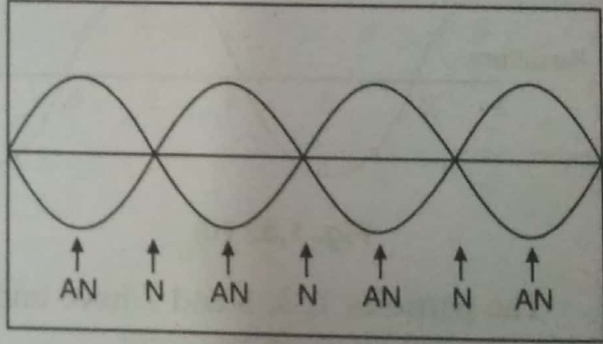

Fig. 1.33

Nodes and antinodes should not be confused with crests and troughs.

Distinction between Travelling Waves and Standing Waves

	Travelling waves	Standing waves
1.	A travelling wave propagates in a medium continuously with a finite velocity.	A standing wave is stationary and does not move in the medium.
2.	A travelling wave transports energy from one location to the other. Hence there is energy flow across every plane in the direction of wave propagation.	There is no energy transfer in a standing wave. There is no flow of energy across any plane. The energy of oscillations periodically transform from kinetic energy to potential energy of the elastically deformed medium and vice versa.
3.	No particle on the wave is permanently at rest.	Nodes are permanently at rest in a standing wave.

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4.	In a travelling wave all the points oscillate with the same amplitude regardless of their location.	All the points of a standing wave between two adjacent nodes oscillate with different amplitudes.
5.	In a travelling wave, different points oscillate with different phases.	In a standing wave, all the points between any pair of nodes oscillate in the same phase.
6.	In a travelling wave, all the particles do not pass through their mean positions or reach the extreme positions simultaneously.	In a standing wave, all the particles pass through their mean positions and reach their extreme positions simultaneously twice in each cycle.

Example 1.10: *Standing waves are produced by the superposition of two waves, $y_1 = 10 \sin (3\pi t - 4x)$ and $y_2 = 10 \sin (3\pi t + 4x)$. Find the amplitude of the motion, at x = 18.*

Solution: The resultant amplitude is given by

$$\begin{aligned} y &= y_1 + y_2 \\ &= 10 \sin (3\pi t - 4x) + 10 \sin (3\pi t + 4x) \\ &= 10[\sin 3\pi t \cos 4x - \cos 3\pi t \sin 4x + \sin 3\pi t \cos 4x + \cos 3\pi t \sin 4x] \\ &= 10[2 \sin 3\pi t \cos 4x] \\ &= 20 \cos 4x \sin 3\pi t \end{aligned}$$

The amplitude of motion is $(20 \cos 4x)$.

When $x = 18$, then $4x = 72 = [72 \times \pi/3.14]$ radians $= 22.9\ \pi$ radians

$\therefore$ Amplitude $= 20[\cos (22.9\ \pi)] = 20\ (0.9673) = 19.35$ units of length.

1.13.1 Harmonics

In general, standing waves form in a bounded medium. For instance, when a string is tied at both ends, standing waves set up, but set up only for a certain discrete set of frequencies. We then say that the system resonates at these frequencies. The standing wave patterns are called **oscillation modes**. Because the ends of the string cannot move, a node of the standing wave pattern must exist at each end of the string. Therefore, the length L of the string must be an integral multiple of $\lambda/2$. The allowed frequencies are then given by

$$\nu = \frac{\upsilon}{\lambda} = n\frac{\upsilon}{2L}, \quad n = 1, 2, 3, \ldots \qquad (1.54)$$

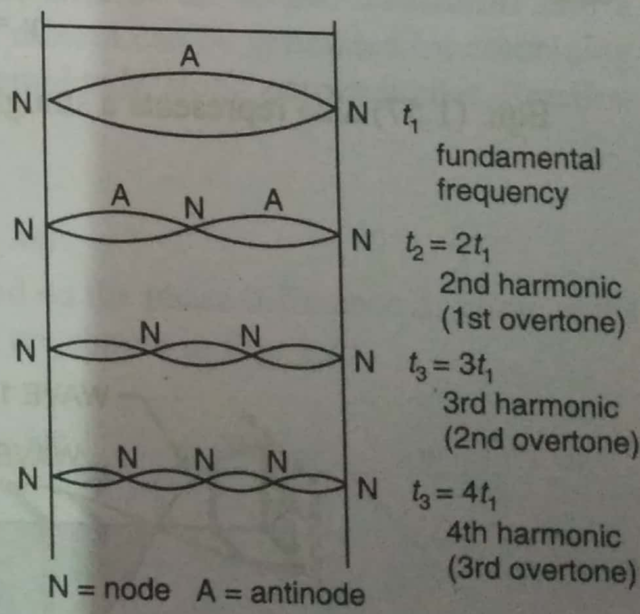

Fig. 1.34

Each frequency is associated with a different standing wave pattern. These frequencies and their associated wave patterns are referred to as **harmonics**. The pattern with two nodes and one antinode is referred to as the first harmonic, that with three nodes and two antinodes is the second harmonic and are depicted in the Fig. 1.34.

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6.4 TECHNIQUES OF OBTAINING INTERFERENCE

The phase relation between the waves emitted by two conventional light sources fluctuates rapidly and therefore they can never be coherent, though they are identical in all respects. However, two coherent sources are derived from a single source by techniques, which can be divided into two broad classes.

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1. **Division of wave front:** One of the methods consists in using a narrow slit as the source and subsequently, the wave front is divided. For example, in the Young's double slit experiment, a wave front emerging from the slit S is divided into two parts by the double slit S_1S_2. Fresnel's biprism, Lloyd's mirror, etc are the other examples where the **division of wave front** method is used.
2. **Division of amplitude:** In this method, amplitude of the light beam is divided by partial reflection into two or more beams. Thin films (wedge, Newton's rings etc), interferometers such as Michelson's interferometer etc utilize this method in producing interference.

6.5 REVIEW OF IMPORTANT CONCEPTS

When studying the topic of interference, we frequently come across the terms optical path and path difference. We should clearly understand these terms and know how to calculate them.

6.5.1 Geometrical Path

Light travels along a straight line path from a point A to another point B and it is known as the *path of the light*. The shortest path between any two points A and B is called the *geometrical path length* (GPL). GPL remains the same whether it is measured in a vacuum or in any medium.

6.5.2 Optical Path

Light travels μ times slower in a medium. Therefore, it takes μ times more time to cover the distance AB in the medium than it takes to cover the same distance in a vacuum. This time delay is accounted for by introducing another distance called *optical path length* (OPL). It is defined as

$$\text{O.P.L.} = \mu \times \text{G.P.L.}$$

or

$$\Delta = \mu L \tag{6.2}$$

The optical path length, Δ signifies the number of wavelengths that are accommodated in a given medium over the corresponding geometrical path length.

6.5.3 Path Difference

Light rays travel along different paths, which may lie in the same medium or in different media. The difference between optical paths of two rays travelling in different directions is known as the optical path difference.

6.5.4 Phase Difference

The phase of a wave arriving at a point depends on the optical path length it traversed. We know that if a wave covers in air a distance of one wavelength, 1λ, its phase changes by 2π radians. Therefore, we compute that if a wave travels a distance L in air, its phase change is given by

$$\delta = \frac{2\pi L}{\lambda} \tag{6.3}$$

When the wave travels the distance L in a medium, then

$$\delta = \frac{2\pi\Delta}{\lambda} = \frac{2\pi\mu L}{\lambda} \tag{6.4}$$

Comparing equs. (6.3) and (6.4), we find that a light path of geometric length L in a medium of refractive index μ produces the same phase change as a light path of length μL in a vacuum. Therefore, ***in the study of optics we always must calculate the optical paths travelled by light rays.***

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The path difference between two in-phase waves may be zero or an integral multiple of a wavelength, λ and the path difference between two opposite-phase waves will be $\lambda / 2$ or an odd integral multiple of $\lambda / 2$.

Optical path difference and the consequent phase difference may arise due to two reasons. One reason is the difference in the optical paths (See Art.5.15) and the other is due to reflections at optical interfaces.

(a) Phase difference due to optical path difference:

Let us consider two sources of light S_1 and S_2, as shown in Fig.6.4. Let us assume that the sources are identical and produce waves of same wavelength and that their vibrations are in the same phase at S_1 and S_2. Light from these sources travel in air along different paths, S_1P and S_2P; and meet at a point P. The path lengths S_1P and S_2P are different and contain different number of waves. The geometric path difference between the waves at P is $(S_2P - S_1P)$ and the optical path difference is $\mu(S_2P - S_1P)$. Since the paths contain different number of waves, the optical path difference will be equal either to a few integral number of waves or an integral number of wavelengths plus a fraction of one wavelength. This optical path difference leads to a phase difference between the waves meeting at P. It means that though the waves started with the same phase, they arrive at P with different phases because they travelled along different path lengths. Using equn.(6.4), the phase difference between the waves at P may be expressed as

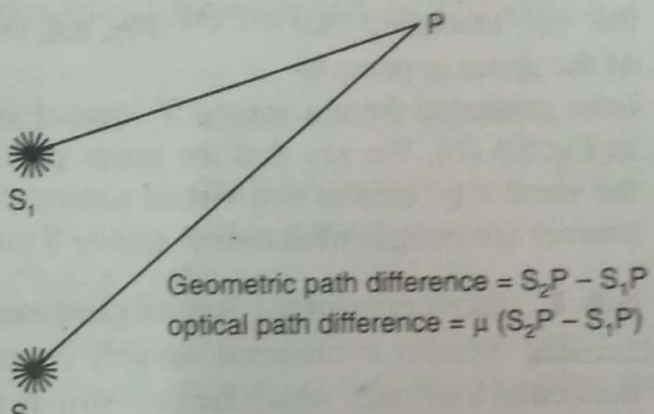

Fig. 6.4

$$\delta = \frac{2\pi}{\lambda}\mu(S_2P - S_1P) \tag{6.5}$$

(b) Phase difference due to reflection at boundaries of optical interfaces:

Light waves may also undergo phase change due to reflection at some point in their path. If the waves are reflected at a rarer-to-denser medium boundary, the reflected waves suffer a phase change of π rad or 180° compared to the incident waves (See Fig. 6.5 a & b). It is seen from equn.(6.3) that a phase change of π rad is equal to a path change of $\lambda/2$. Therefore, we must add (or subtract) $\lambda/2$ in the calculation of true optical path difference whenever a reflection occurs at a denser medium.

