UNIT- 5.

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JOINT PDF and STOCHASTIC PROCESS.

* Joint Probability distribution :-

If x & y are any 2 discrete random variables we define the joint probability distribution of x & y as,

$$P(X = x_i, Y = y_j) = f(x, y).$$

where i) $f(x,y) \geq 0$ (ii) $\sum_x \sum_y f(x,y) = 1$.

If x & y are 2 discrete random variables such that $X = x_i$ (for) $\forall i = 1, 2, --- m$.
$Y = y_j$ $\forall j = 1, 2, ---, n$.

$$J_{ij} = f(x_i, y_j).$$

where $\boxed{J_{ij} \geq 0}$ and $\boxed{\sum_{i,j} J_{ij} = 1.}$

$x \backslash y$	y_1	y_2	----	y_n	Sum.
x_1	J_{11}	J_{12}	----	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	----	J_{2n}	$f(x_2)$
⋮	⋮				⋮
x_m	J_{m1}	J_{m2}	----	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	---	$g(y_n)$	1

Correlation of XY.

$$\rho(XY) = \frac{Cov(XY)}{\sigma_X \sigma_Y} = \frac{-1.2}{3.039} = -0.3948.$$

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* Markov chain :-

1. Find the unique fixed probability vector for the regular stochastic matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}.$$

col of 3x3 matrix.

Sol:- Let $v = [x, y, z]$. such that $x+y+z=1$ — ①.

$vP = v$.

$$[x \; y \; z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

$= [x\times 0 + y \times 1/6 * z \times 0]$

$= \left[\frac{y}{6}, x + \frac{y}{2} + \frac{2z}{3}, \frac{y}{3} + \frac{z}{3}\right] = [x \; y \; z]$

$\frac{y}{6} = x$ $\quad x + \frac{y}{2} + \frac{2z}{3} = y$ $\quad \frac{y}{3} + \frac{z}{3} = z.$

$y = 6x$ — ① $\quad x + \frac{y}{2} + \frac{2z}{3} = y$ $\quad y + z = 3z$ — ③

$y - 2z = 0$

$6x + 3y + 4z = 6y$

$6x - 3y + 4z = 0$

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Soln:-
$$P = \begin{array}{c} \\ M \\ A \\ S \end{array}\begin{array}{c} \begin{array}{ccc} M & A & S. \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$
→ he serves all the 3.

$p^{(0)} = [0 \quad 0 \quad 1]$ → Initial condn Santro was his 1st car

i) 2002.

$p^{(2)} = p^{(0)} \cdot p^2$

$$p^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$p^{(2)} = [0 \quad 0 \quad 1] \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$= \begin{array}{ccc} [1/9 & 4/9 & 4/9] \\ M & A & S. \end{array}$$

In 2002 he has Santro = 4/9.

In 2002 he has Maruti = 1/9.

In 2003.

$p^{(3)} = p^{(0)} \cdot p^3$

$$p^3 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 7/27 & 16/27 \end{bmatrix}$$

$$p^{(3)} = [0 \quad 0 \quad 1] \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 7/27 & 16/27 \end{bmatrix}.$$

$$p^{(3)} = \begin{array}{ccc} M & A & \\ \left[\frac{4}{27}\right., & 7/27, & \left.16/27\right] \end{array}$$

$$4x + z = 4^2 \frac{(3z)}{8}$$

$$4x + z = 6z \Rightarrow 4x - 5z = 0 \quad \text{— (5)}$$

Solving (4) & (5) $\quad x = \frac{1}{3}, \quad z = \frac{4}{15}$

$$y = \frac{3}{2}\left(\frac{4}{15}\right) = \frac{2}{5}$$

$$\therefore V = \left[\frac{1}{3}, \frac{4}{15}, \frac{2}{5}\right]$$

Prove that matrix is irreducible so solve it with reducible only.

Q. A man's smoking habits are as follows:-
If he smoke filter cigarette 1B he switches too non-filter cigarette the next week with probability 0.2. On the other hand if he smokes non-filter cigarette one week there is a probability of 0.7 that he will smoke non-filter cigarette the next week as well. In the long run how often does he smoke filter cigarette.

Soln:-

$$A = \begin{array}{cc} & \begin{array}{cc} F & NF \end{array} \\ \begin{array}{c} F \\ NF \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

$V = [x \ y]$ such that $x + y = 1$

$VP^{\text{or } A} = V$

$$[x \ y] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = [x \cdot y]$$

$$[0.8x + 0.3y \ , \ 0.2x + 0.7y] = [x \ y]$$

$0.8x + 0.3y = x$ $\qquad$ $0.2x + 0.7y = y$

$0.2x - 0.3y = 0$ — ① $\qquad$ $0.2x - 0.3y = 0$ — ②

$$\left.\begin{array}{l} x + y = 1 \\ 0.2x - 0.3y = 0 \end{array}\right\} \text{solving}$$

$x = \frac{3}{5}$, $y = \frac{2}{5}$

$V = \left[\frac{3}{5}, \frac{2}{5}\right]$ $\therefore$ In the long run smokes filter cigarette is 3/5 u

5. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$.

