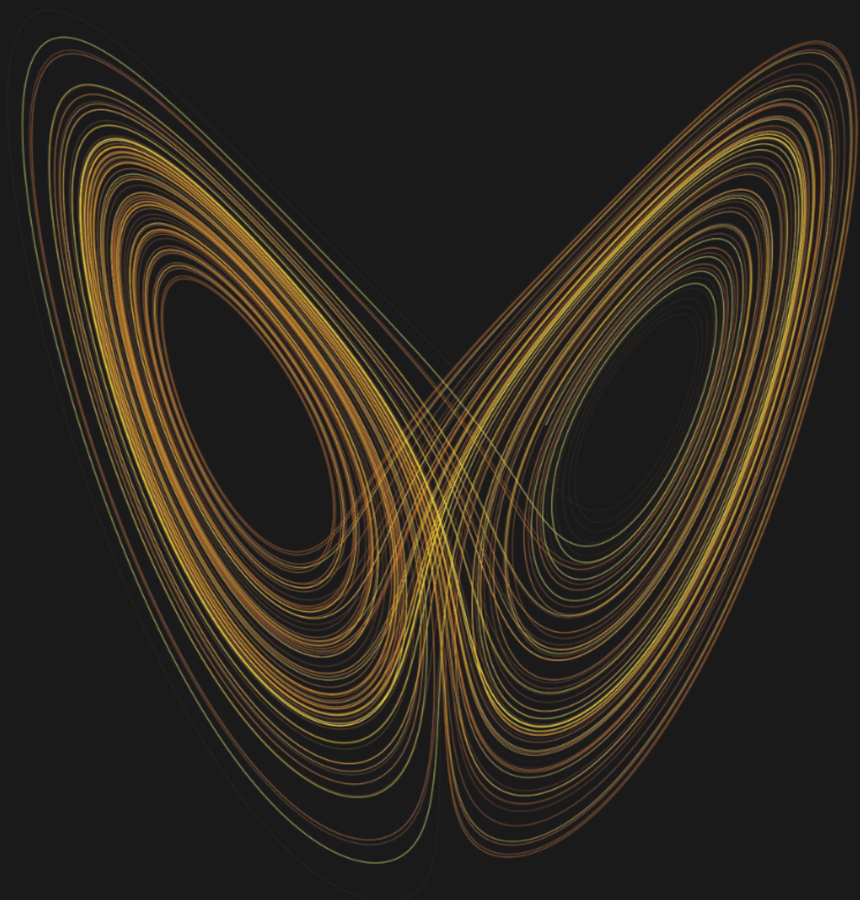
Non Linear Dynamics and Chaos

Applications to Supply-Demand Models



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1 Introduction

1.1 Motivation

It is quite compelling to start this report with a beautiful image of the Mandelbrot set shown below:

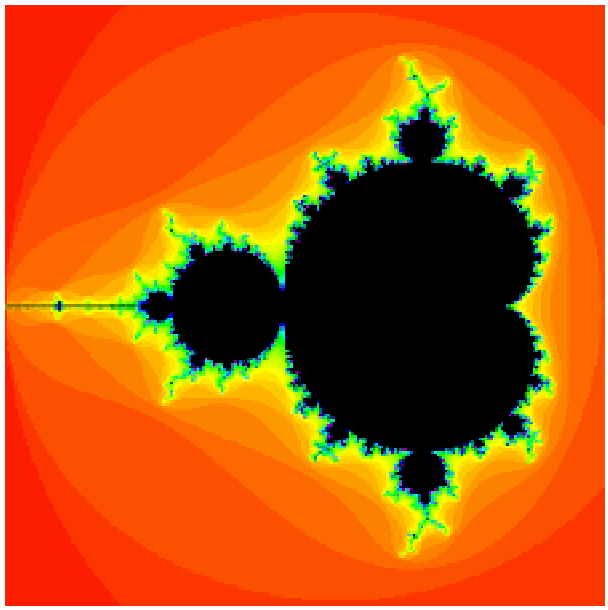


Figure 1: The Mandelbrot Set

Awesome right ? All it takes to produce this set are two complex numbers c and Z_0, and then get the rest of the points using the iterative formula

$$Z_{n+1} = {Z_n}^2 + c \tag{1}$$

The set of all values of c for which the iteration does not diverge constitute the elements of the Mandelbrot set.These points are coloured black in the figure.
The distinctive pattern which you observe is called as a fractal. Fractals are infinitely complex patterns that are self-similar across different scales. Driven by recursion, fractals are images of dynamic systems – the pictures of Chaos.
Chaos and Fractals are a part of even "grander" subject of dynamics. Dynamics deals with change, systems that evolve in time. Merriam Webster defines system as a regularly interacting or interdependent group of items forming a unified whole. A system can be anything right from an atom to an entire galaxy. However, in the scope of daily life problems,some examples could be say a moving car, or the economy of a country, or ecological systems.
The next thing we want to know about systems is that how do they change in time? For eg. If our system is a classical particle, Newtons second law governs the time evolution of the trajectory of the particle. For any dynamical system, if we know the governing differential equation and the initial conditions, we can find everything about that system in the future. The great eighteen century physicist Pierre Simon Laplace laid it out in a famous quote.
We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would knew all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit

these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.
There is one problem however which Laplace missed and that was of precision. No instrument in the world can give us infinitely precise initial conditions. Why is it so is a different matter altogether. Small differences in the initial conditions can give rise to big differences later in time. The figure below displays this.

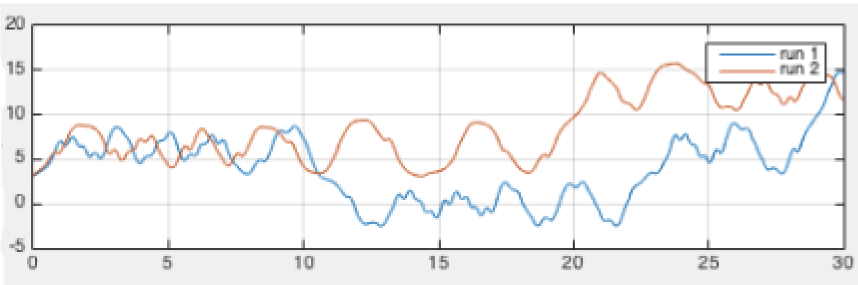


Figure 2: Small Difference Initially leading to big difference later in time

By small difference initially we mean the error in determining the initial values which we feed our computer to simulate the dynamical system. Our simulation maybe say the red curve which has some error from the real blue curve. Initially, our simulation will agree well with reality, but as time progresses, there will be large differences as is evident from the graph. But do all systems follow such a behavior? The answer is no. Only Non-linear systems which are chaotic have this property. Lets us see what we mean by this.