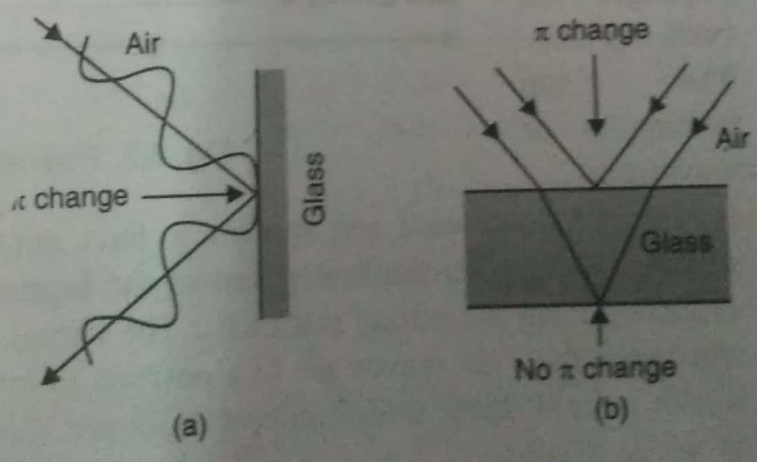

Fig. 6.5

6.8 PLANE PARALLEL FILM

A transparent thin film of uniform thickness bounded by two parallel surfaces is known as a *plane parallel thin film.*

When light is incident on a parallel thin film, a small portion of it gets reflected from the top surface and a major portion is transmitted into the film. Again, a small part of the transmitted component is reflected back into the film by the bottom surface and the rest of it is transmitted from the lower surface of the film. Thin films transmit incident light strongly and reflect only weakly. After two reflections, the intensities of reflected rays drop to a negligible strength. Therefore, we consider the first two reflected rays only. These two rays are derived from the same incident ray but appear to come from two sources located below the film. The sources are virtual coherent sources (see Fig.6.12). The reflected waves 1 and 2 travel along parallel paths and interfere at infinity. This is a case of *two-beam* interference.

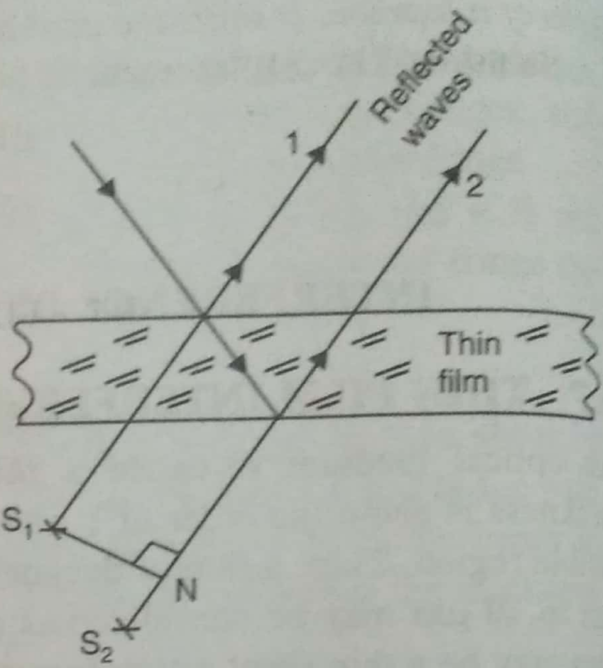

Fig. 6.12

The condition for maxima and minima can be deduced once we have calculated the optical path difference between the two rays at the point of their meeting.

6.8.1 Interference Due to Reflected Light

Let us consider a transparent film of uniform thickness 't' bounded by two parallel surfaces as shown in Fig.6.13. Let the refractive index of the material be μ. The film is surrounded by air on both the sides. Let us consider plane waves from a monochromatic source falling on the thin film at an angle of incidence 'i'. Part of a ray such as AB is reflected along BC, and part of it is transmitted into the film along BF. The transmitted ray BF makes an angle 'r' with the normal to the surface at the point B. The ray BF is in turn partly reflected back into the film along FD while a major part refracts into the surrounding medium along FK. Part of the reflected ray FD is transmitted at the upper surface and travels along DE. Since the film

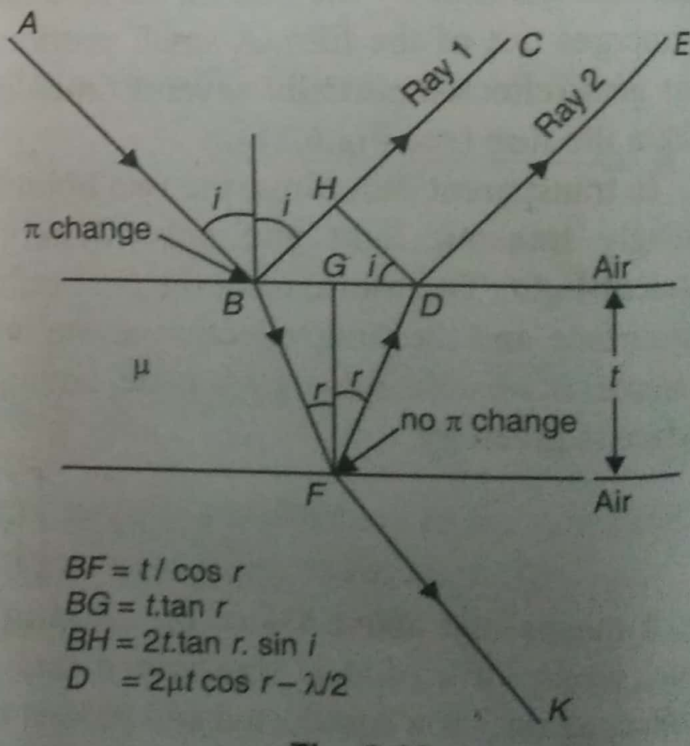

Fig. 6.13

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boundaries are parallel, the reflected rays BC and DE will be parallel to each other. The waves travelling along the paths BC and BFDE are derived from a single incident wave AB. Therefore they are coherent and can produce interference if they are made to overlap by a condensing lens or the eye.

(*i*) **Geometrical Path Difference:** Let DH be normal to BC. From points H and D onwards, the rays HC and DE travel equal path. The ray BH travels in air while the ray BD travels in the film of refractive index μ along the path BF and FD. The geometric path difference between the two rays is

$$BF + FD - BH.$$

(*ii*) **Optical Path Difference:**

Optical path difference $\Delta_a = \mu L$

$$\therefore \quad \Delta_a = \mu\,(BF + FD) - 1(BH) \tag{6.15}$$

In the ΔBFD, $\angle BFG = \angle GFD = \angle r$

$$BF = FD$$

$$BF = \frac{FG}{\cos r} = \frac{t}{\cos r}$$

$$\therefore \quad BF + FD = \frac{2t}{\cos r} \tag{6.16}$$

Also,

$$BG = GD$$

$$\therefore \quad BD = 2BG$$

$$BG = FG \tan r = t \tan r$$

$$\therefore \quad BD = 2t \tan r$$

In the Δ^{le} BHD

$$\angle HBD = (90 - i)$$

$$\angle BHD = 90°$$

$$\therefore \quad \angle BDH = i$$

$$\therefore \quad BH = BD \sin i = 2\,t \tan r \sin i \tag{6.17}$$

From Snell's law,

$$\sin i = \mu \sin r$$

$$\therefore \quad BH = 2t \tan r\,(\mu \sin r) = \frac{2\mu t \sin^2 r}{\cos r} \tag{6.18}$$

Using the equations (6.17) and (6.16) into equ.(6.15), we get

$$\Delta_a = \mu\left[\frac{2t}{\cos r}\right] - \left[\frac{2\mu t \sin^2 r}{\cos r}\right]$$

$$= \frac{2\mu t}{\cos r}\left[1 - \sin^2 r\right]$$

$$= \frac{2\mu t}{\cos r}\cos^2 r$$

$$\therefore \quad \Delta_a = 2\mu\, t \cos r \tag{6.19}$$

(*iii*) **Correction on account of phase change at reflection:** When a ray is reflected at the boundary of a rarer to denser medium, a path-change of λ/2 occurs for the ray BC (see Fig.6.13). There is no path difference due to transmission at D. Including the change in path difference due to reflection in eqn. (6.19), the true path difference is given by

$$\Delta_t = 2\mu\, t \cos r - \lambda/2 \tag{6.20}$$

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6.8.2 Conditions for Maxima (Brightness) and Minima (Darkness)

Maxima occur when the optical path difference $\Delta = m\lambda$. If the difference in the optical path between the two rays is equal to an *integral number of full waves*, then the rays meet each other in phase. The crests of one wave falls on the crests of the others and the waves *interfere constructively*. Thus, when

$$2\mu t \cos r - \frac{\lambda}{2} = m\lambda \tag{6.21}$$

the reflected rays undergo constructive interference to produce brightness or maxima at the point of their meeting.

$$2\mu t \cos r = m\lambda + \lambda/2$$

or

$$2\mu t \cos r = (2m+1)\lambda/2 \qquad \textbf{Condition for Brightness} \tag{6.22}$$

Minima occur when the optical path difference is $\Delta = (2m + 1)\lambda/2$. If the difference in the optical path between the two rays is equal to an *odd integral number of half-waves*, then the rays meet each other in opposite phase. The crests of one wave falls on the troughs of the others and *the waves interfere destructively*. Thus, when

$$2\mu t \cos r - \lambda/2 = (2m+1)\lambda/2 \tag{6.23}$$

the reflected rays undergo destructive interference to produce darkness. Equ.(6.23) may be rewritten as

$$2\mu t \cos r = (m+1)\lambda$$

The phase relationship of the interfering waves does not change if one full wave is added to or subtracted from any of the interfering waves. Therefore $(m + 1)\lambda$ can be as well replaced by $m\lambda$ for simplicity in expression. Thus,

$$2\mu t \cos r = m\lambda \qquad \textbf{Condition for Darkness} \tag{6.24}$$

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6.9 VARIABLE THICKNESS (WEDGE-SHAPED) FILM

A wedge is a thin film of varying thickness having a zero thickness at one end and progressively increasing to a particular thickness at the other end. A thin wedge of air film can be formed by two glass slides resting on each other at one edge and separated by a thin spacer at the opposite edge.