Sol:- $v = [x \ y \ z]$ such that $x+y+z = 1$.

$vP = v$.

$$[x \ y \ z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \ y \ z]$$

$$\left[\frac{z}{2}, \ x + \frac{z}{2}, \ y\right] = [x \ y \ z]$$

$$\frac{z}{2} = x, \quad x + \frac{z}{2} = y, \quad y = z.$$

$2x - z = 0$ — ① $\quad 2x - 2y + z = 0$ — ② $\quad y - z = 0$ — ③

mandatory always take this eqn $\Longrightarrow$

$$x + y + z = 1$$
$$2x - z = 0$$
$$2x - 2y + z = 0$$

$$\begin{array}{cccc} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right] \\ a \quad b \quad c \quad d \end{array}$$

Ean → ②

$$x = \frac{1}{5}, \quad y = \frac{2}{5}, \quad z = \frac{2}{5}$$

$$v = \left[\frac{1}{5}, \ \frac{2}{5} \ \frac{2}{5}\right].$$

$E(Y) = \sum y\, g(y)$
$= -3 \times 0.4 + 2 \times 0.3 + 4 \times 0.3$
$= 0.6.$

$E(XY) = \sum_{ij} x_i y_j J_{ij}.$

$= (1 \times -3 \times 0.1) + (1 \times 2 \times 0.2) + (1 \times 4 \times 0.2) + (3 \times -3 \times 0.3)$
$+ (3 \times 2 \times 0.1) + (3 \times 4 \times 0.1)$
$= 0.$

(iii) Corelation :- $\sigma_x^2 = E(X^2) - [E(X)]^2.$

$E(X^2) = \sum x^2 f(x) = 1^2 \times 0.5 + 3^2 \times 0.5 = 5.$

$E(Y^2) = \sum y^2 g(y) = -3^2 \times 0.4 + 2^2 \times 0.3 + 4^2 \times 0.3 = 9.6.$

$\sigma_x^2 = 5 - (2)^2 = 1 \Rightarrow \sigma_x = \sqrt{1} = 1.$

$\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 9.6 - (0.6)^2 = \cancel{0.36}\ 9.24.$
$\sigma_Y = \sqrt{9.24} = 3.039.$

$\therefore Cov(XY) = E(XY) - E(X) \cdot E(Y)$
$= 0 - 2(0.6)$
$= -1.2.$

4. A Joint Pdf is

$x \backslash y$	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Determine :-

i) Marginal distribution of X & Y

ii) Mean or Expectation of X, Y, XY.

iii) Covariance & Correlation of XY.

Soln :- i) Marginal distribution of X & Y

$x \backslash y$	-3	2	4	Sum
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
Sum	0.4	0.3	0.3	1

i)

x	1	3
$f(x)$	0.5	0.5

y	-3	2	4
$g(y)$	0.4	0.3	0.3.

ii) $E(X) = \Sigma x f(x)$

$= 1 \times 0.5 + 3 \times 0.5$

$= 2.$

In 2003, he has Ambassador $= 7/27$
" " " " Santro $= 16/27$

$v = [x \ y \ z]$.

Solve further

H.W $\Rightarrow$ A salesmen's territory consists of 3 cities A, B, C he never sells in the same city on consequtive days, if he sells in city A then the next day he sells in city B however if he sells in either B or C then the next day he is twice as likely to sell in city A in other city. In the long run how often does he sell in each of these cities.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/4 & 1/2 \end{bmatrix}$$

i) The probability that A has the ball on 4^{th} throw $= 1/4$
∴ The probability that B has the ball on 4^{th} throw $= 1/4$
The probability that C has the ball on 4^{th} throw $= 1/2$

4. Every year a man trades his car for a new car. If he has Maruti he trades it for Ambassador. If he has an Ambassador he trades it for Santro. However if he has a Santro he is just as likely to trade it for a new Santro as to trade it for maruti or an Ambassador. In 2000 he brought his 1^{st} Car which was Santro

i) Find the probability that he has :-
a) 2002 Santro
b) 2002 Maruti
c) 2003 Ambassador.
d) 2003 Santro.

ii) In the long run how often will he ~~sett~~ serve on Santro

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3. 3 boys A, B, C are throwing a ball to each other randomly. A always throws to B and B always to C. C is as lightly to throw to B as to A they never throw to themself. Find the probability:

i) A has

ii) B has

iii) C has, the ball on the fourth throw:

If, C was the 1st to throw the ball.

