Unit-2:

RELATIONS AND FUNCTIONS.

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* Relations:

Consider the sets ~~A1,~~ ~~A~~ $A = \{a_1, a_2, \ldots, a_n\}$ $(A = \{a_i\}, i = 1, 2, \ldots, n)$ and $B = \{b_1, b_2, \ldots, b_m\}$ $(B = \{b_j\}, j = 1, 2, \ldots, m)$ of order n and m respectively. Then R is the relation from set A to set B. is denoted by $a_i R b_j$. $((a_i, b_j) \in R)$ and defined as the subset of $A \times B$ where ~~$A \times B$~~ $A \times B = \{(a_i, b_j) \; \forall \; i = 1, 2, \ldots n, \; j = 1, 2, \ldots m\}$.

i.e $\boxed{R \subseteq A \times B}$

* Zero-One Matrix:

A zero-one matrix of a relation R are defined from set A to set B is denoted by M_R or $M(R)$ and defined by

$$M_R = M(R) = [M_{ij}]_{n \times m}$$

where,

$$M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In particular, if $A = B$, then zero-one matrix of R is $n \times n$ matrix and whose elements are $M_{ij} = \begin{cases} 1, & \text{if } (a_i, a_j) \in R \\ 0, & \text{if } (a_i, a_j) \notin R \end{cases}$

Example:

Let us consider a set $A = \{a, b\}$ and set $B = \{1, 2\}$ and a relation R defined by $R = \{(a,1), (a,2), (b,2)\}$.

$\therefore$ Zero-one matrix on R is,

$$M_R = M(R) = [M_{ij}]_{2 \times 2} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{21} \end{bmatrix}.$$

where,

$M_{11} = 1 \quad \because (a,1) \in R$

$M_{12} = 1 \quad \because (a,2) \in R$

$M_{21} = 0 \quad \because (b,1) \notin R$

$M_{22} = 1 \quad \because (b,2) \in R$

$$\therefore M_R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (\text{or}) \quad M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Edge — Relation
Vertices — Elements.

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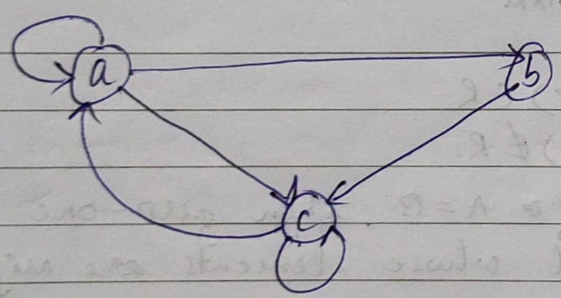
★ Directed Graph (or) Diagraph:
Let R be a relation on a finite set A. Then R can be pictorially represented as small circles for each element of A called vertices (or) nodes and draw an arrow from one vertex 'a' to another vertex 'b' if $(a,b) \in R$. called an edge.
This results, a pictorial representation of relation R called diagraph (or) directed graph of R.

Example:
Let us consider a following relation defined on set $A = \{a,b,c\}$
$R = \{(a,a), (a,b), (a,c), (b,c), (c,a), (c,c)\}$
Diagraph of this relation R is.



★ Origin (source) of an Edge and terminal of an Edge:
In a diagraph, a vertex from which an edge leaves is called origin (source) of an edge and a vertex where edge ends is called terminals of a edge.

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★ loop:
An edge for which the origin and terminal vertex is same vertex called loop.

* Isolated vertex:

A vertex in a diagraph which is neither a origin nor terminals is called isolated vertex.

* In-degree and out-degree:

The number of edges terminating at a vertex is called in-degree of that vertex and number of edges leaving a vertex is called out degree of that vertex.

* Example:

1) Let $A = \{a, b, c\}$ and $B = \{0, 1\}$ and $R = \{(a,0), (b,0), (c,1)\}$ be the relation from set A to B. write down the matrix of this relation.

Sol^n: Let $A = \{\overset{a_1}{a}, \overset{a_2}{b}, \overset{a_3}{c}\}$ & $B = \{\overset{b_1}{0}, \overset{b_2}{1}\}$

Zero one matrix of R is,

$$M_R = M(R) = [m_{ij}]_{3\times 2} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \end{bmatrix} =$$

where,

$M_{11} = 1 \quad (\because (a_1, b_1) = (a,0) \in R)$

$M_{12} = 0 \quad (\because (a_1, b_2) = (a,1) \notin R)$

$M_{21} = 1 \quad (\because (a_2, b_1) = (b,0) \in R)$

$M_{22} = 0 \quad (\because (a_2, b_2) = (b,1) \notin R)$

$M_{31} = 0 \quad (\because (a_3, b_1) = (c,0) \notin R)$

$M_{32} = 1 \quad (\because (a_3, b_2) = (c,1) \in R)$

$$M_R = M(R) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{OR} \quad \begin{array}{c|cc} & 0 & 1 \\ \hline a & 1 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{array}$$

