FOURIER SERIES.

→ Fourier Series is an infinite series representation of a function in terms of sine and cosine

- Dirichlet's Condition.

A function $f(x)$ can be represented as fourier series provided:-

i) $f(x)$ is periodic, finite & single-valued.

ii) $f(x)$ has finite number of discontinuities in any given period.

iii) $f(x)$ has almost finite number of maxima & minima.

~~iv) Euler's formula~~.

- Euler's formula.

Let $f(x)$ be a periodic function with period $2c$ defined in the interval $(\alpha, \alpha+2c)$ can be represented as sum of trignometric series i.e $f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{c}\right)x$

$$+ \sum b_n \sin\left(\frac{n\pi}{c}\right)x$$

where $a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\left(\frac{n\pi}{c}\right)x\,dx$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\left(\frac{n\pi}{c}\right)x\,dx.$$

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1. $f(x) = x(2\pi - x)$ in $(0, 2\pi)$

$(\alpha, \alpha+2c) = (0, 2\pi)$

$\alpha = 0$, $\alpha + 2c = 2\pi$ $\therefore c = \pi$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[2\pi \frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\pi(4\pi^2 - 0) - \frac{1}{3}(8\pi^3 - 0)\right] = \frac{1}{\pi}\left[\frac{4\pi^3}{3}\right]$$

$$= \frac{4\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\cos\left(\frac{n\pi}{\pi}\right)x\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\frac{\sin nx}{n} - (2\pi - 2x)\frac{(-\cos nx)}{n^2} + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\frac{\sin nx}{n} + 2(\pi - x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi n}\left[(2\pi x - x^2)\sin nx + 2(\pi - x)\frac{\cos nx}{n} + \frac{2\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi n}\left[\{0-0\} + \frac{1}{n}\{-2\pi - 2\pi\} + \frac{2}{n^2}\{0\}\right]$$

$$= \frac{1}{\pi n}\left[-\frac{4\pi}{n}\right] = \frac{-4}{n^2}.$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\left(\frac{n\pi}{c}\right)x\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\sin(nx)\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\sin(nx)\,dx.$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{n\pi}\left[-(2\pi x - x^2)\cos nx + 2(\pi - x)\frac{\sin nx}{n} - 2\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{n\pi}\left[-\{0-0\} + \frac{2}{n}\{0-0\} - \frac{2}{n^2}\{1-1\}\right]$$

$$= 0.$$

$$\Rightarrow f(x) = \frac{\cancel{4}\pi^3}{6} + \sum_{n=1}^{\infty} \frac{-4}{n}\cos nx + 0$$

$$= \frac{4\pi^3}{6} - 4\sum_{n=1}^{\infty} \frac{4\cos nx}{n}$$

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2] $f(x) = \frac{\pi - x}{2}$ in $(0, 2\pi)$

$(\alpha, \alpha + 2c) = (0, 2\pi)$

$\alpha = 0, \ c = \pi.$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \pi - x\, dx.$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \cancel{\frac{4\pi^2}{2}}\right]_0^{2\pi.}$$

$$= \cancel{\frac{1}{2\pi} \times \frac{\pi^2}{2}} = \cancel{\pi/4}.\ 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)}{2} \cos\left(\frac{n\pi}{\pi}\right) x\, dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx\, dx.$$

$$= 1/2\pi \int_0^{2\pi} \pi\cos nx - x\cos nx\, dx.$$

$$= \frac{1}{2\pi}\left[\frac{\pi\sin nx}{n} - \left\{\frac{x\sin(nx)}{n} - \left\{-\frac{\cos nx}{n^2}\right\}\right\}\right]_0^{2\pi.}$$

$$= \frac{1}{2\pi}\left[\frac{\pi\sin nx}{n} - \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi.}$$

$$= \frac{1}{2\pi n}\left[\pi\sin nx - x\sin(nx) + \frac{\cos nx}{n}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi n}\left[\{0-0\} - \{0-0\} + \{1-1\}\right] \cdot 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \pi\sin(nx) - x\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\left[\pi\frac{(-\cos nx)}{n} - \left\{x\frac{(-\cos nx)}{n} - \left(\frac{-\sin nx}{n^2}\right)\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\pi\cos nx}{n} + \frac{x}{n}\cos nx - \frac{\sin(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\pi\left\{\frac{1}{n} - \frac{1}{n}\right\} + \frac{1}{n}\left\{2\pi(1) - 0\right\} - \frac{1}{n^2}\left\{0-0\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right] = \frac{1}{n}$$

