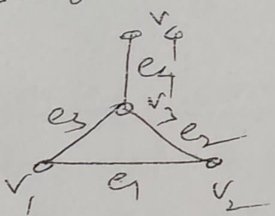
Graph Theory

A graph $G = (V, E)$ consists of a finite non empty set V of vertices (points or nodes) and E a set of edges (lines/arcs). Each edge has either one or two vertices associated with it called as endpoints.

An edge is said to be connect or link its vertices.

Ex



Here $V = \{v_1, v_2, v_3, v_4\}$

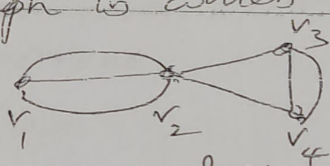
$E = \{e_1, e_2, e_3, e_4\}$

e_2 joins v_2 & v_3 e_4 joins v_3 & v_4 etc.

Self loop : an edge joining vertex to itself

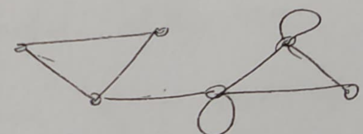
Multiedges : More than one edge joining a pair of vertices

Multigraph : Its a graph in which multiedges are allowed.



Note v_1 & v_2 are joined by 3 edges
v_3 & v_4 by 2 edges.

Pseudograph : A graph containing self loops is called a Pseudograph.

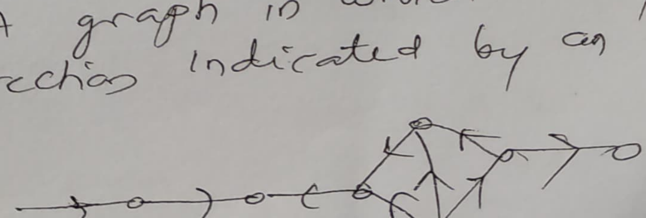


Simple Graph

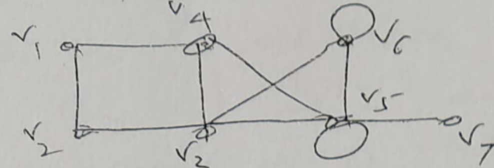
A graph without self loops & multiedges.

Digraph

A graph in which every edges has direction indicated by an arrow over it.



<u>Degree</u> : The no of edges incident on any vertex is called its degree. The degree of a vertex v is taken as $d(v)$. If ~~self~~ loop exists then its counted as 2 as degree.



$\deg v_1 = 2 = \deg v_2$

$\deg v_3 = 4$ $\deg v_4 = 3$

$d(v_5) = 5$ $d(v_6) = 4$

$d(v_7) = 1$

* A vertex of degree 1 is called a pendant vertex.

<u>Handshaking lemma</u>

Statement : Let $G = (V, E)$ be any undirected graph with 'e' edges then

$$\sum_{v \in V(G)} d(v) = 2e$$

Proof : While counting degree of any vertex in a graph ~~then~~ edge is counted twice once from one end vertex & once from other end vertex.

Since it holds for vertices

$$\sum_{v \in V(G)} \deg(v) = 2e$$

we have immediate consequence

Corollary : An undirected graph has an even number of vertices of odd degree.

Proof : Let $V = V_1 \cup V_2$ where V_1 - set of vertices of odd degree & V_2 - set of vertices of even degree

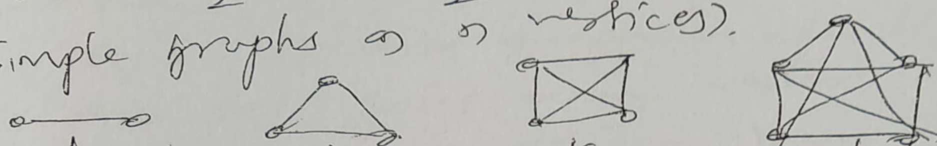
we know that $\sum_{u \in V_1} d(u) + \sum_{v \in V_2} d(v) = 2e$

