Chapter 5

Laplace Transforms

INTRODUCTION

The Laplace transform is a powerful mathematical technique useful to the engineers and scientists, as it enables them to solve linear differential equations with given initial conditions by using algebraic methods. The Laplace transform technique can also be used to solve systems of differential equations, partial differential equations and integral equations. Starting with the definition of Laplace transform, we shall discuss below the properties of Laplace transforms and derive the transforms of some functions which usually occur in the solution of linear differential equations.

Definition

If $f(t)$ is a function of t defined for all $t \geq 0$, $\int_0^\infty e^{-st} f(t)\,dt$ is defined as the *Laplace transform of $f(t)$*, provided the integral exists.

Clearly the integral is a function of the parameters s. This function of s is denoted as $\bar{f}(s)$ or $F(s)$ or $\phi(s)$. Unless we have to deal with the Laplace transforms of more than one function, we shall denote the Laplace transform of $f(t)$ as $\phi(s)$. Sometimes the letter 'p' is used in the place of s.

The Laplace transform of $f(t)$ is also denoted as $L\{f(t)\}$, where L is called the Laplace transform operator.

Thus $$L\{f(t)\} = \phi(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

The operation of multiplying $f(t)$ by e^{-st} and integrating the product with respect to t between 0 and ∞ is called *Laplace transformation*.

The function $f(t)$ is called the *Laplace inverse transform* of $\phi(s)$ and is denoted by $L^{-1}\{\phi(s)\}$.

Thus $$f(t) = L^{-1}\{\phi(s)\}, \quad \text{when} \quad L\{f(t)\} = \phi(s)$$

Note ☑

1. The parameter s used in the definition of Laplace transform is a real or complex number, but we shall assume it to be a real positive number sufficiently large to ensure the existence of the integral that defines the Laplace transform.
2. Laplace transforms of all functions do not exist. For example, $L(\tan t)$ and $L(e^{t^2})$ do not exist. We give below the sufficient conditions (without proof) for the existence of Laplace transform of a function $f(t)$:

Conditions for the existence of Laplace transform If the function $f(t)$ defined for $t \geq 0$ is

(i) piecewise continuous in every finite interval in the range $t \geq 0$, and

(ii) of the exponential order, then $L\{f(t)\}$ exists.

Note ☑

1. A function $f(t)$ is said to be piecewise continuous in the finite interval $a \leq t \leq b$, if the interval can be divided into a finite number of sub-intervals such that (i) $f(t)$ is continuous at every point inside each of the sub-intervals and (ii) $f(t)$ has finite limits as t approaches the end points of each sub-interval from the interior of the sub-interval.
2. A function $f(t)$ is said to be of the exponential order, if $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$ and some constants M and α or equivalently, if $\lim_{t \to \infty} \{e^{-\alpha t} f(t)\} =$ a finite quantity.

Most of the functions that represent physical quantities and that we encounter in differential equations satisfy the conditions stated above and hence may be assumed to have Laplace transforms.

LINEARITY PROPERTY OF LAPLACE AND INVERSE LAPLACE TRANSFORMS

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1 L\{f_1(t)\} \pm k_2 L\{f_2(t)\},$$

where k_1 and k_2 are constants.

Proof:

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = \int_0^\infty \{k_1 f_1(t) \pm k_2 f_2(t)\} e^{-st}\, dt$$

$$= k_1 \int_0^\infty f_1(t) \cdot e^{-st}\, dt \pm k_2 \int_0^\infty f_2(t) \cdot e^{-st}\, dt$$

$$= k_1 \cdot L\{f_1(t)\} \pm k_2 \cdot L\{f_2(t)\}.$$

Thus L is a linear operator.

As a particular case of the property, we get

$$L\{kf(t)\} = kL\{f(t)\}, \quad \text{where } k \text{ is a constant.}$$

If we take $L\{f_1(t)\} = \phi_1(s)$ and $L\{f_2(t)\} = \phi_2(s)$,
the above property can be written in the following form.