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{\pi}\right)x.$$

$$\boxed{\frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)}$$

3] An alternating current (AC) after passing through a rectifier has the form

$$I = \begin{cases} I_0 \sin\theta & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$$

where I_0 = max current, express I as fourier series in the interval $(0, 2\pi)$

$\cos n\pi = (-1)^n$

$(\alpha, \alpha+2c) = (0, 2\pi)$

$c = \pi$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} I_0 \sin\theta \, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} I_0 \sin\theta \, d\theta$$

$$= \frac{I_0}{\pi}\Big[-\cos\theta\Big]_0^{\pi}$$

$$= \frac{I_0}{\pi}\Big[\{-(-1)\} - \{-1\}\Big]$$

$$= \frac{I_0}{\pi}\Big[1+1\Big] = \frac{2I_0}{\pi}$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\left(\frac{n\pi}{c}\right)x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} I_0 \sin\theta\cos n\theta \, d\theta \qquad \frac{1}{2}\{\sin(A+B) - \sin(A-B)\}$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \overset{\sin B}{\sin\theta}\ \overset{\cos A}{\cos n\theta}\, d\theta \qquad \cos A\cos B - \sin A \sin B$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \{\sin(n\theta+\theta) - \sin(n\theta-\theta)\}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin(n+1)\theta - \sin(n-1)\theta \, d\theta$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)\theta}{n+1} + \frac{\cos(n-1)\theta}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\left\{\frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1}\right\} + \left\{\frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right\}\right]$$

$$= \frac{I_0}{2\pi}\left[-\frac{1}{n+1}\left\{(-1)^{n+1} - 1\right\} + \frac{1}{n-1}\left\{(-1)^{n-1} - 1\right\}\right]$$

$$= \frac{I_0}{2\pi}\left[\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\} - \left\{\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}\right]\right.$$

$$= \frac{I_0}{2\pi}\left[\frac{(n-1)-(n+1)}{n^2-1} + (-1)^n\left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\}\right.$$

$$= \frac{I_0}{2\pi}\left[-\frac{2}{n^2-1} + (-1)^n\left\{\frac{-(n-1)+n+1}{n^2-1}\right\}\right.$$

$$= \frac{I_0}{2\pi}\left[\frac{-2}{n^2-1} + (-1)^n \cdot \left\{\frac{2}{n^2-1}\right\}\right]$$

$$= \frac{I_0}{\cancel{2}\pi}\left[\frac{+\cancel{2}}{n^2-1}\right]\left[(-1)^n - 1\right]$$

$$= \frac{I_0\left\{(-1)^n - 1\right\}}{\pi(n^2-1)}$$, according to ma'am $\left\{(-1)^n + 1\right\}$

for $n \neq 1$

$$a_1 = \frac{I_0}{2\pi}\int_0^{\pi} \sin 2\theta \, d\theta$$

$$= \frac{I_0}{2\pi}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} I_0 \sin\theta \sin\left(\frac{n\pi}{\pi}\right)\theta \, d\theta.$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \sin\theta \sin(n\theta) d\theta$$

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$$= \frac{I_0}{\pi}\int_0^\pi \frac{1}{2}\{\cos(n-1)\theta - \cos(n+1)\theta\}\,d\theta.$$

$$= \frac{I_0}{2\pi}\int_0^\pi \{\cos(n-1)\theta - \cos(n+1)\theta\}\,d\theta. \quad -(2)$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(n-1)}{n-1} - \frac{\sin(n+1)\theta}{n+1}\right]_0^\pi.$$

$$b_n = \frac{I_0}{2\pi}\left[\frac{1}{n-1}\{0-0\} - \frac{1}{n+1}\{0-0\}\right]$$

$b_n = 0$ for $n \neq 1$

Put $n=1$ in (2).

$$b_1 = \frac{I_0}{2\pi}\int_0^\pi \{1 - \cos 2\theta\}\,d\theta.$$

$$= \frac{I_0}{2\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi.$$

$$= \frac{I_0}{2\pi}\left[\{\pi - 0\} - \{0-0\}\right]$$

$$= \frac{I_0}{2}.$$

$$f(x) = \frac{\cancel{2}I_0}{\pi} + \sum_{n=2}^{\infty} \frac{I_0\{(-1)^n+1\}}{\pi(n^2-1)}\cos n\theta + \frac{I_0}{2}\sin\theta.$$

