Solutions to Homework 1
CS 4331/5331
Introduction to Quantum Computing

Spring 2022

1. From the condition that the vector

$$\begin{pmatrix} \cos(\theta_1) + \cos(\theta_2) \\ \sin(\theta_1) + \sin(\theta_2) \end{pmatrix} \tag{1}$$

is normalized we obtain

$$(\sin(\theta_1) + \sin(\theta_2))^2 + (\cos(\theta_1) + \cos(\theta_2))^2 = 1. \tag{2}$$

Thus, we have

$$\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) = -\frac{1}{2}, \tag{3}$$

which gives

$$\cos(\theta_1 - \theta_2) = -\frac{1}{2}. \tag{4}$$

Therefore, $\theta_1 - \theta_2 = 2\pi/3$ or $\theta_1 - \theta_2 = 4\pi/3$.

2. (i) Since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$, we have

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|. \tag{5}$$

(ii) For the standard basis, we find

$$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

(ii) For the Hadamard basis, we find

$$U_{NOT} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

We see that the respective matrix representations for the two bases can be different.

3. Using the fact that

$$H^2 = I_2 \tag{8}$$

we have

$$\begin{aligned} V_H(|\psi\rangle) &= \langle\psi| I_2 |\psi\rangle - \langle\psi| H |\psi\rangle^2 & (9) \\ &= 1 - \frac{1}{2}(\cos(2\theta) + \sin(2\theta))^2 & (10) \\ &= \frac{1}{2}(1 - \sin(4\theta)). & (11) \end{aligned}$$

The minimum value is 0, for example when $\theta = \pi/8$. The maximum value is 1, for example when $\theta = 3\pi/8$.

4. We have

$$W_2(|0\rangle \otimes |0\rangle) = (W \otimes W)(|0\rangle \otimes |0\rangle) = W|0\rangle \otimes W|0\rangle. \tag{12}$$

Thus,

$$W_2(|0\rangle \otimes |0\rangle) = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \tag{13}$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle). \tag{14}$$

We see that W_2 generates a linear combination of all states. This is also true for W_n.

5. (i) We obtain

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\imath 2\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{15}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\imath 2\phi}\cos(2\theta) \\ e^{\imath\phi}\sin(2\theta) \\ e^{\imath\phi}\sin(2\theta) \\ -\cos(2\theta) \end{pmatrix} \tag{16}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{\imath 2\phi}\sin(2\theta) \\ e^{\imath\phi}\cos(2\theta) \\ e^{\imath\phi}\cos(2\theta) \\ \sin(2\theta) \end{pmatrix} \tag{17}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{\imath\phi} \\ -e^{\imath\phi} \\ 0 \end{pmatrix}. \tag{18}$$

(ii) The choices $\phi = 0$ and $\theta = 0$ lead to standard basis for $|0\rangle$ and $|1\rangle$, therefor we have

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{19}$$

$$
|\Psi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \tag{20}
$$

6. Direct calculation of

$$
f(P) = c_0 I_n + c_1 P + c_2 P^2 + \cdots + c_{n-1} P^{n-1} \tag{21}
$$

yields the matrix C. Note that $P^2, P^3, \ldots, P^{n-1}$ are permutation matrices.

7. One has

$$
\rho = |\psi\rangle \langle\psi| = \begin{pmatrix} \cos^2(\theta) & \mathrm{e}^{-\imath\phi} \cos(\theta)\sin(\theta) \\ \mathrm{e}^{\imath\phi} \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{pmatrix}. \tag{22}
$$

It can be seen that $\rho^\dagger = \rho$, $\mathrm{tr}(\rho) = \cos^2(\theta) + \sin^2(\theta) = 1$, and $\rho^2 = \rho$. Therefore, it is a density matrix that corresponds to a pure state.

8. We have

$$
\rho_1 - \rho_2 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \tag{23}
$$

that leads to

$$
(\rho_1 - \rho_2)(\rho_1 - \rho_2)^\dagger = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}. \tag{24}
$$

It then follows that

$$D_T(\rho_1, \rho_2) = \frac{1}{2}. \tag{25}$$

9. By using the fact that

$$\mathrm{tr}(\sigma_1\sigma_2) = 0, \qquad \mathrm{tr}(\sigma_2\sigma_3) = 0, \qquad \mathrm{tr}(\sigma_3\sigma_1) = 0 \tag{26}$$

along with repeated use of

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{27}$$
$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B) \tag{28}$$

we find

$$\mathrm{tr}(\rho\sigma_{k_0} \otimes \sigma_{k_1} \otimes \cdots \otimes \sigma_{k_{N-1}}) = c_{k_0k_1\ldots k_{N-1}}. \tag{29}$$

10. It is clear that

$$\rho^\dagger = \rho, \qquad \mathrm{tr}(\rho) = 1. \tag{30}$$

Moreover, it can be verified that

$$\rho^2 = \rho. \tag{31}$$

Thus, ρ is a density matrix (pure state), which given by

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

11. We find

$$\begin{aligned}
\mathrm{tr}_B(\rho_{AB}) &= \left(I_2 \otimes \begin{pmatrix}1\\0\end{pmatrix}\right)^\dagger \rho_{AB} \left(I_2 \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) + \\
&\quad \left(I_2 \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)^\dagger \rho_{AB} \left(I_2 \otimes \begin{pmatrix}0\\1\end{pmatrix}\right) && (33)\\
&= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} && (34)\\
&= \begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}. && (35)
\end{aligned}$$