CHAPTER 6

Time Delay, Plant Uncertainty, and Robust Stability

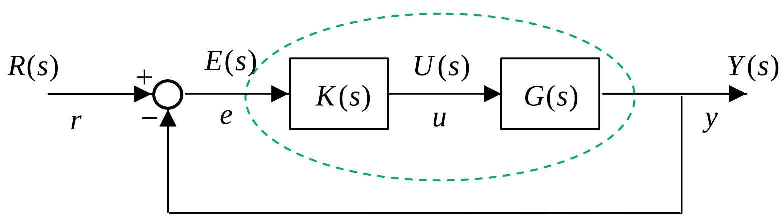
Stability **A system is stable if all the poles are in the left half of the complex plane.**

Time delay system **is an infinite dimensional system, which means the system has infinite number of poles.**

How to make sure that all the infinite number of poles of the system are in the left half of the complex plane?

There was no solution to this important practical problem until 1932 when Harry Nyquist (1889 – 1976) developed the the Nyquist stability criterion based on Cauchy complex integral theorem presented by a French mathematician Augustin-Louis Cauchy (1789 –1857) in 1831.

6.1.1 Time Delay and Stability



$Fig.1$

If the loop transfer function $L(s)$ **is given as**

$$L(s) = G(s)K(s) = \frac{2}{s+1}$$

then the closed-loop characteristic equation will be

$$1 + L(s) = 1 + \frac{2}{s+1} = 0 \quad \rightarrow \quad s+3=0$$

Now, assume the delay time is *T*. Then the loop transfer function will become

$$L(s) = \frac{2}{s+1} e^{-sT}$$

where the term e^{-sT} is the transfer function of a time delay element with delay time *T*. The closed-loop characteristic equation will turn out to be

$$F(s) = 1 + L(s) = \frac{1}{s+1}\left[3 + (1-2T)s + T^2 s^2 - \frac{1}{3}T^3 s^3 + \cdots\cdots\right] = 0$$

which apparently is a polynomial equation with infinite numbers of roots.

6.1.2 Plant Uncertainty and Stability

A feedback control system usually is designed based on a mathmatical model of the system to be controlled, which is called the plant.

In practice, the real system to be controlled is not identical to theideal plant model due to unmodelled plant dynamics, specification tolerance of components, and plant parameter perturbations influenced by the environment conditions

Hence, a feedback control system not only needs to be stable for the nominal system with the ideal plant model, it should be designed to achieve robust stability against all possible plant uncertainties.

The Nyquist approach not only resolves the stability analysis issue of infinite-dimensional feedback control systems, it also provides important concepts and tools for achieving robust stability of feedback control systems.

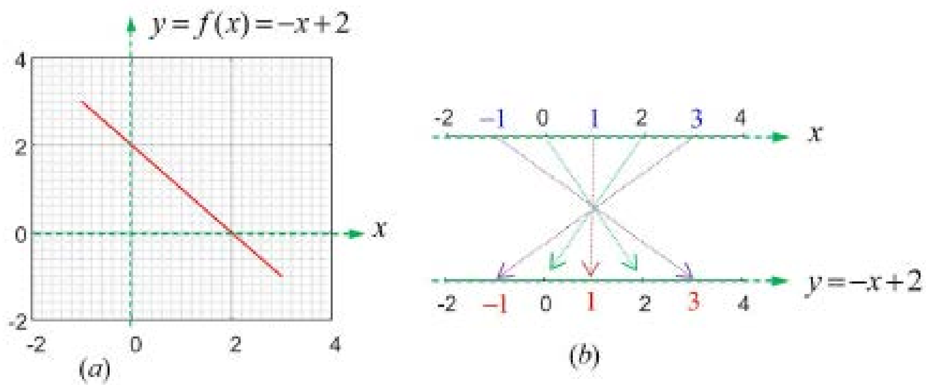
6.2 Contour Mapping and Cauchy's Principle of the Argument

Ex 0: A Simple Real Function Mapping

Consider the simple real function

$$y = f(x) = -x + 2$$

x	-1	0	1	2	3
y	3	2	1	0	-1



$Fig. 2$

Ex 1: An Illustration of Complex Function Contour Mapping

Consider the simple complex function

$$F(s) = \frac{s-0.5}{s+0.5}$$

How the *complex variable* s will affect the value of the *complex function* $F(s)$?

