Probability

The study of probability provides a mathematical framework for such assertions and is essential in every decision making process.

Principle of counting: If an event can happen in 'n_1' ways and thereafter for each of these events a second event can happen in 'n_2' ways and for each of these first and second events a third event can happen for n_3 ways and so on, then the number of ways these 'm' event can happen is given by the product $n_1 \cdot n_2 \cdot n_3 \cdot \ldots n_m$.

Permutations: A permutation of a number of objects is their arrangement in some definite order.

(8) (1a)

The number of permutation of n different thing taken over r at time is

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Combinations: The number of combinations of 'n' different objects taken 'r' at a time is

$$^{n}C_{r} = \frac{^{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!}$$

Basic Definitions:

Exhaustive events: A sets of events is said to be exhaustive, if it includes all the possible events.

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Mutually Exclusive events:

If the occurance of one of the events precludes the occurance of all other, then such a set of events is said to be mutually exclusive.

Equally likely Events:

If one of the events cannot be expected to happen in ~~par~~ preference to another then such events are said to be equally likely.

probability: If n are exhaustive, mutually exclusive and equally likely cases of which 'm' are favourable to an event, then the probability of the happening of event is "m/n"

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Note: ① If E is any event in a sample space 'S', then $0 \leq P(E) \leq 1$

② To the entire sample space S there corresponds $P(S) = 1$

③ If A & B are mutualy exclusive events then $P(A \cup B) = P(A) + P(B)$

④ If A & B are any events in a sample space S, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

⑤ If E is an event and E^c is complement of E then $P(E) = 1 - P(E^c)$

⑥ If A & B are events in a sample space S and $P(A) \neq 0$, $P(B) \neq 0$ then
$$P(A \cap B) = P(A) P(B|A) = P(B) P(A|B)$$

Random Variable's

A random variable X (also called stochastic variable or variate) is a function whose values are real numbers and depend on 'chance' : ~~more precisely, is a function X which has the following properties~~

Discrete Random variables and Discrete Distribution :

A random variable X [X is outcome of some experiment] and the corresponding distribution are said to be distribution.

If the probability that X takes the values x_i is P_i then $P(X = x_i) = P_i, \; i = 1, 2 \ldots$

(i) $P(x_i) > 0 \;\; \forall i$

(ii) $\sum_i P(x_i) = 1$

(3a)

Let $x_1, x_2 \cdots$ be the value for which X has positive probability and let $P_1, P_2 \cdots$ be the corresponding probabilities. Then

(i) $P(x_i) \geq 0$ (ii) $\sum P(x_i) = 1$. and

$$P(a < X \leq b) = \sum_{a < x_i \leq b} P(x_i)$$

Continuous Random Variables and Continuous Distributions;

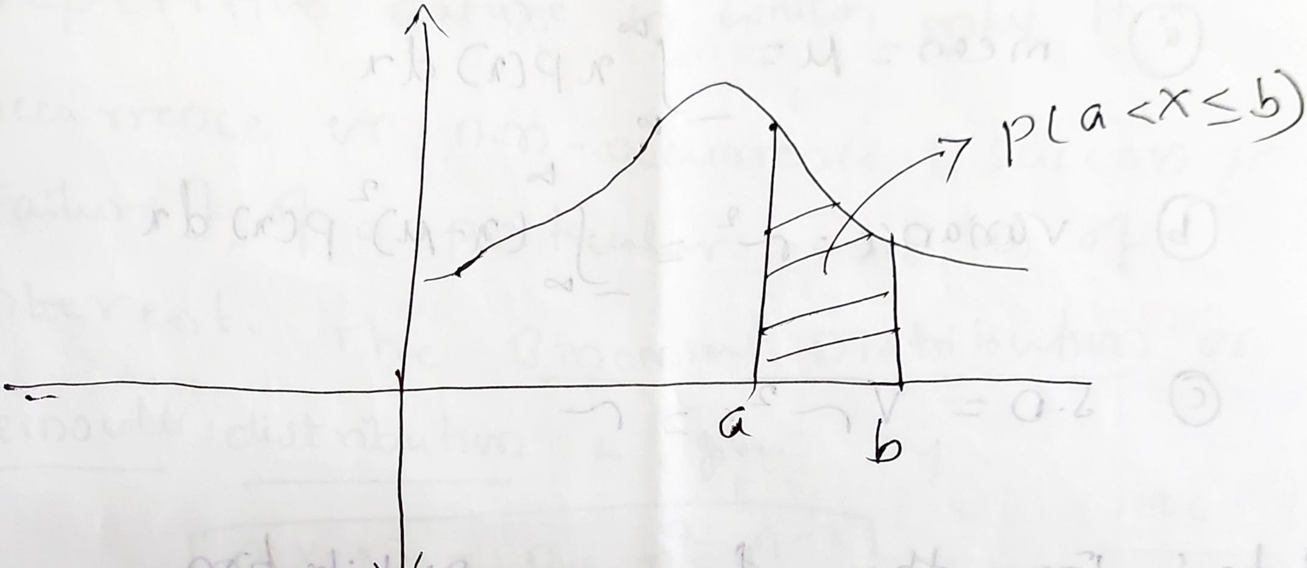
A random variable X and the corresponding distributions are said to be of continuous type, if the distribution function $F(x) = P(X \leq x)$ of X can be represented as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

