Unit-3:

ADVANCED COUNTING TECHNIQUE

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★ **Sterling Number of Second Kind:**

Sterling number of second kind is the number of partition of set of size 'm' into 'n' non-empty subsets.

(OR)

Sterling number of second kind is the number of possible ways to assign 'm' objects into 'n' identical places (boxes) with no place left empty.

★ **Properties:**

1) $S(m,1) = 1, \quad \forall\, m \geq 1$
2) $S(m,m) = 1, \quad \forall\, m \geq 1$

★ **Recurrence Relation of Sterling number of second kind:**

If 'm' objects are kept in 'n' identical places with no place left empty and $m \geq n$, then Recurrence Relation of sterling number of second kind is:

$$\boxed{S(m,n) = S(m-1, n-1) + n\,S(m-1, n)}$$

Proof:

~~If~~ Let $a_1, a_2, a_3, \ldots a_m$ are m objects.

Then $S(m,n)$ is the count of number of ways of distributing m-objects in n-identical places.

⟹ There are $S(m-1, n-1)$ ways of distributing (assigning) $a_1, a_2, \ldots, a_{m-1}$ ($m-1$ objects) into $(n-1)$ places by ~~knowing~~ keeping a_m^{th} object in n^{th} place.

Next, assigning $a_1, a_2, \ldots a_{m-1}$ objects in all n-places has $S(m-1, n)$ ways. and placing a_n^{th} object in n-places has n- no of ways.

$\therefore S(m-1, n)$ is total no of ways in which a_m^{th} object is kept in n-places.

$\therefore$ Total no of ways of assigning m-objects in n-places is

$$\boxed{S(m,n) = S(m-1, n-1) + n\,S(m-1, n)}$$

★ Table consisting of possible Stirling Number of second kind:

$m = 1$	$\overset{1}{S(1,1)}$				
$m = 2$	$\overset{1}{S(2,1)}$	$\overset{1}{S(2,2)}$			
$m = 3$	$\overset{1}{S(3,1)}$	$\overset{3}{S(3,2)}$	$\overset{1}{S(3,3)}$		
$m = 4$	$\overset{1}{S(4,1)}$	$\overset{7}{S(4,2)}$	$\overset{6}{S(4,3)}$	$\overset{1}{S(4,4)}$	
$m = 5$	$\overset{1}{S(5,1)}$	$\overset{15}{S(5,2)}$	$\overset{25}{S(5,3)}$	$\overset{10}{S(5,4)}$	$\overset{1}{S(5,5)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

★ General Formula to find Stirling No of second Kind:

If 'm' objects are kept in 'n' identical cases, then, we have the following formula to find Stirling no of second kind:

$$\boxed{S(m,n) = \frac{1}{n!}\sum_{k=0}^{n} (-1)^k\, {}^nC_k\, (n-k)^m}$$

★ General Formula to find no of onto functions:

Let A and B be two finite sets with $n(A) = m$ & $n(B) = n$, where $m \geq n$. Then number of onto functions from A to B is denoted by $P(m,n)$ and given by the formula

$$P(m,n) = n!\, S(m,n)$$

(OR)

$$P(m,n) = \sum_{k=0}^{n} (-1)^k\, {}^nC_k\, (n-k)^m$$

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★ Examples:

(1) Evaluate $S(8,5)$ and $S(7,4)$.

Sol: Let $S(8,5)$

$\therefore$ $m = 8$, $n = 5$.

$\therefore$ General formula Sterling number of second kind:

$$S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, (n-k)^m.$$

$$S(8,5) = \frac{1}{5!} \sum_{k=0}^{5} (-1)^k \, {}^5C_k \, (5-k)^8.$$

$$S(8,5) = 1050$$

$S(7,4)$; $m = 7$ & $n = 4$

$$S(7,4) = \frac{1}{4!} \sum_{k=0}^{4} (-1)^k \, {}^4C_k \, (4-k)^7.$$

$$S(7,4) = 350$$

(2) A chemist who has 5 assistants is engaged in a research project that calls for 9 compounds must be synthezied. How many ways can the chemist assign these to the 5 assistants, so that each is working on atleast one synthesis.

Sol: Given that, there are 5 assistants,

$\therefore$ $n = 5$

And 9 synthesis.

$m = 9$

$\therefore$ General Formula to find no. of onto functions,

$$P(m,n) = \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, (n-k)^m$$

$$P(9,5) = \sum_{k=0}^{5} (-1)^k \, {}^5C_k \, (5-k)^9.$$

$$P(9,5) = 834120.$$

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③ There are 6 programmers who can assist 8 executives. In how many ways can the executives be assisted so that each programmer assists at least one executive.

Sol: Given that, there are 6 programmers,

$\therefore n = 6.$

And 8 executives

$\therefore m = 8$

$\therefore$ General Formula to find no of onto function.

$$P(m,n) = \sum_{K=0}^{n} (-1)^K \, {}^nC_K \, (n-K)^m$$

$$P(8,6) = \sum_{K=0}^{6} (-1)^K \, {}^6C_K \, (6-K)^8$$

$$P(8,6) = 191520.$$

④ If A & B are 2 finite sets with $n(A) = 5$ & $n(B) = 3$. Then find number of onto functions from set A to B.