1.2 Linear vs Non-Linear

Whether a system behaves linearly or non-linearly depends on the governing dynamical equation of the system. A simple check for linearity is the principle of superposition. Say a dynamical equation f has two solutions ψ_1 and ψ_2, then for any real numbers $a, b \in \mathbb{R}$, we have

$$f(a\psi_1 + b\psi_2) = af(\psi_1) + bf(\psi_2) \tag{2}$$

If the dynamical equation does not satisfy the above equation then the system is said to be non linear. In general non linear systems are harder to analyse and occur more frequently than linear systems. The reason they are harder to solve than linear systems is because linear systems can be broken down into parts and each part can be solved separately and then finally combined to give the final solution. But many things in nature don't act this way. Most of everyday life is nonlinear. Two play two songs you like simultaneously, you wont get double pleasure, superposition fails here. However non-linear systems exhibit spectacular properties which are visually appealing and have a beauty in them (maybe that's why life is beautiful :p). Let us explore these properties of non-linear systems in more detail.

2 Geometrical Way

Henceforth I will use DS to denote Dynamical System and a dot on top a symbol will denote the time derivative of that quantity. Pictures are often more helpful than formulas for analyzing non linear systems. The complete report assumes two dimensional non linear systems. We assume two dynamical variables a and b and two differential equations or difference equations (more about this later) f and g such that

$$\dot{a} = f(a, b) \tag{3}$$

$$\dot{b} = g(a, b) \tag{4}$$

2.1 Phase Portrait

We first start with a **Phase Plane**. Ordinarily while studying functions, we think of an input and an output and while plotting, we plot the input on the x axis and the output on the y axis. Suppose we wish to graphically determine the nature of our DS given by equations [3] and [4]. We consider a and b as two axes, they both are the independent variables (or the inputs if you wish) and at each point, we have the time rate of change of a and b. If we divide equations [3] and [4], we will get the slope of the trajectory followed by our DS. Now we just decide some step size and we can mark out the trajectory starting from an initial point! Also, entire phase plane is filled with trajectories since any point in the phase plane can be chosen as the initial conditions. An example is shown below to give you an idea of what phase portrait looks like.

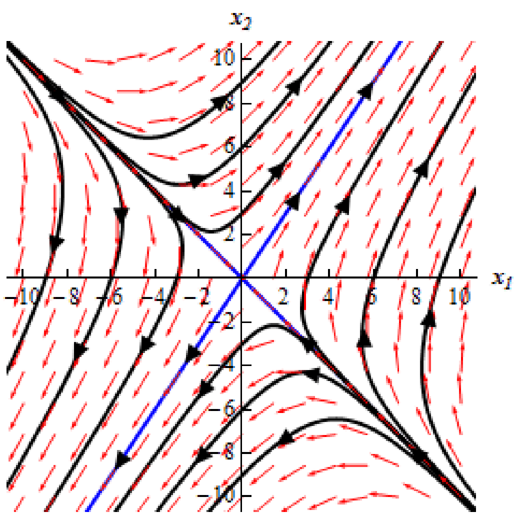


Figure 3: Phase Portrait

We try to determine the qualitative behaviour of our DS from the phase portrait. The most salient features a phase portrait may posses are fixed points and closed orbits. But hang on! Given the initial conditions can we always get a trajectory? I mean the equations are hard to solve right? Turns out, we can, and let me state the theorem without proof.

Existence and Uniqueness Theorem : Consider the initial value problem $\dot{x} = f(x)$, $x(o) = x_0$. Suppose **f** is continuous and that all partial derivatives $\partial f_i/\partial x_i$, i, j = 1,. . ., n, are continuous for **x** in some open connected set $D \in \mathbb{R}^n$. Then for $x_0 \in D$, the initial value problem has a solution**x**(t) on some interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

We see that the phase portrait are much like fields as they have a vector associated with each point in the phase plane. Two important concepts associated with them are isoclines and nullclines. Nullclines are set of points for which $da/db = 0$ and Isoclines are those points for which $da/db = c$ where c is some constant. In the above fig[2] we can see that $y = x$ is a isocline. So now we will see some properties of phase portraits.

2.2 Properties of Phase Portraits

Let me use this example from Strogatz to better explain the properties of Phase Portraits.

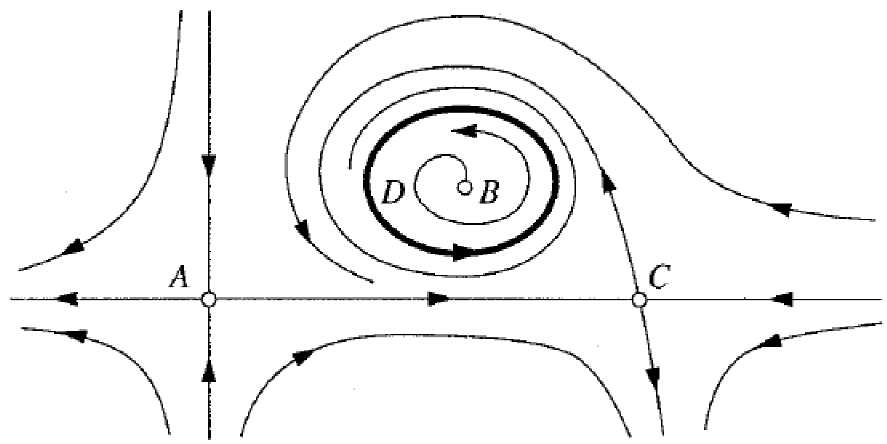


Figure 4: Properties of Phase Portraits

The most salient features of any phase portraits are

- **Fixed Points**
  In the fig.3, A,B, C are fixed points. Fixed points satisfy

$$\mathbf{f(x^*) = 0} \tag{5}$$

  Where $\mathbf{x^*}$ is any fixed point. They correspond to steady states or equilibria of the system.

- **Closed Orbits**
  In the figure, D is a closed orbit. They correspond to periodic solutions. They satisfy

$$f(t + T) = f(t) \tag{6}$$

  Where T denotes period of the trajectory. Notice that we haven't plotted time explicitly on an axis. Instead the arrow directions indicate the "flow of time".

- **Analysis of trajectories near fixed points and closed orbits**
  The flow pattern near A and C is similar but different than that near B.

- **Stability of Fixed Points and Closed Orbits**
  Here, fixed points A, B and C are unstable because nearby trajectories move away from them whereas closed orbit D is stable.

2.3 Limit Cycles

A limit cycle is a isolated closed trajectory. Isolated in the sense that, neighbouring trajectories are not closed ; they either spiral towards or away from the limit cycle. Let us take an example to understand limit cycles. Consider the DS given by the equations below :

$$\dot{x} = (1 - \sqrt{x^2 + y^2})x + y \tag{7}$$

$$\dot{y} = (1 - \sqrt{x^2 + y^2})y - x \tag{8}$$

Now let us transfrom the eqautions into polar coordinates using the substitution $x = \rho cos\theta$ and $y = \rho sin\theta$. We get the following equations:

$$\dot{\rho} = (1 - \rho)\rho \tag{9}$$

$$\dot{\theta} = -1 \tag{10}$$

From eqn[10], we see that in the angular coordinate there is a constant rotational motion everywhere in the phase diagram, in the negative, i.e., clockwise sense. We can easily integrate eqn[9] and we get the following expression for ρ

$$\rho = \frac{1}{1 - Ke^{-t}} \tag{11}$$

We can simply write θ as $\theta = \phi - t$. Again changing back to x and y, we get the time dependent expressions for x and y. We get a plot as shown below. The axes are x and y.