Even and Odd function

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ and an odd function. If $f(-x) = -f(x)$ in the interval $(-c, c)$

Also $f(x) = \begin{cases} \phi(x) & \text{in } -c < x < 0 \\ \psi(x) & \text{in } 0 < x < c \end{cases}$

$\phi(-x) = \psi(x)$; $f(x)$ is even.
$\phi(-x) = -\psi(x)$; $f(x)$ is odd

If $f(x)$ is an even function then :-

$b_n = 0$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi}{c}\right)x\,dx.$$

If $f(x)$ is an odd function :-

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi}{c}\right)x\,dx.$$

1] Obtain the fourier series for the function

$$f(x) \begin{cases} 1+\frac{2x}{\pi}, & -\pi < x < 0 \\ 1-\frac{2x}{\pi}, & 0 < x < \pi. \end{cases}$$

$(-\pi, \pi)$ in $(-c, c)$

$c = \pi$.

$$\phi(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = \psi(x)$$

$\therefore f(x)$ is an even function

$b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right)dx.$$

$$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\left(\frac{x^2}{2}\right)\right]_0^{\pi}.$$

$$= \frac{2}{\pi}\left[(\pi - 0) - \frac{2}{\pi}\left\{\frac{\pi^2}{2}\right\}\right]$$

$$= 0.$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi}{c}\right)x\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi}\cos nx\,dx - \frac{2}{\pi}\int_0^{\pi}\frac{2x}{\pi}\cos nx\,dx.$$

$$= \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi^2}\left[\frac{x\sin(nx)}{n} - \frac{(-\cos(nx))}{n^2}\right]_0^{\pi}.$$

$$= 0 - \frac{4}{\pi^2}\left[\{\pi(0) - 0\} - \left\{-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right\}\right].$$

$$= 0 - \frac{4}{\pi^2}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right\}$$

$$= 0 - \frac{4\{(-1)^n - 1\}}{\pi^2 n^2} = \frac{4\{1 - (-1)^n\}}{\pi^2 n^2}.$$

2] $f(x) = \begin{cases} 1 + 2x & \text{in } -3 < x < 0 \\ 1 - 2x & \text{in } 0 < x < 3. \end{cases}$

Soln: $(-3, 3)$ in $(-c, c)$.

$\phi(-x) = 1 + 2(-x) = 1 - 2x = \psi(x).$

$\therefore f(x)$ is an even function.

$b_n = 0$, $a_0 = \frac{2}{c}\int_0^c f(x)\,dx$

$= \frac{2}{3}\int_0^3 1-2x\,dx.$

$= \frac{2}{3}\left[x - \frac{2x^2}{2}\right]_0^3$

$= \frac{2}{3}\left[x - x^2\right]_0^3$

$= \frac{2}{3}\left[\{3-0\} - \{3^2-0\}\right]$

$= \frac{2}{3}\times(3-9) = -4$

$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi}{c}\right)x\,dx.$

$= \frac{2}{3}\int_0^3 (1-2x)\cos\frac{n\pi}{3}x\,dx.$

$= \frac{2}{3}\int_0^3 \cos\frac{n\pi x}{3} - 2x\cos\frac{n\pi}{3}x\,dx$

$= \frac{2}{3}\int_0^3 \cos\frac{n\pi}{3}x\,dx - \frac{2\times 2}{3}\int_0^3 x\cos\frac{n\pi}{3}x\,dx$

$= \frac{2}{3}\left[\frac{\sin(n\pi/3)x}{(n\pi/3)}\right]_0^3 - \frac{4}{3}\left[\frac{x\sin\left(\frac{n\pi}{3}\right)x}{(n\pi/3)^2} + \frac{\cos\left(\frac{n\pi}{3}\right)x}{(n\pi/3)^2}\right]_0^3$

$= \frac{2}{3}\left[0-0\right] - \frac{4}{3}\left[\{0-0\} + \left\{\frac{(-1)^n}{(n\pi/3)^2} - \frac{1}{(n\pi/3)^2}\right\}\right]$

$= -\frac{4}{3}\left[\left\{\frac{9(-1)^n}{n\pi} - \frac{9}{(n\pi)^2}\right\}\right] = -4\left[\frac{3(-1)^n}{n^2\pi^2} - \frac{3}{n^2\pi^2}\right]$

$= \frac{-12}{n^2\pi^2}\left[(-1)^n - 1\right]$

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$= \frac{12}{n^2\pi^2}\left[\cancel{\cos\pi}\,1 - (-1)^n\right]$