The arrangement for observing the interference pattern in a wedge shaped air-film is shown in Fig.6.16. If a parallel beam of *monochromatic* light illuminates the wedge from above, the rays reflected from its two bounding surfaces will not be parallel. They appear to diverge from a point near the film. The path difference between the rays reflected from the

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upper and lower surfaces of the air film varies along its length due to variation in thickness. Therefore, alternate bright and dark fringes are observed on its top surface (Fig.6.17). The fringes are localized at the top surface of the film.

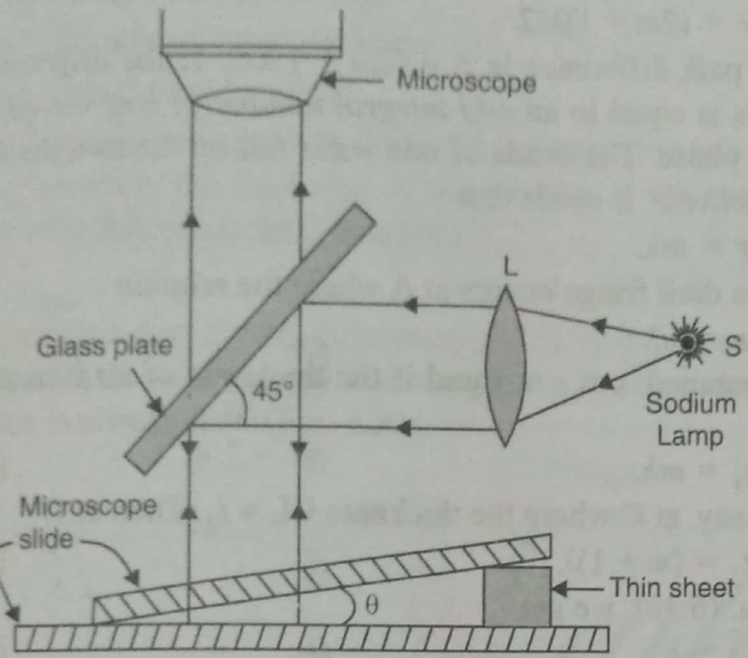

Fig. 6.16

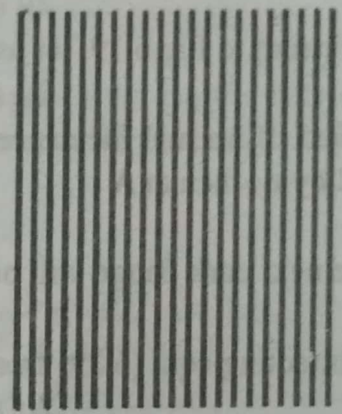

Fig. 6.17

When the light is incident on the wedge from above, it gets partly reflected from the glass-to-air boundary at the top of the air film. Part of the light is transmitted through the air film and gets reflected partly at the air-to-glass boundary, as shown in Fig.6.18. The two rays BC and FE, thus reflected from the top and bottom of the air film, are coherent as they are derived from the same ray AB through *division of amplitude*. The rays are close enough if the thickness of the film is of the order of a wavelength of light. For small film thickness the rays interfere producing darkness or brightness depending on the phase difference. The thickness of the glass plates is large compared with the wavelength of the incident light. Hence, the observed interference effects are entirely due to the wedge-shaped air film.

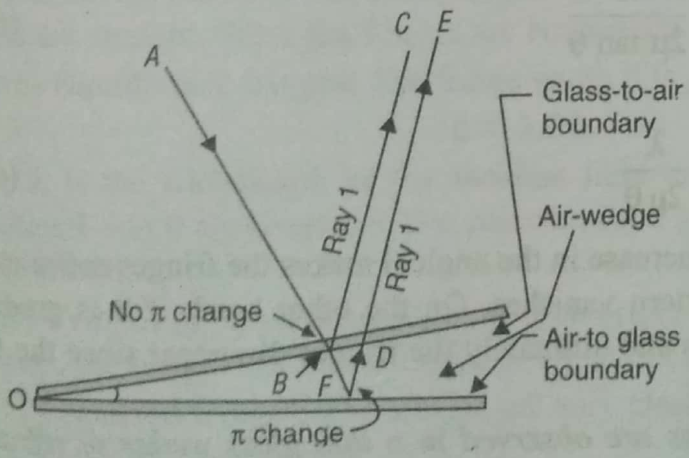

Fig. 6.18

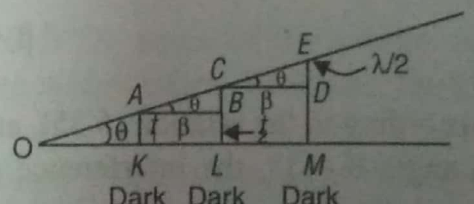

Fig. 6.19

The optical difference between the two rays BC and FE is given by

$$\Delta = 2\mu\, t \cos r - \lambda/2$$

where $\lambda/2$ takes account the gain of half–wave due to the abrupt jump of π radians in the phase of the wave reflected from the bottom boundary of air – to – glass.

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Maxima occur when the optical path difference $\Delta = m\lambda$. If the difference in the optical path between the two rays is equal to an *integral number of full waves*, then the rays meet each other in phase. The crests of one wave falls on the crests of the others and the waves *interfere constructively*. This needs that

$$2\mu t \cos r = (2m + 1)\lambda/2$$

Minima occur when the optical path difference is $\Delta = (2m + 1)\lambda/2$. If the difference in the optical path between the two rays is equal to an *odd integral number of half-waves*, then the rays meet each other in opposite phase. The crests of one wave fall on the troughs of the others and *the waves interfere destructively*. It needs that

$$2\mu t \cos r = m\lambda.$$

Referring to Fig.6.19, let us say a dark fringe occurs at A where the relation

$$2\mu t \cos r = m\lambda$$

is satisfied. If normal incidence is assumed, $\cos r = 1$ and if the thickness of air film at A is denoted by t_1, then at A

$$2\mu t_1 = m\lambda \tag{6.29}$$

The next dark fringe will occur, say, at C where the thickness $CL = t_2$. Then at C

$$2\mu t_2 = (m+1)\lambda \tag{6.30}$$

Subtracting equ. (6.29) from equ. (6.30), we get

$$2\mu(t_2 - t_1) = \lambda \tag{6.31}$$

But $$(t_2 - t_1) = BC$$

$\therefore$ $$2\mu(BC) = \lambda$$

or $$BC = \frac{\lambda}{2\mu} \tag{6.32}$$

From the $\Delta^{le}ABC$, $\angle CAB = \theta$ and $BC = AB \tan\theta$

$\therefore$ $$(AB)\tan\theta = \frac{\lambda}{2\mu} \tag{6.33}$$

AB is the distance between successive dark fringes and it also equals the separation of the successive bright fringes. It is, therefore, called the **fringe width**, β. That is $AB = \beta$. We may write equ. (6.33) as

$\therefore$ $$\beta = \frac{\lambda}{2\mu\tan\theta} \tag{6.34}$$

For small values of θ, $\tan\theta \approx \theta$.

$\therefore$ $$\beta = \frac{\lambda}{2\mu\theta} \tag{6.35}$$

According to the relation (6.35), an increase in the angle θ makes the fringes move closer. At an angle $\theta \approx 1°$, the interference pattern vanishes. On the other hand, if θ is gradually decreased, the fringe separation increases and ultimately the fringes disappear since the faces of the film become parallel at $\theta = 0°$.

Example 6.5: *Fringes of equal thickness are observed in a thin glass wedge of refractive index 1.52. The fringe spacing is 0.1 mm, wavelength of light being 5893 Å. Calculate the wedge angle.*

Solution: The fringe width $\beta = \lambda/2\mu\theta$.

$\therefore$ $$\theta = \frac{\lambda}{2\mu\beta} = \frac{5893\times10^{-10}\,\text{m}}{2\times1.52\times10^{-4}\,\text{m}} = 1.938 \text{ rad} = \mathbf{0.11°}.$$

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surface of the film.

6.11 NEWTON'S RINGS

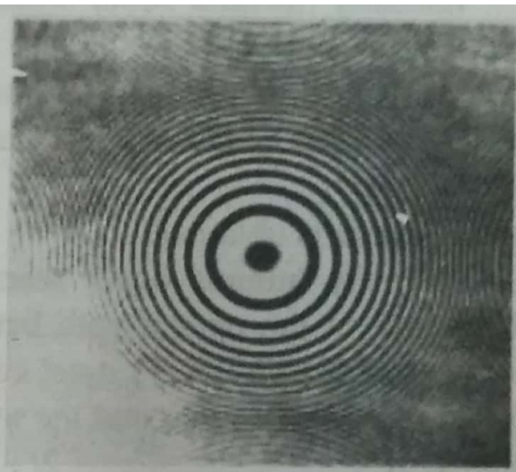

Fig. 6.22

Newton's rings are another example of fringes of equal thickness. Newton's rings are formed when a plano-convex lens L of a large radius of curvature placed on a sheet of plane glass AB. The combination forms a thin circular air film of variable thickness in all directions around the point of contact of the lens and the glass plate. The locus of all points corresponding to specific thickness of air film falls on a circle whose centre is at O. Consequently, interference fringes are observed in the form of a series of concentric rings with their centre at O (Fig. 6.22). Newton originally observed these concentric circular fringes and hence they are called Newton's rings.