The function $f(x)$ is called probability density function

And $P(a < X \le b) = \int_a^b f(x)\,dx$

$= F(b) - F(a)$



Mean and variance of Distribution:

① Discrete Distribution:

ⓐ Mean $= \mu = \sum x_i P(x_i)$

ⓑ Variance $= \sigma^2 = \sum x_i^2 P(x_i) - \mu^2$

or $= \sum (x_i - \mu)^2 P(x_i)$

(c) Standard Deviation $= \sqrt{\sigma^2} = \sigma$

2. Continuous Distribution :

(a) mean $= \mu = \int_{-\infty}^{\infty} x\, p(x)\, dx$

(b) variance $= \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, p(x)\, dx$

(c) S.D $= \sqrt{\sigma^2} = \sigma$

Note: For the frequency Distribution

(i) $N = \Sigma f_i$

(ii) mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

~~Bin~~ Binomial Distribution:

It is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence [success or failure], of a particular event is of interest. The Binomial Distribution or Bernoulli distribution is given by

$$\boxed{P(X=x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}}$$

where 'p' is the probability of success in one trial and q is failure.

If 'n' independent trials constitute one experiment and this experiment be repeated N times then the Binomial frequency Distribution is

$$N\, {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

mean of Binomial Distribution

$\boxed{\bar{x} = \mu = np}$ or $\boxed{\bar{x} = np}$

variance of Binomial Distribution

$\boxed{\sigma^2 = npq}$

$\boxed{S.D = \sqrt{\sigma^2} = \sigma}$

Poisson Distribution:

It is a distribution related to the probabilities of events which have a large number of independent opportunities for occurence.

It is the limiting case of Binomial Distribution as $p \to 0$ and $n \to \infty$

The discrete distribution with the probability function

$$P(X=x) = \frac{m^x e^{-m}}{x!} \quad \text{or}$$

$$P(X=x) = \frac{\mu^x e^{-\mu}}{x!}$$

is called Poisson Distribution.

The mean and S.D. of poisson Distribution

$$\boxed{\text{Mean} = \mu = np} = m$$

$$\boxed{\text{variance} = \sigma^2 = \mu = m}$$

$$\boxed{S.D = \sqrt{\sigma^2} = \sigma}$$

Poisson frequency Distribution

$$= N P(x) = N \frac{m^x e^{-m}}{x!}$$

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Note:

① prove that the probability of x success in n- Bernoulli trails is $n_{C_x} p^x q^{n-x}$ ②

proof: Let x be the number of success.

given: In an experiment of n- Bernoulli

$\Rightarrow (n-x)$ are failures. can be

Here x success & $(n-x)$ failures can be obtained in n_{C_x} ways.

In each of these ways, the probability of x success and $(n-x)$ failures is

$= P(x \text{ success}) \cdot P((n-x) \text{ failures})$

$= \underbrace{p \cdots\cdots p}_{x \text{ times}} \times \underbrace{q\,q \cdots q}_{(n-x) \text{ times}}$

