Fourier Transform

The fourier transformation of a fn. $f(x)$ in $(-\infty, \infty)$ denoted by $F(u)$ & defined as

$$F(u) = F\{f(x)\} = \int_{-\infty}^{\infty} e^{iux} f(x)\, dx$$

where u is the parameter.

The inverse fourier transform denoted by $f(x)$ is defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} F(u)\, du$$

Properties of Fourier transform:

1) Linearity transform -

If $F(u)$ & $G(u)$ are fourier transform of $f(x)$ & $g(x)$, then

$$F\{af(x) + bg(x)\} = aF(u) + bG(u)$$

where a & b are constants.

2) Change of scalar property -

If fourier transform of $f(x) = F(u)$, then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{u}{a}\right), \quad a \neq 0$$

3) Shifting property -

If $f(x) = F(u)$, then

$$F\{f(x-a)\} = e^{iua} \cdot F(u)$$

4) Modulation theorem -

If $f(x) = F(u)$, then

$$F\{f(x)\cos ax\} = \frac{1}{2}\{F(u+a) + F(u-a)\}$$

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* Whenever deduce is asked, inverse is asked.

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Formula:

1) $e^{i\theta} = \cos\theta + i\sin\theta$

2) $|i| = 1$

3) $\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$

4) $\frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta$

5) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$; if $f(x)$ is even

$= 0$, " " " odd

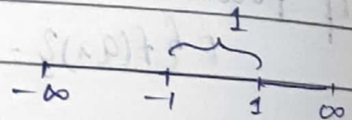
6) $\frac{e^{x} + e^{-x}}{2} = \cosh x$

7) $\frac{e^{x} - e^{-x}}{2} = \sinh x$

1) Express the function $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

& hence evaluate the integral $\int_{0}^{\infty} \frac{\sin x}{x}\,dx$

→ $F(u) = \int_{-\infty}^{+\infty} e^{iux} f(x)\,dx$



$= \int_{-1}^{1} e^{iux} \cdot 1 \cdot dx$

$= \left[\frac{e^{iux}}{iu}\right]_{-1}^{1}$

$= \frac{1}{iu}\left[e^{iu} - e^{-iu}\right]$

$= \frac{1}{iu}(2\sin u)$

$= \frac{2\sin u}{u}$

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$$F(u) = \frac{2\sin u}{u}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} \cdot F(u) \cdot du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-iux} \cdot \frac{\sin u}{u} \cdot du$$

put $x=0$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin u}{u} \cdot du$$

$$1 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin u}{u} \cdot du \qquad \because \text{ It is an even fn}$$

$$\int_0^{\infty} \frac{\sin u}{u} \cdot du = \frac{\pi}{2} \cdot \text{ (or) } \int_0^{\infty} \frac{\sin x}{x} \cdot dx = \frac{\pi}{2}$$

2) Find Fourier transform $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$

$$\text{P.T. } \int_{-\infty}^{\infty} \frac{\sin x - x\cos x}{x^3} \cdot dx = \frac{\pi}{4}$$

→

$$F(u) = \int_{-\infty}^{+\infty} e^{iux} \cdot f(x) \cdot dx$$

$$= \int_{-a}^{a} \cancel{a^2 - a^2} \cdot e^{iux}(a^2 - x^2) \cdot dx$$

$$= \int_{-a}^{a} (\cos ux + \cancel{i\sin ux})(a^2 - x^2) \cdot dx \qquad 0 \; [\because \text{even}]$$

$$= 2\int_{0}^{a} \cos ux (a^2 - x^2) \cdot dx$$

$$= 2\left[\cancel{(a^2-x^2)\left(\frac{\sin ux}{u}\right)}^{0} - (-2x)\left(\frac{-\cos ux}{u^2}\right) + (-2)\left(\frac{-\sin ux}{u^3}\right)\right]_0^a$$

$$= 2\left[\frac{-2}{u^2}\cos ux + \frac{2}{u^2}\sin ux\right]_0^a$$

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$$= 2\left\{\left[-\frac{2\cos ua}{u^2} + \frac{2}{u^3}\sin ua\right] - 0\right\}_a^a$$

$$= \frac{4}{u^3}\left[\sin au - ua\cos au\right]_{\!/\!/}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(u).du$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-isx}\left(\frac{\sin au - au\cos au}{u^3}\right).du$$

put $x=0$, $a=1$

$$1 = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin u - u\cos u}{u^3}\right).du$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin u - u\cos u}{u^3}\right).du \qquad (\because \text{ It is an even } f^{\underline{n}})$$

$$\frac{\pi}{4} = \int_{0}^{\infty}\left(\frac{\sin u - u\cos u}{u^3}\right).du$$

$$\therefore \int_{0}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right).dx = \frac{\pi}{4}_{\!/\!/}$$

