CHAPTER 9

State-space Solutions to H_2 and H_∞ Optimal Control Problems

J. Doyle, K. Glover, P. Khargonekar and B. Francis, "State-space solutions to standard H_2 and H_∞ optimal control problems," *IEEE Transactions on Automatic control*, Vol. 33, pp. 831-847, 1989.

K. Glover and J. Doyle, "State-space formulae for all stabilizing controllers that satisfy an L_∞-norm bound and relations to risk sensitivity," *Systems Control Letters*, Vol. 11, pp. 167-172, 1988.

DGKF paper: p. 834

The realization of the transfer matrix G is taken to be of the form

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

The following assumptions are made.

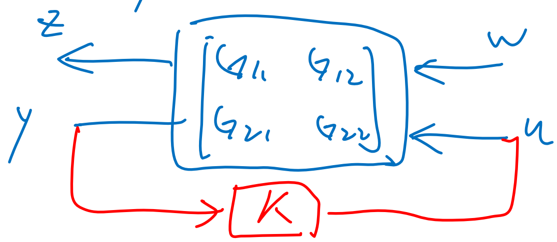
i) (A, B_1) is stabilizable and (C_1, A) is detectable.

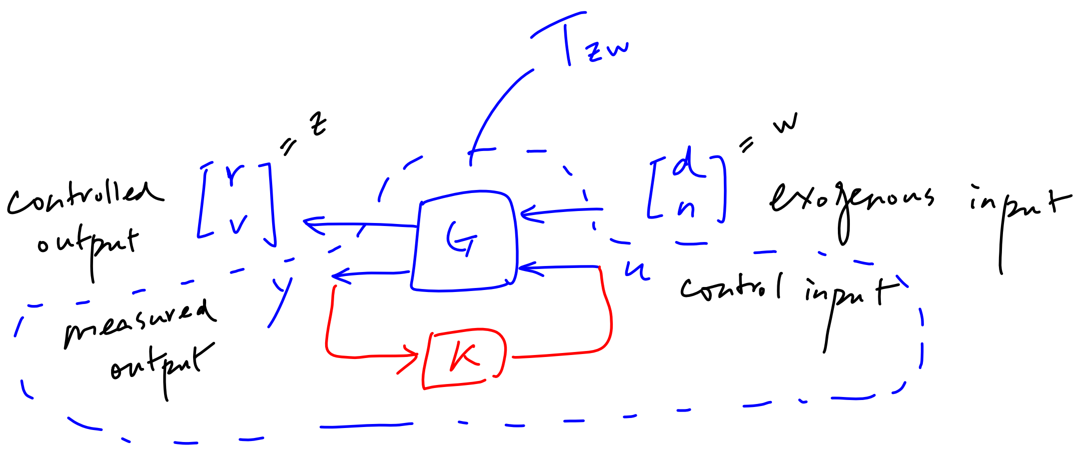
ii) (A, B_2) is stabilizable and (C_2, A) is detectable.

iii) $D_{12}'[C_1 D_{12}] = [0 \;\; I]$.

iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$





H_2 control problem :

Find K s.t. the closed-loop system is stable,

and $\|T_{zw}\|_2$ is minimized .

Hamiltonian matrices

$$H_2 := \begin{bmatrix} A & -B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}, \quad J_2 := \begin{bmatrix} A' & -C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}$$

belong to dom(Ric) and, moreover, $X_2 := \mathrm{Ric}(H_2)$ and $Y_2 := \mathrm{Ric}(J_2)$ are positive semidefinite. Define $F_2 := -B_2'X_2$, $L_2 := -Y_2C_2'$, and

$$A_{F_2} := A + B_2F_2, \qquad C_{1F_2} := C_1 + D_{12}F_2$$

$$A_{L_2} := A + L_2C_2, \qquad B_{1L_2} := B_1 + L_2D_{21}$$

$$\hat{A}_2 := A + B_2F_2 + L_2C_2$$

$$G_c(s) := \left[\begin{array}{c|c} A_{F_2} & I \\ \hline C_{1F_2} & 0 \end{array}\right], \qquad G_f(s) := \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline I & 0 \end{array}\right].$$

Theorem 1: The unique optimal controller is

$$K_{\mathrm{opt}}(s) := \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

Moreover, $\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 = \|G_cL_2\|_2^2 + \|C_1G_f\|_2^2$.