Sol: Given that,

$n(A) = 5$

$n(B) = 3.$

General formula to find no of onto function.

$$P(m,n) = \sum_{K=0}^{n} (-1)^K \, {}^nC_K \, (n-K)^m$$

$$P(5,3) = \sum_{K=0}^{3} (-1)^K \, {}^3C_K \, (3-K)^5$$

$$\therefore P(5,3) = 150$$

★ ~~Pigeonhole~~ Principle of Inclusion and Exclusion:

Statement: Let A and B are any two sets. Then the number of elements in union of A and B is, the sum of the numbers of elements of A and B minus the number of elements in the intersection

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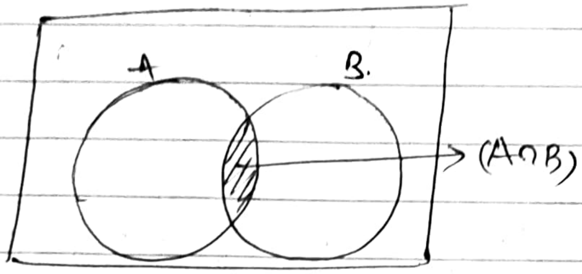
of A and B.

$$\boxed{n(A\cup B) = n(A) + n(B) - n(A\cap B)}$$



Similarly,

If A, B and C are any three sets, then the number of elements in union of A, B & C is,

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C).$$

★ Pigenhole Principle:
Statement:

If m pigeons occupy n-pigeonholes and $m>n$, then two or more occupy pigeon same pigeonhole.
(OR)
If 'm' ~~pigeonhole~~ pigeon's occupy 'n' pigeonholes & $m>n$. Then atleast one pigeonhole must contain two or more- pigeons in it.

★ Generalized Pigeonhole Principle:
Statement:

If m- pigeo~~nholes~~ occupy n- pigeonholes then atleast one pigeonhole must contain $[(m-1)/n]+1$ or more pigeons where [ ] stands for greatest integer function.

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Proof:
We need to prove this theorem by method of contradiction.
Suppose every pigeonhole must contain $\left[\frac{m-1}{n}\right]$ or less no of pigeons in it.
$\therefore$ Total no of pigeons in 'n' pigeonholes is

$$n\left[\frac{m-1}{n}\right] \leq n\left(\frac{m-1}{n}\right) = (m-1).$$

$\therefore$ Total no of pigeons are less then $(m-1)$.
It contradicts the fact that there are 'm' pegions.
$\therefore$ At least one pigeonhole must contain $\left[\frac{m-1}{n}\right]+1$

or more pegions in it.

* Examples.

① Prove that in a set of 13 children atleast 2 have the birthday during same month.

Sol: Let us consider 13 children as pegions.

$m = 13.$

And 12 months as pegionholes.

$n = 12.$

$\therefore$ By Generalized pegionhole principle atleast one month, must be the birth month of $\left[\frac{m-1}{n}\right]+1$ or more children.

$$\therefore \left[\frac{m-1}{n}\right]+1 = \left[\frac{13-1}{12}\right]+1 = \left[\frac{12}{12}\right]+1 = [1]+1 = 1+1 = 2.$$

② If 7 cars carry 26 passengers prove that atleast one car must have 4 or more passengers.

Sol: Given that,

$m = 26$ $\quad$ $n = 7$ (cars)

$\therefore$ By generalized pegion hole principle atleast one car must contain $\left\lceil \frac{m-1}{n} \right\rceil + 1$ or more passengers.

$\therefore \left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{26-1}{7} \right\rceil + 1 = \left\lceil \frac{25}{7} \right\rceil + 1 = \lceil 3.57 \rceil + 1 = 3 + 1 = \boxed{4}$.

Imp.

(3) What should be the minimum number of students that atleast two students have their last name starts with same english letter.

Sol^n: Given that,

$n = 26$ (english alphabet)

$m = ?$

Also given that 2 or more students have their last name starts with same english letter.

$\therefore$ By generalized pigeonhole principle,

$\left\lceil \frac{m-1}{n} \right\rceil + 1 \geq 2.$

$\left\lceil \frac{m-1}{26} \right\rceil + 1 \geq 2.$

$\left\lceil \frac{m-1}{26} \right\rceil \geq 2 - 1$

$\left\lceil \frac{m-1}{26} \right\rceil \geq 1$

$\left(\frac{m-1}{26} \right) \geq \cancel{2}1$ [If it is -26 then inequality changes i.e $\leq$.

$m - 1 \geq 26.$

$\boxed{m \geq 27}$

$\therefore$ minimum 27 students should be there.

selection : ^nC_r
combinations : ^nP_r

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4) Find least number of ways of choosing 3 different no's. from 1 to 10 so that all choices have have the same sum.

Sol^n: No. of ways of chosing 3-digits from 1 to 10 is $^{10}C_3$.