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3) Find the fourier transform of $f(x) = \begin{cases} x & , |x| < a \\ 0 & , |x| > a \end{cases}$

→ $F(u) = \int_{-\infty}^{\infty} e^{+iux} \cdot f(x) \cdot dx$

$= \int_{-a}^{a} e^{iux} \cdot x \cdot dx$

$= \int_{-a}^{a} (\cos ux + i \sin ux) \cdot x \cdot dx$

$= \cancel{\underbrace{\int_{-a}^{a} x \cdot \cos ux \, dx}_{odd}}^{0} + \underbrace{\int_{-a}^{a} x i \sin ux \, dx}_{even}$

$= \int_{-a}^{a} x i \sin ux \cdot dx$

$= 2 \int_{0}^{a} x i \sin ux \cdot dx$

$= 2i \left[x \left(\frac{-\cos ux}{u} \right) - 1 \left(\frac{-\sin ux}{u^2} \right) \right]_0^a$

$= 2i \left[-\frac{a \cos au}{u} - 0 + \left(\frac{\sin au}{u^2} - 0 \right) \right]$

$= 2i \left[\frac{\sin au - au \cos au}{u^2} \right]$

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$\frac{1}{i} = -i \quad i^2 = -1 \quad i^4 = 1$

4) Find F.T. of $f(x) = \begin{cases} 1-|x| & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$

& hence P.T. $\int_0^\infty \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2}$

→ $F(u) = \int_{-\infty}^{\infty} e^{iux} \cdot f(x) \, dx$

$= \int_{-1}^{1} e^{iux} \cdot (1-|x|) \, dx$

$= \int_{-1}^{1} (\cos ux + i \sin ux)(1-|x|) \, dx$

$= \int_{-1}^{0} (\cos ux + i \sin ux)(1+x) \, dx$

$+ \int_0^1 (\cos ux + i \sin ux)(1-x) \, dx$

$-1 \to 0 \quad 1-|x| = 1+x \quad 1-|x| = 1-x$

$=$

$F(u) = \int_{-\infty}^{\infty} e^{iux} \cdot f(x) \, dx$

$= \int_{-1}^{1} e^{iux} \cdot (1-|x|) \, dx$

$= \left\{ \int_{-1}^{0} e^{iux} (1+x) \, dx + \int_0^1 e^{iux} (1-x) \, dx \right.$

$= \left\{ \left[(1+x)\left(\frac{e^{iux}}{iu}\right) - 1 \left(\frac{e^{iux}}{i^2u^2}\right) \right]_{-1}^{0} + \right.$

$\left. \left[(1-x)\left(\frac{e^{iux}}{iu}\right) - (-1)\left(\frac{e^{iux}}{i^2u^2}\right) \right]_0^1 \right\}$

$= \left\{ \left[\left(\frac{1}{iu} + \frac{1}{u^2}\right) - \frac{e^{-iu}}{u^2} \right] + \left[\left(0 - \frac{e^{iu}}{u^2}\right) - \left(\frac{1}{iu} - \frac{1}{u^2}\right) \right] \right\}$

$= \frac{2}{u^2} - \frac{1}{u^2}\left(e^{iu} + e^{-iu}\right)$

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$$= \frac{2}{u^2} - \frac{2}{u^2}\cos u$$

$$= \frac{2}{u^2}(1-\cos u)$$

$$= \frac{2}{u^2}\left(2\sin^2\frac{u}{2}\right)$$

$$= \frac{4\sin^2(u/2)}{u^2}$$

$$= \frac{\sin^2(u/2)}{(u/2)^2}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}F(u)\,.du$$

put $x=0$

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2(u/2)}{(u/2)^2}\,.du$$

put $\frac{u}{2} = t$

$u = 2t$

$du = 2dt$

$$\therefore 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\,.2dt$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\,.dt \quad \because \text{it is an even } f^n$$

$$1 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\,.dt$$

$$\therefore \int_{0}^{\infty} \frac{\sin^2 t}{t^2}\,.dt = \frac{\pi}{2}$$

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5) Find the complex F.T. of $e^{-a^2x^2}$, $a>0$ & hence deduce that $e^{-x^2/2}$ gives self reciprocal.