Q) find $f(x) = x$ in $(-l, l)$

$f(x)$ is odd function

$a_0 = 0$

$a_n = 0$

$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi}{c}\right)x\,dx.$

$= \frac{2}{l}\int_0^l x\sin\left(\frac{n\pi}{l}\right)x\,dx.$

$= \frac{2}{l}\left[\frac{x\left(-\cos\left(\frac{n\pi}{l}\right)x\right)}{n\pi/l} + \frac{\sin\left(\frac{n\pi}{l}\right)x}{(n\pi/l)^2}\right]_0^l$

$= \frac{2}{l}\left[\left\{\frac{-l \times (-1)^n}{n\pi/l} - (-0)\right\} - \{0 - 0\}\right]$

$= \frac{2}{\cancel{l}}\left[\frac{-(-1)^n\, l}{n\pi}\right] \times \cancel{l}\, l = \frac{-2}{n\pi}\{(-1)^n\}\, l$

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{c}\right)x.$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{n\pi}\sin\left(\frac{n\pi}{l}\right)x}$$

Half-range sine and cosine series.

Many a times it is required to obtain fourier expansion of $f(x)$ for the range $(0, c)$. As it is unmaterial of whatever the function may be outside the range $(0,c)$ we extend the function to cover the range $(-c, c)$ so that the new function may be even or odd.

Half range sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{c}\right)x$$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi}{c}\right)x \, dx$$

Half range of cos series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{c}\right)x$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi}{c}\right)x \, dx$$

1] Construct half range cosine series for $f(x) = x(l-x)$ in $(0, l)$

$$f(x) = x(l-x). \qquad c = l.$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x(l-x)\,dx = \frac{2}{l}\int_0^l lx - x^2\,dx$$

$$= \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{2}{l}\left[\left\{\frac{l^3}{2} - \frac{l^3}{3}\right\}\right]$$

$$= \frac{2}{l}\left[\frac{3l^3 - 2l^3}{6}\right]$$

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$$= \frac{2}{l} \times \frac{l^3}{6} = \frac{l^2}{3}$$

$$a_n = \frac{2}{c}\int_0^{c} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\int_0^{l} x(l-x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\int_0^{l} xl\cos\left(\frac{n\pi}{l}\right)x - x^2\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\left[l\left\{ \frac{x\sin(n\pi/l)x}{(n\pi/l)} + \frac{\cos(n\pi/l)x}{(n\pi/l)^2}\right\} - \left\{ \frac{x^2\sin(n\pi/l)x}{(n\pi/l)} + \frac{2x\cos(n\pi/l)x}{(n\pi/l)^2} + \frac{2\sin(n\pi/l)x}{(n\pi/l)^3}\right\}\right]_0^{l}$$

$$= \frac{2}{l}\left[\left(l\left\{\frac{l\sin(n\pi)}{(n\pi/l)} + \frac{\cos n\pi}{(n\pi/l)^2}\right\} - \left\{\frac{l^2\sin n\pi}{(n\pi/l)} + \frac{2l\cos n\pi}{(n\pi/l)^2} + \frac{2\sin n\pi}{(n\pi/l)^3}\right\}\right) - \left(l\left\{0 + \frac{1}{(n\pi/l)^2}\right\} - \left\{0 + 0 + 0\right\}\right)\right]$$

$$= \frac{2}{l}\left[\left(l\left\{0 + \frac{l^2}{n^2\pi^2}(-1)^n\right\} - \left\{0 + \frac{2l\times l^2}{n^2\pi^2}(-1)^n + 0\right\}\right) - \left(\frac{l\times l^2}{n^2\pi^2}\right)\right]$$

$$= \frac{2}{l}\left[\frac{l^3}{n^2\pi^2}(-1)^n - \frac{2l^3}{n^2\pi^2}(-1)^n - \frac{l^3}{n^2\pi^2}\right] = \frac{2}{l}\left[\frac{-l^3(-1)^n}{n^2\pi^2} - \frac{l^3}{n^2\pi^2}\right]$$

$$= \frac{-2l^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

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Practical Harmonic Analysis

Sometimes only tabulated value of the function $y = f(x)$ and the corresponding values of x are given.

Hence the fourier co-efficients can not be evaluated by integration, Hence the process of Harmonic Analysis is used to find the constant terms and first few terms of sine and cosine.