$= p^x q^{n-x}$

∴ The probability of x success in n- Bernoulli trails is

$$P(X=x) = n_{C_x} p^x q^{n-x}$$

(2) prove that mean and S.D. of Bernoulli's Distribution [or Binomial Distribution] are np & $\sqrt{npq}$

proof: Let μ be the mean of Binomial Distribution

w.k.t

$$\mu = \sum x_i P(x_i)$$

$$= \sum_{i=1} x_i \, {}^nC_{x_i} \, p^{x_i} \, q^{n-x_i}$$

since $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, ---

$$\mu = (0)\, {}^nC_0 \, p^0 q^{n-0} + (1)\, {}^nC_1 \, p^1 q^{n-1} + (2)\, {}^nC_2 \, p^2 q^{n-2} + -$$

$$-- + n \, {}^nC_n \, p^n q^{n-n}$$

$$= np[q^{n-1} + pq^{n-2} + --- + p^{n-1}]$$

$$\mu = np[p+q]^{n-1}$$

since $\boxed{p+q=1}$

$$\therefore \boxed{\mu = np}$$

(7c)

wkt

$$var = \sum x_i^2 P(x_i) - \mu^2$$

$$var = \sum_{i=1}^{n} x_i^2 P(x_i) - (np)^2$$

$$= \sum_{i=1}^{n} [x_i(x_i-1) + x_i] P(x_i) - \mu^2$$

$$= \sum_{i=1}^{n} x_i(x_i-1) P(x_i) + \sum_{i=1}^{n} x_i P(x_i) - \mu^2$$

$$= \sum_{i=1}^{n} x_i(x_i-1)\, {}^nC_{x_i}\, p^{x_i} q^{n-x_i} + \mu - \mu^2$$

$$= \sum_{i=1}^{n} x_i(x_i-1) \frac{n!}{x_i!\,(n-x_i)!} p^{x_i} q^{n-x_i} + \mu - \mu^2$$

$$= \sum_{i=1}^{n} \frac{n(n-1)(n-2)\cdots}{(x_i-2)!\,(n-x_i)!} p^{x_i} q^{n-x_i} + \mu - \mu^2$$

$$= n(n-1)p^2 \sum_{i=1}^{n} \frac{(n-2)!}{(x_i-2)!\,(n-x_i)!} p^{x_i-2} q^{n-x_i} + \mu - \mu^2$$

$$= n(n-1)p^2 (p+q)^{n-2} + np - (np)^2$$

$$= n(n-1)p^2 + np - (np)^2 = n^2p^2 - np^2 + np - (np)^2$$

$$\therefore \boxed{var = np(1-p) = npq} \qquad \boxed{SD = \sqrt{npq}}$$

(3) prove that $p(x) = \frac{m^x e^{-m}}{x!}$

Consider the Binomial Distribution

$$p(x) = {}^nC_x \, p^x q^{n-x} \longrightarrow (1)$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!} p^x q^{n-x}$$

$$p(x) = \frac{n \cdot n^{x-1} (1-1/n)(1-2/n)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!} p^x q^{n-x} \longrightarrow (2)$$

wkt $np = m$ $\quad q^n = (1-p)^n$

$$\boxed{p = m/n}$$

$$\therefore q^n = \left[1 - \frac{m}{n}\right]^n = \left[\left[1 - \frac{m}{n}\right]^{-n/m}\right]^{-m}$$

wkt $\lim_{k \to 0} (1+k)^{1/k} = e$

$\therefore$ as $n \to \infty$ $\left[1-(m/n)\right]^{-1/(m/n)} \longrightarrow e$

$\therefore$ as $n \to \infty$ $q^n \longrightarrow e^{-m}$

from (2)

$$\therefore P(x) = \frac{(np)^x (1 - 1/n)(1 - 2/n) \cdots \left(1 - \frac{(x-1)}{n}\right)}{x!} q^{n-x}$$

as $n \to \infty$

$$\boxed{P(x) = \frac{m^x e^{-m}}{x!}}$$

(4) P.T mean and S.D of Poisson Distribution are m.

Proof: For D.R.V mean is given by

$$\text{mean} = \mu = \sum x_i P(x_i)$$

$$= \sum_i x_i \frac{m^{x_i} e^{-m}}{x_i!}$$

$$= m e^{-m} \sum_i \frac{m^{x_i - 1}}{(x_i - 1)!}$$

$$= m e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right]$$

$$= m e^{-m}[e^m] = m$$

$$\therefore \boxed{\text{mean} = m = \mu}$$

$$\text{variance} = V = \sum x^2 p(x) - \mu^2$$

$$= \sum [(x-1)x + x]p - \mu^2$$

$$= \sum x(x-1)p + \sum xp - \mu^2$$

$$= \sum x(x-1)\frac{m^x e^{-m}}{x!} + \sum xp - \mu^2$$

$$= \sum \frac{m^x e^{-m}}{(x-2)!} + \sum xp - \mu^2$$

$$= m^2 e^{-m} \sum \frac{m^{(x-2)}}{(x-2)!} + \sum xp - \mu^2$$

$$= m^2 e^{-m}[e^m] + m - m^2 = m$$

$$\therefore \boxed{V = \sigma^2 = m}$$

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Continuous probability Distribution

If x belonging to the range of a continuous random variable X, we assign a real number $f(x)$ satisfying the conditions

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x) = 1$

$f(x)$ is called continuous probability function or probability density function.