The experimental arrangement for observing Newton's rings is shown in Fig. 6.23.

Monochromatic light from an extended source S is rendered parallel by a lens L'. It is incident on a glass plate inclined at 45° to the horizontal, and is reflected normally down onto a plano-convex lens placed on a flat glass plate. Part of the light incident on the system is reflected from the glass-to-air boundary, say from point D (Fig.6.24).The remainder of the light is transmitted through the air film. It is again reflected from the air-to-glass boundary, say from point J. The two rays reflected from the top and bottom of the air film are derived through division of amplitude from the same incident ray CD and are therefore coherent. The rays 1 and 2 are close to each other and interfere to produce darkness or brightness. The condition of brightness or darkness depends on the path difference between the two reflected light rays, which in turn depends on the thickness of the air film at the point of incidence.

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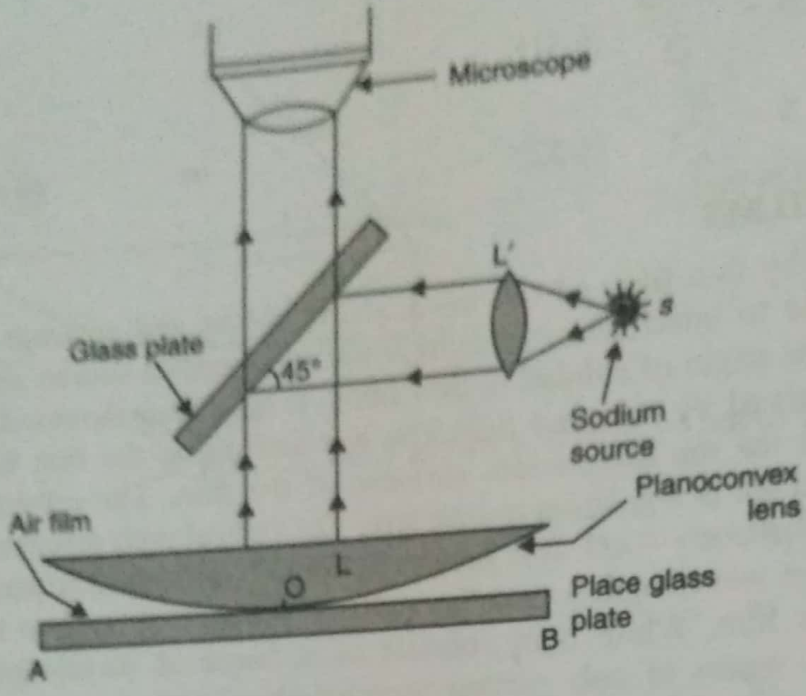

Fig. 6.23

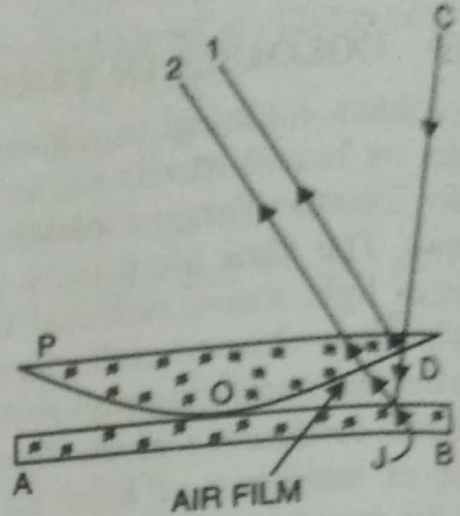

Fig. 6.24

6.11.1 Condition for Bright and Dark Rings

The optical path difference between the rays is given by $\Delta = 2\mu t \cos r - \lambda/2$. Since $\mu = 1$ for air and $\cos r = 1$ for normal incidence of light,

$$\Delta = 2t - \lambda/2 \tag{6.48}$$

Intensity maxima occur when the optical path difference $\Delta = m\lambda$. If the difference in the optical path between the two rays is equal to an *integral number of full waves*, then the rays meet each other in phase. The crests of one wave falls on the crests of the others and the waves *interfere constructively*. Thus, if $2t - \lambda/2 = m\lambda$

$$2t = (2m + 1)\lambda/2 \tag{6.49}$$

bright fringe is obtained.

Intensity minima occur when the optical path difference is $\Delta = (2m + 1)\lambda/2$. If the difference in the optical path between the two rays is equal to an *odd integral number of half-waves*, then the rays meet each other in opposite phase. The crests of one wave fall on the troughs of the other and *the waves interfere destructively.*

Hence, if $$2t - \lambda/2 = (2m + 1)\lambda/2$$

$$2t = m\lambda \tag{6.50}$$

and dark fringe is produced.

6.11.2 Circular Fringes

In Newton's ring arrangement, a thin air film is enclosed between a plano-convex lens and a glass plate. The thickness of the air film at the point of contact is zero and gradually increases as we move outward. The locus of points where the air film has the same thickness then fall on a circle whose centre is the point of contact. Thus, the thickness of air film is constant at points on any circle having the point of lens–glass plate contact as the centre. The fringes are therefore circular.

6.11.3 Radii of Dark Fringes

Let R be the radius of curvature of the lens (Fig. 6.25). Let a dark fringe be located at Q. Let the thickness of the air film at Q be PQ = t. Let the radius of the circular fringe at Q be OQ = r_m. By the Pythagorus theorem,

$$PM^2 = PN^2 + MN^2$$

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$$R^2 = r_m^2 + (R-t)^2$$

$$\therefore \quad r_m^2 = 2Rt - t^2 \qquad (6.51)$$

As $R >> t$, $2Rt >> t^2$.

$$\therefore \quad r_m^2 \cong 2Rt \qquad (6.52)$$

The condition for darkness at Q is that

$$2t = m\lambda$$

$$\therefore \quad r_m^2 \cong m\lambda R$$

$$r_m = \sqrt{m\lambda R} \qquad (6.53)$$

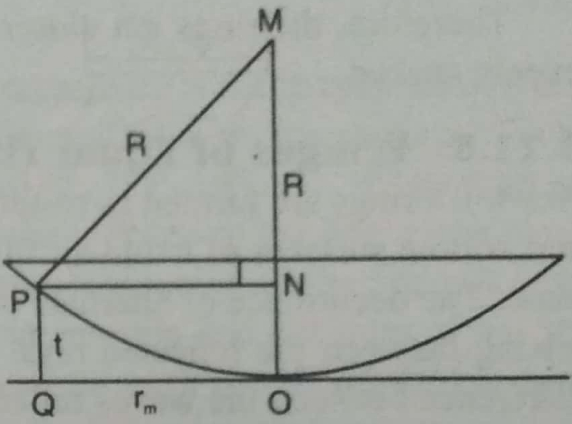

Fig. 6.25

The radii of dark fringes can be found by inserting values 1,2,3, ... for m.

Thus,

$$r_1 = \sqrt{1\lambda R} \quad \text{or} \quad r_1 \propto \sqrt{1}$$

$$r_2 = \sqrt{2\lambda R} \quad \text{or} \quad r_2 \propto \sqrt{2}$$

$$r_3 = \sqrt{3\lambda R} \quad \text{or} \quad r_3 \propto \sqrt{3} \text{ and so on}$$

It means that *the radii of the dark rings are proportional to square root of the natural numbers*.

The above relation also implies that $r_m \propto \sqrt{\lambda}$

Thus, *the radius of the m^{th} dark ring is proportional to square root of wavelength.*

Ring Diameter:

Diameter of m^{th} dark ring $\quad D_m = 2r_m$

$$D_m = 2\sqrt{2Rt}$$

or

$$D_m = 2\sqrt{m\lambda R} \qquad (6.54)$$

Example 6.6: *In a Newton's rings experiment, the diameter of 10^{th} dark ring due to wavelength 6000 Å in air is 0.5 cm. Find the radius of curvature of the lens.*

Solution: Radius of curvature, $R = \dfrac{(D/2)^2}{m\lambda} = \dfrac{(0.5\times10^{-2}/2)^2 \text{m}^2}{10\times6000\times10^{-10}\text{m}} =$ **104 cm.**

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6.13 MICHELSON'S INTERFEROMETER

An **interferometer** is an instrument in which the phenomenon of interference is used to make precise measurements of wavelengths or distances. Michelson designed an ingenious interferometer which utilizes the thin film interference.

6.13.1 Principle

A beam of light from an extended source is divided into two coherent beams of equal intensities by partial reflection and refraction. These beams travel in two mutually perpendicular directions and come together after reflection from plane mirrors. The beams overlap on each other and produce interference fringes. The fringes are *circular* if the reflecting mirrors are exactly perpendicular to each other or *straight* it the mirrors are inclined at a small angle.