$y = f(x)$ in (a, b)

Fourier coefficient over the intervals $(\alpha, \alpha + 2c)$ are given by.

$$a_0 = 2 \times (\text{Mean of } f(x)).$$

$$a_n = 2 \times \left(\text{Mean of } f(x) \cos\left(\frac{n\pi}{c}\right)x\right)$$

$$b_n = 2 \times \left(\text{Mean of } f(x) \sin\left(\frac{n\pi}{c}\right)x\right)$$

1] For the periodic function $f(x)$ of a period six specified by the following table over the interval $(0,6)$. Find the coefficients a_0, a_1, b_1

x	0	1	2	3	4	5	6
$f(x)$	9	18	24	28	26	20	9.

$\alpha = 0$ $\quad c = 3$

$a_0 = 2 \times (\text{Mean of } f(x) \text{ in } (0,6))$

$a_1 = 2 \times \left(\text{Mean of } f(x) \cos \frac{\pi}{3} x \text{ in } (0,6)\right)$

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$b_1 = 2 \times$ (Mean of $f(x) \sin \pi/_3 x$ in $[0,6]$).

Mean of $f(x)$ $= \frac{125}{6} = 20.833$

$a_0 = 2 \times 20.833$
$= 41.67$.

x	$f(x)$	$\cos(\pi/_3)x$	$f(x)\cos\pi/_3$	$\sin(\pi/_3)x$	$f(x)\sin\pi/_3$
0	9	1	9	0 ~~0.0183~~	0
1	18	$1/2$	9	0.86602	15.58
2	24	$-1/2$	-12	0.86602	20.78
3	28	-1	-28	0	0
4	26	$-1/2$	-13	-0.86602	-22.51
5	20	$1/2$	10	-0.86602	-17.32
			-25		$-2\sqrt{3}$

$a_1 = 2 \times \left(\frac{-25}{6}\right)$

$= -8.33$

$b_1 = 2 \times \left(\frac{-2\sqrt{3}}{6}\right) = -1.1547$

2] The following table gives the value of periodic current I measured at 12 equidistant values of θ covering a period of 2π.

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$
I	0	2.3	5.5	8.9	10.8	11.4	9.9	4.8

θ	$4\pi/3$	$3\pi/2$	$5\pi/3$	2π
I	0	0	0	0

Find the Fourier expansion of I for Ist Harmonic.

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$a_1 = 4.098$ $b_1 = 9.5$

$a_0 = 2 \times (\text{Mean of } f(\theta) \text{ in } (0, 2\pi))$

$a_1 = 2 \times (\text{Mean of } f(\theta) \cos nx$

$b_1 = 2 \times (\text{Mean of } f(\theta) \sin nx)$

θ	$f(\theta)$	$\cos\theta$	$f(\theta)\cos\theta$	$\sin\theta$	$f(\theta)\sin\theta$
0	0	1	0	0	
$\pi/6$	2.3	$\sqrt{3}/2$	1.9918		
$\pi/3$	5.5	$1/2$	2.75		
$\pi/2$	8.9	0	0		
$2\pi/3$	10.8	$-1/2$	-5.4		
$5\pi/6$	11.4	$-\sqrt{3}/2$	-9.873		
π	9.9	-1	-9.9	0	
$7\pi/6$	4.8	$-\sqrt{3}/2$	-4.157		
$4\pi/3$	0	$-1/2$	0	-	0
$3\pi/2$	0	0	0	-	0
$5\pi/3$	0	$1/2$	0	-	0
2π	0	1	0	-	0
	53.6		-24.59		27.46
Mean	4.467				

3] Obtain first 3 co-efficients in fourier cosine series for y where y is given by the following table. find a_0, a_1, a_2, a_3

x :	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol^n. $(0,6) = (0,c)$

$\therefore c = 6$

$a_0 = 2(\text{Mean of } f(x) \text{ in } (0,6))$

$a_1 = 2(\text{Mean of } f(x) \cos(\frac{\pi}{6})x \text{ in } (0,6))$

$a_2 = 2 \times (\text{Mean of } f(x) \cos(\frac{\pi}{3})x \text{ in } (0,6))$

x	$f(x)=y$	$\cos(\frac{\pi}{6})x$	$f(x)\cos(\frac{\pi}{6})x$	$\cos(\frac{\pi}{3})x$	$f(x)\cos(\frac{2\pi}{3})x$
0	4	1	4	1	4
1	8	0.866	6.928	0.5	4
2	15	0.5	7.5	-0.5	-7.5
3	7	0	0	-1	-7
4	6	-0.5	-3	-0.5	-3
5	2	-0.866	-1.732	0.5	1
Summation =	42		13.696		-8.5
Mean =	7		= 2.2827		-1.4167

$a_0 = 2 \times 7 = 14$

$a_1 = 2 \times 2.2827 = 4.565$

$a_2 = -1.4167 \times 2 = -2.834$

4) The following values of y gives the displacement inches of a certain machine part for the rotation x of the fly wheel. expand y as fourier series.

x :	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$
y :	0	9.2	14.2	14.8	17.3	11.7

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FOURIER TRANSFORMS.