$= \frac{10!}{7!\,3!} = 120.$

$\therefore \boxed{m = 120.}$

The smallest sum of 3-digits by chosing 1, 2 and 3, i.e $1+2+3 = 6.$

The largest sum of 3-digits by chosing 8, 9 and 10, i.e $8+9+10 = 27$

$\therefore$ Sum of 3-digits lies b/w 6 and 27 is 22 in number

$\therefore \boxed{n = 22}$

$\therefore$ By generalized pigeonhole principle, atleast $\left[\frac{m-1}{n}\right]+1$ choices have the same sum.

$\therefore \left[\frac{m-1}{n}\right]+1 = \left[\frac{120-1}{22}\right]+1 = \left[\frac{119}{22}\right]+1.$

$= [5.40]+1$

$= 5+1$

$= 6.$

$\therefore$ 6 different choices have same least sum.

★ Recurrence Relation:

A recurrence relation for the sequence $\{a_n\}$ (or) $(a_0, a_1, \ldots\ldots a_{n-1}\ldots)$ is an equation that comprises a_n in terms of one or more of previous terms of the sequence, namely $a_1, a_2, a_3 \ldots a_n$ for all integers $n \geq 0$.

★ **Solution of Recurrence Relation:**

A sequence $\{a_n\}$. $(a_1, a_2, a_3 \dots a_n \dots)$ is called a solution of recurrence relation if its terms satisfy the recurrence relation.

Example:

The sequence 1, 3, 7, 15, 31, ... is the solution of recurrence relation $\boxed{a_n = 2a_{n-1} + 1}$

Note: The recurrence relation,

$$4a_{n+3} - a_{n+2} + 11a_{n+1} - 6a_n = 0.$$

can also be written as,

$$4Y_{n+3} - Y_{n+2} - 11Y_{n+1} - 6Y_n = 0.$$

OR

$$4U_{n+3} - U_{n+2} - 11U_{n+1} - 6U_n = 0.$$

OR

$$4f(x+3) - f(x+2) + 11f(x+1) - 6f(x) = 0.$$

★ **Order of a recurrence relation:**

It is the difference b/w the largest and smallest subscript appearing in its relation.

Example:

$$4a_{n+3} - a_{n+2} + 11a_{n+1} - 6a_n = 0.$$

order is $n+3-n = 3$.

★ **Linear Recurrence Relation with constant co-efficients:**

The linear recurrence relation with constant co-efficients of order K is of the form

$$\boxed{C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots\dots + C_K a_{n-K} = f(n).}$$

where $f(n)$ is the function of variable n only and

$c_0, c_1, c_2, \ldots c_k$ are constants.

* Homogeneous and Non-homogeneous linear recurrence relation:
The linear recurrence relation.
$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$
is said to be homogeneous linear recurrence relation if $f(n) = 0$. Otherwise it is called non-homogeneous linear recurrence relation.

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* Generating function:
Let $\{a_n\}$ or $(a_0, a_1, a_2, \ldots)$ be a sequence of real no's then the generating function of $\{a_n\}$ is denoted by $G(a,z)$ and defined by,

$$\boxed{G(a,z) = a_0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n=0}^{\infty} a_n z^n}$$

where z is a variable.

* Some Standard Generating function:
① If $\{a_n\}$ is a sequence and $a_n = c$, $\forall\, n \geq 0$. where c is constant, then find generating function of $\{a_n\}$.
Sol^n: Let $\{a_n\}$ be a sequence when $a_n = c$, $\forall\, n \geq 0$.
By generating function.

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n.$$

$$G(a,z) = \sum_{n=0}^{\infty} c z^n.$$

$$G(a,z) = c + cz + cz^2 + cz^3 + \cdots$$

$$\therefore G(a,z) = c(1 + z + z^2 + z^3 + \cdots).$$

$$G(a,z) = c\left(\frac{1}{1-z}\right)$$

$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$

$$\therefore \boxed{G(a,z) = \frac{c}{1-z}}$$

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(2) If $\{a_n\}$ is a sequence and $a_n = b^n$, $\forall\, n \geq 0$, then find generating function of $\{a_n\}$.

Solⁿ: Let $\{a_n\}$ be a sequence when $a_n = b^n$, $\forall\, n \geq 0$.

By generating function,

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n.$$

$$= \sum_{n=0}^{\infty} b^n z^n.$$

$$= 1 + bz + b^2z^2 + b^3z^3 + \cdots$$

$$= 1 + (bz) + (bz)^2 + (bz)^3 + \cdots$$

$$\boxed{G(a,z) = \frac{1}{1-bz}.}$$

Note: If $\{a_n\}$ is a sequence & $a_n = c.b^n$, $\forall\, n \geq 0$ then.

$$\boxed{G(a,z) = \frac{c}{1-bz}}$$

(3) If $\{a_n\}$ is a sequence & $a_n = n$ $\forall\, n \geq 0$, then find generating function of $\{a_n\}$.

Solⁿ: Let $\{a_n\}$ be sequence when $a_n = n$, $\forall\, n \geq 0$.