6.13.2 Construction

The schematic of a simple Michelson interferometer is shown in Fig.6.35 (a). It consists of a beam splitter G_1, a compensating plate G_2, and two plane mirrors M_1 and M_2. The beam *splitter* G_1 is a partially silvered plane parallel glass plate and it has the property that it transmits half the incident light and reflects the rest. The compensating plate G_2 is a simple plane parallel glass plate having the same thickness as G_1. The two plates G_1 and G_2 are held parallel to each other and are inclined at an angle of 45° with respect to the mirror M_2. The mirror M_1 is mounted on a carriage and can be moved exactly parallel to itself with the help of a micrometer screw. The distance through which the mirror M_1 is moved can be read with the help of a graduated drum D attached to the screw. Displacements of the order of 0.1 mm can be easily read. The plane mirrors M_1 and M_2 can be made perfectly perpendicular with

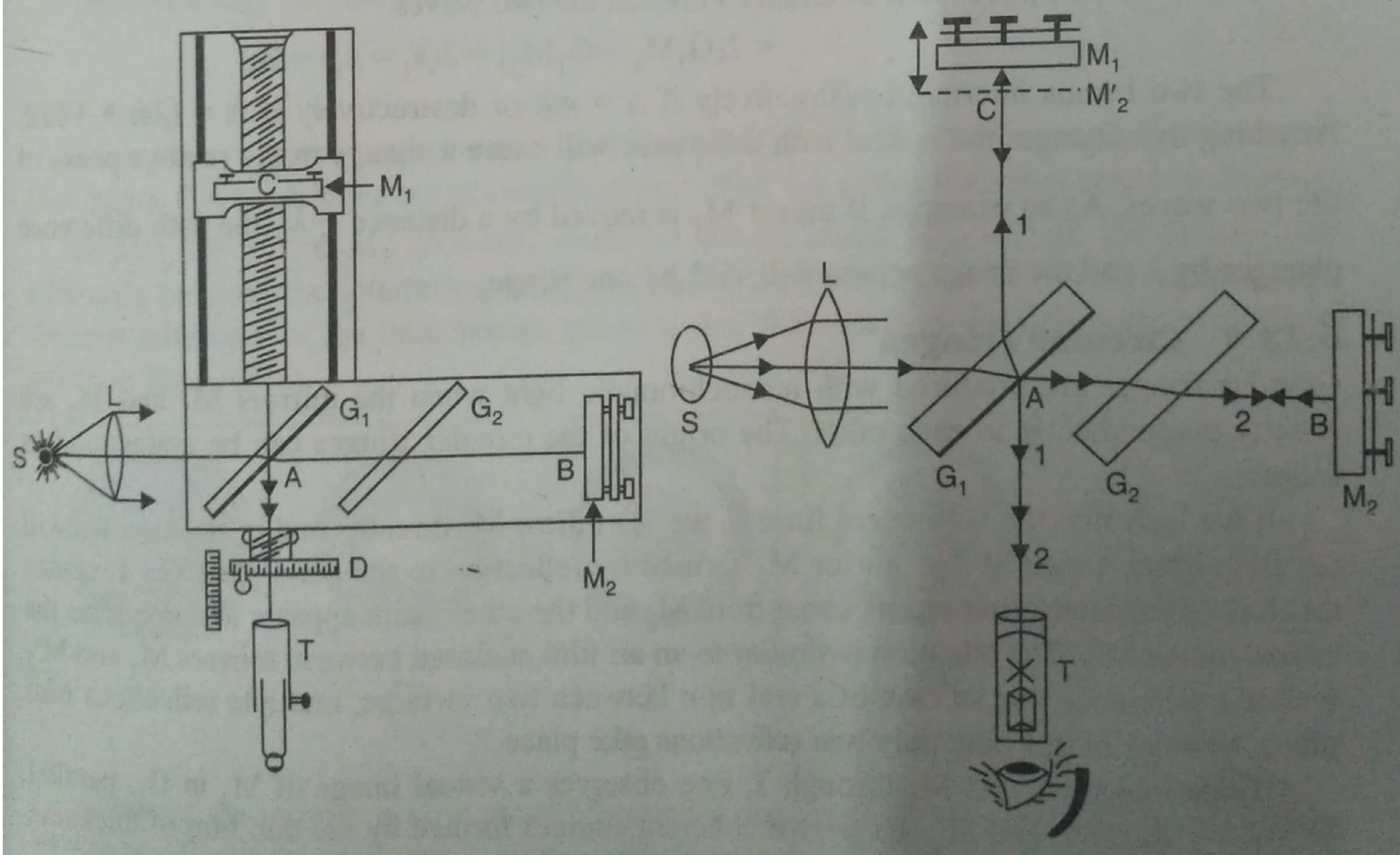

Fig. 6.35

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the help of the fine screws attached to them. The interference bands are observed in the field of view of the telescope T.

6.13.3 Working

Monochromatic light from an extended source is rendered parallel by means of a collimating lens L and is made incident on the beam splitter G_1. It is partly reflected at the back surface of G_1 along AC and partly transmitted along AB. The beam AC travels normally towards the plane mirror M_1 and is reflected back along the same path and comes out along AT. The transmitted beam travels toward the mirror M_2 and is reflected along the same path. It is reflected at the back surface of G_1 and proceeds along AT. The two beams received along AT are produced from a single source through division of amplitude and are hence *coherent*. The superposition of these beams leads to interference and produces interference fringes.

From the Fig.6.35 (b) it is clearly seen that a light ray starting from the source S and undergoing reflection at the mirror M_1 passes through the glass plate G_1 three times. On the other hand, in the absence of plate G_2, the ray reflected at M_2 travels through the glass plate G_1 only once. For compensating this path difference, a compensating plate G_2 of the same thickness as that of G_1 is inserted into the path AB and is held exactly parallel to G_1.

If we look into the instrument from T, we see mirror M_1 and in addition we see a virtual image, M'_2, of mirror M_2. Depending on the positions of the mirrors, image M'_2 may be in front of, or behind, or exactly coincident with mirror M_1.

Optical path difference between the two waves at T:

The optical path of the wave that travelled along

$$LG_1M_1T = 2G_1M_1 + \lambda/2 + \lambda/2 = 2G_1M_1 + \lambda.$$

The optical path of the wave that travelled along

$$LG_1M_2T = 2G_1M_2 + \lambda/2 + \lambda/2 = 2G_1M_2 + \lambda.$$

Therefore, the optical path difference between the two waves

$$= 2(G_1M_1 - G_1M_2) = 2(x_1 - x_2) = 2d.$$

The two beams interfere constructively if $\Delta = m\lambda$ or destructively if $\Delta = (2m + 1)\lambda/2$. Anything that changes the optical path difference will cause a change in the relative phase of the two waves. As an example, if mirror M_1 is moved by a distance $\frac{1}{2}\lambda$, the path difference changes by λ and the fringe pattern will shift by one fringe.

6.13.4 Circular Fringes

Circular fringes are produced with monochromatic light when the mirrors M_1 and M_2 are exactly perpendicular to each other. The origin of the circular fringes can be understood as follows.

If we look into the instrument from T, we see mirror M_1 directly, and in addition we will see the virtual image M'_2 of mirror M_2 formed by reflection in the glass plate G_1. It means that one of the interfering beams come from M_1 and the other beam appears to come from the virtual image M'_2. The situation is similar to an air film enclosed between mirrors M_1 and M'_2 with the difference that in case of a real film between two surfaces, multiple reflections take place, whereas in this case only two reflections take place.

If one looks towards M_1 through T, one observes a virtual image of M_2 in G_1, parallel to M_1, say M'_2. M_1 and M'_2 act as two coherent sources formed by the thin film of thickness $M_1M'_2$.

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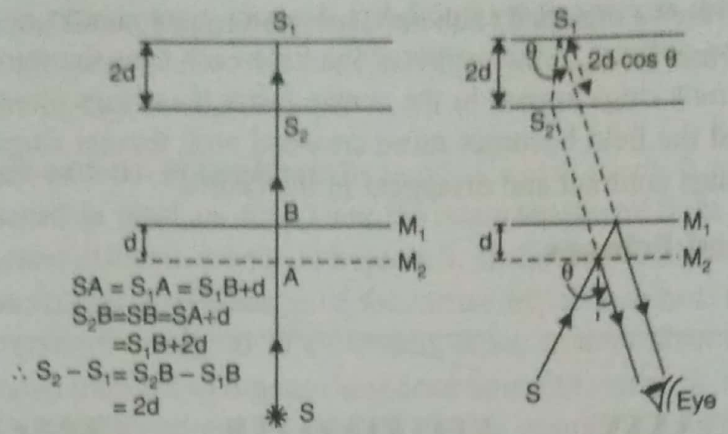

Fig. 6.36

$$M_1M'_2 = G_1M_1 - G_1M'_2 = G_1M_2 = (x_1 - x_2)$$

If the two arms of the interferometer are equal in length, and image M'_2 coincides with mirror M_1. If M'_2 and M_1 do not coincide, the distance between them is finite, $M'_2\ M_1 = (x_1 - x_2) = d$. The effect is that of light from a point source S falling on a uniformly thick film of air whose thickness is equal to d. Now, a light ray is reflected by both M'_2 and M_1 and the observer will see two virtual images- S_1 due to reflection at M'_2 and S_2 due to reflection at M_1. The virtual images are separated by a distance $2d$.

If the observer looks into the system at an angle θ, the path difference between the two beams will be $2d \cos \theta$. The light that comes from M_2 and goes to T undergoes rare-to-dense reflection and therefore a π-phase change occurs. In view of this, the total path difference between the two beams is given by

$$\Delta = 2d \cos \theta + \lambda / 2. \tag{6.72}$$

The condition for obtaining brightness

$$2d \cos \theta + \lambda / 2 = m\lambda \tag{6.73}$$

where $m = 0, 1, 2, ...$.

For a given mirror separation d, a given wavelength λ and order m, angle θ is constant. This means that the fringes are of circular shape. They are called *fringes of equal inclination*.

In case the mirror M_1 coincides with the virtual image M'_2, $d = 0$, the optical path difference between the interfering beams will be λ/2 (Refer to equ.6.73). Consequently, we obtain a minimum at the coincidence position and the centre of the field will be dark, as shown in Fig.6.37 (a).

