Solutions to Homework 2
CS 4331/5331
Introduction to Quantum Computing

Spring 2022

1. The truth table is given by

x_1	x_2	x_3	x'_1	x'_2	x'_3
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	1
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	1	1	0
1	1	0	1	0	0
1	1	1	1	0	1

From the truth table we see that the transformation is invertible, that is, we have a 1-to-1 map.

2. (i)

$$\mathrm{NOT}(a) = \mathrm{BIT3}(F(a,1,0)). \tag{1}$$

(ii)

$$\mathrm{AND}(a,b) = \mathrm{BIT3}(F(a,0,b)). \tag{2}$$

3. We find the truth table

x_1	x_2	x_3	x_1'	x_2'	x_3'
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	1
0	1	1	0	1	0
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	1	0
1	1	1	1	0	0

From the truth table we see that the gate is reversible.

4. It can be directly verified that $MM^\dagger = I_4$, and hence M is unitary.

5. For $n = 3$, the quantum Fourier transform becomes

$$U_{QFT} = \frac{1}{2\sqrt{2}} \sum_{j=0}^{7} \sum_{k=0}^{7} \mathrm{e}^{-\imath 2\pi kj/8} \left|k\right\rangle \left\langle j\right| . \tag{3}$$

By using the Euler's identity

$$\frac{\mathrm{e}^{\imath\theta} + \mathrm{e}^{-\imath\theta}}{2} = \cos(\theta), \tag{4}$$

we have

$$U_{QFT}\left|\psi\right\rangle = \frac{1}{4\sqrt{2}} \sum_{i,j,k=0}^{7} \mathrm{e}^{-\imath 2\pi kj/8} \cos(2\pi i/8) \left|k\right\rangle \left\langle j|i\right\rangle \tag{5}$$

$$= \frac{1}{4\sqrt{2}} \sum_{j,k=0}^{7} \mathrm{e}^{-\imath 2\pi kj/8} \cos(2\pi j/8) \left|k\right\rangle \tag{6}$$

$$= \frac{1}{8\sqrt{2}} \sum_{j,k=0}^{7} \mathrm{e}^{-\imath 2\pi kj/8} \left(\mathrm{e}^{2\imath\pi j/8} + \mathrm{e}^{-2\imath\pi j/8}\right) \left|k\right\rangle \tag{7}$$

$$= \frac{1}{8\sqrt{2}} \sum_{j,k=0}^{7} \left(\mathrm{e}^{2\imath\pi j(1-k)/8} + \mathrm{e}^{-2\imath\pi j(1+k)/8}\right) \left|k\right\rangle \tag{8}$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{7} (\delta_{k1} + \delta_{k7}) \left|k\right\rangle \tag{9}$$

$$= \frac{1}{\sqrt{2}} (\left|1\right\rangle + \left|7\right\rangle). \tag{10}$$

6. We obtain

$$(I_2 \otimes U_H) U_{CNOT} (I_2 \otimes U_H) \left|jk\right\rangle \tag{11}$$

$$= \frac{1}{\sqrt{2}} (I_2 \otimes U_H) U_{CNOT} \left|j\right\rangle \otimes (\left|0\right\rangle + (-1)^k \left|1\right\rangle) \tag{12}$$

$$= \begin{cases} \frac{1}{\sqrt{2}} (I_2 \otimes U_H) \left|j\right\rangle \otimes (\left|0\right\rangle + (-1)^k \left|1\right\rangle), & j = 0 \\ \frac{1}{\sqrt{2}} (I_2 \otimes U_H) \left|j\right\rangle \otimes (\left|1\right\rangle + (-1)^k \left|0\right\rangle), & j = 1 \end{cases} \tag{13}$$

$$= \frac{1}{\sqrt{2}} (I_2 \otimes U_H) \left|j\right\rangle \otimes (-1)^{j\cdot k} (\left|0\right\rangle + (-1)^k \left|1\right\rangle) \tag{14}$$

$$= (-1)^{j\cdot k} \left|jk\right\rangle, \tag{15}$$

which is $U_P(\pi)$.

7. By the fact that

$$\langle \phi_j | \phi_k \rangle = \delta_{jk}, \tag{16}$$

we have

$$
\begin{aligned}
UU^\dagger &= \left(\sum_{k=0}^{N-2} |\phi_k\rangle \langle\phi_{k+1}| + |\phi_{N-1}\rangle \langle\phi_0|\right) \times \\
&\quad \left(\sum_{k=0}^{N-2} |\phi_{k+1}\rangle \langle\phi_k| + |\phi_0\rangle \langle\phi_{N-1}|\right) \qquad (17) \\
&= \sum_{k=0}^{N-1} |\phi_k\rangle \langle\phi_k| = I_N. \qquad (18)
\end{aligned}
$$

8. Since $\sigma_1^\dagger = \sigma_1$, we obtain

$$
U(\alpha)^\dagger = \frac{\mathrm{e}^{-\imath\alpha}}{\sqrt{2}}(I_2 \otimes I_2 \otimes I_2 - \imath\sigma_1 \otimes \sigma_1 \otimes \sigma_1). \qquad (19)
$$

Thus, we have

$$
\begin{aligned}
U(\alpha)^\dagger U(\alpha) &= \frac{1}{2}\left(I_2 \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes I_2\right) \qquad (20) \\
&= I_2 \otimes I_2 \otimes I_2 \qquad (21) \\
&= I_8, \qquad (22)
\end{aligned}
$$

where we have used the fact that $\sigma_1^2 = I_2$.

9. It can be easily verified that $\Pi^\dagger = \Pi$ and $\Pi^2 = \Pi$, and thus Π is a projection matrix.

10. (i)

$$
|\langle 0|\psi\rangle|^2 = \cos^2(\theta/2). \qquad (23)
$$

(ii)

$$
|\langle 1|\psi\rangle|^2 = \sin^2(\theta/2). \qquad (24)
$$

Therefore, we see that the parameters σ and ϕ can not be detected in this measurement model.

11. Since $\langle 0|0\rangle = 1$, $\langle 0|1\rangle = 0$, and $I_2 |1\rangle = |1\rangle$, we obtain

$$(\langle 0| \otimes I_2) |\psi\rangle = \frac{1}{\sqrt{2}} |1\rangle \,. \tag{25}$$