If $f(x)$ is a function, then define the interval $-\infty$ to $+\infty$ then the fourier transform is denoted by $F(s)$ & is defined as

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx.$$

for $F(s)$, the inverse fourier transform is given by,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} F(s)\,ds.$$

1] Find the fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & \text{otherwise} \end{cases}$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx. \qquad -a \le x \le a$$

$$= \int_{-\infty}^{-a} e^{isx} f(x)\,dx + \int_{-a}^{a} e^{isx} f(x)\,dx + \int_{a}^{\infty} e^{isx} f(x)\,dx$$

$$= 0 + \int_{-a}^{a} e^{isx}(a^2 - x^2)\,dx + 0$$

$$= \left[(a^2 - x^2)\frac{e^{isx}}{is} - \frac{(-2x)e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3} \right]_{-a}^{a}$$

$$= \left[\{0 - 0\} + \left\{ \frac{2ae^{isa}}{i^2 s^2} - \frac{2(-a)e^{-isa}}{i^2 s^2} \right\} - \left\{ \frac{2e^{isa}}{i^3 s^3} - \frac{2e^{-isa}}{i^3 s^3} \right\} \right]$$

$$= \left\{ \frac{2a}{i^2 s^2}\{e^{isa} + e^{-isa}\} - \frac{2}{i^3 s^3}\{e^{isa} - e^{-isa}\} \right.$$

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$e^{i\theta} = \cos\theta + i\sin\theta$.

$$= -\frac{2a}{s^2}\{2\cos(sa)\} - \frac{2i}{s^3}\{2i\sin(sa)\}.$$

$$= -\frac{4a}{s^2}\cos(sa) + \frac{4}{s^3}\sin(sa)$$

2] $f(x) = \begin{cases} x, & |x| \le \alpha \\ 0, & |x| > \alpha. \end{cases}$ where α is constant.

$\rightarrow -\alpha < x < \alpha$. $\rightarrow \alpha < x < -\alpha$, $\alpha > x > \alpha$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-a}^{a} x e^{isx}\,dx.$$

$$= \left[\frac{xe^{isx}}{is} - \frac{e^{isx}}{(is)^2}\right]_{-\alpha}^{\alpha}$$

$$= \left[\left\{\frac{\alpha e^{isa}}{is} + \frac{a e^{-isa}}{is}\right\} - \left\{\frac{e^{is\alpha}}{(is)^2} - \frac{e^{-isa}}{(is)^2}\right\}\right]$$

$$= \frac{\alpha}{is}\left\{e^{is\alpha} + e^{-is\alpha}\right\} - \frac{1}{i^2 s^2}\left\{e^{is\alpha} - e^{-is\alpha}\right\}$$

$-\frac{i\alpha}{s}[2\cos(s\alpha)]$

$+\frac{1}{s^2}[2i\sin(s\alpha)]$

3] Find the F.T of $f(x) = e^{-a|x|}$; $a > 0$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx.$$

$$= \int_{-\infty}^{0} e^{isx} e^{-a(-x)}\,dx + \int_{0}^{\infty} e^{isx} e^{-ax}\,dx$$

$$= \int_{-\infty}^{0} e^{isx+ax}\,dx + \int_{0}^{\infty} e^{isx-ax}\,dx.$$

$$= \int_{-\infty}^{0} e^{(is+a)x}\,dx + \int_{0}^{\infty} e^{-(a-is)x}\,dx.$$

$$= \left[\frac{e^{(is+a)x}}{(is+a)}\right]_{-\infty}^{0} - \left[\frac{e^{-(a-is)}}{a-is}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{is+a} - 0\right] - \left[0 - \frac{1}{a-is}\right]$$

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$$= \frac{1}{(a+is)} + \frac{1}{(a-is)}$$

$$= \frac{a-is+(a+is)}{a^2-i^2s^2}$$

$$= \frac{2a}{a^2+s^2} = \frac{2a}{a^2+s^2}$$

4] $f(x) = e^{-|x|}$, Find F.T.