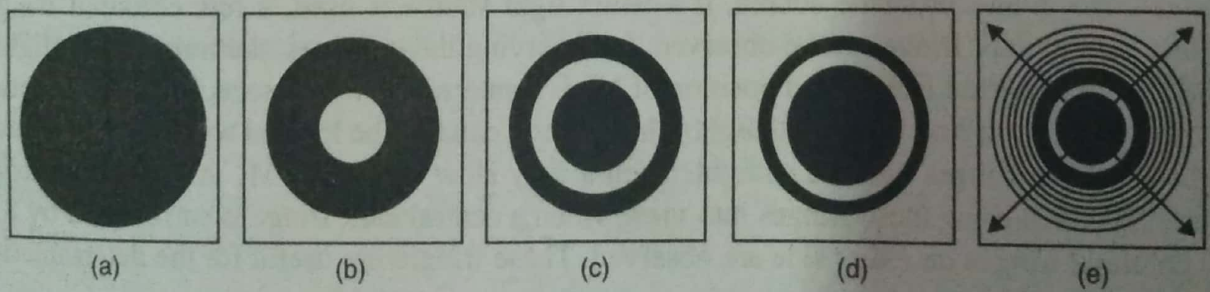

Fig. 6.37

If one of the mirrors is now moved through a distance λ/4, the path difference changes by λ/2 and therefore a maximum is obtained. By moving the mirror through another λ/4, a minimum is obtained; moving it by another $\lambda/4$ again a maximum is obtained and so on. Therefore, a new ring appears in the centre of the field each time the mirror is moved through $\lambda/2$. As d increases new rings appear in the centre faster than rings already present disappear in the periphery; and the field becomes more crowded with thinner rings. Conversely, as d is made smaller, the rings contract and disappear in the centre.

6.13.5 Localized Fringes

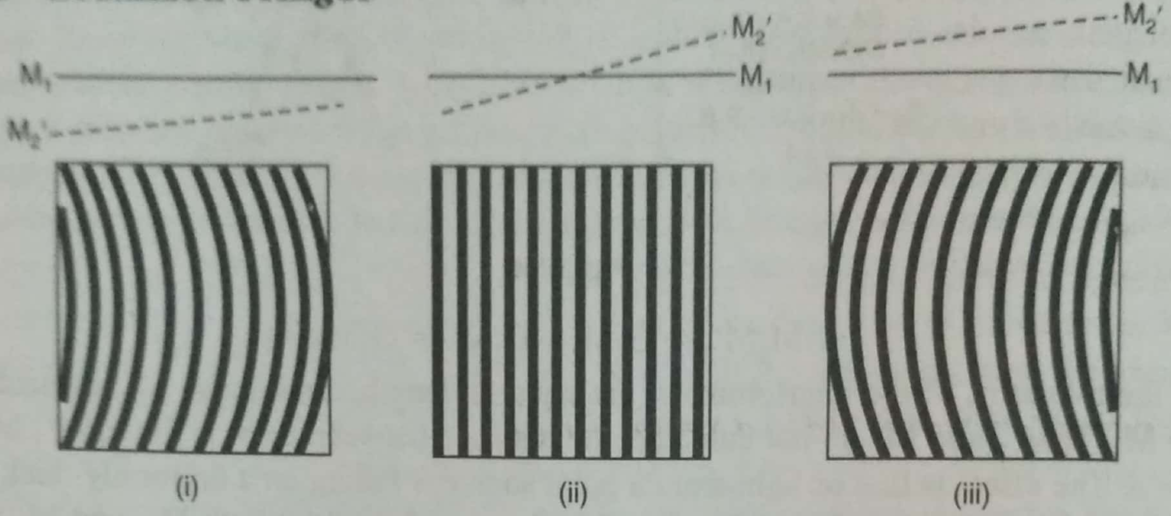

Fig. 6.38

When the two mirrors are tilted, they are not exactly perpendicular to each other and therefore the mirror M_1 and the virtual image M'_2 are not parallel. In this case the air path between them is wedge-shaped and the fringes appear to be straight. If one of the mirrors is moved, the fringes move across the field. The position of any particular bright fringe is taken up by the one next to it. The fringes can be counted as they pass a reference mark. If m fringes move across the field of view when M_1 moves through a distance d, then

$$d = m\,\lambda/2$$

or

$$\lambda = \frac{2d}{m} \tag{6.74}$$

Example 6.13: *In a Michelson interferometer 200 fringes cross the field of view when the movable mirror is moved through 0.0589 mm. Calculate the wavelength of light used.*

Solution:

$$\lambda = \frac{2d}{m} = \frac{2\times 0.0589\times 10^{-3}\,\text{m}}{200} = 5890\ \text{Å}$$

6.13.6 White Light Fringes

Instead of a monochromatic source, if a white light source is used, a few coloured fringes with a central dark fringe can be observed. In observing these fringes, the mirrors are slightly tilted as for localised fringes and position of M_1 is found where it intersects M'_2. This position is often difficult to find with white light. The position can best be located with monochromatic light when the fringes become straight. Then a very slow motion of M_1 in this region using white light will bring these fringes into view, when a central dark fringe is surrounded by 8 to 10 coloured fringes on either side are observed. These fringes are useful for the determination of zero path difference.

6.14 APPLICATIONS OF MICHELSON INTERFEROMETER

Michelson interferometer can be used to determine (i) the wavelength of a given monochromatic source of light (ii) the difference between the two neighbouring wavelengths or

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resolution of the spectral lines, (iii) refractive index and thickness of various thin transparent materials and (iv) for measurement of the standard metre in terms of the wavelength of light. We study here only the first three applications.

6.14.1 Measurement of Wavelength

Michelson interferometer is used to determine the wavelength of light from a monochromatic source. The monochromatic source is kept at S. If the mirrors M_1 and M_2 are exactly perpendicular, circular fringes are obtained. If the mirror M_1 is moved forward or backward, the circular fringes appear or disappear at the centre. Now, as the mirror is moved through a known distance d and the number of fringes disappearing at the centre is counted. Suppose d_1 is the initial thickness of the air film between the mirror M_1 and the image of M_2 corresponding to the bright fringe of order m_1 and d_2 is the final thickness of the air film corresponding to a bright fringe of order m_n in the same position. Then,

$$2d_1 = m_1\lambda$$

and
$$2d_2 = m_n\lambda$$

By subtraction, we get $2(d_2 - d_1) = (m_n - m_1)\lambda$

$$\therefore \quad 2d = N\lambda$$

$$\therefore \quad \lambda = \frac{2d}{N} \tag{6.75}$$

6.14.2 Determination of the Difference in the Wavelength of Two Waves

If a source of light consists of two wavelengths λ_1 and λ_2, which differ slightly, then two sets of fringes corresponding to the two wavelengths are produced in a Michelson interferometer. By adjusting the position of the mirror M_1 of the interferometer, the position is found when the fringes are very bright. In this position, the bright fringe due to λ_1 coincides with the bright fringes due to λ_2. When the mirror M_1 is moved, the two sets of fringes get out of step because their wavelengths are different. When the mirror M_1 has been moved through a certain distance, the bright fringe due to one set will coincide with the dark fringe due to the other set and no fringes will be seen in this case. Again by moving the mirror M_1, a position is reached when a bright fringe of one set falls on the bright fringe of the other and the fringes are again distinct. This is possible when the m^{th} order of the longer wavelength coincides with the $(m + 1)^{th}$ order of the shorter wavelength.

Let m_1 and m_2 be the changes in the order at the centre of the field when the mirror M_1 is displaced through a distance d between two consecutive positions of maximum distinctness of the fringes.

$$\therefore \quad 2d = m_1 \lambda_1 = m_2 \lambda_2$$

If λ_1 is greater than λ_2

$$m_2 = m_1 + 1$$

$$\therefore \quad 2d = m_1 \lambda_1 = (m_1 + 1) \lambda_2$$

$$\therefore \quad m_1 = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

$$\therefore \quad 2d = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}$$

or
$$\lambda_1 - \lambda_2 = \frac{\lambda_1\lambda_2}{2d}$$

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Taking λ as the mean wavelength of the two wavelengths λ_1 and λ_2, the small difference $\Delta\lambda$ is given by

$$\Delta \lambda = \lambda_1 - \lambda_2 = \frac{\lambda^2}{2d} \qquad (6.76)$$

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7.3 DISTINCTION BETWEEN INTERFERENCE AND DIFFRACTION

The main differences between interference and diffraction are as follows:

Interference	*Diffraction*
1. Interference is the result of interaction of light coming from different wave fronts originating from the source.	1. Diffraction is the result of interaction of light coming from different parts of the same wave front.
2. Interference fringes may or may not be of the same width.	2. Diffraction fringes are not of the same width.
3. Regions of minimum intensity are perfectly dark.	3. Points of minimum intensity are not perfectly dark.
4. All bright bands are of same intensity.	4. All bright bands are not of same intensity.

7.4 THE TWO TYPES OF DIFFRACTION

The diffraction phenomena are broadly classified into two types: Fresnel diffraction and Fraunhoffer diffraction.

1. **Fresnel diffraction:** In this type of diffraction, the source of light and the screen are effectively at finite distances from the obstacle (see Fig. 7.4a). Lenses are not used to make the rays parallel or convergent. The incident wave front is not planar. As a result, the phase of secondary wavelets is not the same at all points in the plane of the obstacle. The resultant amplitude at any point of the screen is obtained by the mutual interference of secondary wavelets from different elements of unblocked portions of wave front. It is experimentally simple but the analysis proves to be very complex.