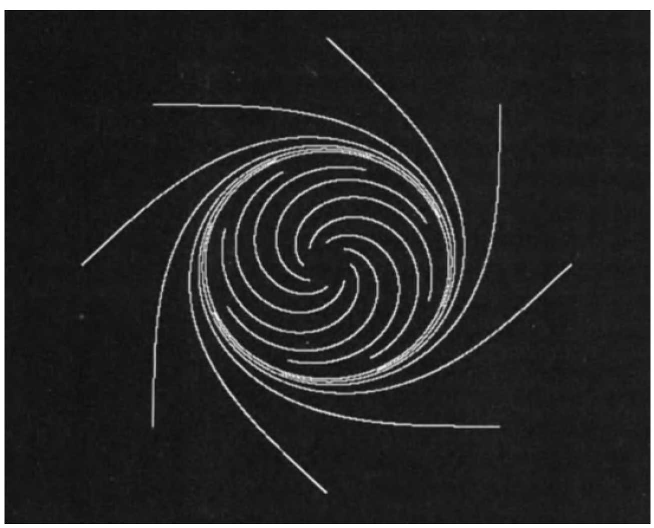


Figure 5: Final Plot of x and y

We can see a limit cycle (the circle with center at the origin) and a fixed point at x = y = 0 (This can be easily verified using equations [7] and [8]). We see that the closed trajectory is containing a fixed point in its interior. Does this always happen? The answer is yes. How do prove existence of limit cycles? Poincare-Brendixson Theorem to the rescue!

Poincare Brendixson Theorem : Suppose that

1. R is a closed bounded subset of a plane.
2. $\dot{x} = f(x)$ is a continuously differential vector field on an open set containing R.
3. R does not contain any fixed points.
4. There exists a trajectory C that is “confined” in R, in the sense that it starts in R and stays in R for for all future time.

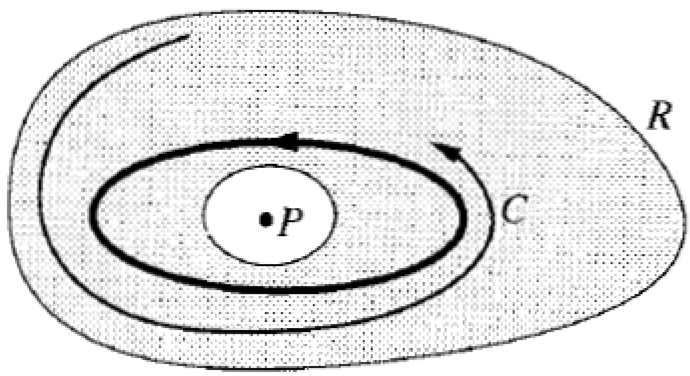


Figure 6: Poincare Brendixson Theorem

Then, either C is a closed orbit or it spirals towards one as $t \to \infty$. In either case R contains a closed orbit.

2.4 Stability of Fixed Points

We saw that in case of fixed points, in some cases the arrows are going away from them whereas in some cases they are converging to them. How do we determine the stability of fixed points given we have the governing differential equations of the system? We do it by considering the linearization of the vector field at the fixed point.
Suppose we have DS given by eqns (3) and (4) and say $a = a'$ and $b = b'$ is a fixed point. Then the linearization at this fixed point is given by the matrix

$$M = \begin{bmatrix} \frac{df}{da}(a', b') & \frac{df}{db}(a', b') \\ \frac{dg}{da}(a', b') & \frac{dg}{db}(a', b') \end{bmatrix} \tag{12}$$

This matrix is called as the Jacobian of the DS. A fixed point is stable if both of the eigenvalues of this matrix have a negative real part and is unstable even if one of them has a positive real part.

We saw that the geometric picture helps understand a lot about the qualitative behaviour of the DS.In the next section, we will see how the phase portraits change as we tweak the parameters of the governing differential equation.

3 Bifurcations

Suppose a DS follows the following differential equation

$$\frac{dx}{dt} = f(x, \mu) \tag{13}$$

Here, μ is a parameter in the function f, it is constant in time. Say, for a given μ_0, the DS will have some specific dynamical behaviour. If μ changes, the dynamical behaviour may change drastically. The μ_t at which this occurs is called as a Bifurcation.
Why are we interested in this? Well because the differential equation followed by our DS may change. For eg: Consider the population of a region. The rate of change of population will definitely depend on the rate at which the species mate. Call this α. This will act as a parameter. Suppose due to some reason, this rate changes. This will affect the population so, the phase portrait will change. We are interested in how the phase portrait changes as we tweak the parameters in the governing differential equation.
Bifurcations are easier to understand with the help of examples. For each type of bifurcation, we will illustrate examples for better understanding.

3.1 Saddle Node Bifurcation

The saddle node bifurcation is the basic mechanism for creation and destruction of fixed points. As a parameter is varied, two fixed points move towards each other, collide and mutually annihilate. Here's a prototypical example :

$$\dot{x} = \mu - x^2 \tag{14}$$

$$\dot{y} = -y \tag{15}$$

In y, it is decaying exponentially. In x, observe that $x =_{-} \mu$ and $y = 0$. Is a fixed point. Now as $\lim_{\mu \to 0}$, we will see the two fixed points move towrds each other. Assume that initially $\mu > 0$.They collide at $\mu = 0$ and they cease to exist for $\mu < 0$. The figure below depicts this process. $\mu = 0$ is a bifurcation point in this example.

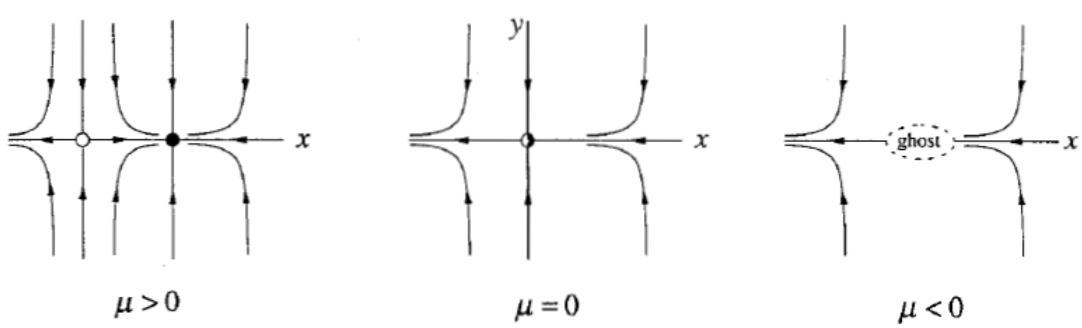


Figure 7: Saddle Node Bifurcation

Note that even after the fixed points have annihilated each other, they continue to influence the flow. They leave a ghost, a bottleneck region that kind of sucks in trajectories at one end and delays them before allowing the passage from the other end.