RHS = even & $\sum_{v \in V_2} d(v)$ - also even

$\Rightarrow \sum_{u \in V_1} d(u) =$ even since $d(u)$ - odd sum will be even if there are even vertices of odd degree. \#

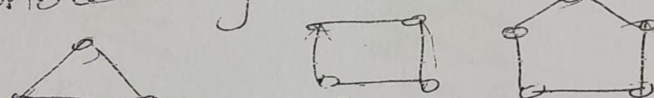
Some Special Graphs

Complete Graph : A simple graph in which every pair of vertices are joined (connected by an edge) is called complete graph. A complete graph on 'n' vertices is denoted by K_n. It has $nC_2 = \frac{n(n-1)}{2}$ edges (Largest among all Simple graphs on n vertices).



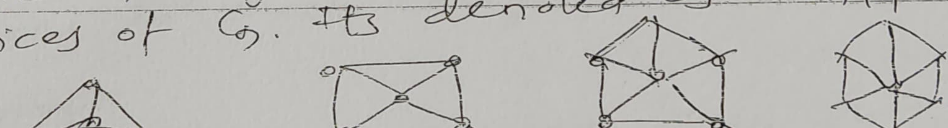
K_2 K_3 K_4 K_5.

Cycle : A cycle is a graph in which every vertex has degree 2. A cycle on n vertices is denoted by C_n. It has 'n' edges



C_3 C_4 C_5

Wheel : Wheel is obtained by taking a cycle C_n & joining a new vertex to all vertices of C_n. Its denoted as W_{n+1} or $W_{1,n}$



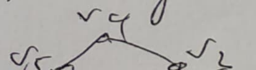
W_4 ($W_{1,3}$) W_5 ($W_{1,4}$) W_6 ($W_{1,5}$) W_7 ($W_{1,7}$)

* A regular graph is a graph in which every vertex has same degree. If G is a graph with every vertex having degree 'r' then its said to be 'r' regular. By handshaking Lemma it has $\frac{nr}{2}$ edges.

Ex cycle, complete graph.

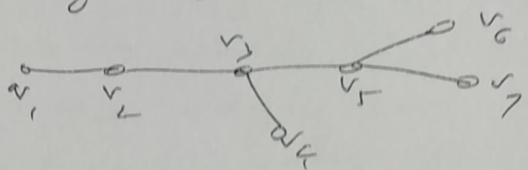
Bipartite Graph

A simple graph G is said to be bipartite if its vertex set V can be partitioned into two disjoint subsets V_1 & V_2 such that every edge in graph G joins a vertex in V_1 with a vertex in V_2. (no edge joins a vertex in the same set)



Bipartite with $V_1 = \{v_1\ v_3\ v_5\}$

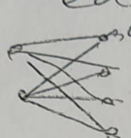
In general every cycle of even order is bipartite. Any acyclic graph (tree) is also bipartite.



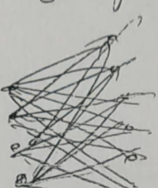
Here $V_1 = \{v_1\ v_3\ v_6\ v_7\}$
$V_2 = \{v_2\ v_4\ v_5\}$

Complete Bipartite Graph.

Its a bipartite graph in which every vertex in one set joins every vertex in the other set. If $|V_1| = m$ $|V_2| = n$ then $K_{m,n}$ denotes complete bipartite graph which has '$m+n$' vertices & 'mn' edges.



$K_{2,4}$



$K_{4,6}$

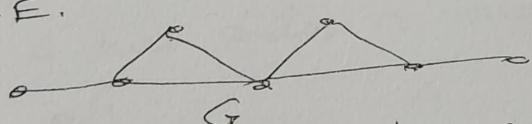
Job Assignments / work assignments → Employee / Students / Customers

Utility to houses / establishments

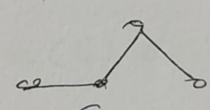
belongs to such category of graphs as models.

Subgraph

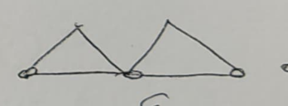
Let $G = (V, E)$ be a graph then $G_1 = (V_1, E_1)$ is said to be subgraph of G if $V_1 \neq \phi \subseteq V$ & $E_1 \subseteq E$.