By generating function

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} n z^n$$

$$= 0 + 1z + 2z^2 + 3z^3 + \cdots$$

$$= z(1 + 2z + 3z + \cdots)$$

$$G(a,z) = z\left(\frac{1}{(1-z)^2}\right)$$

$$\boxed{G(a,z) = \frac{z}{(1-z)^2}}$$

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Note: If $\{a_n\}$ is sequence & $a_n = (n+1)$, $\forall n \geq 0$ then

$$G(a,z) = \frac{1}{(1-z)^2}$$

Sl. No	Sequence $\{a_n\}$	Generating Function $G(a,z)$.
1.	$a_n = c$, $\forall n \geq 0$	$\frac{c}{1-z}$
2.	$a_n = b^n$, $\forall n \geq 0$	$\frac{1}{1-bz}$
3.	$a_n = n$, $\forall n \geq 0$	$\frac{z}{(1-z)^2}$
4.	$a_n = (n+1)$, $\forall n \geq 0$	$\frac{1}{(1-z)^2}$
5.	$a_n = c \cdot b^n$, $\forall n \geq 0$.	$\frac{c}{1-bz}$.

★ Solution of Homogeneous Linear Recurrence Relation By Generating Function:

Consider a homogeneous linear recurrence relation

$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$ — (1), $\forall n \geq k$.

Then following steps to be followed to solve (1) are:

Step 1:

Multiply z^n on both sides of eq^n (1) & take summation from $n = k$ to ∞.

Step 2:

Write each term in the form of $G(a,z)$.

Step 3:

Solve $G(a,z)$ by using standard generating function for the sequence $\{a_n\}$.

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Examples:

(1) Solve homogeneous linear recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, $\forall n \geq 2$ given initial condition as $a_1 = 5$ and $a_2 = 3$ by using generating function.

Sol^n: Let,

$$a_n = 3a_{n-1} - 2a_{n-2} \quad \forall n \geq 2.$$

$$a_n - 3a_{n-1} + 2a_{n-2} = 0 \quad \text{——— (1)}$$

Put $n = 2$ in (1), we get.

$$a_2 - 3a_1 + 2a_0 = 0$$

$$3 - 3(5) + 2a_0 = 0$$

$$3 - 15 + 2a_0 = 0$$

$$-12 + 2a_0 = 0$$

$$2a_0 = 12$$

$$\boxed{a_0 = 6}$$

Multiply z^n on both sides of eq(1) & take summation from $n = 2$ to ∞

$$\sum_{n=2}^{\infty} (a_n - 3a_{n-1} + 2a_{n-2}) z^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n z^n - 3\sum_{n=2}^{\infty} a_{n-1} z^n + 2\sum_{n=2}^{\infty} a_{n-2} z^n = 0.$$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} a_n z^n + a_0 + a_1 z}_{} - a_0 - a_1 z - 3\sum_{n=2}^{\infty} a_{n-1} z^{n-1} \cdot z^1 +$$

$$2\sum_{n=2}^{\infty} a_{n-2} z^{n-2} \cdot z^2 = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - 6 - 5z - 3z\left(\sum_{n=2}^{\infty} a_{n-1} z^{n-1} + a_0 - a_0\right)$$

$$+ 2z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0.$$

By step

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - 6 - 5z - 3z\left(\sum_{n=0}^{\infty} a_n z^n - 6\right) + 2z^2\left(\sum_{n=0}^{\infty} a_n z^n\right) = 0.$$

$$\Rightarrow G(a,z) - 6 - 5z - 3z\,(G(a,z) - 6) + 2z^2 G(a,z) = 0.$$

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$$G(a,z) - 3zG(a,z) + 18z + 2z^2G(a,z) = 6 + 5z.$$

$$G(a,z)\,[1 - 3z + 2z^2] + 18z = 6 + 5z.$$

$$G(a,z)\,[1 - 3z + 2z^2] = 6 + 5z - 18z.$$

$$G(a,z)\,[2z^2 - 3z + 1] = 6 - 13z.$$

$$G(a,z)\,[2z^2 - 2z - z + 1] = 6 - 13z.$$

$$G(a,z)\,[2z(z-1) - 1(z-1)] = 6 - 13z.$$

$$G(a,z)\,[(2z-1)(z-1)] = 6 - 13z.$$

$$G(a,z) = \frac{6 - 13z}{(2z-1)(z-1)} \quad \text{—— (2)}.$$

By partial function.

$$\frac{6 - 13z}{(1-z)(1-2z)} = \frac{A}{(1-z)} + \frac{B}{(1-2z)} \quad \text{—— (3)}.$$

$$6 - 13z = A(1-2z) + B(1-z) \quad \text{—— (4)}.$$

put $z=1$ in (4), we get.

$$6 - 13(1) = A(1-2(1)) + B(1-1)$$

$$6 - 13 = -A$$

$$\boxed{A = 7}$$

put $z = 1/2$ in (4), we get

$$6 - 13\left(\tfrac{1}{2}\right) = A(1-2(1/2)) + B(1-1/2)$$

$$6 - \frac{13}{2} = \frac{B}{2}$$

$$-1 = B; \quad \boxed{B = -1}$$

$\therefore$ eq^n (3) becomes,

$$G(a,z) = \frac{7}{(1-z)} - \frac{1}{(1-2z)}$$

$\therefore$ By using standard generating function,

$$\boxed{a_n = 7 - 2^n}$$

$\because \frac{c}{1-z} \Rightarrow a_n = c \; : \frac{7}{1-z} ; a_n = 7$

$\frac{c}{1-bz} \Rightarrow a_n = c.b^n ; \frac{1}{1-2z} ; a_n = 2^n$

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(2) Solve homogeneous linear recurrence relation $a_n = 2a_{n-1} - a_{n-2}$, $\forall n \geq 2$, given that $a_1 = 1$ and $a_2 = 4$ by generating function.