3.2 Transcritical

There are some situations where a fixed point must exist for all the values of the parameter and can never be destroyed. However, such a fixed point may change its stability as the parameter is varied. Transcritical bifurcation is the standard mechanism for such changes in the stability.
Consider a one dimensional example given by

$$\dot{x} = rx - x^2 \tag{16}$$

Here, irrespective of the parameter r, $x = 0$ will remain a fixed point. The figure below shows how the stability of fixed point changes as r is varied from negative to zero to positive.

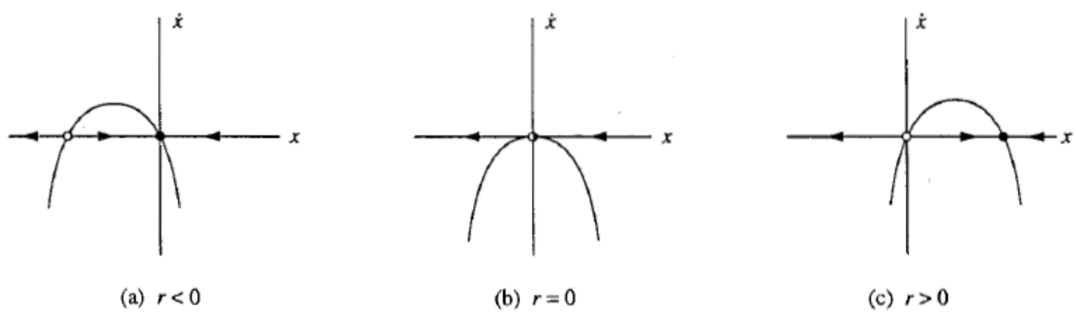


Figure 8: Transcritical Bifurcation

3.3 Pitchfork Bifurcations

This bifurcation is common in systems which have a symmetry. Many examples have spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs.
There are two different types of pitchfork bifurcation - subcritical and supercritical.
Supercritical : The normal form of supercritical bifurcation is

$$\dot{x} = rx - x^3 \tag{17}$$

This equation is invariant under the transformation $x \to -x$. This invariance is a mathematical expression of left-right symmetry mentioned earlier. Figure below shows how fixed point $x = 0$ changes as r is varied.

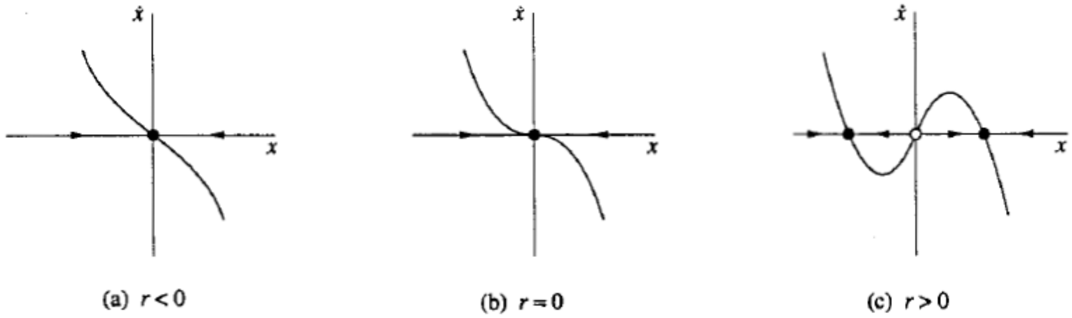


Figure 9: Supercritical Pitchfork Bifurcation

When $r < 0$, the origin is a stable point. At $r = 0$, it is still stable, but much more weakly so. When $r > 0$, two new fixed points appears and $x = 0$ no longer remains a stable fixed point. The reason this is called a pitchfork bifurcation is because of the graph of stable points vs r.

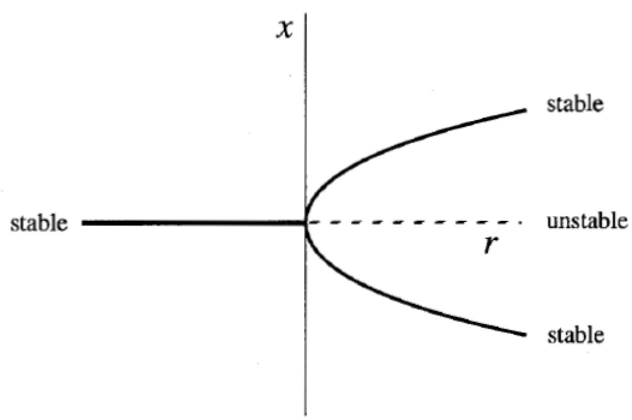


Figure 10: Plot of r vs x^* for Supercritical Pitchfork

Dotted line indicates unstable fixed point.

Subcritical Pitchfork: The normal form in this case is

$$\dot{x} = rx + x^3 \tag{18}$$

In this case, we get a plot for x^* vs r as shown below

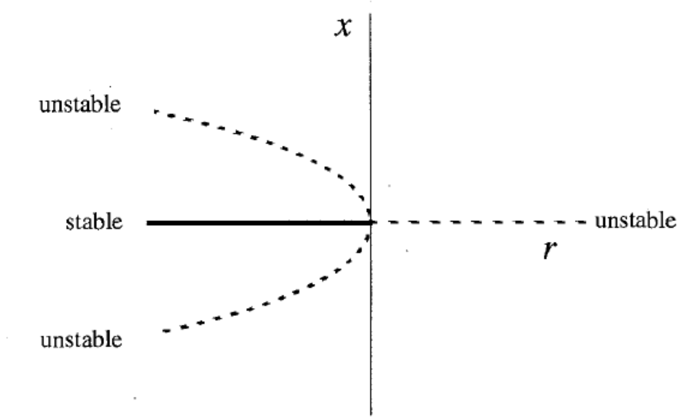


Figure 11: Plot of r vs x^* for Subcritical Pitchfork

Note that for $r < 0$, the two newly generated fixed points are unstable, whereas $x = 0$ is a stable point. For $r > 0$, only $x = 0$ remains as a fixed point which is unstable.

3.4 Hopf Bifurcation

Hopf Bifurcation occurs when a critical point switches from stable to unstable and a periodic trajectory occurs. More precisely, it is a local bifurcation where a fixed point looses stability as a pair of complex conjugate eigenvalues - of the linearization around the fixed point - crosses the complex plane imaginary axis. Recall in section 2.4, we discussed stability of fixed point from the linearization matrix. Depending on whether the limit cycle is stable or unstable the bifurcation is termed as super-critical and sub-critical hopf bifurcation respectively.