G



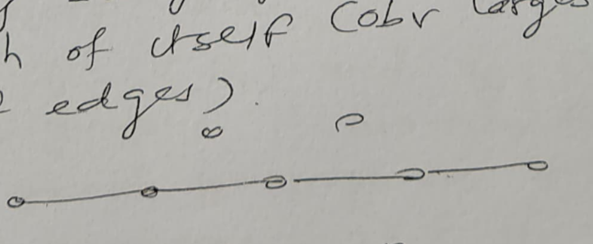
G_1



G_2

A subgraph may be connected or disconnected (G_2 disconnected).

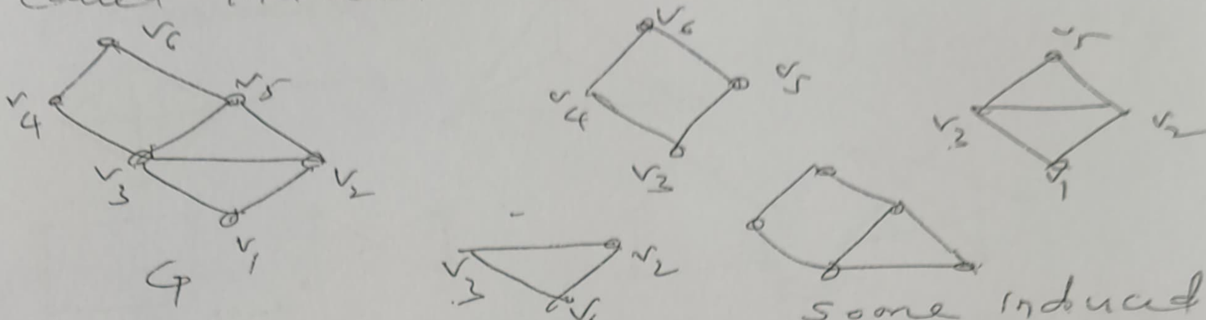
If $V_1 = V$ then subgraph is said to be spanning subgraph. Every graph is spanning subgraph of itself (or largest possible in terms of edges).



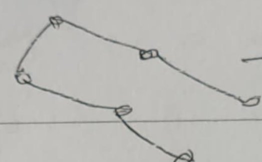
All are spanning first is disconnected

Induced subgraph

Let $G = (V, E)$ be a graph & $U \subseteq V$ then, a subgraph of G in which U is the vertex set & having all the edges corr to U as in G is called induced subgraph.



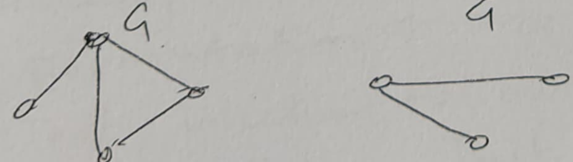
Some induced subgraphs.

*  → is not induced (although spanning) because many edges are not taken.

Complement of a Graph

Let G be a loop free undirected graph on 'n' vertices. The complement of G denoted by $\bar{G}$ is the graph obtained by removing edges in G & adding edges which are not in G.
Means a pair of vertices adjacent (joined by an edge) in G are not in $\bar{G}$ & vice versa.

If G has order n then $E(G) + E(\bar{G}) = {}^nC_2$
$= E(K_n)$



$n = 4 \quad {}^nC_2 = 6$

$G \to 4$ edges $\quad \bar{G} - 2$ edges

Isomorphism

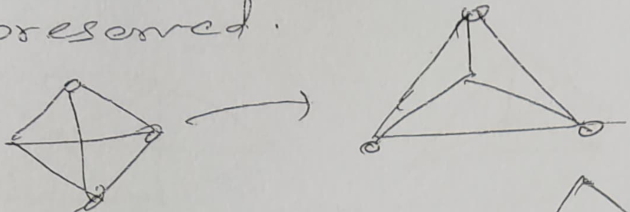
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. A function $f : V_1 \to V_2$ is called a graph isomorphism if

i) f is one-one & onto

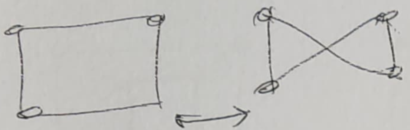
ii) f is edge preserving that is if $(a, b) \in E(G_1)$ then $(f(a), f(b)) \in E(G_2)$