Soln:-

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1^{(A\to B)} & 0 \\ B & 0 & 0 & 1^{(B\to C)} \\ C & 1/2 & 1/2 & 0 \end{array}$$

lightly to A & B.

$P^{(0)} = [0 \quad 0 \quad 1]$ [$\because$ C was the 1st to throws the ball].

$P^{(3)} = P^{(0)} \cdot P^3$ $\quad\quad$ $\boxed{P^{(n)} = P^{(0)} \cdot P^{n}}$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$\therefore P^{(3)} = P^{(0)} \cdot P^{(3)}$

long run → probability vector.

1. A Student's study habits are as follows. If he studies one night he is 70% sure not to study the next night. On the other hand if he does not study one night he is 60% sure not to study next night. In the long run how often does he study.

S → study
SN → not study

Sol:-

$$A = \begin{array}{c c} & \begin{array}{cc} S & NS \end{array} \\ \begin{array}{c} S \\ NS \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

x → study
y → not study.

$V = [x \; y]$. Such that $x + y = 1$.

or A.

$VP = V$.

$$[x \; y] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \; y]$$

$$[0.3x + 0.4y \quad 0.7x + 0.6y] = [x \; y]$$

$-0.3x$
$= 0.7x$

$0.3x + 0.4y = x$ $\qquad$ $0.7x + 0.6y = y$

$0.7x + 0.4y = 0$ $\qquad$ $0.7x - 0.4y = 0$

$$\left.\begin{array}{l} x + y = 1 \\ 0.7x - 0.4y = 0 \end{array}\right\} \text{solving this.} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0.7 & -0.4 & 0 \end{bmatrix}$$

$x = \frac{4}{11}$, $y = \frac{7}{11}$

$$V = \left[\frac{4}{11}, \frac{7}{11}\right]$$

In the long run he studies $= 4/11\%$.

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mode-5- Equ $\rightarrow$ 1.

$x+y+z=1$

$x+6x+z=1 \Rightarrow 7x+z=1$ — (4). $\begin{bmatrix} 7 & 1 & 1 \\ 6 & -2 & 0 \end{bmatrix}$

from (3) $6x-2z=0$ — (5).

$\therefore x=\frac{1}{10}, \ y=\frac{3}{5}, \ z=\frac{3}{10}$. $x=1/10$, $y=3/10$ we get the x & z. then put the $x+y+z=1$ we get y.

Solving (4) & (5).

$x=\frac{1}{10}$. $z=\frac{3}{10}$.

$\therefore y=6\left(\frac{1}{10}\right)=\frac{3}{5}$. $\therefore V=\left[\frac{1}{10}, \frac{3}{5}, \frac{3}{10}\right]$.

2. $A=\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$.

Sol:- $V=[x \ y \ z] \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} = [x \ y \ z]$

$\left[\frac{y}{2}+\frac{z}{2}, \ x+\frac{z}{4}, \ \frac{y}{2}+\frac{z}{4}\right] = [x \ y \ z]$

$\frac{y}{2}+\frac{z}{2}=x, \quad x+\frac{z}{4}=y, \quad \frac{y}{2}+\frac{z}{4}=z$.

$y+z=2x$. — (1). $4x+z=4y$ — (2) $2y+z=4z$.

$y=\frac{3z}{2}$ — (3)

W.K.T $x+y+z=1$.

$x+\frac{3z}{2}+z=1 \rightarrow 2x+3z+2z=2$.

$2x+5z=2$ — (4)

from (2)

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x×y.

Soln :-

x\Y	1	2	3
0	0.02	0.08	0.1
1	0.08	0.32	0.40

3. The Joint pdf of two random variable X & Y is

x\Y	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Determine :- (i) Marginal distribution of X & Y.
(ii) Co-variance and Co-relation of X & Y.
(iii) Are X & Y independent ?

Soln :-

x\Y	2	3	4	Sum
1	0.06	0.15	0.09	0.3
2	0.14	0.35	0.21	0.7
Sum	0.2	0.5	0.3	1

i)

(Sum)

X	1	2
f(x)	0.3	0.7

(Sum).

Y	2	3	4
g(y)	0.2	0.5	0.3

Sol:- $S = \{HHH, HHT, THH, HTH, TTH, HTT, THT, TTT\}$.

at 1st occurance $H \to 1$ & $T \to 0$.

$X = \{1, 1, 0, 1, 0, 1, 0, 0\}$.

$Y = \{0, 1, 1, 1, 2, 2, 2, 3\}$.

X\Y	0	1	2	3	Sum
0	0	1/8	1/4	1/8	1/2
1	1/8	1/4	1/8	0	1/2
Sum	1/8	3/8	3/8	1/8	1

Marginal distribution of X & Y.

X	0	1
(sum) f(X)	1/2	1/2.