2) Determine the relation R from a set A to set B as represented by the following matrix.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solⁿ: Given that,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 4} \end{array}$$

Let us consider elements of set A & B as follows,

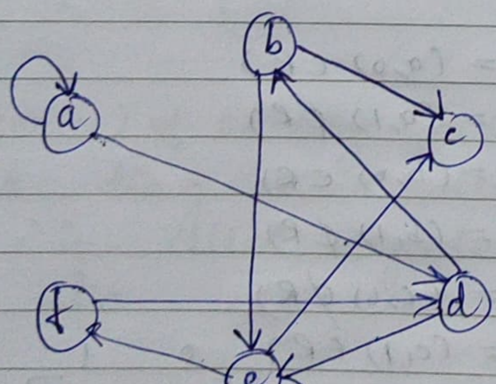
$A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$

∴ Relation R from set A to B is,

$R = \{(a,1), (b,2), (c,1), (c,2), (c,3), (d,2), (d,4)\}$.

3) Let $A = \{a, b, c, d, e, f\}$ and R be the relation on set A defined by $R = \{(a,a), (a,d), (b,c), (b,e), (d,b), (d,e), (e,c), (e,f), (f,d)\}$. Draw the digraph of R. Also determine the in-degree and out-degree of each vertex in digraph.

Solⁿ: Digraph of relation R is,



Vertex	In-degree	Out-degree
a	1	2
b	1	2
c	2	0
d	2	2
e	2	2
f	1	1

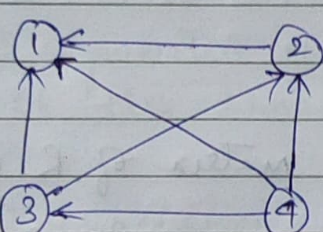
4) If $R = \{(x,y),\ x > y\}$ is relation defined on set $A = \{1,2,3,4\}$ write down the matrix and digraph of a relation R. Also find in-degree and out-degree of each value of a digraph.

Solⁿ: $R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

Zero-one matrix of R is

$$M_R = M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Diagraph of relation R is,



Vertex	In-degree	Out-degree
1	3	0
2	2	1
3	1	2
4	0	3.

5) Find the relation R in a set A and write down its diagraph given that $A = \{a, b, c, d, e\}$ and matrix of R is,

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solⁿ: Given that

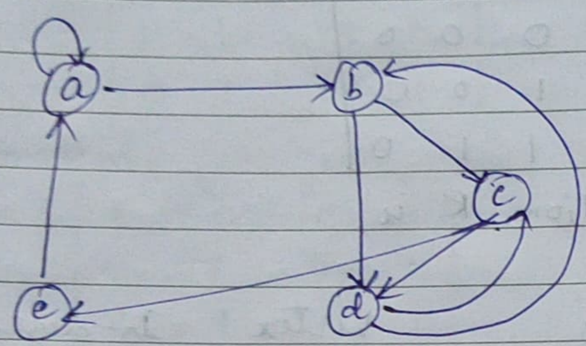
$A = \{a, b, c, d, c\}$.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation R defined by M_R is,

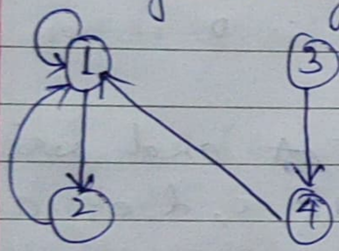
$R = \{(a,a), (a,b), (b,c), (b,d), (c,d), (c,e), (d,b), (d,e), (c,a)\}$.

diagraph of relation R is,

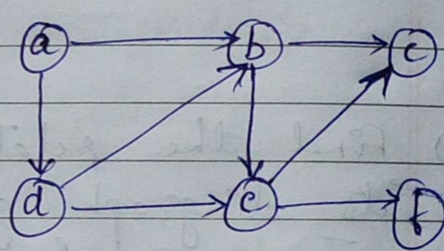


6) Find the relation R and matrix of R determined by the following diagraphs.

i)



ii)



Sol^n:

Sol^n:

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* Types of Relation (Properties of Relation).
If R be the relation defined on non-empty set A, then,

i) Reflexive Relation:
R is reflexive of $(a,b) \in R$, $\forall a \in R$ (or) (aRa, $\forall a \in R$).