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1\, \phi_1(s) \pm k_2\, \phi_2(s).$$

$$\therefore \quad L^{-1}\{k_1\, \phi_1(s) \pm k_2\, \phi_2(s)\} = k_1 \cdot f_1(t) \pm k_2 \cdot f_2(t)$$

$$= k_1 \cdot L^{-1}\{\phi_1(s) \pm k_2 \cdot L^{-1}\{\phi_2(s)\}$$

Thus L^{-1} is also a linear operator

As a particular case of this property, we get

$$L^{-1}\{k\,\phi(s)\} = k\,L^{-1}\{\phi(s)\}, \quad \text{where } k \text{ is a constant.}$$

Note ☑

1. $L\{f_1(t) \cdot f_2(t)\} \neq L\{f_1(t)\} \cdot L\{f_2(t)\}$ and

$$L^{-1}\{\phi_1(s) \cdot \phi_2(s)\} \neq L^{-1}(\phi_1(s)\} \times L^{-1}\{\phi_2(s)\}$$

2. Generalising the linearity properties,

we get (i)
$$L\left\{\sum_{r=1}^{n} k_r\, f_r\,(t)\right\} = \sum_{r=1}^{n} k_r \cdot L\{f_r(t)\}$$

(ii)
$$L^{-1}\left\{\sum_{r=1}^{n} k_r\, \phi_r\,(s)\right\} = \sum_{r=1}^{n} k_r \cdot L^{-1}\{\phi_r(s)\}$$

Using (i), we can find Laplace transform of a function which can be expressed as a linear combination of elementary functions whose transforms are known.

Using (ii), we can find inverse Laplace transform of a function which can be expressed as a linear combination of elementary functions whose inverse transforms are known.

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

1. $L\{k\} = \dfrac{k}{s}$, $\quad s > 0$, where k is a constant,

$$L\{k\} = \int_0^\infty k\,e^{-st}\,dt, \text{ by definition.}$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{k}{-s}\,(0-1)\,[\because e^{-st} \longrightarrow 0 \text{ as } t \longrightarrow \infty, \text{ if } s > 0]$$

$$= \frac{k}{s}.$$

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$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1 L\{f_1(t)\} \pm k_2 L\{f_2(t)\},$$

where k_1 and k_2 are constants.

Proof:

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = \int_0^\infty \{k_1 f_1(t) \pm k_2 f_2(t)\} e^{-st}\, dt$$

$$= k_1 \int_0^\infty f_1(t) \cdot e^{-st}\, dt \pm k_2 \int_0^\infty f_2(t) \cdot e^{-st}\, dt$$

$$= k_1 \cdot L\{f_1(t)\} \pm k_2 \cdot L\{f_2(t)\}.$$

Thus L is a linear operator.

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$$L\{kf(t)\} = kL\{f(t)\}, \quad \text{where } k \text{ is a constant.}$$

If we take $L\{f_1(t)\} = \phi_1(s)$ and $L\{f_2(t)\} = \phi_2(s)$,
the above property can be written in the following form.

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1\, \phi_1(s) \pm k_2\, \phi_2(s).$$

$$\therefore \quad L^{-1}\{k_1\, \phi_1(s) \pm k_2\, \phi_2(s)\} = k_1 \cdot f_1(t) \pm k_2 \cdot f_2(t)$$

$$= k_1 \cdot L^{-1}\{\phi_1(s) \pm k_2 \cdot L^{-1}\{\phi_2(s)\}$$

Thus L^{-1} is also a linear operator

As a particular case of this property, we get

$$L^{-1}\{k\, \phi(s)\} = k\, L^{-1}\{\phi(s)\}, \quad \text{where } k \text{ is a constant.}$$

Note

1. $L\{f_1(t) \cdot f_2(t)\} \neq L\{f_1(t)\} \cdot L\{f_2(t)\}$ and

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LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

1. $L\{k\} = \dfrac{k}{s}$, $s > 0$, where k is a constant,

$$L\{k\} = \int_0^{\infty} k\, e^{-st}\, dt, \text{ by definition.}$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{k}{-s}(0-1)\ [\because e^{-st} \longrightarrow 0 \text{ as } t \longrightarrow \infty, \text{ if } s > 0]$$

$$= \frac{k}{s}.$$

In particular, $L(0) = 0$ and $L(1) = \frac{1}{s}$

$$\therefore \quad L^{-1}\left\{\frac{1}{s}\right\} = 1.$$

2. $L\{e^{-at}\} = \dfrac{1}{s+a}$, where a is a constant,

$$L\{e^{-at}\} = \int_0^\infty e^{-st} \cdot e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{-(s+a)}(0-1), \quad \text{if } (s+a) > 0$$

$$= \frac{1}{s+a}, \quad \text{if } s > -a.$$

Inverting, we get $L^{-1}\left(\dfrac{1}{s+a}\right) = e^{-at}$.