$$F[f(x)] = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-\infty}^{0} e^{isx} e^{-(-x)}\,dx + \int_{0}^{\infty} e^{isx} e^{-x}\,dx$$

$$= \int_{-\infty}^{0} e^{isx+x}\,dx + \int_{0}^{\infty} e^{isx-x}\,dx$$

$$= \int_{-\infty}^{0} e^{(is+1)x}\,dx + \int_{0}^{\infty} e^{-(1-is)x}\,dx$$

$$= \frac{1}{is+1} + \frac{1}{(1-is)} = \frac{(is-1)-(is+1)}{(is)^2-1^2}$$

$$= \frac{2}{1+s^2}$$

Fourier Sine and Cosine Transform

- Fourier Cosine transform of $f(x)$ is given by,

$$F_c\{f(x)\} = F_c(s) = \int_0^{\infty} f(x).\cos(sx)\,dx$$

Inverse cosine Transform, $f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s)\cos(sx)\,ds$

- Fourier sine transform of f(x) is given by,

$$F_s\{f(x)\} = F_s(s) = \int_0^\infty f(x)\sin(sx)\,dx$$

Inverse sine Transform, $f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin(sx)\,dx$.

1] Find Fourier Cosine Transform $f(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & x>4. \end{cases}$

$$F_c\{f(x)\} = F_c(s) = \int_0^\infty f(x)\cos(sx)\,dx$$

$$= \int_0^1 4x\cos(sx)\,dx + \int_1^4 (4-x)\cos(sx)\,dx + 0.$$

$$= 4\int_0^1 x\cos(sx)\,dx + 4\int_1^4 \cos(sx)\,dx - \int_1^4 x\cos(sx)\,dx.$$

$$= 4\left[x\frac{\sin(sx)}{s} - \frac{(-\cos(sx))}{s^2}\right]_0^1 + 4\left[\frac{\sin(sx)}{s}\right]_1^4$$

$$- \left[\frac{x\sin(s)}{s} + \frac{\cos(sx)}{s^2}\right]_1^4$$

$$= 4\left[\left\{\frac{\sin(s)}{s} - 0\right\} + \left\{\frac{\cos(s)}{s^2} - \frac{1}{s^2}\right\}\right] + 4\left[\frac{\sin(4s)}{s} - \frac{\sin s}{s}\right]$$

$$= -\left[\left\{\frac{4\sin(s)}{s} - \frac{\sin(s)}{s}\right\} + \left\{\frac{\cos(4s)}{s^2} - \frac{\cos(s)}{s^2}\right\}\right]$$

$$= \cancel{\frac{4}{s}\sin(s)} + \frac{4\cos(s)}{s^2} - \frac{4}{s^2} + \frac{4\sin(4s)}{s} - \cancel{\frac{4}{s}\sin(s)} -$$

$$\frac{3}{s}\sin(s) - \frac{\cos(4s)}{s^2} + \frac{\cos(s)}{s^2} =$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$= 5\frac{\cos(s)}{s^2} - \frac{3}{s}\sin(s) + \frac{4\sin(4s)}{s} - \frac{4\cos(4s)}{s^2} - \frac{4}{s^2}$$

2] Find the f.C.T of $f(x) = \frac{e^{-ax}}{x}$; $a > 0$; $x \neq 0$

3] $\int_0^\infty f(\theta)\cos\alpha\theta\,d\theta = \begin{cases} 1-\alpha, & 0<\alpha<1 \\ 0, & \alpha>1 \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$.

$$\int_0^\infty f(\theta)\cos\alpha\theta\,d\theta = F_c(\alpha) = \phi(\alpha).$$

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha\theta\,d\alpha.$$

$$= \frac{2}{\pi}\int_0^1 (1-\alpha)\cos(\alpha\theta)\,d\alpha.$$

$$= \frac{2}{\pi}\left[(1-\alpha)\frac{\sin\alpha\theta}{\theta} - (-1)\left(\frac{-\cos\alpha\theta}{\theta^2}\right)\right]_0^1$$

$$= \frac{2}{\theta\pi}\left[(1-\alpha)\sin\alpha\theta - \frac{1}{\theta}\{\cos\alpha\theta\}\right]_0^1$$

$$= \frac{2}{\pi\theta}\left[\{0-0\} - \frac{1}{\theta}\{\cos\theta - 1\}\right]$$