Example : Consider the system given by

$$\dot{y_1} = \beta y_1 - y_2 + \sigma y_1({y_1}^2 + {y_2}^2) \tag{19}$$

$$\dot{y_2} = \beta y_1 + y_2 + \sigma y_1({y_1}^2 + {y_2}^2) \tag{20}$$

As we vary the parameter β and σ we get the following super and sub-critical hopf bifurcations

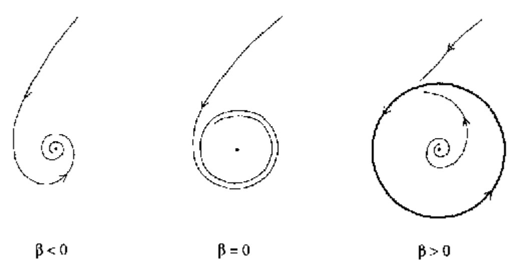


Figure 12: Supercritical Hopf Bifurcation when $\sigma = 1$

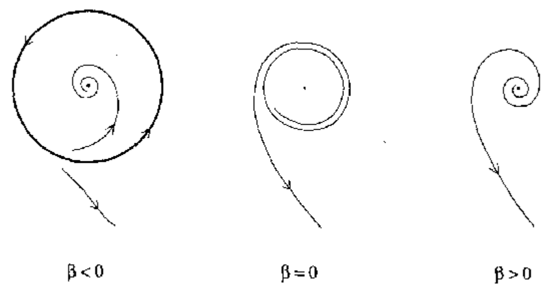


Figure 13: Subcritical Hopf Bifurcation when $\sigma = -1$

3.5 Summary

We saw various types of bifurcations with examples, although we needed to look at the phase portraits to determine which kind of bifurcation it was. Can we determine it before hand just by examining the governing differential equations ? Turns out, we can and we summarize the results as shown below.

Saddle Node
System : $f(x', \mu') = 0$
Conditions : $[\partial f/\partial x] = 0; [\partial f/\partial \mu] > 0; [\partial^2 f/\partial x^2] > 0$
Transcritical
System : $f(x', \mu') = 0$
Conditions : $[\partial f/\partial x] = 0; [\partial^2 f/\partial x^2] > 0; [\partial^2 f/\partial x \partial \mu] > 0$
Pitchfork
System : $f(-x, \mu) = -f(-x, \mu)$
Conditions : $[\partial f/\partial x] = 0; [\partial^3 f/\partial x^3] < 0; [\partial^2 f/\partial x \partial \mu] > 0$
Hopf
System : $f(x', \mu') = 0$
Condition : For the eigenvalues of the Jacobian, we need atleast one eigenvalue to pass the imaginary axis as the parameter μ is varied.
We now turn to the most interesting aspect of learning NLD, that is Chaos.

4 Chaos

Poincaré was the first to understand the possibility of completely irregular, or "chaotic," behavior of solutions of nonlinear differential equations that are characterized by an extreme sensitivity to initial conditions. He said : *Given slightly different initial conditions, from errors in measurements for example, solutions can grow exponentially apart with time, so the system soon becomes effectively unpredictable, or 'Chaotic'.* This property of Chaos is called the 'Butterfly Effect'. Lorenz rediscovered this effect while working in the meteorology department in the early 1960 s. The question of defining chaos is basically the question what makes a dynamical system chaotic rather than non chaotic. Stephen Kellert defines chaos theory as "the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems". This is a very qualitative definition in the sense there are no mathematically precise criteria given for the unstable and aperiodic nature of the behavior in question. But for now, it will suffice. We can define Chaos mathemtically once we are familiar with the concepts of Lyapunov Exponent, Logistic Maps, etc.

4.1 Logistic Maps

So far, we have been dealing with differential equations where time was a continuous entity. What will we do, it want to automate the 'solving of differential equations' ? Say we are given a differential equation and the initial conditions? How can we generate the entire trajectory from this data? As you might have guessed, we need to discretize the differential equation, in other words, we view dx as some Δx and dt as Δt. Then we fix a 'timestep' as the value for Δt. Notice that there is certain tradeoff here. Smaller the value of Δt, more accurate our trajectory will be. Time discretized into various timesteps is called stroboscopic time. Now basically we have a difference equation relation the $x's$ from adjacent time steps. Let is summarize all this.

$$\frac{dx}{dt} = f(x) \text{ converts to } x_{n+1} = x_n + \Delta t f(x_n)$$

Where $\Delta t = h$, some constant. This is called a differnce equation. Their study is important because this is the way computers generate solutions to differential equations. Now consider the difference equation

$$x_{n+1} = \mu x_n(1 - x_n); \quad x_n \in [0, 1]; \quad 1 < \mu < 4 \tag{21}$$

This is called the logistic map used by P. F. Verhulst in 1845 to model the development of a breeding population whose generations do not overlap. The density of the population at time n is x_n. The linear term simulates the birth rate and the nonlinear term the death rate of the species in a constant environment controlled by the parameter μ. The quadratic function $f_\mu(x) = \mu x(1 - x)$ is chosen because it has one maximum in the interval [0 , 1] and zero at endpoints. Intuitively, one might guess that the iteration might converge to some value. This will happen if $x_n \to x_{n+1}$. For any initial value x_0, $0 < x_0 < 1$, the x_i converge towards the fixed point x' or **attractor**.

$$\mu x'(1 - x') = x'$$

$$\implies x' = 1 - \frac{1}{\mu}$$

The process is shown in the figure on the next page.

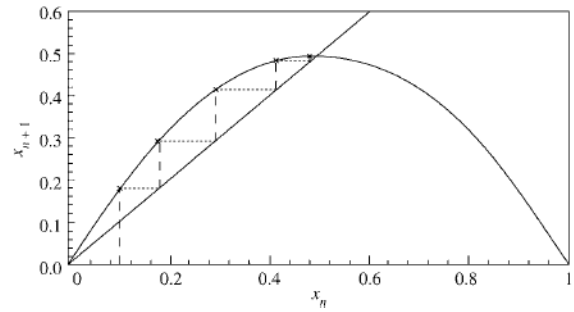


Figure 14: Cycle for the logistic map for $\mu = 2$

The interval (0,1) defines a basin of attraction for the fixed point x'. The attractor is **stable** provided the slope $|f'_\mu(x')| = |2 - \mu| < 1$ or $1 < \mu 3$. If this happens, then the next iterate x_{n+1}, lies closer to $x*$ than does x_n implying convergence to and stability of the fixed point. Note that, for the fixed point, we put the condition $x_{n+1} = f_\mu(x_n) = x_n$. If this is true, then certainly, $x_{n+2} = f_\mu(x_{n+1} = x_{n+1})$, but putting $n+1$ instead of n in the previous equation. This also implies that $x_{n+2} = f_\mu(f_\mu(x_n)) = x_{n+1} = x_n$. We look at

$$f_\mu(f_\mu(x')) = x'$$

Here, we dropped the subscipt as all the points have converged to fixed point $x*$.

$$x' = \mu^2 x'(1 - x')[1 - \mu x'(1 - x')] \tag{22}$$

This equation has roots given by

$$x' = \frac{1}{2\mu}(\mu + 1 \pm \sqrt{(\mu + 1)(\mu - 3)}$$

See when $\mu > 3$, instead of one, we get two fixed points. In other words, it bifurcates. One may recall the pitchfork bifurcation from the earlier section. Just that here, we have discretized everything. We did $f_\mu(f_\mu(x')) = x'$, we can also do $f_\mu(f_\mu(f_\mu(x'))) = x'$ or for that matter for any number of times $f_\mu(f_\mu(f_\mu(....(x'))..) = x'$. Depending on the value of μ, we may get either two solutions,four solutions and so on. This is where chaos begins. Suppose we graph μ vs x'. We get a plot as shown below