Assume s is moving along the simple closed contour Γ_s : $s = j\omega$, where $\omega = -1 \to 0 \to 1$, and then $s = e^{j\phi}$, where $\phi = \pi/2 \to 0 \to -\pi/2$.

The $F(s)$ mappings of segments (1) and (2), respectively, will be

$$F(j\omega)=\frac{j\omega-0.5}{j\omega+0.5}=\frac{e^{j(\pi-\theta)}}{e^{j\theta}}=e^{j(\pi-2\theta)},\quad \text{where}\quad \theta=\tan^{-1}\frac{\omega}{0.5},\quad \omega=-1\rightarrow 0\rightarrow 1$$

and

$$F(e^{j\phi})=\frac{e^{j\phi}-0.5}{e^{j\phi}+0.5}=\frac{0.75+j\sin\phi}{(\cos\phi+0.5)^2+\sin^2\phi},\quad \text{where}\quad \phi=\frac{\pi}{2}\rightarrow 0\rightarrow \frac{-\pi}{2}$$

The mapping relationship between s and $F(s)$ can be represented by the following tabulated chart:

s	$-j$	$\rightarrow$	$-j0.5$	$\rightarrow$	0	$\rightarrow$	$j0.5$	$\rightarrow$	j	$\rightarrow$	e^{j0}
$F(s)$	e^{-j53°	$\rightarrow$	$e^{-j\pi/2}$	$\rightarrow$	-1	$\rightarrow$	$e^{j\pi/2}$	$\rightarrow$	e^{j53°	$\rightarrow$	$1/3$

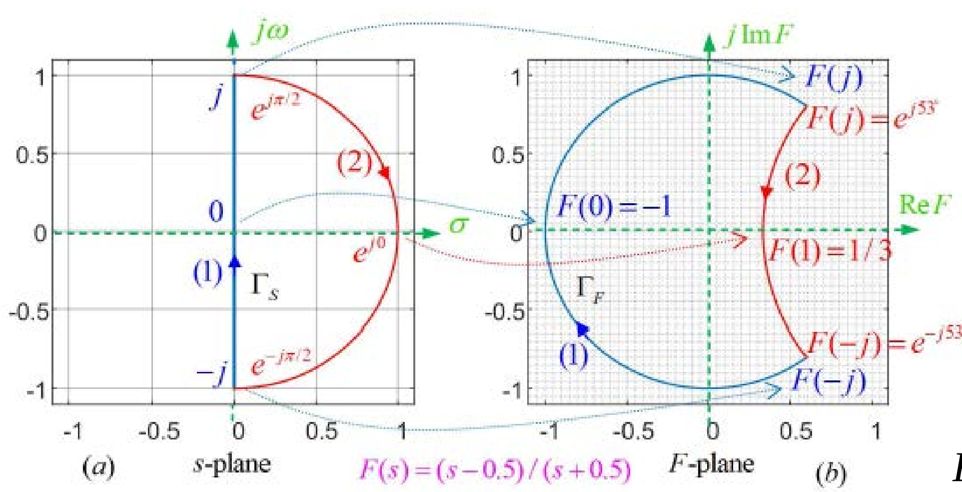


Fig. 3

After obtaining the complex contour mapping Γ_F, we are particularly interested in the number and direction of the encirclements of the origin by Γ_F on the F-plane.

It can be seen that Γ_F encircles the origin once clockwise, which is in the same direction of the Γ_s contour in the s-plane. Therefore, the number of encirclement is N = 1.

6.2.2 Cauchy's Principle of the Argument

Theorem 1: Cauchy's Principle of the Argument

Let Γ_s be a simple closed curve in the (complex) s-plane, as shown in the left graph of Figure 3.