$$= \frac{-2}{\pi\theta}\left[\frac{1}{\theta}\{\cos\theta - 1\}\right] = \frac{2}{\pi\theta^2}(\cos\theta - 1)$$

$$= \frac{2}{\pi\theta^2}\cdot\{2\sin^2\theta/2\}$$

$$= \frac{4}{\pi\theta^2}\sin^2\theta/2.$$

put $\theta = 2t$. $d\theta = 2dt$

$$\therefore f(t) = \frac{1}{\pi t^2}\sin^2 t.$$

$$= \frac{1}{\pi}\left(\frac{\sin^2 t}{t^2}\right)$$

$$F_c(\alpha) = \int_0^\infty f(\theta)\cos\alpha\theta \, d\theta.$$

$$F_c(\alpha) = \frac{2}{\pi}\int_0^\infty \frac{\sin^2 t}{t^2}\cos 2\alpha t \, dt.$$

For $\alpha = 0$.

$F_c(\alpha) = 1-\alpha$

$F_c(0) = 1.$

$$1 \times \frac{\pi}{2} = \int_0^\infty \frac{\sin^2 t}{t^2} \cdot 1 \cdot dt.$$

$$\therefore \int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \pi/2.$$

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4] Find the function whose fourier cosine Transform is,

$$F(\alpha) = \begin{cases} a - \alpha/2 & 0 \le \alpha \le 2a \\ 0 & \alpha > 2a. \end{cases}$$

Soln. $F_c(s) = \int_0^{\infty} f(x) \cos sx \, dx.$

Inverse FCT,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos(sx) \, ds.$$

$$f(x) = \frac{2}{\pi} \left[\int_0^{2a} \left(a - \frac{\alpha}{2}\right) \cos \alpha x \, d\alpha + \int_{2a}^{\infty} 0 \, d\alpha \right]$$

$$= \frac{2}{\pi} \left[\left(a - \frac{\alpha}{2}\right) \frac{\sin \alpha x}{x} - \left(\frac{-1}{2}\right) \frac{(-\cos \alpha x)}{x^2} \right]_0^{2a}$$

$$= \frac{2}{\pi} \left[\frac{1}{x} \{0 - 0\} - \frac{1}{2x^2} \{\cos 2ax - 1\} \right]$$

$$f(x) = \frac{1}{\pi x^2} \{1 - \cos 2ax\}$$

5] Solve integral eq,

$$\int_0^{\infty} f(x) \sin \alpha x \, dx = \begin{cases} 10 & 0 \le \alpha \le 1 \\ 20 & 1 \le \alpha \le 2 \\ 0 & \alpha > 2. \end{cases}$$

Soln: $\int_0^{\infty} f(x) \sin \alpha x = F_s(\alpha) = \phi(\alpha)$

Inverse F.S.T, $f(x) = \frac{2}{\pi} \int_0^{\infty} f(\alpha) \sin \alpha x \, d\alpha$

$e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

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$$f(x) = \frac{2}{\pi}\left[\int_0^1 10\sin\alpha x\,d\alpha + \int_1^2 20\sin\alpha x\,d\alpha + 0\right]$$

$$= \frac{2}{\pi}\left[10\left\{-\frac{\cos\alpha x}{x}\right]_0^1 + 20\left[-\frac{\cos\alpha x}{x}\right]_1^2\right]$$

$$= \frac{2}{\pi}\left[-\frac{10}{x}\{\cos(x)-1\} - \frac{20}{x}\{\cos 2x - \cos x\}\right]$$

$$= \frac{2}{x\pi}\left[10 + 10\cos x - 20\cos 2x\right].$$

6] Find the F.S.T of $\frac{e^{-ax}}{x}$.

$$F_s(s) = \int_0^\infty f(x)\sin sx\,dx.$$

$$= \int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx.$$

Diff w.r.t 's' on both sides

$$\frac{d}{ds}F_s(s) = \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\sin sx\right)dx.$$

$$= \int_0^\infty \frac{e^{-ax}}{\cancel{x}}(\cos sx)\,\cancel{x}\,dx.$$

$$= \int_0^\infty e^{-ax}\cos(sx)\,dx.$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}\left\{-a\cos sx + s\sin sx\right\}\right]_0^\infty$$

$$= \left[0 - \left\{\frac{1}{a^2+s^2}\{-a\}\right\}\right] = \frac{a}{a^2+s^2}$$

$$F_s(s) = \int_0 \frac{a}{a^2+s^2}\,ds.$$

$$F_s(s) = \frac{a}{a}\tan^{-1}(s/a) + c$$