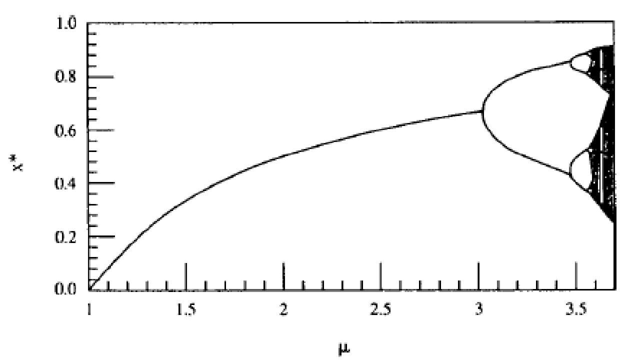


Figure 15: Bifurcation Plot x' vs μ

The kind of behaviour reflected in the above graph, where the number of fixed points growing exponentially, is the true essence of chaos. There is more to this. We saw that the first bifurcation occurs at $\mu = 3$, the next one occurs at $\mu = 3.45$. As $n \to \infty$, the ratio of spacings between the μ_n converges to δ.

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta = 4.66920161.... \tag{23}$$

This special number δ is called as Feigenbaum number. The Feigenbaum number δ is universal for the route to chaos via period doublings for all maps with a quadratic maximum similar to the logistic map

4.2 Lyapunov Exponent

We saw that the logistic map shows fixed points depending on the parameters in the governing differential equation. Is this chaos ? To be called chaotic, a system should also show sensitive dependence on initial conditions in the sense that neighbouring orbits separate exponentially fast on average. Given some initial conditions, x_0, consider a nearby point $x_0 + \delta_0$ where the initial separation is extremely small. Let δ_n be the separation after n iterates. If, $|\delta_n| \approx |\delta_0|e^{n\lambda}$, then λ is called the lyapunov exponent named after the Russian Mathematician Aleksandr Lyapunov. **A positive lyapunov exponent is a signature of chaos**. A more precise and computationally efficient formula for λ can be stated as follows :

$$\lambda \approx \frac{1}{n} ln\left|\frac{\delta_n}{\delta_0}\right|$$

Note that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$, where f^n is just the iterating function applied n times. Then,

$$\lambda = \frac{1}{n} ln\left|\frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0}\right|$$

Now as $\delta_0 \to 0$,

$$\lambda = \frac{1}{n} ln|(f^n)'(x_0)|$$

Now a subtle mathematical trick. The term inside the logarithm can be expanded using the chain rule.

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

This imples

$$\lambda \approx \frac{1}{n} ln\left|\prod_{i=0}^{n-1} f'(x_i)\right|$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} ln|f'(x_i)|$$

Now as $n \to \infty$, we define that limit to be the lyapunov exponent for orbit starting at x_0,

$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} ln|f'(x_i)|\right) \tag{24}$$

Note that λ depends on x_0, however, it is the same in the basin of attraction of a given attractor. For stable fixed points and cycles, λ is negative. For chaotic attractors, it is positive.

4.3 Fractals

Roughly speaking, fractals are complex geometric shapes with a fine structure at arbitrarily small scales. Usually they have some degree of self-similarity. In words, if we magnify a tiny part of a fractal, we will see features reminiscent of the whole. Sometimes the similarity is exact, other times, it is approximate or statistical. Let us see what we mean by this.
There is this common notion that they are self similar shapes (you get the same shape when you zoom in) which is not wrong, however, Benoit Mandelbrot (father of fractal geometry) had a much more generalized definition in mind. He wanted to capture 'roughness' present in nature, like say the coastline which is generally indented. Fratcals have a certain dimension associated to them. Generally, for abstract, perfectly self-similar shapes, this dimensionality can be found explicitly. For other shpaes, (real life snow-flakes, coastlines, 'rough' objects) statistical methods are adopted. To begin with, think about a line segment, of length l. Say it has a mass m. Divide that line segment to get one with length $l/2$. Obviosly, it will have a mass $m/2$. Now consider a square of side l and mass m. Divide this square to get a square of side $l/2$. We will get 4 such squares from the original one, and the mass will be reduced to $m/4$. Now think of a cube of side l and mass m. If we divide it to get a cube of size $l/2$, we will get 8 such pieces each with mass $m/8$. Can you now see why a line segment is called one dimensional, a sqaure two dimensional and a cube three dimensional ? Say if the length is scaled by a factor f and the object is D dimensional, then the mass will be scaled by a factor of f^D. Let us see how this can be applied for fractals. Consider an example of the Sierpinski triangle as shown below.

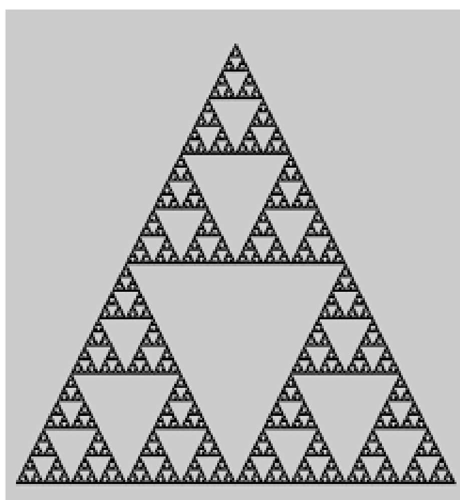


Figure 16: Sierpinski Traingle

Now say the triangle has a side l and mass m. If we get a similar shape of length $l/2$, it is easy to see that its mass will be $m/3$ as the original sierpinski traingle is constructed by 3 smaller sierpinski triangles. Using our earlier definition, we can find the dimensionality of sierpinski triangle.

$$\left(\frac{1}{2}\right)^D = \frac{1}{3}$$

$$\implies D = log_2(3) \approx 1.584$$

Here $D \approx 1.584$ is called the fractal dimension of the sierpinski triangle. It may not always be the case that we may be able to figure out how much the mass is scaled. In such cases (say an imperfect fractal), statistical techniques are applied. But, why are fractals related to chaos anyway ? See how are fractals constructed - by repeating over a simple process over and over in an ongoing feedback loop. Sounding somewhat similar to difference equations ? Fractals generally 'arise' from a chaotic system. They are maps of chaos. Since most of the systems in the real world are chaotic, we say fractals all around. Best example is freezing of ice. It is due to the non linear, chaotic behaviour of freezing (i.e. the differential equation governing the shape of freezing is non-linear and chaotic), that we observe the fractal like structure of a snowflake (also called as Van-Koch Snowflake). In the introduction we saw the Mandelbrot set which also an example of a fractal. There, the iteration was governed by the equation $Z_{n+1} = {Z_n}^2 + c$ We iterate this over and over and for all those c's which makes the iteration not diverge, belong to the mandelbrot set. Here are a two examples from real life which have fractal like appearance -