F(s) is a rational function having no poles or zeros on Γ_s. Let Γ_F be the image of Γ_s under the map F(s). Then,

$$N = Z - P$$

> ***N** is the number of clockwise encirclements of the origin by Γ_F as s traverses Γ_s once in the clockwise direction;*
>
> ***Z** is the number of zeros of F(s) enclosed by Γ_s, counting multiplicities; and*
>
> ***P** is the number of poles of F(s) enclosed by Γ_s, counting multiplicities*

Proof:

$$F(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$F(s) = K\frac{\rho_{z1}e^{j\theta_{z1}}\rho_{z2}e^{j\theta_{z2}}\cdots\rho_{zm}e^{j\theta_{zm}}}{\rho_{p1}e^{j\theta_{p1}}\rho_{p2}e^{j\theta_{p2}}\cdots\rho_{pn}e^{j\theta_{pn}}} = \frac{K\rho_{z1}\rho_{z2}\cdots\rho_{zm}}{\rho_{p1}\rho_{p2}\cdots\rho_{pn}}e^{j[(\theta_{z1}+\theta_{z2}+\cdots+\theta_{zm})-(\theta_{p1}+\theta_{p2}+\cdots+\theta_{pn})]} \qquad (*)$$

where

$$\rho_{zi} = |s-z_i|, \quad i=1,\cdots,m \qquad \text{and} \qquad \rho_{pj} = |s-p_j|, \quad j=1,\cdots,n$$
$$\theta_{zi} = \angle(s-z_i), \quad i=1,\cdots,m \qquad \text{and} \qquad \theta_{pj} = \angle(s-p_j), \quad j=1,\cdots,n$$

In view of Equation (*), as s traverses Γ_s once, its image Γ_F encircles the origin only if at least one of the angles θ_j undergoes a change of 2π radians.

Any pole or zero outside of Γ_s does not produce any angle change through a circuit of Γ_s. On the other hand, a pole or zero inside of Γ_s does produce a 2π angle change. Equation (*) implies that for a complete clockwise transverse of Γ_s, each zero inside of Γ_s produces a clockwise 2π angle change and each pole inside of Γ_s produces a counterclockwise 2π angle change. Hence, the net number of clockwise encirclements of the origin by Γ_F is $Z - P$.

Ex 2: Illustration of the Principle of the Argument

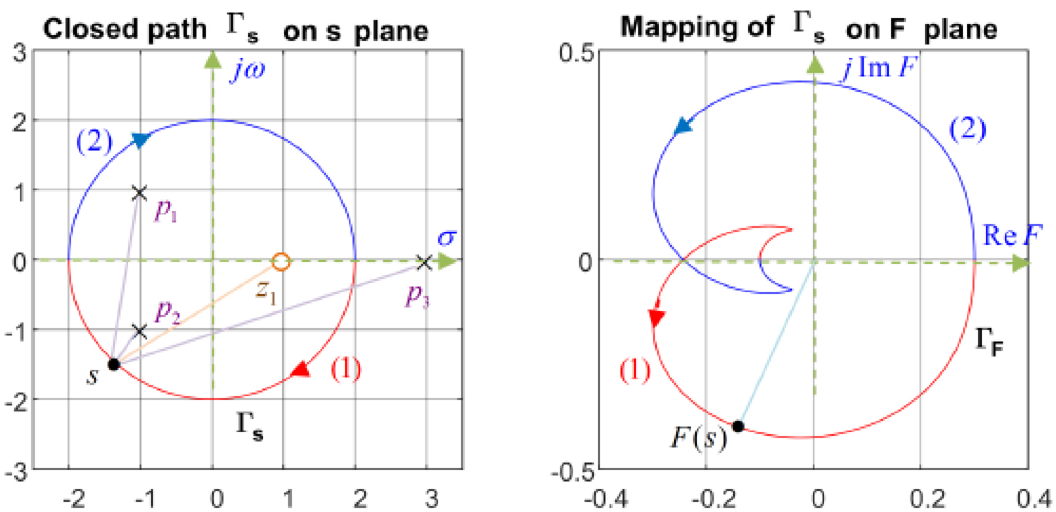
Consider the complex rational function

$$F(s) = \frac{s - z_1}{(s - p_1)(s - p_2)(s - p_3)} = \frac{s - 1}{(s + 1 - j)(s + 1 - j)(s - 3)} = \frac{s - 1}{s^3 - s^2 - 4s - 6}$$

and the simple closed path Γ_s is a circle centered at the origin of the *s*-plane with radius equals to 2. For clarity, Γ_s is partitioned into two segments:

Segment (1) is in red, which starts from $s = 2$, clockwise along the semicircle to $s = 2e^{-j135^\circ}$ and ends at $s = -2$.

Immediately, segment (2), which is in blue, begins from $s = -2$, moving along the upper semicircle to go back to $s = 2$ to complete one revolution of the circle.



$Fig.\,4$

The image of segment (1) of Γ_s is shown in the right graph of $Fig.\,4$ as segment (1) of Γ_F, also in red. It can be seen that the starting point $s = 2$ is mapped to $F(2) = -0.1$, $s = 2e^{-j135^\circ}$ to $F(2e^{-j135^\circ}) = -0.16212 - j0.38819$, and $s = -2$ finds its image at $F(\text{-}2) = 0.3$ on Γ_F.

Similarly, the three points on segment (2) of Γ_s: $s = -2$, $s = 2e^{j135^\circ}$, and $s = 2$ are mapped to $F(-2) = 0.3$, $F(2e^{j135^\circ}) = -0.16212 + j0.38819$, and $F(2) = -0.1$, respectively, on Γ_F.

Note that Γ_s is symmetrical with respect to the real axis, and it encircles the origin of the *F*-plane once counterclockwise. Hence, the number of clockwise encirclements of the origin by Γ_F is $N=-1$.

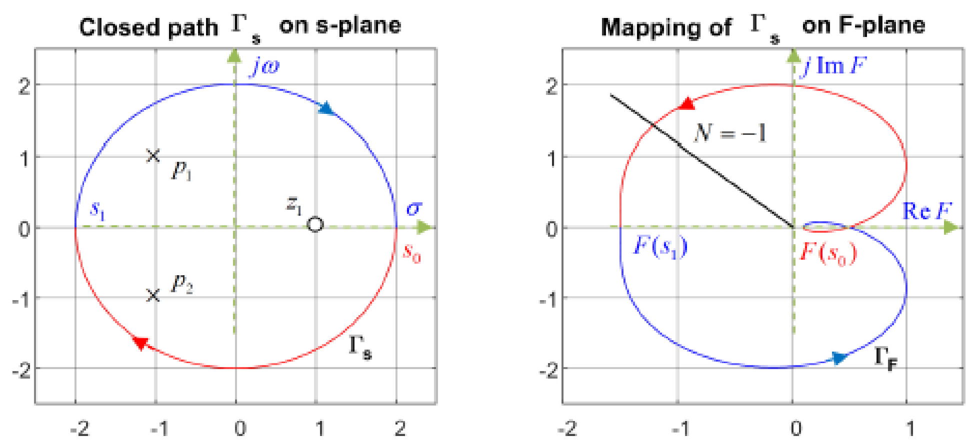
Recall that the number of poles of $F(s)$ enclosed by Γ_s is $P=2$, and the number of zeros of $F(s)$ enclosed by Γ_s is $Z=1$.

Therefore, $N=-1=Z-P=1-2$, which is consistent with the result of Theorem 1.

Ex 3: A Pole or Zero Outside Γ_s Does Not Affect the Encirclement Number N, but Will Change the Shape of Contour Mapping Γ_F

The complex rational function $F(s)$ here is almost the same as that considered in the previous example, except that the pole p3 outside Γ_s is removed. That is,

$$F(s)=\frac{s-z_1}{(s-p_1)(s-p_2)}=\frac{s-1}{(s+1-j)(s+1-j)}=\frac{s-1}{s^2+2s+2}$$



$Fig.5$

The simple closed path Γ_s is still the same, which is a circle centered at the origin of the *s*-plane with radius equal to 2.