Sol: Let $a_n = 2a_{n-1} - a_{n-2}$ —— (1).

put $n=2$ in (1).

$a_2 = 2a_1 - a_0$.

~~$a_0 = a_2 - 2a_1 + 1$~~ $\qquad 4 = 2 - a_0$.

~~$a_0 = 4 - 2(1) + 1$~~ $\qquad a_0 = 2 - 4$

$\cancel{\boxed{a_0 = 3}}$ $\qquad \boxed{a_0 = -2}$

Multiply z^n on both sides of equation (1) and take summation from $n=2$ to ∞

$$\Rightarrow \sum_{n=2}^{\infty} (a_n - 2a_{n-1} + a_{n-2}) z^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n z^n - 2\sum_{n=2}^{\infty} a_{n-1} \cdot z^n + \sum_{n=2}^{\infty} a_{n-2} \cdot z^n = 0.$$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} a_n z^n + a_0 + a_1 z}_{} - a_0 - a_1 z - 2 \cdot \sum_{n=2}^{\infty} a_{n-1} z^{n-1} \cdot z + \sum_{n=2}^{\infty} a_{n-2} z^{n-2} \cdot z^2 = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - a_0 - a_1 z - 2z\left(\underbrace{\sum_{n=2}^{\infty} a_{n-1} z^{n-1} + a_0}_{} - a_0\right) + z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n + 2 - z - 2z\left(\sum_{n=0}^{\infty} a_n z^n + 2\right) + z^2 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\Rightarrow G(a,z) + 2 - z - 2z\left(G(a,z) + 2\right) + z^2 G(a,z) = 0$$

$$G(a,z) + 2 - z - 2zG(a,z) - 4z + z^2 G(a,z) = 0.$$

$$G(a,z)(1 - 2z + z^2) = 5z - 2.$$

$$G(a,z)(1-z)^2 = 5z - 2.$$

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$$G(a,z) = \frac{5z-2}{(1-5z)^2} \quad \text{—— (2)}.$$

$\therefore$ By partial function we have.

$$\frac{5z-2}{(1-z)^2} = \frac{A}{(1-z)} + \frac{B}{(1-z)^2} \quad \text{—— (3)}.$$

$$\frac{5z-2}{\cancel{(1-z)^2}} = \frac{A(1-z)+B}{\cancel{(1-z)^2}}$$

$$5z-2 = A(1-z) + B. \quad \text{—— (4)}.$$

put $z=1$ in eq (4), we get

$$5(1)-2 = A(1-1)+B$$

$$\boxed{B=3}$$

put $z=0$ in eq (4), we get

$$5(0)-2 = A(1-0)+B$$

$$-2 = A+3$$

$$\boxed{A=-5}$$

$\therefore$ Equation (2) becomes

$$G(a,z) = \frac{-5}{(1-z)} + \frac{3}{(1-z)^2}$$

$\therefore$ By standard generating function, of a_n,

$$a_n = -5 + 3(n+1)$$

$$a_n = -5 + 3n + 3$$

$$\therefore \boxed{a_n = 3n-2}$$

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★ Solution of Non Homogeneous linear recurrence relations by using generating function.

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Example:

(1) Solve non-homogeneous linear recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $\forall\, n \geq 0$ given that $a_0 = 2$ & $a_1 = 1$ by generating function.

Sol^n: Let $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $\forall\, n \geq 0$ —— (1).

Multiply z^n on both sides of eq (1). and take summation from $n=0$ to ∞

$$\therefore \sum_{n=0}^{\infty} (a_{n+2} - 2a_{n+1} + a_n) z^n = \sum_{n=0}^{\infty} 2^n z^n.$$

$a_n = b^n$, $\frac{1}{1-bz}$.