LQG problem → Standard H2 Problem

$$\begin{cases} \dot{x} = Ax + Bu + W_d d \\ y = Cx + W_n n \end{cases}$$

$A, W_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $W_n \in \mathbb{R}^{p\times p}$

(A, B) controllable, (C, A) observable
stabilizable detectable

$E[d(t)] = 0$, $E[d(t)d^T(t+\tau)] = I_n \delta(\tau)$

$E[n(t)] = 0$, $E[n(t)n^T(t+\tau)] = I_p \delta(\tau)$

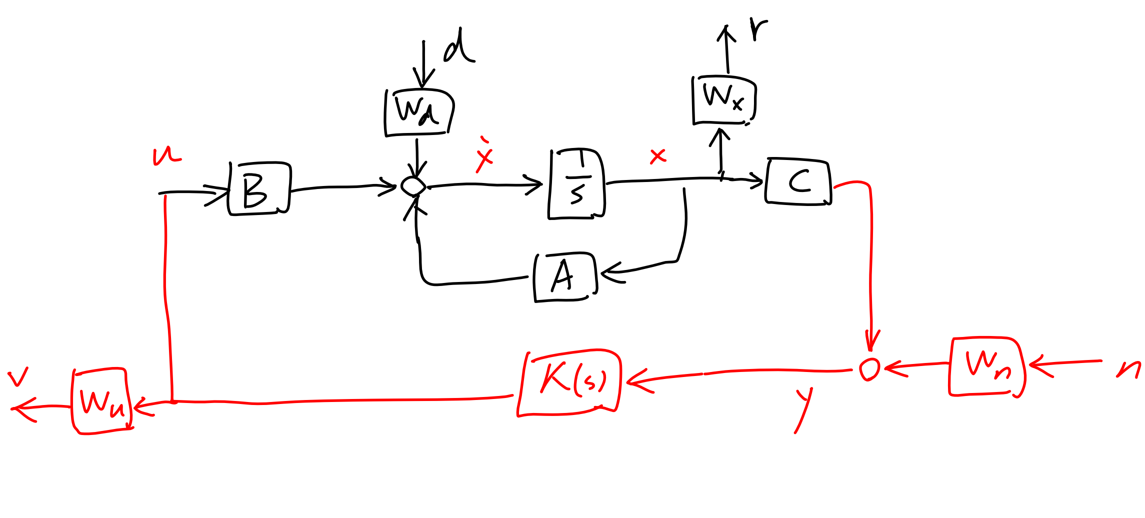
LQG problem:

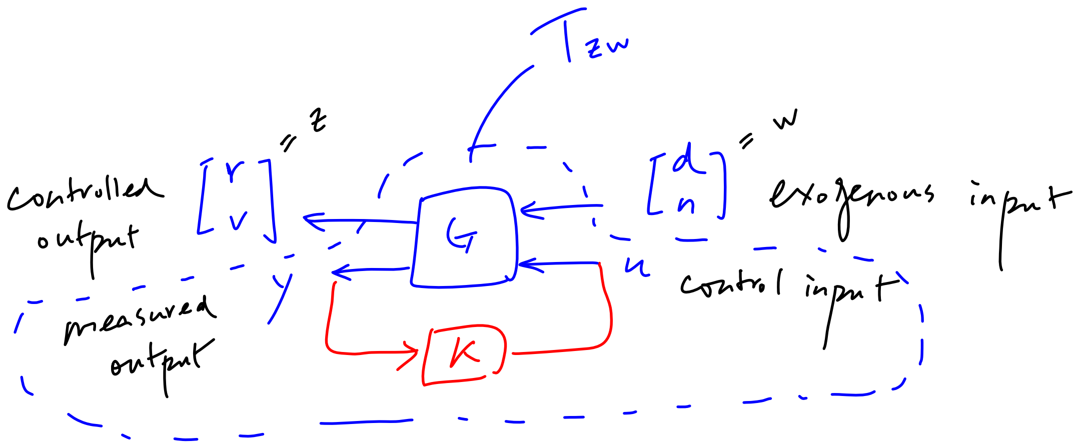
Find a controller $u(s) = K(s)y(s)$

which stabilizes the system and minimizes

$$PI = E\left[\int_0^\infty \left(x^T W_x^T W_x x + u^T W_u^T W_u u\right) dt\right]$$

where $W_x^T W_x \geq 0$, $W_u^T W_u > 0$





H_2 control problem:

Find K s.t. the closed-loop system is stable, and $\|T_{zw}\|_2$ is minimized.

$$G = \left[\begin{array}{c|cc:c} A & W_d & 0 & B \\ \hline W_x & 0 & 0 & 0 \\ 0 & 0 & 0 & W_u \\ \hdashline C & 0 & W_n & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

To satisfy assumptions (iii) and (iv)