(a) Van Koch Snowflake

(b) Nearly fractal coastline

Figure 17: Example Fractals

5 Supply Demand Models

Before delving into the economic models, we will digress a bit and formally define supply and demand, markets. For any given commodity (goods or services), people in general can classified into two categories - producers (who produce it using raw materials) and consumers who purchase that commodity in exchange for money.
Supply : Supply is a fundamental economic concept that describes the total amount of a specific good or service that is available to consumers
Demand : Demand is an economic principle referring to a consumer's desire to purchase goods and services and willingness to pay a price for a specific good or service.
Markets : A market is a place where two parties can gather to facilitate the exchange of goods and services.
Two things should be kept in mind about supply and demand. If the **prices for a commodity increase, its demand decreases** as people do not wish to spend more.
Whereas, if the **given price increase, supply increases** as the producers seek to increase profit.
An equilibrium is said to exist if the supply for a commodity matches its demand.

5.1 Cobweb Model (with Adaptive Expectations)

The cobweb model describes the temporary market equilibrium prices in a single market with one lag in supply. By one lag in supply, we mean that the amount to be produced must be chosen before the prices are observed. The following example in the agriculture market explains it well.
Suppose that for a particular year, due to bad weather conditions, the production was drastically affected reducing the supply. However, the demand does not necessarily adjust to this reduction which causes the prices to rise. The following year, the farmers expect this price to continue and produce the crops accordingly. However, increase in supply does not cause increase in demand and thus the farmers are forced to reduce the prices to match the demand. Here we see that the farmers (producers) are having a predetermined quantity of supply which they are going to produce keeping in mind the previous equilibrium prices. We call this as adaptive expectations. Let us now formulate the cobweb model. First, some terminology -
Subscript t denotes at time t. p_t for price, $\hat{p}_t$ for expected price, ${q_t}^d$ for demand and ${q_t}^s$ for supply. Then the equations describing the cobweb model are -

$${q_t}^d = D(p_t) \qquad \text{(Demand)}$$

$${q_t}^s = S(\hat{p}_t) \qquad \text{(Supply)}$$

$${q_t}^d = {q_t}^s \qquad \text{(Temporary Equilibrium)}$$

$$\hat{p}_{t+1} = \hat{p}_t + w(p_t - \hat{p}_t) \qquad \text{(Adaptive Expectations)}$$

Here, w is called the expectations weight factor. $w = 1$ corresponds to the **traditional cobweb model** with naive price expectations. We start with p_t and get $\hat{p}_{t+1}$.
We now look at the nature of D and S as they decide the stability of equilibrium in a cobweb model. The figure on the next page shows how supply and demand curves vary w.r.t the price level and quantity demanded/supplied. Note that it clearly depicts the supply and demand law stated earlier.

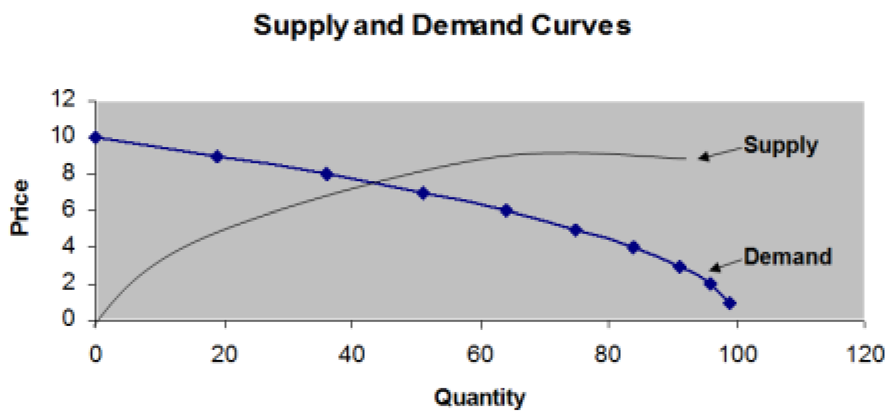
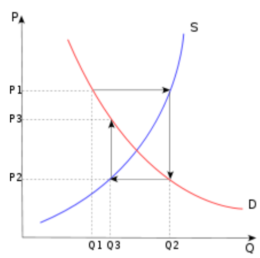
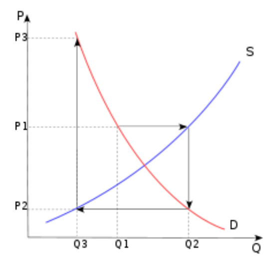


Figure 18: Typical Demand and Supply Curve

For the traditional case, $w = 1$ and hence $\hat{p}_{t+1} = p_t$.
A note here on the stability analysis of temporary equilibrium prices. Figure below shows two situations which can arise (here w = 1).



(a) Converging Case



(b) Diverging Case

Figure 19: Example of Cobweb model with w = 1

Notice how the dotted arrows which follow the prices create a cobweb like structure from which the model derives its name. From the figure it is evident that the derivatives of the supply and demand curves play a major role in deciding the stability of the prices. We state the conditions without proof here.

$$\text{For Convergent Case} \qquad \frac{\frac{dQ^S}{Q}}{\frac{dP^S}{P}} < \left| \frac{\frac{dQ^D}{Q}}{\frac{dP^D}{P}} \right| \tag{25}$$

$$\text{For Divergent Case} \qquad \frac{\frac{dQ^S}{Q}}{\frac{dP^S}{P}} > \left| \frac{\frac{dQ^D}{Q}}{\frac{dP^D}{P}} \right| \tag{26}$$

Interestingly, in case of equality, we will observe limit cycles. The state variables in the case of the market system are the price and the quantity. In case of limit cycle, the price will never go to the equilibrium and there will always be either shortage or excess of goods. This is of course in assumption with naive expectation model.

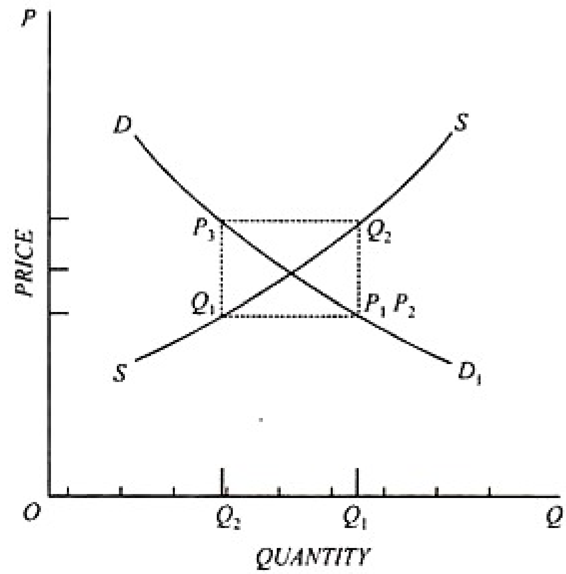


Figure 20: Cobweb Model with limit cycle

5.2 Simulations

For our simulations, we vary the parameter w and try to see the bifurcation diagram for any chaotic behaviour. Consider the Supply function as

$$S(\hat{p_t}) \ = \hat{p_t}^2 \tag{27}$$

and demand function as

$$D(p_t) = 4 - 3p_t \tag{28}$$

We will try to see the effects of non linear supply. From the cobweb model equations, we get at equilibrium, ${q_t}^d = {q_t}^s$

$$\implies p_t = D^{-}S(\hat{p_t})$$

$$\implies p_t = \frac{4 - 3\hat{p_t}^2}{3}$$

And from adaptive expectations, we get

$$\hat{p_{t+1}} = \hat{p_t} + w\left(\frac{4 - 3\hat{p_t}^2}{3} - \hat{p_t}\right)$$

Now, we have a difference equation with a parameter w. Note that this is very similar to the logistic map. We try to get a bifurcation diagram of this equation by simulating it in MATLAB. The graph is shown on the next page.