$\rightarrow$ $F(u) = \int_{-\infty}^{\infty} e^{iux} . f(x) . dx$

$=$

1) $F_s(u) = \frac{u}{1+u^2}$

$F(x) = \frac{2}{\pi} \int_0^{\infty} \frac{u}{1+u^2} \sin ux . du$

put $x = 1$

$e^{-1} = \frac{2}{\pi} \int_0^{\infty} \frac{u}{1+u^2} \sin u . du$

$\int_0^{\infty} \frac{u \sin u}{1+u^2} . du = \frac{\pi e^{-1}}{2}$

$\therefore \int_0^{\infty} \frac{x \sin x}{1+u^2} . dx = \frac{-e^{-1}}{2}$

2) Solve $F_s(s) = \int_0^{\infty} f(x) = \cos x . dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}$

evaluate $\int_0^{\infty} \frac{\sin^2 t}{t^2} . dt$

$F(x) = \frac{2}{\pi} \int_0^{\infty} F(x) . \cos \alpha x . \underbrace{d\alpha}_{1-\alpha}$

$= \frac{2}{\pi} \int_0^1 (1-\alpha) \cos \alpha x . dx$

$= \frac{2}{\pi} \left\{ (1-\alpha) \left[\frac{\sin \alpha x}{x} \right] - (-1) \left[\frac{-\cos \alpha x}{x^2} \right] \right\}_0^1$

$= \frac{2}{\pi} \left\{ \left[0 - \frac{\cos x}{x^2} \right] - \left[0 - \frac{1}{x^2} \right] \right\}$

$$= \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{2\sin^2 x/2}{x^2}\right] = \frac{1}{\pi}\frac{\sin^2(x/2)}{(x/2)^2}$$

$$F_c(\alpha) = \int_0^\infty f(x).\cos\alpha x.dx$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin^2(x/2)}{(x/2)^2}.\cos\alpha x.dx$$

put $\alpha = 0$

$$1 = \frac{1}{\pi}\int_0^\infty \frac{\sin^2(x/2)}{(x/2)^2}.dx$$

put $t = \frac{x}{2} \Rightarrow x = 2t \Rightarrow dx = 2dt$

$$1 = \frac{1}{\pi}\int_0^\infty \frac{\sin^2 t}{t^2}.2dt$$

$$\int_0^\infty \frac{\sin^2 t\,dt}{t^2} = \frac{\pi}{2}$$

3) Find Fourier Sine transform of $\frac{e^{-ax}}{x}$, $a>0$

→ $$F_s(u) = \int_0^\infty f(x).\sin ux.dx$$

$$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x}.\sin ux.dx$$

Using Labneitz rule under integral sign

$$F_s'(u) = \int_0^\infty \frac{e^{-ax}}{x}\frac{\partial(\sin ux)}{\partial u}.dx$$

$$F_s'(u) = \int_0^\infty \frac{e^{-ax}}{x}.\cos ux.x.dx$$

$$F_s'(u) = \int_0^\infty e^{-ax}\cos ux.dx$$

$a=-a \quad b=u$

$$= \frac{e^{-ax}}{a^2+u^2}\left[-a\cos ux + u\sin ux\right]\Big]_0^\infty$$

$$= \frac{1}{a^2+u^2}[0-(-a)]$$

$$F_s'(u) = \frac{a}{a^2+u^2}$$

On integrating,

$$F_s(u) = \int \frac{a}{a^2+u^2}\,du + C$$

$$= \tan^{-1}(u/a) + C \quad \text{--- (2)}$$

put $u=0$

$$0 = \tan^{-1}(0) + C \Rightarrow \boxed{C=0}$$

$\therefore$ (2) becomes,

$$F_s(u) = \tan^{-1}(u/a)$$

4) Find the inverse fourier sine transform of $\frac{e^{-as}}{s}$, $a>0$

$$F_s(s) = \frac{e^{-as}}{s}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\cdot \sin sx \,.\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} \cdot \sin sx \,.\, ds$$