If (a,b) is a subinterval of the space X (range) then

$$P(a \leq x \leq b) = \int_a^b f(x)\, dx$$

Cumulative Distribution Function:

If X is a continuous random variable with probability density function $f(x)$ then

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

Note: If r is any real number then

$$p(X \geq r) = \int_{r}^{\infty} f(x)\,dx$$

$$p(X < r) = 1 - \int_{r}^{\infty} f(x)\,dx$$

Mean and variance:

If X is continuous random variable with prob. density function $f(x)$. Then mean and variance of cont. Dist.

$$\text{mean} = \mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$\text{var} = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

(100)

Exponential Distribution:

The continuous probability distribution having the prob. density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \alpha > 0.$$

clearly $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \alpha e^{-\alpha x}\,dx = \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^{\infty}$$

$$= \alpha \left[\cancel{\tfrac{1}{\alpha}}\, 0 + \frac{1}{\alpha} \right] = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x) = 1$$

Mean and standard Deviation :

P.T $\mu = 1/\alpha$, $S.D = 1/\alpha$

wkt:
$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$\mu = \alpha \int_{0}^{\infty} x\, e^{-\alpha x}\, dx = \alpha \left[x \left(\frac{e^{-\alpha x}}{-\alpha} \right) - 1 \frac{e^{-\alpha x}}{(-\alpha)^2} \right]_0^{\infty}$$

$$= \alpha \left\{ [0-0] - \left[0 - \frac{1}{\alpha^2}\right] \right\} = \alpha \left[\frac{1}{\alpha^2} \right]$$

$\therefore$ $\boxed{\mu = \frac{1}{\alpha}}$

$$\text{var} = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{0}^{\infty} (x-\mu)^2 . \alpha . e^{-\alpha x}\, dx$$

$$= \alpha \left[(x-\mu)^2 \frac{e^{-\alpha x}}{(-\alpha)} - 2(x-\mu) \frac{e^{-\alpha x}}{(-\alpha)^2} + 2 \frac{e^{-\alpha x}}{(-\alpha)^3} \right]_0^{\infty}$$

$$= \alpha \left\{ [0 - 0 + 0] - \left[\frac{\mu^2}{-\alpha} - 2(-\mu) \frac{1}{\alpha^2} + 2 \frac{1}{-\alpha^3} \right] \right\}$$

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Put $\mu = \frac{1}{\alpha}$

$$= \cancel{\alpha \left\{ 0 - \left(\frac{\alpha^2}{-\alpha} + 2\frac{\alpha}{\alpha^2} - \frac{2}{\alpha^3} \right) \right\}}$$

$$= \alpha \left[\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3} \right] = \frac{1}{\alpha^2}$$

$$\therefore \quad \sigma^2 = \frac{1}{\alpha^2} \Rightarrow \boxed{S.D = \sigma = \frac{1}{\alpha}}$$

Normal Distribution:

This is the most important continuous distribution because in applications many random variables are normal random variables.

The normal Distribution is defined as the distribution with the density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right]$$