3. $L\{e^{at}\} = \dfrac{1}{s-a}$, where a is a constant, if $s - a > 0$ or $s > a$.

Changing a to $-a$ in (2), this result follows. The corresponding inverse result is

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

4. $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$, if $s > 0$ and $n > -1$.

$$L(t^n) = \int_0^\infty e^{-st} \cdot t^n\, dt$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}, \quad \text{on putting } st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, \quad \text{if } s>0 \text{ and } n+1>0$$

[by definition of Gamma function]

In particular, if n is a positive integer,

$$\Gamma(n+1) = n!$$

$\therefore$
$$L(t^n) = \frac{n!}{s^{n+1}}, \quad \text{if } s > 0 \text{ and } n \text{ is a positive integer.}$$

Inverting, we get $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$ or

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{n!}t^n$$

Changing n to $n-1$, we get $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$, if n is a positive integer.

If $n > 0$, then
$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{\Gamma(n)}t^{n-1}$$

In particular,
$$L(t) = \frac{1}{s^2} \quad \text{or} \quad L^{-1}\left(\frac{1}{s^2}\right) = t.$$

5. $L(\sin at) = \frac{a}{s^2 + a^2}$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= -\frac{s}{s^2 + a^2}\left(e^{-st} \sin at\right)_0^\infty - \frac{a}{s^2 + a^2}\left(e^{-st} \cos at\right)_0^\infty$$

$$= \frac{a}{s^2 + a^2}$$

[$\because$ $e^{-st} \sin at$ and $e^{-st} \cos at$ tend to zero at $t \longrightarrow \infty$, if $s > 0$]

Inverting this result we get $L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at.$

6. $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^\infty$$

$$= -\frac{s}{s^2 + a^2} \left(e^{-st} \cos at\right)_0^\infty + \frac{a}{s^2 + a^2} \left(e^{-st} \sin at\right)_0^\infty$$

$$= \frac{s}{s^2 + a^2}, \quad \text{as per the results stated above.}$$

Inverting the above result we get $L^{-1}\left(\dfrac{s}{s^2 + a^2}\right) = \cos at$.

Aliter

$$L(\cos at + i \sin at) = L(e^{iat})$$

$$= \frac{1}{s - ia}, \quad \text{by result (3).}$$

$$= \frac{s + ia}{s^2 + a^2}$$

i.e., $L(\cos at) + iL(\sin at) = \dfrac{s}{s^2 + a^2} + i \dfrac{a}{s^2 + a^2}$, by linearity property.

Equating the real parts, we get $L(\cos at) = \dfrac{s}{s^2 + a^2}$.

Equating the imaginary parts, we get $L(\sin at) = \dfrac{a}{s^2 + a^2}$

7. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

$$L(\sinh at) = L\left[\frac{1}{2}\left(e^{at} - e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right], \quad \text{by linearity property.}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right), \quad \text{if } s > a \quad \text{and} \quad s > -a.$$

$$= \frac{a}{s^2 - a^2}, \quad \text{if} \quad s > |a|$$

Aliter

$$L(\sinh at) = -iL(\sin i\,at)\ [\because \sin i\theta = i \sinh\theta]$$

$$= -i \cdot \frac{i\,a}{s^2 + i^2 a^2}, \quad \text{by result (5)}$$

$$= \frac{a}{s^2 - a^2}$$

Inverting the above result we get $L^{-1}\left\{\frac{a}{s^2 - a^2}\right\} = \sinh at.$

8. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

$$L(\cosh at) = L\left[\frac{1}{2}\left(e^{at} + e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[L\left(e^{at}\right) + L\left(e^{-at}\right)\right], \quad \text{by linearity property,}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right), \quad \text{if } s > |a|.$$

$$= \frac{s}{s^2 - a^2}, \quad \text{if } s > |a|.$$

Aliter

$$L(\cosh at) = L(\cos iat) \quad [\because \cos i\theta = \cosh\theta]$$

$$= \frac{s}{s^2 + i^2 a^2}, \quad \text{by result (6)}$$

$$= \frac{s}{s^2 - a^2}$$

Inverting the above result we get $L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \cosh at.$

PROPERTIES OF LAPLACE TRANSFORMS

1. Change of Scale Property

If $L\{f(t)\} = \phi(s)$, then $L\{f(at)\} = \dfrac{1}{a}\,\phi\left(\dfrac{s}{a}\right)$ and $L\left\{f\left(\dfrac{t}{a}\right)\right\} = a\,\phi(as)$.

Proof

By definition, $\phi(s) = L\{f(t)\} = \displaystyle\int_0^{\infty} e^{-st} f(t)\,dt$ (1)

and $L\{f(at)\} = \displaystyle\int_0^{\infty} e^{-st} f(at)\,dt$ [$\because f(at)$ is a function of t]

$= \displaystyle\int_0^{\infty} e^{-s\frac{x}{a}} f(x)\,\frac{dx}{a}$, putting $x = at$ and making necessary changes.

$= \dfrac{1}{a}\displaystyle\int_0^{\infty} e^{-(s/a)x}\cdot f(x)\,dx$ (2)

$= \dfrac{1}{a}\displaystyle\int_0^{\infty} e^{-(s/a)t}\cdot f(t)\,dt$, changing the dummy variable x as t.

Now, comparing (1) and (2), we note that the integral in (2) is the same as the integral in (1) except that 's' in integral in (1) is replaced by $\left(\dfrac{s}{a}\right)$ in the integral in (2).

$\therefore$ When the integral in (1) is equal to ϕ (s), that in (2) is equal to $\phi(s/a)$.

Thus $L\{f(at)\} = \dfrac{1}{a}\,\phi(s/a)$ (3)

Changing a to $\dfrac{1}{a}$ in (3) or proceeding as in the proof given above, we have

$L\{f(t/a)\} = a\,\phi(as)$.

2. First Shifting Property

If $L\{f(t)\} = \phi(s)$, then $\quad L\{e^{-at} f(t)\} = \phi(s+a)$

and $\quad L\{e^{at} f(t)\} = \phi(s-a).$

Proof

By definition, $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t = \phi(s) \qquad (1)$

and $\quad L\{e^{-at} f(t)\} = \int_0^\infty e^{-st}\, [e^{-at} f(t)]\, \mathrm{d}t$

[$\because$ $e^{-at} f(t)$ is a function of t]

$$= \int_0^\infty e^{-(s+a)t} f(t)\, \mathrm{d}t \qquad (2)$$

$= \phi(s+a)$, comparing the integrals in (1) and (2)

Changing a to $-a$ in the above result,

we get $\quad L\{e^{at} f(t)\} = \phi(s-a).$

Note ☑

1. The above property can be rewritten as a working rule (formula) in the following way:

$$L\{e^{-at} f(t)\} = \phi(s+a)$$
$$= [\phi(s)]_{s \longrightarrow s+a}$$
$$= L\{f(t)\}_{s \longrightarrow s+a}$$

'$s \longrightarrow s+a$' means that s is replaced by $(s+a)$.

Thus, to find the Laplace transform of the product of two factors, one of which is e^{-at}, we ignore e^{-at}, find the Laplace transform of the other factor as a function of s and change s into $(s+a)$ in it.

Similarly,

$$L\{e^{at} f(t)\} = L\{f(t)_{s \longrightarrow s-a}$$

Theorem

If $\quad L\{f(t)\} = \phi(s), \quad$ then $L\{t\,f(t)\} = -\phi'(s)$.