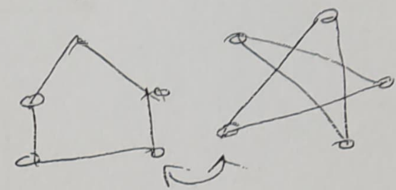
In that case G_1 is said to be isomorphic to G_2 denoted by $G_1 \cong G_2$.

In other words, A pair of graphs G_1 and G_2 are isomorphic if there is one to one correspondence between their vertex sets such that adjacency is preserved.



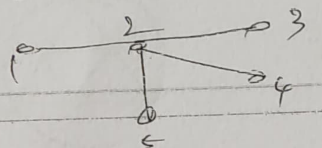
} Isomorphic





Adjacency Matrix of a graph G denoted by $A(G)$ is a matrix in which a_{ij} (ij^{th} entry) is 1 if $\exists$ edge joining v_i to v_j). So G has n vertices $A(G)$ is $n \times n$ matrix

Ex :



$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 \end{array}$$

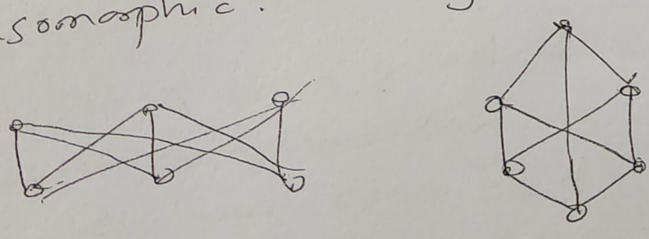
Each row sum gives the degree of corr vertex.

Matrix is symmetric

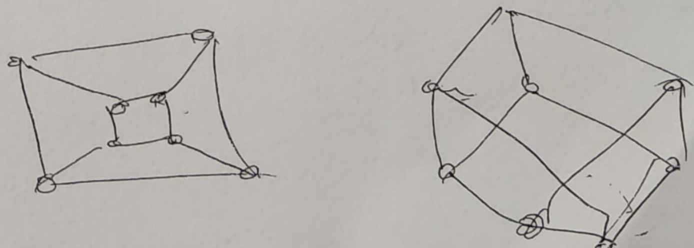
For Isomorphic graphs the corresponding adjacency matrices are either identical or permutation of one another.

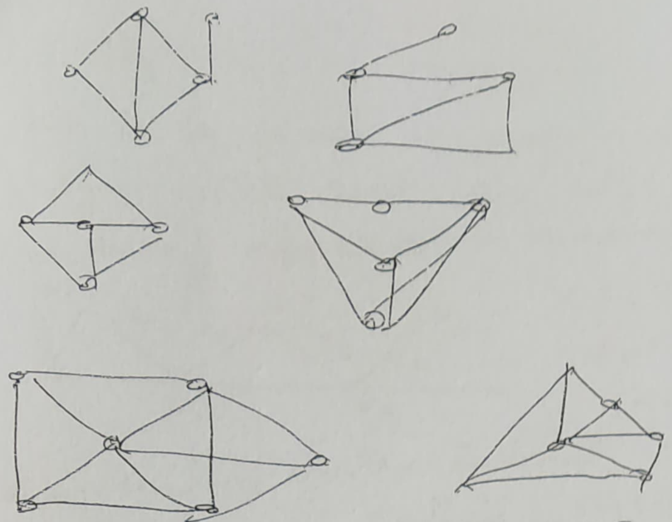
Test whether following pairs of graphs are isomorphic. Justify if they are NOT.