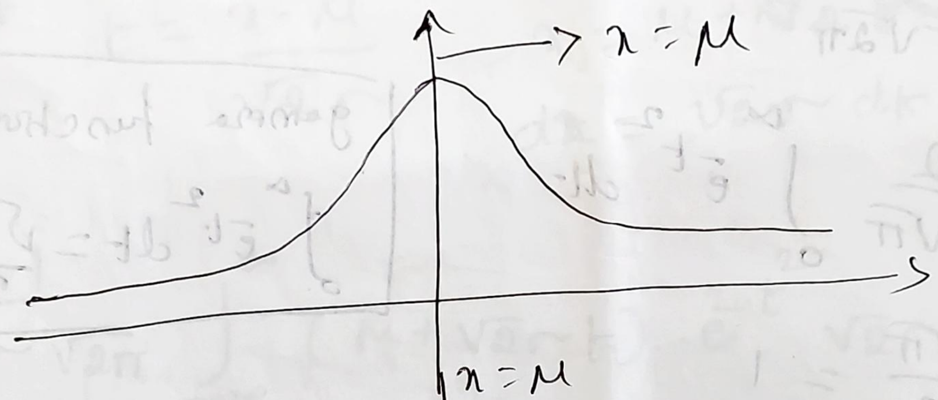
where $\sigma > 0$

Note: ① μ is mean and σ S.D

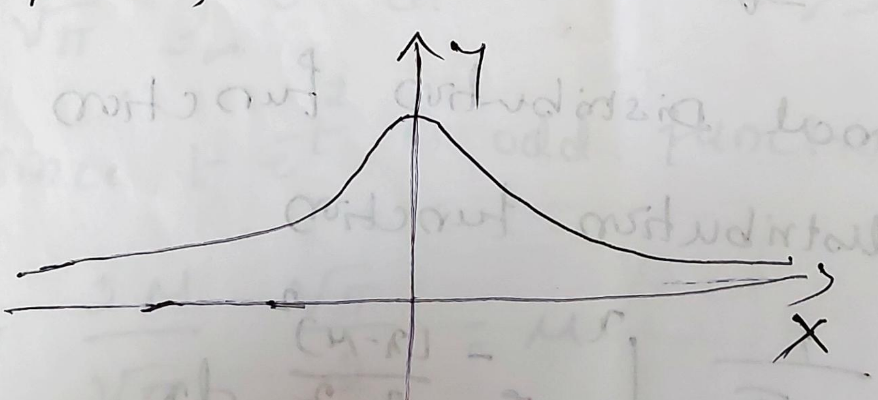
② $\frac{1}{\sigma\sqrt{2\pi}}$ is a constant that makes the area under the curve equal to 1

i.e $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

③ $f(x)$ is symmetric about $x = \mu$



for $\mu = 0$, $f(x)$ is symmetric about y axis



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consider

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{\sigma^2}}\,dx$$

put $t = \dfrac{x-\mu}{\sqrt{2}\,\sigma}$ $\qquad x = \mu + \sqrt{2}\,\sigma t$

$\qquad dx = \sqrt{2}\,\sigma\,dt$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-t^2} \sqrt{2}\,\sigma\,dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\,dt$$

$$\boxed{\text{gamma function} \quad \int_0^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}}$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Note: Normal Distribution function has the distribution function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Mean and S.D of Normal Distribution

W.K.T.

$$\text{mean} = \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

put $t = \dfrac{x-\mu}{\sqrt{2}\,\sigma}$ $\quad x = \mu + \sqrt{2}\,\sigma t$

$dx = \sqrt{2}\,\sigma\, dt$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\,\sigma t)\, e^{-t^2}\, \sqrt{2}\,\sigma\, dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt + \sigma\sqrt{\frac{2}{\pi}} \cancel{\int_{-\infty}^{\infty} t\, e^{-t^2}\, dt} \to 0$$

since $t\, e^{-t^2}$ is odd function.

$$= \frac{2\mu}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \mu$$

$\therefore \boxed{\text{mean} = \mu}$

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$$\text{var} = V = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

put $\quad t = \dfrac{x-\mu}{\sqrt{2}\sigma}$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} \sqrt{2}\sigma \, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} \, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cancelto{0}{\left[t \, \frac{e^{-t^2}}{-1} \right]_{-\infty}^{\infty}} + \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \, \frac{\sqrt{\pi}}{2} = \sigma^2$$

$$\therefore \boxed{S.D = \sigma}$$

Standard Normal Distribution:

The distribution function $F(x)$ of the normal distribution with any μ & σ is related to the standardised distribution function $\phi(z)$.

w.k.t

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

Put $t = \frac{u-\mu}{\sigma} \Rightarrow dt = \frac{du}{\sigma}$

at $u = x \Rightarrow t = \frac{x-\mu}{\sigma}$

$$\therefore F = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\left(\frac{x-\mu}{\sigma}\right)} e^{-t^2/2} dt$$

$$= \phi\left[\frac{x-\mu}{\sigma}\right]$$

put $\boxed{z = \frac{x-\mu}{\sigma}}$

$\therefore \phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2}\, dz$

$\frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is called standard normal probability density function.

If $\mu = 0$ & $\sigma = 1$ then normal probability density function is the standard normal probability density function.