The image of segment (1) of Γ_s is shown in the right graph of *Fig*. 5 as segment (1) of Γ_F, also in red. It can be seen that the starting point $s=2$ is

mapped to $F(2)=0.1$, $s=-j2$ **to** $F(2e^{-j90^\circ})=0.5$, **and** $s=2e^{-j136.4^\circ}$ **to** $F(2e^{-j136.4^\circ})=j1.978$, **and** $s=-2$ **finds its image at** $F(\text{-}2)=\text{-}1.5$ **on** Γ_F.

Similarly, the four points on segment (2) of Γ_s**:** $s=-2$, $s=2e^{j136.4^\circ}$, $s=j2$, **and** $s=2$ **are mapped to** $F(-2)=-1.5$, $F(2e^{j136.4^\circ})=-j1.978$, $F(j2)=F(2e^{-j90^\circ})=0.5$ **and** $F(2)=0.1$, **respectively, on** Γ_F.

Note that Γ_F **is symmetrical with respect to the real axis, and it encircles the origin of the *F*-plane once, counterclockwise. Hence, the number of clockwise encirclements of the origin by** Γ_F **is** $N=-1$. **Recall that the number of poles of** $F(s)$ **enclosed by** Γ_s **is** $P=2$, **and the number of zeros of** $F(s)$ **enclosed by** Γ_s**, is** $Z=1$. **Therefore,** $N=-1=Z-P=1-2$, **which is consistent with the result of Theorem 1.**

Note that the *N*, *Z*, and *P* numbers are the same as those in Example 2, but the shape of contour mapping Γ_F **is very different from that in Example 2.**

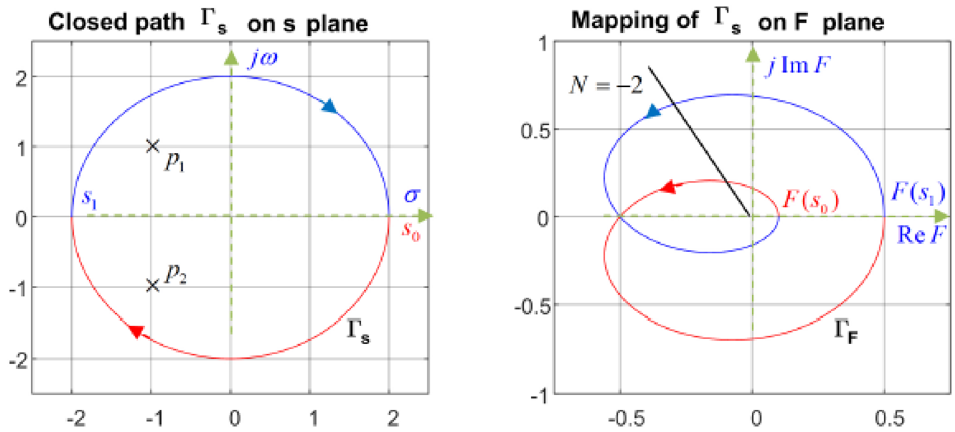
In the next example, we will observe how the contour mapping Γ_F **and its clockwise encirclement number around the origin will be affected if the zero,** z_1**, inside** Γ_s **is removed.**

Ex 4: A Change of the Number of Poles or Zeros Inside Γ_s Will Affect the Encirclement Number *N* of Γ_F Around the Origin)

The complex rational function $F(s)$ **here is almost the same as that considered in the previous example, except that the zero** z_1 **inside** Γ_s **is removed. That is,**

$$F(s)=\frac{1}{(s-p_1)(s-p_2)}=\frac{1}{(s+1-j)(s+1-j)}=\frac{1}{s^2+2s+2}$$

The simple closed path Γ_s is still the same, which is a circle centered at the origin of the s-plane with radius equal to 2.