Let $\hat{u} = W_u u$ and $\hat{n} = W_n n$

Then
$$\begin{bmatrix} \dot{x} \\ r \\ v \\ y \end{bmatrix} = \begin{bmatrix} A & W_d & 0 & B \\ W_x & 0 & 0 & 0 \\ 0 & 0 & 0 & W_u \\ C & 0 & W_n & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ n \\ u \end{bmatrix}$$

$$= \begin{bmatrix} A & W_d & 0 & BW_u^{-1} \\ W_x & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ C & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ \hat{n} \\ \hat{u} \end{bmatrix}$$

i.e., $G = \left[\begin{array}{c|cc|c} A & W_d & 0 & BW_u^{-1} \\ \hline W_x & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline C & 0 & I & 0 \end{array}\right]$

Computations of H_2 norm in time domain

Given $\hat{G}(s)$ = {$A,B,C,0$} with A stable. Let L_c and L_o be the controllability and observability grammians respectively. That is,

$$L_c = \int_0^\infty e^{At} BB^T e^{A^T t} dt \quad \text{which satisfies} \quad AL_c + L_c A^T + BB^T = 0$$

$$L_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt \quad \text{which satisfies} \quad A^T L_o + L_o A + C^T C = 0$$

Theorem:

$$\left\|\hat{G}\right\|_2 = \sqrt{trace\left\{CL_c C^T\right\}} = \sqrt{trace\left\{B^T L_o B\right\}}.$$

Pf:

The impulse response function

$$G(t) = \mathcal{L}^{-1}[\hat{G}(s)] = Ce^{At}B, \quad t > 0$$

by Parseval's Theorem:

$$\|\hat{G}\|_2^2 = \|G\|_2^2 = tr \int_0^\infty Ce^{At}B \cdot B^T e^{A^T t} C^T dt$$

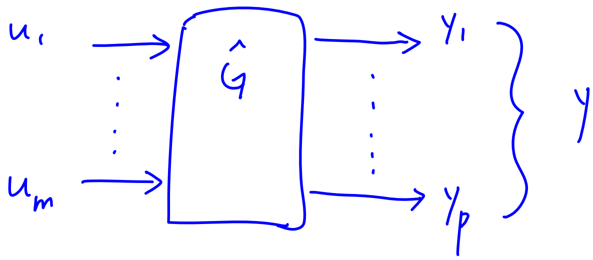
$$= tr\left(C \int_0^\infty e^{At} BB^T e^{A^T t} dt \, C^T\right) = tr\left(CL_c C^T\right)$$

Theorem:

$$L_c = \int_0^\infty e^{At} BB^T e^{A^T t} dt \quad \text{satisfies} \quad AL_c + L_c A^T + BB^T = 0$$

Pf:

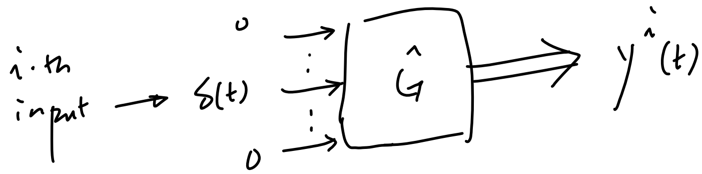
H_2 norm



$\hat{y} = \hat{G}\,\hat{u}$

$\hat{G} = [\hat{G}_1\ \hat{G}_2\ \cdots\ \hat{G}_m]$

$$\|\hat{G}\|_2^2 = \|G\|_2^2 = \|y^1\|_2^2 + \|y^2\|_2^2 + \cdots + \|y^m\|_2^2$$



$$\|y\|_2^2 = \sum_{i=1}^{m} \|y^i\|_2^2 = \sum_{i=1}^{m} \int_0^\infty y^{iT}(t)\cdot y^i(t)\,dt$$

$$= \sum_{i=1}^{m} \frac{1}{2\pi}\int_{-\infty}^{\infty} y^{iT}(j\omega)\, y^i(j\omega)\,d\omega = \sum_{i=1}^{m} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}_i^T(j\omega)\, \hat{G}_i(j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{i=1}^{m} \hat{G}_i^T(j\omega)\, \hat{G}_i(j\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left[\hat{G}^T(j\omega)\, \hat{G}(j\omega)\right] d\omega$$

$$= \|\hat{G}\|_2^2$$

Linear Quadratic Regulator (LQR) Problem

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

$$\min_{u(t)} \int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right) dt$$

$$where \quad Q^T = Q \geq 0, \quad R^T = R > 0$$

Without a loss of generality, we assume $R = I$.