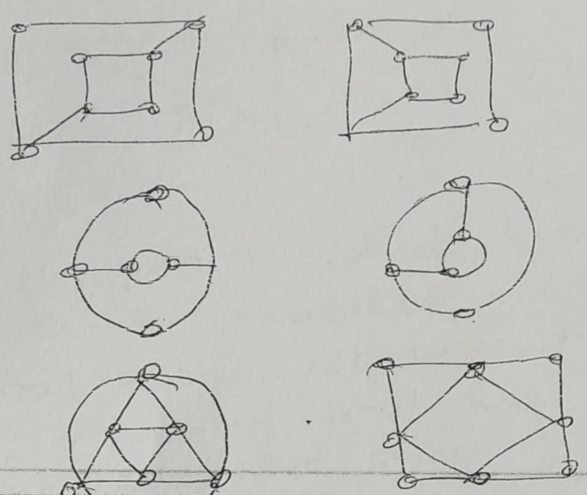
①

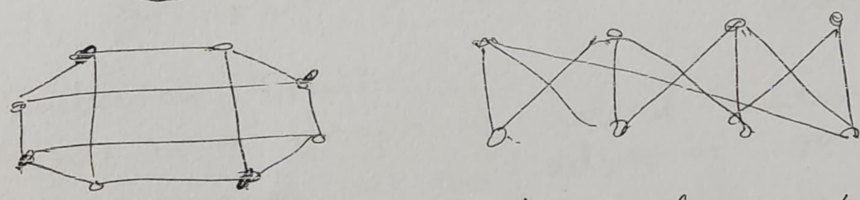


②









A graph G is said to be self complementary if $G \cong \overline{G}$. Ex : C_5 is self complementary.

Theorem : If G is a self complementary graph of order 'n' then $n = 4k$ or $4k+1$, $k \in \mathbb{Z}^+$

Proof: Let G be a graph of order 'n' which is self complementary i.e. $G \cong \overline{G}$. Then both will have same no. of edges also.

$$E(G) + E(\overline{G}) = 2E(G) = {}^{n}C_2 = \frac{n(n-1)}{2}$$

$\Rightarrow E(G) = \frac{n(n-1)}{4}$ $\quad$ $E(G)$ – integer so $\frac{n(n-1)}{4}$ – integer

$\Rightarrow 4 / n$ or $4 / n-1$ $\quad$ $n = 4k$

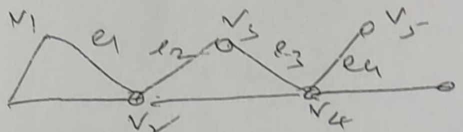
$n-1 = 4k$ or $n = 4k+1$

Connectivity

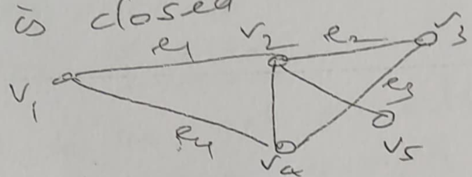
Path in a graph is sequence of distinct vertices and edges such that an edge between a pair of vertices appears in between that pair.

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ - Path

Path length = 5 (no. of edges appearing)



When initial vertex & final vertex coincide then path is closed called cycle.



$v_1 v_2 v_3 v_4 v_1$ → cycle assuming edges to be present in between both vertices and

Walk is a path in which edges can be repeated.

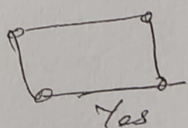
A trail is a walk without repeating any edges. A closed trail has its starting and ending vertex are same.

Walk > Trail > Path — in terms of edges.

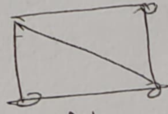
Eulerian Trail

A trail in a graph is called eulerian if it passes through all the edges of graph exactly once.

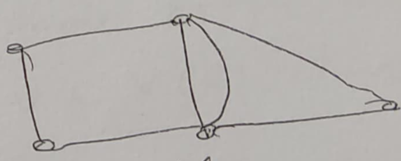
A graph with an eulerian trail is called as eulerian graph.