In perticular we have

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2}\, dz$$

(*) $z = \frac{x-\mu}{\sigma}$ is called standard normal varaite

Note :

① The probability that a normal random variable is x with μ & σ.

let $a < x \leq b$ then

$$P(a < x \leq b) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

i.e at $x = a$ $\quad z_1 = \frac{a-\mu}{\sigma}$

$x = b$ $\quad z_2 = \frac{b-\mu}{\sigma}$

$$P(a < x \leq b) = F(b) - F(a) = \phi(z_1) - \phi(z_2)$$

$$\Rightarrow P(z_1 < z < z_2) = \phi(z_1) - \phi(z_2)$$

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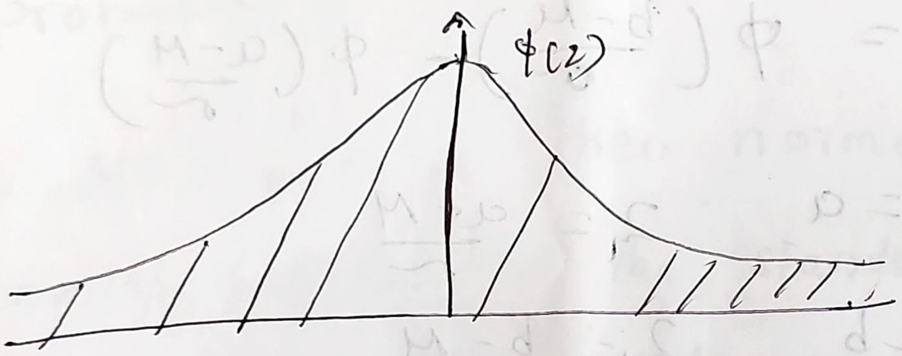
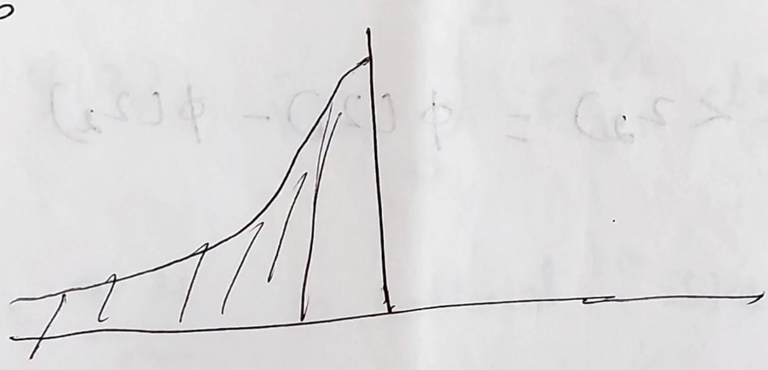
Note: Let $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} \, dz$

then

① $P(-\infty \le z \le \infty) = \int_{-\infty}^{\infty} \phi(z) \, dz = 1$



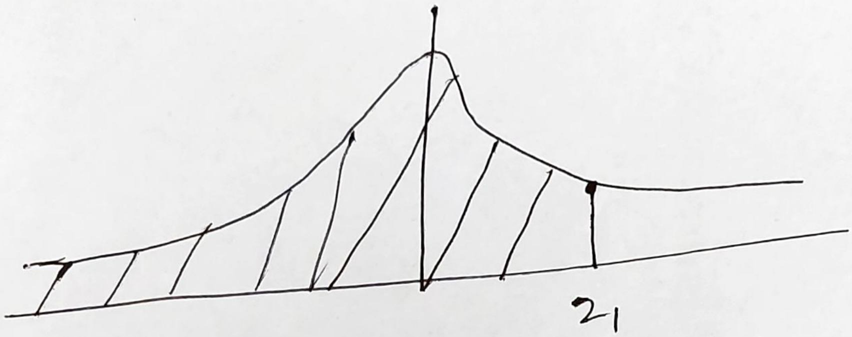
② $\int_{-\infty}^{0} \phi(z) \, dz = P(-\infty \le z \le 0) = 1/2$



③ $\int_0^{\infty} \phi(z) \, dz = P(0 \le z \le \infty) = 1/2$

$P(-\infty < z < z_1) = P(-\infty < z \le 0) + P(0 < z < z_1)$

$P(z < z_1) = 0.5 + \phi(z_1)$



(h) $P(z > z_2) = P(z \ge 0) - P(0 \le z \le z_2)$

$= 0.5 - \phi(z_2)$