ii) Symmetric Relation:
R is symmetric of $(a,b) \in R$, then $(b,a) \in R$ (or)
(if aRb then bRa).

iii) Transitive Relation:
R is transitive if $(a,b) \in R$ and $(b,c) \in R$ then, $(a,c) \in R$
(or) (if aRb and bRc then aRc)

iv) Antisymmetric Relation:
A relation R on a set A is said to be antisymmetric if whenever $a \neq b$, then either $(a,b) \notin R$ (or) $(b,a) \notin R$.

v) Equivalence Relation:
A relation R on a set A is said to be ~~antisymmet~~ equivalence Relation if ~~whenever~~ a it satisfies relfexive, symmetric & transitive relation.

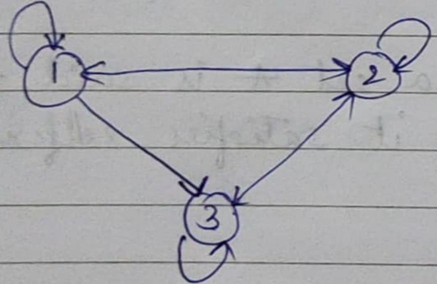
* Closure of a Relation:
If 'R' is a relation that does not possess a particular property (like reflexive, symmetric, transition) then we may wish to add related ordered pairs to R until we get a relation that does have the required property is called closure of a relation.

1) Reflexive Closure:
The reflexive closure of a Relation R on set A is the smallest reflexive relation on set A containing relation R.

2) Symmetric Closure:
The symmetric closure of a relation R on set A is the smallest symmetric relation on set A containing relation R.

3) Transitive Closure:
The transitive closure of a relation R on set A is smallest transitive relation on set A containing relation R.

* Examples:
1) The diagraph of a relation 'R' on set $A=\{1,2,3\}$ is as given below. Determine whether R is an equivalence relation or not.



Sol^n: Relation R defined by above diagraph is
$R=\{(1,1), (1,2), (1,3), (2,1), (2,3), (2,2), (3,2), (3,3)\}$

i) Reflexive:
Since $(1,1), (2,2)$ & $(3,3) \in R$
R is reflexive relation.

ii) Symmetric:
Since $(1,3) \in R$ & $(3,1) \notin R$
R is not symmetric relation.

$\therefore$ R is not an equivalence relation.

2) A relation R on set $A = \{a, b, c\}$ is represented by the following matrix. Determine whether R is an equivalence relation or not

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sol^n: Let $A = \{a, b, c\}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Relation R defined by above matrix is,

$R = \{(a, a), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$.

i) Reflexive:

Since $(a, a), (b, b), (c, c) \in R$

$\therefore$ R is reflexive relation

ii) Symmetric:

Since $(a, c) \in R$ & $(c, a) \in R$ & $(b, c) \in R$ & $(c, b) \in R$.

$\therefore$ R is symmetric relation

iii) Transitive:

Since $(a, c) \in R$, $(c, b) \in R$ & $(a, b) \notin R$.

$\therefore$ R is not transitive relation

$\therefore$ R is not equivalence relation.

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3) Find the reflexive, symmetric and transitive closure of a relation $R = \{(a, b), (b, c), (a, c)\}$ defined on set $A = \{a, b, c\}$

Sol^n: Let $A = \{a, b, c\}$

$R = \{(a, b), (b, c), (a, c)\}$.

i) Reflexive closure:

Make a union of following relation R', with R.

$R' = \{(a, a), (b, b), (c, c)\}$.

$\therefore R_1 = R \cup R' = \{(a,b), (b,c), (a,c), (a,a), (b,b), (c,c)\}$.

ii) Symmetric closure:

To make R symmetric closure, we need to make union of following relation R'' with R,

$R'' = \{(b,a), (c,b), (c,a)\}$.

$\therefore R_2 = R \cup R''$

$= \{(a,b), (b,c), (a,c)\} \cup \{(b,a), (c,b), (c,a)\}$.

$R_2 = \{(a,b), (b,c), (a,c), (b,a), (c,b), (c,a)\}$.

iii) Transitive closure:

Since R is transitive relation defined on set A,

$\therefore$ R is transitive closure of itself.

★ Equivalence Classes:

Let 'R' be the equivalence relation on a set A and $a \in A$. Then the set of all elements x of A, which are related to a by R. (i.e xRa (or) $(x,a) \in R$). is called equivalence class of 'a' and it is denoted by $[a]$ or $R(a)$.

$\therefore a = R(a) = \{x : xRa\}$.

(or) $a = R(a) = \{x : (x,a) \in R\}$.