Yes



No

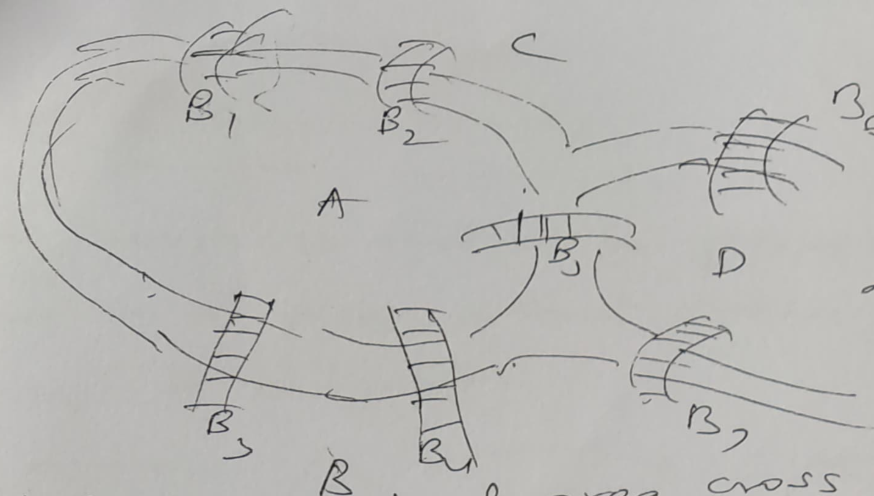


Yes.

In a eulerian graph starting from any vertex we can travel along all the edges exactly once & return to the initial vertex.

Theorem: A graph G is Eulerian if and only if every vertex of G has even degree.

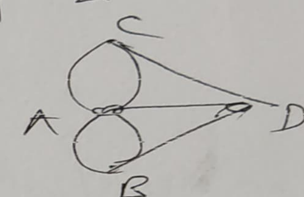
Proof: — [illegible]



East Prussia
River Pregel
divided 4 land
areas by 7
bridges

start from any land area cross all the edges (bridges) exactly once & return to initial point was the Problem.

Proved to be not possible by Leonard Euler in 1736 using graph model & showing it to be non eulerian.



Hamiltonian Graph

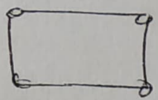
A spanning cycle (closed path) in a graph is called a Hamiltonian cycle. Its path if last vertex is not first vertex. Graph with Hamiltonian cycle is called Hamiltonian graph.

For $n \geq 3$ every complete graph K_n is Hamiltonian. Every cycle / cycle with extra edges added also are Hamiltonian.

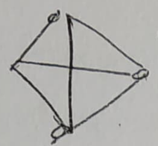
Dirac's Theorem : If G is a simple graph with n vertices & $d(v) \geq \frac{n}{2}$ $\forall v \in V(G)$ then G is Hamiltonian.

Planar Graphs

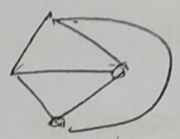
A graph is said to be planar if it can be drawn in a plane without crossing of edges (except at end points).



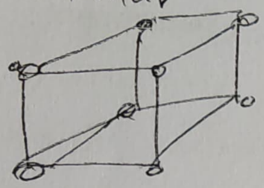
Planar

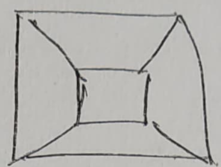


– Planar



Planar embedding





What about K_5, K_{33} ?

K_5 – Smallest non planar graph with least no' of vertices (all graphs upto 4 are planar)

$K_{3,3}$ – Smallest non planar graph on least no' of edges (all graphs upto 8 edges are planar).

Eulers Formula

Let G be a connected simple planar graph with 'e' edges and n vertices. Let r be the number of regions in a planar representation of G. Then

$r = e - n + 2$.

Proof : By induction on 'n'.