Y	0	1	2	3
g(Y)	1/8	3/8	3/8	1/8

i) $E(X+Y) = \Sigma\, (x+y) \times J_{ij}$.

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3. Verify that A is regular. [Till all entries are ≥ 0]

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}.$$

mode → matrix
3×3.
add all number.
then AC
shift → 4
: Mat A then x^2.
then AC → MatAns
× Mat A
A^3 ↑.

Sol:- $$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0.125 & 0.3125 & 0.5625 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.1562 & 0.6406 & 0.2031 \\ 0.125 & 0.3125 & 0.5625 \end{bmatrix} //$$

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4. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ Show that A is regular and also find the associated unit fixed probability vector.

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \quad A^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \quad A^5 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.125 & 0.375 & 0.5 \end{bmatrix}.$$

1/8 3/8.

∴ A is regular since all the entries are non-zero

10) (ii) $Cov(XY) = E(XY) - E(X)E(Y)$ —— ①.

$E(X) = \sum x f(x)$

$= 1\times 0.3 + 2\times 0.7$

$= 0.3 + \cancel{0.14}\ 1.4$

$= 1.7$

$\begin{array}{r} 0.7 \\ \times 2 \\ \hline 1.4 \end{array}$

$E(Y) = \sum y\, g(y)$

$= 2\times 0.2 + 3\times 0.5 + 4\times 0.3$

$= 0.4 + 1.5 + \cancel{0.12}\ 1.2$

$= 3.1$

$\begin{array}{r} 0.3 \\ \times 4 \\ \hline 1.2 \end{array}$

$E(XY) = \sum_{ij} x_i y_j \cdot J_{ij}$

$= (1\times 2\times 0.06) + (1\times 3\times 0.15) + (1\times 4\times 0.09) + (2\times 2\times 0.14) + (2\times 3\times 0.35) + (2\times 4\times 0.21)$

$= 5.27$

Sub in ①, we get

$Cov(XY) = 5.27 - 1.7\times 3.1$

$= 0$

0/anything = zero

$\rho(XY) = \dfrac{Cov(XY)}{\sigma_x \sigma_y} = 0$

$\therefore X$ & Y are independent.

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i) $E(X+Y) = 0 + (0+1)\times\frac{1}{8} + (0+2)\times\frac{1}{4} + (0\times 3)\times\frac{1}{8} +$

$(1+0)\times\frac{1}{8} + (1+1)\times\frac{1}{4} + (1+2)\times\frac{1}{8} + 0$

$= 2.$

$E(XY) = \Sigma xy\, J_{ij}.$

$= 0 + (0\times1)\times\frac{1}{8} + (0\times2)\times\frac{1}{4} + (0\times3)\times\frac{1}{8} + (1\times0)\times\frac{1}{8}$

$+ 1\times\frac{1}{4} + 1\times2\times\frac{1}{8} + 0$

$= \frac{1}{2}.$

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2. The distribution of two stochastically independent random variables X & Y are:

Marginal is given.

X	0	1		Y	1	2	3
$f(x)$	0.2	0.8		$g(y)$	0.1	0.4	0.5

Find Joint probability distribution of X & Y.

Note :- If Co-variance of X & Y = 0 then X & Y are independent.

[Variance] :-

$$\boxed{\sigma_x^2 = E(X^2) - [E(X)]^2}$$

$$\boxed{\sigma_y^2 = E(Y^2) - [E(Y)]^2}$$

Covariance of X & Y :-

$$\boxed{Cov(X, Y) = E(XY) - E(X)E(Y)}$$

Correlation of X & Y :-

$$\boxed{\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}}$$

1. A coin is tossed thrice. X = 0 or 1 accordingly tail ~~and~~ or head occurs on the 1st toss.
6-7M Y = No. of tails.
Determine :- i) Marginal distribution of X & Y.
ii) Join probability distribution of X & Y.
iii) Expected value (X + Y) & (XY).

Marginal distribution of x & Y.

x	x_1	x_2	----	x_n
$f(x)$	$f(x_1)$	$f(x_2)$		$f(x_n)$

Y	y_1	y_2	----	y_n
$g(Y)$	$g(y_1)$	$g(y_2)$		$g(y_n)$

v.imp. Note:- X & Y are said to be independent if
$$\boxed{f(x_i)\, g(y_j) = J_{ij}}$$

* Mean, Variance & Standard Deviation:-
[mean = Expectation]

Imp

$$E(X) = m_X = \Sigma x f(x) = \text{mean of } X$$

$$E(Y) = m_Y = \Sigma y\, g(y) = \text{mean of } Y.$$

$$E(XY) = m_{XY} = \Sigma\, xy\, J_{ij} = \text{mean of } XY.$$

$$E(X+Y) = m_{X+Y} = \Sigma\, x+y\, J_{ij}.$$