$$f'(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\,\frac{\partial}{\partial x}(\sin sx)\,.\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{\cancel{s}}[\cos sx]\,\cancel{s}\;.\,ds$$

$$= \frac{2}{\pi}\,\frac{e^{-as}}{a^2+x^2}\Big[-a\cos sx + x\sin sx\Big]_0^\infty$$

$$= \frac{2}{\pi}\,\frac{1}{a^2+x^2}[0-(-a)]$$

$$f'(x) = \frac{2}{\pi}\,\frac{a}{a^2+x^2}$$

on integrating

$$f(x) = \frac{2}{\pi}\int \frac{a}{a^2+x^2}\,dx + C$$

$$= \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right) + C \quad \text{--- (2)}$$

put $s=0$

$$0 = \frac{2}{\pi}\tan^{-1}(0) + C \Rightarrow C = 0$$

$\therefore$ (2) becomes,

$$f(x) = \frac{2}{\pi}\tan^{-1}(x/a)$$

5) Find F.C.T. of $\frac{1}{1+x^2}$

$\Rightarrow$

$$f(x) = \frac{1}{1+x^2}$$

$$F_c(u) = \int_0^\infty f(x).\cos ux.dx \quad \text{--- (1)}$$

$$= \int_0^\infty \frac{1}{1+x^2}.\cos ux.dx$$

using Lebnetz's rule

$$F_c'(u) = \int_0^\infty \frac{1}{1+x^2}\frac{\partial(\cos ux)}{\partial u}.dx$$

$$= \int_0^\infty \frac{1}{1+x^2}x(-\sin ux).dx$$

$$= -\int_0^\infty \frac{x\sin ux.dx}{1+x^2}$$

~~again using L'rule~~

$$\cancel{F_c''(u) = -\int_0^\infty \frac{x}{1+x^2}\frac{\partial(\sin ux)}{\partial u}dx}$$

$$\cancel{= -\int_0^\infty \frac{x^2\cos ux\,dx}{1+x^2}}$$

$$F_c'(u) = -\int_0^\infty \frac{x^2}{x(1+x^2)} \cdot \sin ux \, dx$$

$$= -\int_0^\infty \frac{[(1+x^2)-1]}{x(x+1)^2} \cdot \sin ux \, dx$$

$$= -\int_0^\infty \left[\frac{1}{x} - \frac{1}{x(1+x^2)}\right] \sin ux \, dx$$

$$= -\int_0^\infty \frac{\sin ux}{x} \, dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx$$

$$F_c'(u) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx \quad \text{---} (2)$$

using Lebneitz's rule

$$F_c''(u) = 0 + \int_0^\infty \frac{\cos ax \cdot x}{x(1+x^2)} \, dx$$

$$F_c''(u) = + F_c(u) \qquad \text{from } (1)$$

$$F_c''(u) - F_c(u) = 0$$

$$(D^2-1) F_c(u) = 0$$

i.e. $m^2 - 1 = 0$

$m = \pm 1$

Let m_1, m_2 be roots

1) If roots are real & distinct.

$$C.F. = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

2) If roots are equal

$$m_1 = m_2$$

$$C.F. = (C_1 + C_2) x e^{m_1 x}$$

3) If roots are complex $\alpha \pm \beta$

$$C.F. = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$$

$$C.F. = F_c(u) = C_1 e^u + C_2 e^{-u} \quad \text{---} (3)$$

diff (3) w.r.t. (4)

$$F_c'(u) = C_1 e^u - C_2 e^{-u} \quad \text{---} (4)$$

put $u = 0$ in (3)

6) Find the fourier transform of e^{-x^2}

→ $F_c(u) = \int_0^\infty f(x) \cdot \cos ux \cdot dx$

$= \int_0^\infty e^{-x^2} \cdot \cos ux \cdot dx$

using L'rule

$F_c'(u) = \int_0^\infty e^{-x^2} \cdot \frac{\partial}{\partial u} (\cos ux) \cdot dx$

$= \int_0^\infty e^{-x^2} (-\sin ux) x \cdot dx$

$= -\int_0^\infty x e^{-x^2} \cdot \sin ux \cdot dx$

$=$