$$x^TQx + u^Tu = \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \geq 0$ can be factored as

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$$

and then

$$\int_0^\infty (x^TQx + u^Tu)\,dt = \int_0^\infty \|Cx + Du\|_2^2\,dt$$

Therefore, the LQR problem becomes:

$$\begin{cases} \dot{x} = Ax + Bu, & x(0) = x_0 \\ z = Cx + Du \end{cases}$$

$$\min_{u(t) \in \mathcal{L}_2[0,\infty]} \|z\|_2^2$$

with the assumptions

$$C^TD = 0, \quad D^TD = I$$

Assume (A, B) stabilizable
(C, A) detectable

Then the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \in \mathrm{dom\,Ric}$$

and $X = \mathrm{Ric}(H) \geq 0$ satisfies —— $(*)$

$$A^TX + XA + C^TC - XBB^TX = 0$$

and $A - BB^TX$ is stable.

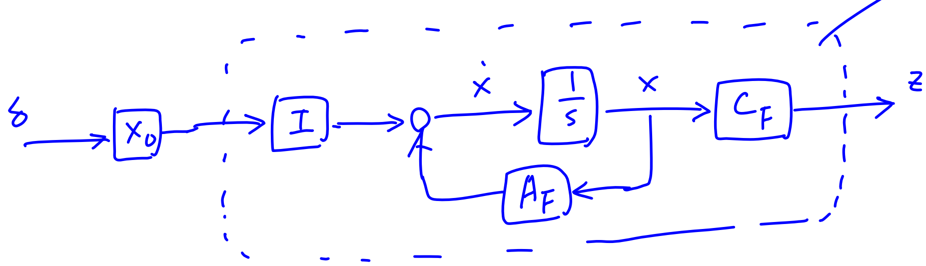
Define $F = -B^TX$, then $A+BF$ is stable

Apply the state feedback $u = Fx$, then

$$\begin{cases} \dot{x} = Ax + Bu \\ z = Cx + Du \end{cases} \xrightarrow{u = Fx} \begin{cases} \dot{x} = (A+BF)x := A_F x, \quad x(0) = x_0 \\ z = (C+DF)x := C_F x \end{cases}$$

i.e., $\begin{cases} \dot{x} = A_F x + x_0 \delta \\ z = C_F x \end{cases}$

$$\hat{G}_c(s) = \left[\begin{array}{c|c} A_F & I \\ \hline C_F & 0 \end{array}\right]$$



$$\hat{z} = \hat{G}_c(s) \cdot x_0$$

$$\|z\|_2^2 = \|\hat{z}\|_2^2 = \|\hat{G}_c x_0\|_2^2 = \|G_c x_0\|_2^2 = \mathrm{tr} \int_0^\infty x_0^T e^{A_F^T t} C_F^T C_F e^{A_F t} x_0 \, dt$$

$$= \mathrm{tr}\left(x_0^T \int_0^\infty e^{A_F^T t} C_F^T C_F e^{A_F t} dt \; x_0 \right) = \mathrm{tr}(x_0^T L_o x_0) = x_0^T L_o x_0$$

The Riccati equation (*) can be rearranged as

$$(A+BF)^T X + X(A+BF) + (C+DF)^T(C+DF) = 0$$

i.e., $A_F^T X + X A_F + C_F^T C_F = 0$ - - - $(**)$

Note that X is the observability gramian of (C_F, A_F).

Hence, $\|z\|_2^2 = x_0^T X x_0$

Theorem: For the LQR problem there exists a unique optimal solution

$$u(t) = F\,x(t) = -B^T X\,x(t)$$

where X is the stabilizing solution of (*)

Moreover,

$$\min_u \|z\|_2^2 = x_0^T X x_0$$

Pf: Differentiate $x^T(t) X x(t)$ w.r.t. t

$$\frac{d}{dt} x^T(t) X x(t) = \dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t)$$

$u = Fx$

$$= (Ax+Bu)^T X x + x^T X (Ax+Bu)$$

$$= [(A+BF)x + B(u-Fx)]^T X x + x^T X [(A+BF)x + B(u-Fx)]$$

$$= x^T(A_F^T X + X A_F)x + 2\langle B(u-Fx), Xx \rangle$$

$$= -x^T C_F^T C_F x + 2\langle u-Fx, B^T X x \rangle \quad \cdots \text{ from } (**)$$

$$= -\|Cx + DFx\|^2 - 2\langle u - Fx, Fx \rangle$$

$$= -\|Cx\|^2 - \|Fx\|^2 - 2\langle u, Fx\rangle + 2\|Fx\|^2 \qquad \because D^TC = 0,\ D^TD = I$$