Example:

Let $A = \{a,b,c\}$ and equivalence relation R defined on set A is $R = \{(a,a), (a,b), (b,a), (b,b), (c,c)\}$.

$\therefore$ Equivalence class of elements of A are,

$[a] = R(a) = \{a,b\}$

$[b] = R(b) = \{a,b\}$

$[c] = R(c) = \{c\}$.

Remark:

1. Equivalence classes defined on set A are either similar or distinct.

(OR)

Theorem:

Any two equivalence classes on set A are disjoint (or) identical

★ Partition of a set:

Let A be any non-empty set. Suppose there exists non-empty sub-sets $a_1, a_2, \ldots, a_n$ of set A called partition of A if following conditions hold:

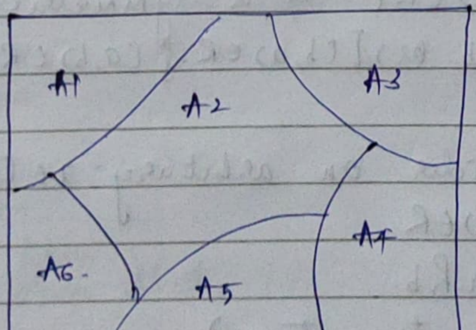
i) $A = A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

ii) Any two sub-sets of $A_1, A_2, \ldots A_n$ are disjoint.
i.e $A_i \cap A_j = \phi$, $i \neq j$.

Remark:

1. Evidently, ~~equival~~ distinct equivalence classes defined on set A makes a partition of set A.

A



∴ A1, A2, A3, A4, A5 & A6 makes a partition of set A.

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★ Theorem:

Let R be an equivalence relation on set A. Then,

i) $\forall a \in A$, $[a] \neq \phi$.

ii) if $b \in [a]$, then $[a] = [b]$ when $a, b \in A$;

iii) $\forall a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \phi$;

iv) A is the union of all equivalence closure with respect to R.

i.e $\quad A = \bigcup_{a \in A} [a]$.

Proof:

Given that R is equivalence Relation

i.e R is reflexive, symmetric and transitive.

i) Let us consider an arbitary $a \in A$,

$\Rightarrow aRa$ (or) $(a,a) \in R$ ($\because R$ is reflexive).

$\Rightarrow a \in [a]$, $\forall a \in A$.

$\Rightarrow [a] \neq \phi$, $\forall a \in A$.

ii) Given that, $b \in [a]$.

$\Rightarrow bRa$ (or) $(b,a) \in R$ ($\because$ By def^n of equivalence class).

$\Rightarrow aRb$ (or) $(a,b) \in R$ ($\because R$ is symmetric)

$\Rightarrow bRa$ and aRb. (or) $[(b,a) \in R$ & $(a,b) \in R]$.

Next, let us consider an arbitary $x \in [a]$ — ①

$\Rightarrow xRa$ (or) $(x,a) \in R$.

$\Rightarrow xRa$ and aRb.

$\Rightarrow xRb$ ($\because R$ is transitive)

$\Rightarrow x \in [b]$. — ②

from ① and ②, we get

$[a] \subseteq [b]$. — ③

And, let us consider an arbitary $y \in [b]$ — (4)

$\Rightarrow yRb$ (or) $(y,b) \in R$.

$\Rightarrow yRb$ and bRa

$\Rightarrow yRa$ ($\because R$ is transitive).

$\Rightarrow y \in [a]$. — (5)

from (4) & (5), we get

$[b] \subseteq [a]$ — (6).

From (3) & (6), we get

$[a] = [b]$

iii) Let us consider $a, b \in A$.

Suppose $[a] \cap [b] \neq \phi$.

$\Rightarrow \exists\, x \in A \ni x \in [a] \cap [b]$.

$\Rightarrow x \in [a]$ and $x \in [b]$.

By the proof of (ii), we get

$\Rightarrow [a] = [x]$ and $[b] = [x]$.

$\Rightarrow [a] = [b] = [x]$.

$\Rightarrow [a] = [b]$, $\forall a, b \in A$.

iv) Let us consider $a \in A$

$\Rightarrow a \in [a]$. ($\because R$ is reflexive relation).

$$\Rightarrow a \in \bigcup_{a \in A} [a]$$

$$\Rightarrow A \subseteq \bigcup_{a \in A} [a] \quad \text{— (7)}$$

Now, we ~~known~~ know that,

$[a] \subseteq A$.

$$\bigcup_{a \in A} [a] \subseteq A \quad \text{— (8)}$$

from (7) & (8), we get

$$\boxed{A = \bigcup_{a \in A} [a]}$$