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2} z^n - 2\sum_{n=0}^{\infty} a_{n+1} z^n + \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} 2^n z^n.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2} z^{n+2} z^{-2} - 2\sum_{n=0}^{\infty} a_{n+1} z^{n+1} z^{-1} + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}.$$

$$\Rightarrow \frac{1}{z^2}\left\{\sum_{n=0}^{\infty} a_{n+2} z^{n+2} + a_0 + a_1 z - a_0 - a_1 z\right\} - \frac{2}{z}\left\{\sum_{n=0}^{\infty} a_{n+1} z^{n+1} + a_0 - a_0\right\}$$
$$+ \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}.$$

$$\Rightarrow \frac{1}{z^2}\left\{\sum_{n=0}^{\infty} a_n z^n - 2 - 1z\right\} - \frac{2}{z}\left\{\sum_{n=0}^{\infty} a_n z^n - 2\right\} + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}.$$

send z^2 to RHS, take lcm

$$\sum_{n=0}^{\infty} a_n z^n - 2 - z - 2z\left[\sum_{n=0}^{\infty} a_n z^n - 2\right] + z^2 \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z} \times z^2$$

$$G(a,z) - 2 - z - 2z\,G(a,z) + 4z + z^2 G(a,z) = \frac{z^2}{1-2z}.$$

$$G(a,z)\left[1 - 2z + z^2\right] = \frac{z^2}{1-2z} + (2-3z)$$

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$$G(a,z)(1-z)^2 = \frac{z^2 + (2-3z)(1-2z)}{(1-2z)}.$$

$$G(a,z)(1-z)^2 = \frac{z^2 + 2 - 4z - 3z + 6z^2}{(1-2z)}.$$

$$G(a,z) = \frac{7z^2 - 7z + 2}{(1-2z)(1-z)^2} \quad \text{---(2)}.$$

By partial fraction :

$$\frac{7z^2 - 7z + 2}{(1-2z)(1-z)^2} = \frac{A}{(1-2z)} + \frac{B}{(1-z)} + \frac{C}{(1-z)^2}$$ $\because$ LCM is $(1-z)^2(1-2z)$. ---(3).

$$7z^2 - 7z + 2 = A(1-z)^2 + B(1-2z)(1-z) + C(1-2z) \quad \text{---(4)}$$

Put $z = 1/2$ in eq (4), we get

$$\frac{7}{4} - \frac{7}{2} + 2 = A(1-1/2)^2$$

$$\frac{7-14+8}{\cancel{4}} = A/\cancel{4}$$

$$\boxed{A = 1}$$

put $z = 1$ in eq (4)

$$7 - 7 + 2 = C(-1)$$

$$\boxed{C = -2}$$

put $z = 0$ in eq (4)

$$2 = A(1)^2 + B(1) + C(1)$$

$$B = 2 - A - C$$

$$B = 2 - 1 + 2$$

$$\boxed{B = 3}$$

$\therefore$ Equation (3) becomes,

$$G(a,z) = \frac{1}{1-2z} + \frac{3}{1-z} - \frac{2}{(1-z)^2}.$$

$\therefore$ By generating function

$$a_n = 2^n + 3 - 2(n+1)$$

$$a_n = 2^n + 3 - 2n - 2$$

$$\boxed{a_n = 2^n - 2n + 1}$$

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Order $= n-(n-k)$
$= k.$

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* Solution of Homogeneous linear Recurrence Relation by Characteristic Root Method:

Consider homogeneous linear recurrence relation:

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$ —— (1).

Characteristic equation (or auxillary eq^n) of degree k.

$C_0 r^n + C_1 r^{n-1} + C_2 r^{n-2} + \cdots + C_k r^{n-k} = 0.$

$r^{n-k}[C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \cdots + C_k] = 0.$

$\boxed{C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \cdots + C_k = 0}$ —— (2)

Roots of (2) are $\alpha_1, \alpha_2, \cdots \alpha_k$.

Solution of (1) has following case.

Case 1: Roots are real and unequal than sol^n of (1) is,

$\boxed{a_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + \cdots + A_k\alpha_k^n}$

Case 2: Roots are equal, $\alpha_1 = \alpha_2$, than sol^n of (1) is,

$\boxed{a_n = (A_1 + nA_2)\alpha_1^n + A_3\alpha_3^n + \cdots + A_k\alpha_k^n}$

Case 3: Roots are complex.

$\boxed{a_n = A_1(\alpha_1 + i\beta_1)^n + A_2(\alpha_1 - i\beta_1)^n + \cdots}$

Case 4: Two complex roots are equal i.e $\alpha_1 \pm i\beta_1 = \alpha_2 \pm i\beta_2$.

$\boxed{a_n = (A_1 + nA_2)(\alpha_1 + i\beta_1)^n + (A_3 + nA_4)(\alpha_1 - i\beta_1)^n + \cdots}$

* Examples.

(1) Solve homogeneous linear recurrence relation $a_{n+2} - 3a_{n+1} + 2a_n = 0$, given that $a_0 = 1$, $a_1 = 3$ by characteristic root method.

Sol^n: Let, $a_{n+2} - 3a_{n+1} + 2a_n = 0$ —— (I).

Replace a_{n+2} by r^{n+2}, a_{n+1} by r^{n+1} & a_n by r^n.

$r^{n+2} - 3r^{n+1} + 2r^n = 0$

$r^n(r^2 - 3r^1 + 2) = 0.$

$\boxed{r^2 - 3r + 2 = 0}$ — (2) (characteristic eq^n).

$\therefore$ Roots are $\alpha_1 = 1$ & $\alpha_2 = 2$.

$\therefore$ General solution of eq^n (1) is,

$a_n = A_1\alpha_1^n + A_2\alpha_2^n$

$a_n = A_1(1^n) + A_2(2^n).$

$\boxed{a_n = A_1 + A_2 2^n}$ — (3).