$$= -\|Cx\|^2 + \|Fx\|^2 - 2\langle u, Fx\rangle$$

$$= -\|Cx\|^2 + \|u - Fx\|^2 - \|u\|^2 \qquad \left(\begin{array}{l} \because z = Cx + Du \\ D^TC = 0,\ D^TD = I \end{array}\right)$$

$$= -\|z\|^2 + \|u - Fx\|^2$$

Integrate

$$\frac{d}{dt} x^T(t) X x(t) = -\|z\|^2 + \|u - Fx\|^2$$

from 0 to ∞,

$$x^T(t) X x(t) \Big|_0^\infty = -\int_0^\infty \|z(t)\|^2 dt + \int_0^\infty \|u(t) - Fx(t)\|^2 dt$$

i.e., $\quad -x_0^T X x_0 = -\|z\|_2^2 + \|u - Fx\|_2^2$

$\Rightarrow$ the unique optimal solution $u = Fx$.

Def: A transfer function $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ in RH_∞ is inner if $G^T(-s)G(s) = I$.

Lemma:

Suppose there exists $X = X^T \in \mathbb{R}^{n\times n}$ s.t.

i) $A^TX + XA + C^TC = 0$

ii) $B^TX + D^TC = 0$

Then $G^T(-s)G(s) = D^TD$

Ex: $G(s) = \frac{s-1}{s+1}$ is inner

Pf:
$$G^T(-s)G(s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \cdot \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ -C^TC & -A^T & -C^TD \\ \hline D^TC & B^T & D^TD \end{array}\right]$$

Applying similarity transform $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$, we have

$$G^T(-s)G(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -(A^TX + XA + C^TC) & -A^T & -(XB + C^TD) \\ \hline B^TX + D^TC & B^T & D^TD \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^T & 0 \\ \hline 0 & B^T & D^TD \end{array}\right] = D^TD$$

Define $U(s) = \left[\begin{array}{c|c} A_F & B \\ \hline C_F & D \end{array}\right]$, $A_F = A + BF$, $C_F = C + DF$

Lemma: $U(s)$ is inner.

Pf: $A_F^T X + X A_F + C_F^T C_F = 0$

$B^T X + D^T C_F = -F + D^T(C + DF) = 0$ $\quad \left(\because \begin{array}{l} D^T C = 0 \\ D^T D = I \end{array}\right)$

$\Rightarrow U^T(-s) U(s) = D^T D = I$

Lemma $U^T(-s) \cdot G_c(s) \in RH_2^\perp$ anti stable purely unstable

Pf: $U^T(-s)\, G_c(s) = \left[\begin{array}{c|c} -A_F^T & -C_F^T \\ \hline B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A_F & I \\ \hline C_F & 0 \end{array}\right]$

$$= \left[\begin{array}{cc|c} -A_F^T & -C_F^T C_F & 0 \\ 0 & A_F & I \\ \hline B^T & F & 0 \end{array}\right] = \left[\begin{array}{cc|c} -A_F^T & 0 & -X \\ 0 & A_F & I \\ \hline B^T & 0 & 0 \end{array}\right]$$

similarity transf $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$

$$= \left[\begin{array}{c|c} -A_F^T & -X \\ \hline B^T & 0 \end{array}\right]$$ which belongs to $RH_2^\perp$

Alternate proof of the LQR Theorem :

$$\begin{cases} \dot{x} = Ax + Bu \\ z = Cx + Du \end{cases} \xrightarrow{u = v + Fx} \begin{cases} \dot{x} = (A+BF)x + Bv, \quad x(0) = x_0 \\ z = (C+DF)x + Dv \end{cases}$$

$$z = \underbrace{\left[\begin{array}{c|c} A_F & I \\ \hline C_F & 0 \end{array}\right]}_{G_c(s)} x_0 + \underbrace{\left[\begin{array}{c|c} A_F & B \\ \hline C_F & D \end{array}\right]}_{U(s)} v$$

$$z = G_c x_0 + U v$$

Let $v \in H_2$. By the previous Lemma the functions $G_c x_0$ and Uv are orthogonal (i.e., $\langle Uv, G_c x_0 \rangle = 0$)

Hence

$$\|z\|_2^2 = \|G_c x_0\|_2^2 + \|Uv\|_2^2$$

Since U is inner, we have

$$\|z\|_2^2 = \|G_c x_0\|_2^2 + \|v\|_2^2$$

The optimum occurs when $v = u - Fx = 0$, i.e. $u = Fx$.