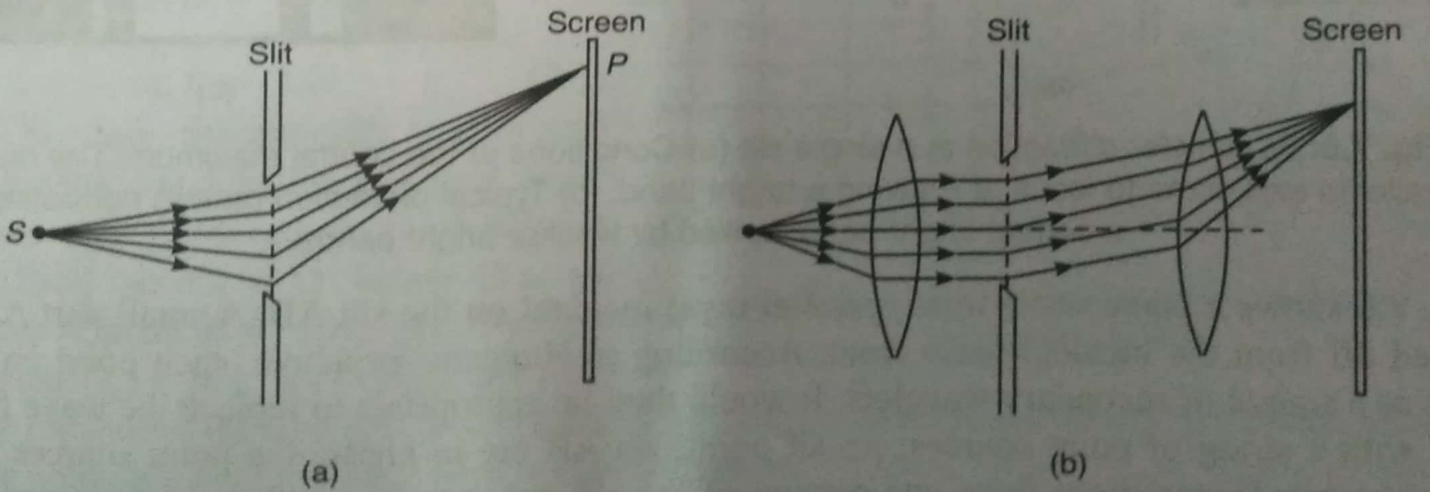

Fig. 7.4: Conditions for (a) Fresnel diffraction and (b) Fraunhoffer diffraction

2. **Fraunhoffer diffraction:** In this type of diffraction, the source of light and the screen are effectively at infinite distances from the obstacle (see Fig. 7.4b). The conditions required for Fraunhoffer diffraction are achieved using two convex lenses, one to make the light from the source parallel and the other to focus the light

after diffraction on to the screen. The incident wave front as such is plane and the secondary wavelets, which originate from the unblocked portions of the wave front, are in the same phase at every point in the plane of the obstacle. The diffraction is produced by the interference between parallel rays, which are brought into focus with the help of a convex lens. This problem is simple to handle mathematically because the rays are parallel.

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7.7 DIFFRACTION DUE TO N-SLITS—DIFFRACTION GRATING (NORMAL INCIDENCE)

Let us now consider the diffraction pattern produced by N-slits, each of width a. The separation between consecutive slits is $d = a + b$, where a is the width of the open portion and b is the width of the opaque portion. Such a device consisting of a large number of parallel slits of equal width and separated from one another by equal opaque spaces is called a **diffraction grating**. The distance d between the centres of the adjacent slits is known as the **grating period**.

Rowland (1848-1901) produced **transmission gratings** by ruling extremely close, equidistant and parallel lines on optically plane glass plates with a diamond point. The rulings (diamond scratch) scatter light and are effectively opaque while the parts without ruling transmit light and act as slits.

Because of the expenses and difficulty involved in fabrication, commonly used gratings are reproduced from the original ruled gratings. The **replica gratings** are made by pouring a thin layer of collodion solution over the surface of a ruled grating and the solution is allowed to harden. The collodion film is peeled carefully afterwards from the grating. The film retains the impression of the rulings of the original grating in the form of ridges. The ruled lines, which scatter light, act as opaque spaces whereas the spaces between them which transmit incident light act as parallel slits. The film is mounted between glass plates and it acts as a *plane transmission grating*. The number of lines on a plane transmission grating is of the order of 6000 lines per cm.

7.8 PLANE DIFFRACTION GRATING - THEORY

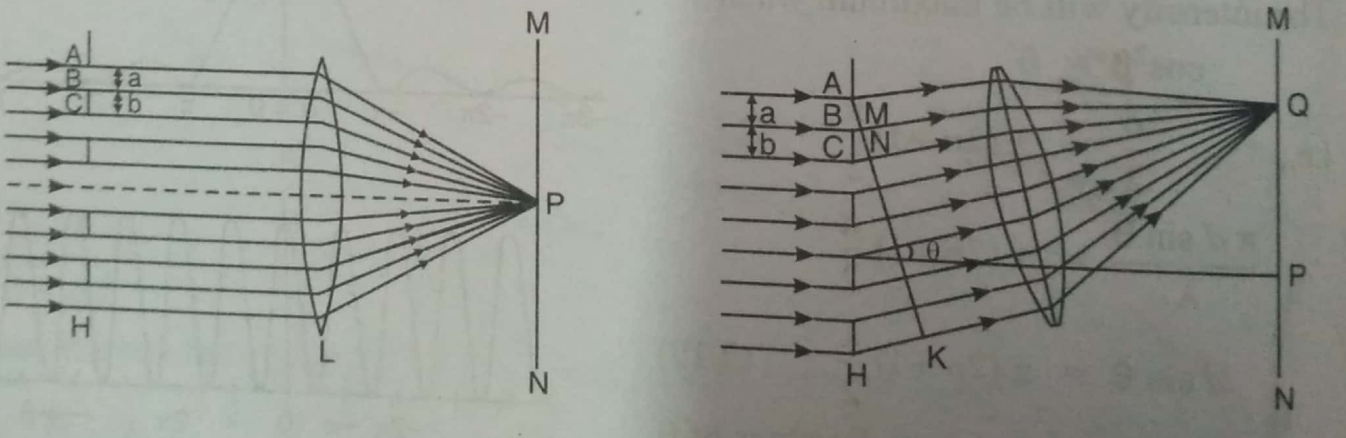

Fig. 7.15

Let us consider the plane transmission grating held normal to the plane of the page (Fig. 7.15 a) and represented by the section ABC…H. Let the width of the transparent portion AB be equal to a and opaque portion BC be equal to b. The distance $(a + b) = d$ and is called the *grating constant* or *grating period*. Let a parallel beam of monochromatic light of wavelength λ be incident normally on the grating surface. Then all the secondary waves travelling in the same direction as that of the incident light will come to focus at the point P on the screen. The screen is placed at the focal plane of the collecting lens, L. The point P where all the secondary waves reinforce one another corresponds to the position of the *central bright maximum*.

Now let us consider the secondary waves travelling in a direction inclined at an angle θ with the direction of the incident light (Fig. 7.15 b). The waves travel different distances and it is obvious that there is a path difference between the waves coming out from each slit and bending at an angle θ. These secondary waves come to focus at the point Q on the screen. The intensity at Q will depend on the path difference between the secondary waves originating from the corresponding points A and C of two neighbouring slits. In the Fig. 7.15 (b), AB = a and BC = b. The path difference between the secondary waves starting from A and C is equal to AC sin θ.

But

$$AC = AB + BC = a + b$$

$$\text{Path difference} = AC \sin\theta$$

$$= (a+b)\sin\theta$$

The point Q will be of maximum intensity if this path difference is equal to integral multiples of λ. It means that all the secondary waves originating from the corresponding points of the neighbouring slits reinforce one another and the angle θ gives the direction of maximum intensity. In general

$$(a+b)\sin\theta_m = m\lambda \tag{7.20}$$

where θ_m is the direction of the m^{th} principal maximum. If $(a + b)\sin\theta = \lambda$, we obtain maximum intensity at Q. When $(a + b)\sin\theta = 2\lambda$, there will be again a maximum and so on. Between the central maximum P and the first maximum at Q there will be minimum intensity and so on.

Similar maxima and minima are obtained on the other side of central maximum. Thus, on each side of the central maximum at P, principal maxima and minimum intensity are observed due to diffracted light. The position of m^{th} minimum is given by

$$(a+b)\sin\theta_m = (2m+1)\lambda/2. \tag{7.21}$$

7.8.1 Intensity Distribution

When illuminated by a beam of monochromatic radiation, the system produces N wavelets at an angle θ, each of the amplitude $A_\theta = A_0 \dfrac{\sin\alpha}{\alpha}$. The phase difference between successive wavelets is $\delta = \dfrac{2\pi d \sin\theta}{\lambda}$.

The resultant amplitude can be expressed as

$$y = A_\theta[\cos\omega t + \cos(\omega t + \delta) + \cos(\omega t + 2\delta) + \ldots + \cos(\omega t + N\delta)]$$

Expressing the amplitude terms as real parts of complex numbers, we have

$$y = A_\theta e^{j\omega t}\left[1 + e^{j\delta} + e^{2j\delta} + \ldots\ldots\right] = A_\theta e^{j\omega t}\left[\frac{1-e^{jN\delta}}{1-e^{j\delta}}\right]$$

We get the intensity by multiplying the amplitude with its complex conjugate.

$$I = A^2 = A_\theta^2\left[\frac{(1-e^{jN\delta})(1-e^{-jN\delta})}{(1-e^{j\delta})(1-e^{-j\delta})}\right] = A_\theta^2\left[\frac{1-\cos N\delta}{1-\cos\delta}\right]$$

$$= A_\theta^2\frac{\sin^2\frac{N}{2}\delta}{\sin^2\frac{\delta}{2}} = A_\theta^2\frac{\sin^2\left[\frac{N\pi d\sin\theta}{\lambda}\right]}{\sin^2\left[\frac{\pi d\sin\theta}{\lambda}\right]}$$

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or $$I = I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)\frac{\sin^2\left[\dfrac{N\pi d\sin\theta}{\lambda}\right]}{\sin^2\left[\dfrac{\pi d\sin\theta}{\lambda}\right]}$$

or $$I = I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)\frac{\sin^2 N\gamma}{\sin^2\gamma} \tag{7.22}$$

where $$\gamma = \frac{\pi d\sin\theta}{\lambda}.$$

The expression for intensity is a product of two terms. The term $I_0\left(\dfrac{\sin^2\alpha}{\alpha^2}\right)$ represents the intensity distribution due to a single slit diffraction. The second term $\dfrac{\sin^2 N\gamma}{\sin^2\gamma}$ represents the distribution of intensity due to interference produced by waves from N equally spaced point sources.