Put $n=0$ in (3). and $a_0 = 1$

$a_0 = A_1 + A_2 . 2^0.$

$1 = A_1 + A_2.$ — (4).

Put $n=1$ in (3) and $a_1 = 3$.

$a_1 = A_1 + A_2 2^1.$

$3 = A_1 + A_2(2).$ — (5).

Solving (4) and (5).

$1 = \cancel{A_1} + A_2.$

$3 = \cancel{A_1} + 2A_2$

$-2 = -A_2$

$\boxed{A_2 = 2}$

Put A_2 in eq^n (4)

$\boxed{A_1 = -1}$

$\therefore$ Particular solution of (1) is

$a_n = (-1) + (2)2^n$

$\boxed{a_n = 2^{n+1} - 1}$

(2) Solve homogeneous linear recurrence relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$, given that $a_2 = 6$, and $a_3 = 80$ by characteristic root method.

Sol^n: Let $a_n - 8a_{n-1} + 16a_{n-2} = 0.$ — (1)

Replace a_n by r^n, a_{n-1} by r^{n-1} & a_{n-2} by r^{n-2}.

$$r^n - 8r^{n-1} + 16r^{n-2} = 0$$

$$r^{n-2}(r^2 - 8r + 16) = 0$$

$$\boxed{r^2 - 8r + 16 = 0} \quad \text{—— (2)}$$

$\therefore$ Roots are $\alpha_1 = \alpha_2 = 4$.

$\therefore$ General solution of eq^n (1) is:

$$a_n = (A_1 + nA_2)\alpha_1^n$$

$$\boxed{a_n = (A_1 + nA_2)4^n} \quad \text{—— (3)}$$

Put $n = 2$ in (3) and $a_2 = 6$.

$$a_2 = (A_1 + 2A_2)4^2$$

$$6 = (A_1 + 2A_2)16.$$

$$6 = 16A_1 + 32A_2 \quad \text{—— (4)}$$

Put $n = 3$ in (3) and $a_3 = 80$.

$$a_3 = (A_1 + 3A_2)4^3$$

$$80 = 64A_1 + 192A_2. \quad \text{—— (5)}$$

Solving (4) and (5) gives,

$$A_1 = \frac{-11}{8} \qquad A_2 = \frac{7}{8}$$

$\therefore$ Particular solution of (1) is,

$$a_n = \left(\frac{-11}{8} + \frac{7n}{8}\right)4^n$$

$$\boxed{a_n = (-11 + 7n)\frac{4^{n-1}}{2}}$$

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* Solution of Non-Homogeneous linear recurrence relation by characteristic root method:

Consider a non-homogeneous linear recurrence relation of order k,

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n) \quad \text{--- (1)}.$$

General solution of recurrence relation (1) is,

$$\boxed{a_n = a_n^{(h)} + a_n^{(P)}} \quad \text{--- (2)}$$

where $a_n^{(h)}$ is the general solution of the homogeneous part of the recurrence relation (1) and $a_n^{(P)}$ is particular solution of (1).

The following are the cases to find $a_n^{(P)}$.

Case 1: If $f(n)$ is a polynomial of degree q and 1 is not a root of characteristic equation of homogeneous part of (1), then $a_n^{(P)}$ is as follows:

$$\boxed{a_n^{(P)} = B_0 + B_1 n + B_2 n^2 + \cdots + B_q n^q}$$

Case 2: If $f(n)$ is a polynomial of degree q and 1 is a root of multiplicity m of characteristic equation of homogeneous part of (1), then $a_n^{(P)}$ is as follows:

$$\boxed{a_n^{(P)} = n^m (B_0 + B_1 n + B_2 n^2 + \cdots + B_q n^q)}$$

Case 3: If $f(n) = \alpha b^n$, where 'α' and 'b' are constants and 'b' is not a root of characteristic equation of homogeneous part of (1), then $a_n^{(P)}$ is as follows:

$$\boxed{a_n^{(P)} = B_0 . b^n}$$

Case 4: If $f(n) = \alpha b^n$, where α and b are constants and 'b' is a root of multiplicity m of characteristic equation of homogeneous part of (1), then $a_n^{(P)}$ is as follows:

$$\boxed{a_n^{(P)} = B_0 n^m b^n}$$

where $B_0, B_1, B_2, \dots B_q$ are constants to be evaluated by using the fact then $a_n = a_n^{(P)}$ satisfying the recurrence relation (I).

* Example:

(1) Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8$, $n \geq 2$ and given that $a_0 = 1$ & $a_1 = 2$.

Sol^n: Let $a_n + 4a_{n-1} + 4a_{n-2} = 8$ —— (1).

Characteristic equation of (1) is,

$r^n + 4r^{n-1} + 4r^{n-2} = 0.$

$r^{n-2}[r^2 + 4r + 4] = 0$

$r^2 + 4r + 4 = 0.$ —— (2).