If G is a connected planar simple graph and v vertices $v \geq 3$ then $e \leq 3v - 6$

K_5 is non planar : Let K_5 be a planar graph

$e = 10$ $\quad v = 5$ $\quad 3v - 6 = 9$ but $10 > 9$ ⋕

If G is a connected simple planar graph having e edges & $v \geq 3$ with no circuits of length 3 then $e \leq 2v - 4$

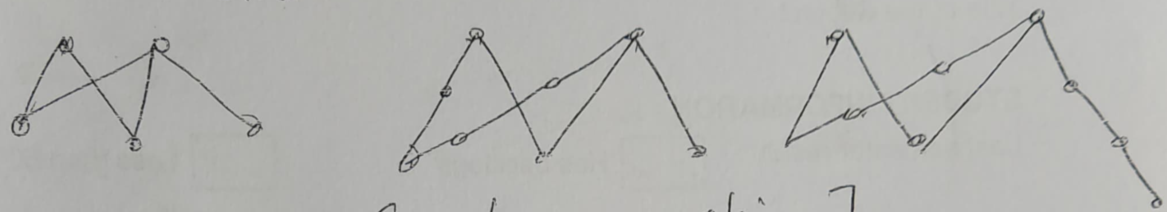
$K_{3,3}$ $\quad e = 9$ $\quad v = 6$ but $2v - 4$ (8) $(9 > 8)$ ⋕

non planar

Kuratowski's Theorem

A graph is nonplanar if and only if it contain a subgraph homeomorphic to $K_{3,3}$ or K_5.

If a graph is planar any graph obtained by removing an edge $\{u,v\}$ and adding a new vertex w together with edges $\{u,w\}$ & $\{v,w\}$ is called elementary subdivision. The pair of resulting graphs are said to be homeomorph to each other.



$G_1 \equiv G_2 \equiv G_3$ homeomorphic

Coloring

A proper coloring of a graph means coloring the vertices of a graph with colors such that adjacent vertices do not have same color. obviously for any graph of order 'n' a proper coloring with n colors is trivial. Give any color to one vertex, second to next & so on.

The minimum number of colors needed to properly color G is called the chromatic number of G is written as $\chi(G)$.

$\chi(K_n) = n$ $\quad \chi(C_n) = \lfloor \frac{n}{2} \rfloor$ $\quad \chi(W_{1,n}) = \lfloor \frac{n}{2} \rfloor + 1$

For a bipartite graph G $\chi(B) = 2$

A planar graph is 4 colorable

$\chi(G) = 4$ G - planar

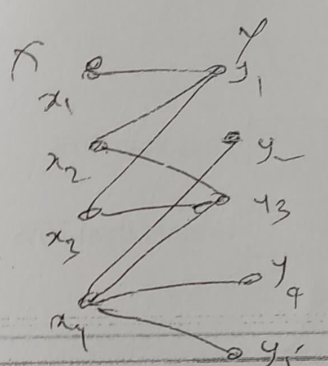
\# 4 color conjecture : Proved by Appel & Haken in 1970

Matching

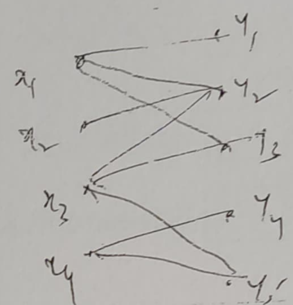
Let $G = (V, E)$ be a bipartite graph with V partitioned as $X \cup Y$ (each edge of E has $\{x, y\}$: $x \in X$ $y \in Y$)

a) A matching in G is a subset of E such that no two edges share a common vertex in X or Y

b) A complete matching of X into Y is a matching in G such that every $x \in X$ is the endpoint of an edge



No complete matching



Complete match
(x_1, y_1) $(x_2\ y_2)$ $(x_3\ y_3)$ $(x_4\ y_4)$

Theorem : Let $G = (V, E)$ be a bipartite graph with V partitioned as $X \cup Y$. A complete matching of X into Y exists if and only if every subset A of X, $|A| \leq |R(A)|$ where $R(A)$ is the subset of Y consisting of those vertices each of which is adjacent to atleast one vertex in A.

Applications to Assignment Problem, Marriage Problem

* Coloring application
storing chemical compounds in a warehouse keeping in mind some of them are volatile incompatible (reactable)
So condition becomes chemical reagents to be stored in seperate compartments
Take the compounds as vertice draw the edge between if they are incompatible. Clearly chromatic compartments