Proof:

Given: $\quad L\{f(t)\} = \phi(s)$

i.e.
$$\int_0^\infty e^{-st} f(t)\,dt = \phi(s) \qquad \cdots (1)$$

Differentiating both sides of (1) with respect to s,

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \frac{d}{ds}\phi(s) \qquad \cdots (2)$$

Assuming that the conditions for interchanging the two operations of integration with respect to t and differentiation with respect to s in (2) are satisfied, we have

$$\int_0^\infty \frac{d}{ds}\{e^{-st}\} f(t)\,dt = \phi'(s)$$

i.e. $$\int_0^\infty -t\, e^{-st} f(t)\,dt = \phi'(s)$$

i.e. $$\int_0^\infty e^{-st}\,[t\, f(t)]\,dt = -\phi'(s)$$

i.e. $$L\{t\, f(t)\} = -\phi'(s)$$

Corollary

Differentiating both sides of (1) n times with respect to s, we get

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\phi(s) \quad \text{or} \quad (-1)^n \phi^{(n)}(s).$$

Note ☑

1. The above theorem can be rewritten as a working rule in the following manner

$$L\{t f(t)\} = -\frac{d}{ds}\phi(s)$$

$$= -\frac{d}{ds} L\{f(t)\}$$

Thus, to find the Laplace transform of the product of two factors, one of which is 't', we ignore 't' and find the Laplace transform of the other factor as a function of s; then we differentiate this function of s with respect to s and multiply by (-1). Extending the above rule,

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\} \text{ and in general}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}.$$

Theorem

If $L\{f(t)\} = \phi(s)$, then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} \phi(s)\, ds$, provided $\lim_{t \to 0}\left\{\frac{1}{t} f(t)\right\}$ exists.

Proof:

Given: $$L\{f(t)\} = \phi(s)$$

i.e. $$\int_0^{\infty} e^{-st} f(t)\, dt = \phi(s) \tag{1}$$

Integrating both sides of (1) with respect to s between the limits s and ∞, we have

$$\int_s^{\infty}\left[\int_0^{\infty} e^{-st} f(t)\, dt\right] ds = \int_s^{\infty} \phi(s)\, ds \tag{2}$$

Assuming that the conditions for the change of order of integration in the double integral on the left side of (2) are satisfied, we have

$$\int_0^{\infty}\left[\int_s^{\infty} e^{-st}\, ds\right] f(t)\, dt = \int_s^{\infty} \phi(s)\, ds$$

i.e. $$\int_0^{\infty}\left[\frac{e^{-st}}{-t}\right]_{s=s}^{s=\infty} f(t)\, dt = \int_s^{\infty} \phi(s)\, ds$$

i.e. $$\int_0^{\infty} -\frac{1}{t}\,(0 - e^{-st}) f(t)\, dt = \int_s^{\infty} \phi(s)\, ds,$$

assuming that $s > 0$

i.e. $$\int_0^{\infty} e^{-st}\left[\frac{f(t)}{t}\right] dt = \int_s^{\infty} \phi(s)\, ds$$

i.e. $$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \phi(s)\, ds$$

Corollary

$$L\left\{\frac{1}{t^2} f(t)\right\} = L\left[\frac{1}{t}\left\{\frac{1}{t} f(t)\right\}\right]$$

$$= \int_s^{\infty}\left[\int_s^{\infty} \phi(s)\, ds\right] ds$$

$$= \int_s^{\infty}\int_s^{\infty} \phi(s)\, ds\, ds$$

Generalising this result, we get

$$L\left\{\frac{1}{t^n} f(t)\right\} = \int_s^{\infty}\int_s^{\infty} \cdots \int_s^{\infty} \phi(s)\, (ds)^n$$

Note ☒

1. The above theorem can be rewritten as a working rule as given below:

$$L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} L\{ f(t)\} \, ds$$

Thus, to find the Laplace transform of the product of two factors, one of which is $\frac{1}{t}$, we ignore $\frac{1}{t}$, find the Laplace transform of the other factor as a function of s and integrate this function of s with respect to s between the limits s and ∞.
Extending the above rule. We get;

$$L\left\{\frac{1}{t^2} f(t)\right\} = \int_s^{\infty}\int_s^{\infty} L\{ f(t)\} \, ds \, ds \text{ and in general}$$

$$L\left\{\frac{1}{t^n} f(t)\right\} = \int_s^{\infty}\int_s^{\infty}\cdots\int_s^{\infty} L\{ f(t)\} \, (ds)^n.$$