$(r+2)^2 = 0.$

$r = -2$ and $r = -2.$

$\therefore$ Roots are $\alpha_1 = \alpha_2 = -2$

$\therefore$ Solution of homogeneous part of (1) is,

$a_n^{(h)} = (A_1 + nA_2)\alpha_1^n$

$\boxed{a_n^{(h)} = (A_1 + nA_2)(-2)^n}$ —— (3).

Particular solution $a_n^{(P)}$ of eq^n (1). ($f(n) = 8$ is a polynomial of degree 0 and 1 is not a root of characteristic eq (2))

$\boxed{a_n^{(P)} = B_0}$ —— (4).

To find B_0, we use the fact that $a_n = a_n^{(P)} = B_0$. which satisfies recurrence relation (1).

$B_0 + 4B_0 + 4B_0 = 8.$ ($\because$ a_n is independent of n. i.e B_0)

$9B_0 = 8$

$\boxed{B_0 = 8/9}$ —— (5).

$\therefore$ General solution of recurrence relation (I) is,

$\boxed{a_n = (A_1 + nA_2)(-2)^n + 8/9}$ —— (6).

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Put $n=0$ in (6) and use $a_0=1$.

$a_0 = (A_1)(-2)^0 + 8/9$

$1 = A_1 + 8/9$

$\boxed{A_1 = 1/9}$

Put $n=1$ in (6) and use $a_1=2$

$a_1 = (A_1 + A_2)(-2)^1 + 8/9.$

$2 = -2A_1 - 2A_2 + 8/9.$

$2 = -\frac{2}{9} + \frac{8}{9} - 2A_2$

$2A_2 = \frac{6}{9} - 2$

$2A_2 = \frac{-12}{9}$

$\boxed{A_2 = -\frac{2}{3}}$

$\therefore$ Solution (6) becomes.

$$\boxed{a_n = \left(\frac{1}{9} - \frac{2n}{3}\right)(-2)^n + \frac{8}{9}}$$

(2) Solve the recurrence relation $a_{n+2} + 3a_{n+1} + 2a_n = 3^n, n \geq a.$
and given that $a_0 = 0$ and $a_1 = 1$.

Soln: Let $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$. —— (1)

Characteristic eqn of (1) is,

$r^{n+2} + 3r^{n+1} + 2r^n = 3^n$

$r^n(r^2 + 3r + 2) = \cancel{3^n}\ 0$

$r^2 + 3r + 2 = \cancel{3}\ 0.$ —— (2).

$r = -1$ and $r = -2$

$\therefore$ Roots are $\alpha_1 = -1$ and $\alpha_2 = -2$.

$\therefore$ Solution of homogeneous part of (1) is,

$a_n^{(h)} = A_1\alpha_1^n + A_2\alpha_2^n$

$\boxed{a_n^{(h)} = A_1(-1)^n + A_2(-2)^n}$ —— (3)

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Particular solution $a_n^{(P)}$ of eq^n ① ($f(n) = 3^n$ is a polynomial of degree 0 and 1 is not a root of characteristic equation ②)

$$\boxed{a_n^{(P)} = B_0 . b^n} \quad —④.$$

To find B_0, we use the fact that $a_n = a_n^{(P)} = B_0 . b^n$ which satisfies recurrence relation ①.

$$\cancel{B_0\, b^{(n+2)} + B_0\, b^{(n+1)} + B_0 = 3^n}$$

$$\cancel{B_0 (b^{(n+2)} + b^{(n+1)} + 1) = 3^n}$$

$$\cancel{B_0 . b^n (b^2 + b + 1) = 3^n}$$

$$B_0 . 3^{n+2} + 3\, B_0 3^{n+1} + 2 B_0 3^n = 3^n$$

$$B_0 3^n (3^2 + 3.3 + 2) = 3^n$$

$$\boxed{B_0 = 1/20}$$

$$\therefore \boxed{a_n^{(P)} = \frac{3^n}{20}}$$

∴ General solution of recurrence relation ① is,

$$\boxed{a_n = A_1 (-1)^n + A_2 (-2)^n + \frac{3^n}{20}} \quad — ⑤$$

Put $n = 0$ in ⑤ and use $a_0 = 0$.

$$a_0 = A_1 (-1)^0 + A_2 (-2)^0 + \frac{3^0}{20}$$

$$0 = A_1 + A_2 + \frac{1}{20}$$

$$\therefore A_1 + A_2 = -1/20 \quad — ⑥$$

Put $n = 1$ in ⑤ and use $a_1 = 1$

$$a_1 = A_1 (-1)^1 + A_2 (-2)^1 + \frac{3^1}{20}$$

$$1 = -A_1 - 2A_2 + \frac{3}{20}$$

$$A_1 + 2A_2 = \frac{3}{20} - 1$$

$$A_1 + 2A_2 = -\frac{17}{20} \quad — ⑦$$

$$\boxed{A_1 = 3/4} \qquad \boxed{A_2 = -4/5}$$

Solution ⑤ becomes,

$$\boxed{a_n = \left(\frac{3}{4}\right) (-1)^n + \left(-\frac{4}{5}\right) (-2)^n + \frac{3^n}{20}}$$