Principal Maxima

When $\sin\gamma = 0$, that is $\quad \gamma = \pm n\pi \quad (n = 0, 1, 2, 3, ...)$

We have $\sin N\gamma = 0$ and hence $\dfrac{\sin N\gamma}{\sin\gamma}$ becomes indeterminate.

According to L'Hospital's rule,

$$\underset{\alpha\to m\pi}{\text{Lt}}\ \frac{\sin N\gamma}{\sin\gamma} = \pm N$$

Therefore, $$\underset{\alpha\to \pm m\pi}{\text{Lt}}\left[\frac{\sin N\gamma}{\sin\gamma}\right]^2 = N^2$$

Substituting this value into equ.(7.22), we get $I = I_0\left(\dfrac{\sin^2\alpha}{\alpha^2}\right)N^2$, which is maximum.

Thus the resultant intensity of maxima is

$$I = I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)N^2. \tag{7.23}$$

Fig. 7.16 shows the intensity distribution determined by the factor $\left(\dfrac{\sin^2\alpha}{\alpha^2}\right)$ and $\dfrac{\sin^2 N\gamma}{\sin^2\gamma}$ respectively. The resultant of these curves is obtained by multiplying the ordinates of first two curves at every point.

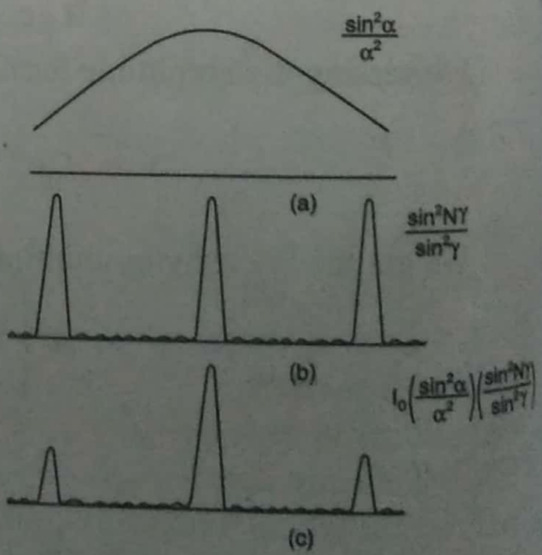

Fig. 7.16

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As the maxima are very intense, they are called **principal maxima**. In order to find the resultant intensity of any of the principal maxima in the diffraction pattern, we have to multiply the square of the number of the slits (N^2) with the factor $I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)$, which is the intensity due to a single slit.

The **direction of the principal maxima** are given by $\sin\gamma = 0$, that is $\gamma = \pm n\pi$.

or
$$d\sin\theta = \pm n\lambda \tag{7.24}$$

This equation is known as the **grating equation**.

If we put $n = 0$, we get $\theta = 0$. This is the direction in which waves from all slits arrive in phase and produce a bright central image. This maxima is called the **zero order principal maxima**. If we put $n = 1, 2, 3,....$ we obtain the directions of the first, second, third order principal maxima respectively. Therefore, the direction of occurrence of principal maxima is given by

$$\sin\theta_n = \frac{n\lambda}{d} = \frac{n\lambda}{a+b}$$

or
$$\sin\theta_n = nN\lambda \tag{7.25}$$

where $N = \frac{1}{a+b}$ is the number of ruled lines per unit width of the grating.

Minima

The intensity is zero when $\sin N\gamma = 0$, or $N\gamma = \pm m\pi$

or
$$N\gamma = \frac{N\pi d\sin\theta}{\lambda} = \pm m\pi$$

or
$$N\cdot d\sin\theta = \pm m\lambda \tag{7.26}$$

Here, m can take all integral values except 0, N, $2N$, $3N$, etc because these values give the positions of principal maxima. The positive and negative signs indicate that the minima of a given order lie symmetrically on both the sides of the central principal maxima.

It is seen from equ.(7.26) that $m = 0$ gives principal maximum of zero order while $m = 1, 2, 3,....(N-1)$ give the minima. Then $m = N$ gives principal maximum of first order. Thus, between zero order, and first order principal maxima we have $(N-1)$ minima. Similarly, it can be shown that there are $(N-1)$ minima between first order and second order principal maxima and so on. Between two such consecutive minima, the intensity has to be maximum, and these maxima are known as **secondary maxima**. The secondary maxima are not visible in the grating spectrum, as the number of slits is very large.

7.8.2 Missing of Orders

Under certain conditions, it is possible that grating forms the first and third order spectra while the second order spectrum is missing. Such a situation arises when for a given angle of diffraction θ_1 the path difference between the diffracted rays from the extreme ends of one slit is equal to an integral multiple of λ. Suppose the path difference is λ. Then each slit can be considered to be made up of two halves, and the path difference between the secondary waves from the corresponding points in the two halves will be $\lambda/2$. This results in destructive interference.

Mathematically, we express this as

$$(a+b)\sin\theta = n\lambda \tag{7.27}$$

For single slit diffraction, the condition for minima is

$$a\sin\theta = m\lambda \tag{7.28}$$

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If both the above conditions are satisfied simultaneously, the principal maxima will not be present in that direction. Dividing equ. (7.26) by (7.27), we obtain

$$\frac{a+b}{a} = \frac{n}{m}$$

i.e.,
$$n = \frac{a+b}{a} m \tag{7.29}$$

The above is the condition for the n^{th} order spectrum to be absent.

If we wish to suppress the second order spectrum, then $n = 2m = 2(\because m = 1)$.

Then,
$$\frac{a+b}{a} = \frac{2m}{m} = 2$$

or
$$a + b = 2a$$

$$a = b$$

Thus, if the width of each slit a is equal to the width b of the ruling, the second order spectrum will be absent.

7.8.3 Maximum Number of Orders Possible

The grating equation $d \sin\theta = \pm n\lambda$ may be rewritten as

$$n = \frac{d \sin\theta}{\lambda} = \frac{\sin\theta}{N\lambda}$$

The maximum value that θ can take is 90° and hence the maximum possible value of sin θ is 1. It implies that

$$(n)_{max} \leq \frac{1}{N\lambda} \tag{7.30}$$

The above relation (7.30) gives the maximum number of orders that would be seen in the spectrum produced by a plane transmission grating having N lines per unit width.

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7.10 RESOLVING POWER OF A PLANE TRANSMISSION GRATING

One of the important properties of a diffraction grating is its ability to resolve spectral lines, which have nearly the same wavelength. The spectral resolving power of a grating is defined in terms of the smallest wavelength interval ($d\lambda$) that can be detected by it. It is given by $\lambda/d\lambda$, where λ is the average of the two wavelengths and $d\lambda$ is their difference.

$$\text{Resolving Power} = \frac{\lambda}{d\lambda} = \frac{\lambda}{d\theta} \cdot \frac{d\theta}{d\lambda} \tag{7.33}$$

Let us now find the values of $\frac{d\theta}{d\lambda}$ and $\frac{\lambda}{d\theta}$.

The diffraction grating equation is

$$(a+b)\sin\theta = m\lambda.$$

Differentiating the above equation both sides, we get

$$(a+b)\cos\theta \cdot d\theta = m\, d\lambda$$

$$\therefore \quad \frac{d\theta}{d\lambda} = \frac{m}{(a+b)\cos\theta} \tag{7.34}$$

where $d\theta$ is the angle between the two diffracted beams whose difference in wavelength is $d\lambda$.

The light diffracted from the grating enters the objective of a telescope in a grating spectrometer. If the diffracted beam completely fills the objective then width of the beam equals the diameter d of the objective lens. The angular limit of resolution of a telescope objective is given by

$$d\theta = \frac{\lambda}{d}$$

Now $d = AB\cos\theta = l\cos\theta$ where l is the length of the grating.

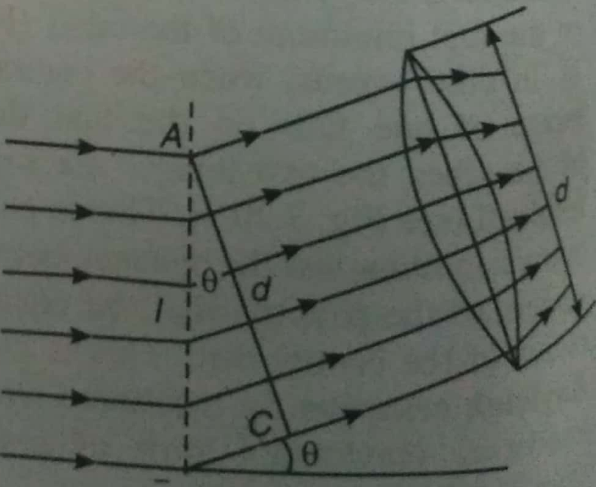

Fig. 7.22

$$\therefore \quad d\theta = \frac{\lambda}{l\cos\theta}$$

or

$$\frac{\lambda}{d\theta} = l\cos\theta$$

$$\text{Resolving Power, } R = \frac{\lambda}{d\lambda} = \frac{\lambda}{d\theta}\cdot\frac{d\theta}{d\lambda}$$

$$= l\cos\theta \times \frac{m}{(a+b)\cos\theta}$$

$$= \frac{ml}{(a+b)}$$

$$= m\,N \tag{7.35}$$

where $N = l/(a + b)$ = number of rulings on the grating surface and m is the order of the spectrum. Hence, the resolving power of a grating is given by the simple expression,

$$\text{R.P.} = mN$$

Example 7.7: *The sodium yellow doublet has wavelengths 5890Å and 5896Å. What should be the resolving power of a grating to resolve these lines?*

Solution: Mean wavelength $\lambda = (5890 + 5896)/2 = 5893$Å.

Wavelength difference , $d\lambda = 5896$ Å – 5890 Å = 6Å

Therefore, the resolving power R required for a grating to resolve these lines is given by

$$R = \frac{\lambda}{d\lambda} = \frac{5893}{6} = \mathbf{982.}$$

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