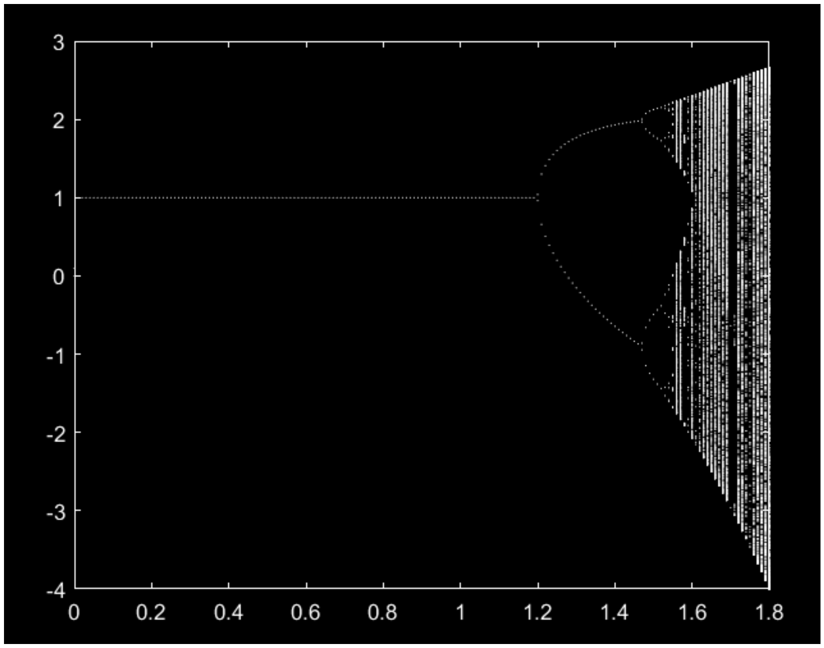


Figure 21: Bifurcation of Expected price

Here, x axis is w and the y axis is fixed points in $\hat{p}_t$. We see that until $w \approx 1.2$, there is just a single fixed point, namely, $\hat{p}_{t\to\infty} = 1$. After that pitchfork bifurcation is seen. After, w exceeds 1.6, the behavior is chaotic. We also plot the lyapunov exponent at steady state (i.e at $t \to \infty$). We get a plot as shown below

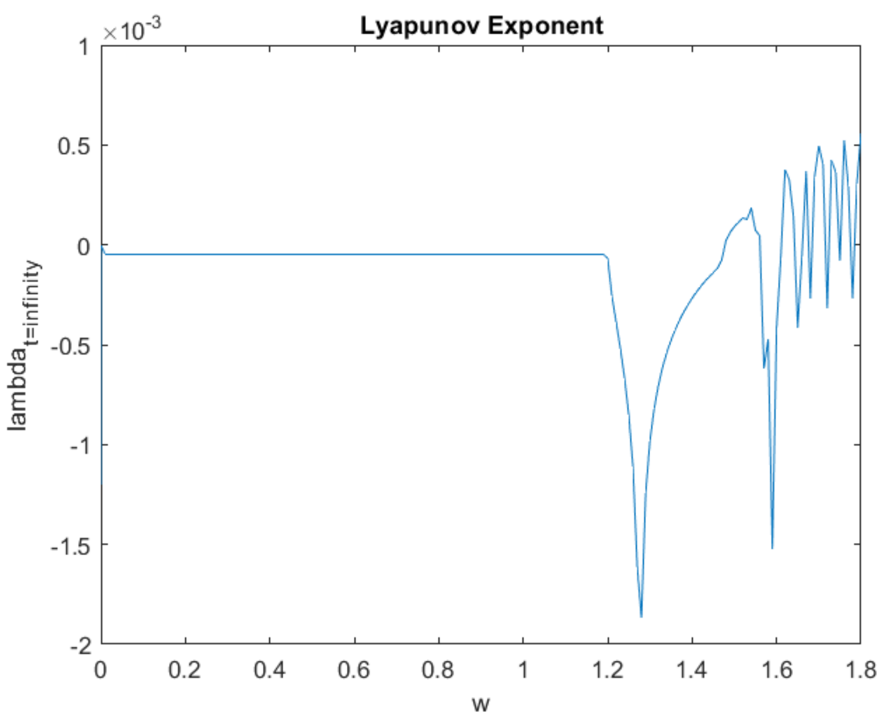


Figure 22: Steady State lyapunov Exponent

We see that the lyapunov exponent is less than 0 until the chaotic behaviour starts at around $w = 1.6$. As we saw in section 4.2, a positive lyapunov exponent indicates chaos is rightly inferred from the bifurcation plot. After 1.6, it fluctuates a lot. The MATLAB codes used for the plotting these graphs can be found here.

5.3 Conclusion

The quadratic supply function is taken from Ronald Shone - Economic Dynamics and the cobweb model dynamics is taken from Homes - Dynamics of Cobweb model with adaptive expectations-1994. Although in the paper, Hommes uses different non linear supply functions, we have demonstrated that using a simple quadratic supply function can also lead to chaos.
Generally, the subject of non linear dynamics and chaos is studied using examples from classical physics (say double pendulum, or say the navier stokes equation in fluid mechanics). For this study, a somewhat offbeat example is chosen to explain NLD. All of this was an attempt to demonstarte the wide applicability of the field and frequent occurrence of Non-Linear Phenomenon in everyday life.

6 References

This is the final report as a part of study in Non Linear Dynamics conducted in the Summer of 2020. Most of the theoretical part is covered from **Steven H. Strogatz - Nonlinear Dynamics and Chaos With Applications to Physics, Biology, Chemistry, and Engineering** -Westview Press (1994) and **Dr. Tonu Puu - Nonlinear Economic Dynamics**-Springer-Verlag Berlin Heidelberg (1997).Logistic Maps is covered from **Arfken, Weber - Mathematical Methods for Physicists, 6^{th} Edition**. Some part of Cobweb Model is covered from **Ronald Shone - Economic Dynamics-Phase diagrams and their economic application**-Cambridge University Press (2003).

A complete list of references can be found here