$Fig.6$

The image of segment (1) of Γ_s is shown in the right graph of *Fig*. 6 as segment (1) of Γ_F, also in red. It can be seen that the starting point $s=2$ is mapped to $F(2)=0.1$, $s=-j2$ to $F(2e^{-j90^\circ})=0.1+j0.2$, $s=2e^{-j120^\circ}$ to $F(2e^{-j120^\circ})=-0.5$, and $s=2e^{-j144^\circ}$ to $F(2e^{-j144^\circ})=-j0.688$, and $s=-2$ finds its image at $F(\text{-}2)=0.5$ on Γ_F.

Similarly, the image of segment (2) of Γ_s can be found as the conjugate of the image of segment (1), as shown in the right graph of *Fig*. 6.

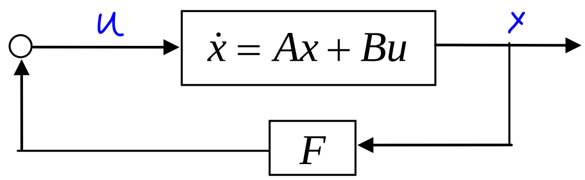
Note that Γ_F is symmetrical with respect to the real axis, and it encircles the origin of the *F*-plane twice, counterclockwise. Hence, the number of clockwise encirclements of the origin by Γ_F is $N=-2$.

Recall that the number of poles of $F(s)$ enclosed by Γ_s is $P=2$, and the number of zeros of $F(s)$ enclosed by Γ_s is $Z=0$.
Therefore, $N=-2=Z-P=0-2$, which is consistent with the result of Theorem 1.

Review of Chapter 5 ***MEM634 Chapter 5***

State-space stabilization approach

Full state feedback



$$\begin{cases} \dot{x} = Ax + Bu \\ u = Fx \end{cases}$$

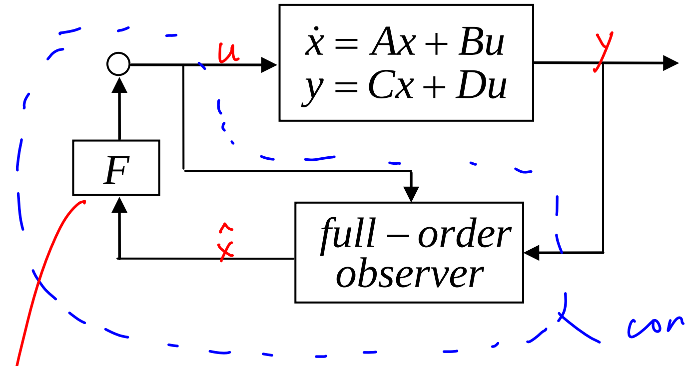
$$\dot{x} = Ax + BFx$$
$$\dot{x} = (A + BF)x$$

System is stable iff $A+BF$ is stable

i.e., $A+BF$ has no eigenvalues in the closed RHP.

Output feedback

F can be chosen s.t. $A+BF$ is stable iff (A, B) is stabilizable.

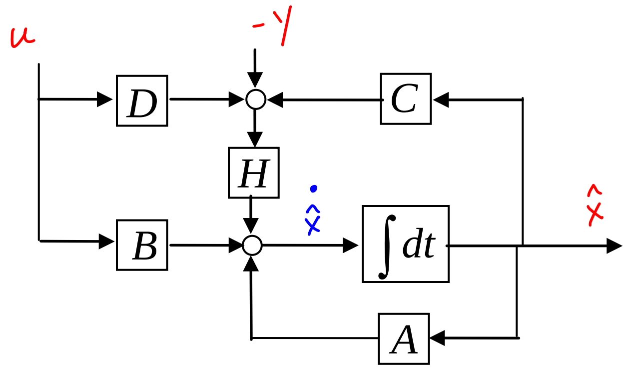


controller K

F : state feedback gain

Structure of a full-order observer

H : observer gain



$(B+HD)u - Hy$

$= \begin{bmatrix} B+HD & -H \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$

$$\dot{\hat{x}} = A\hat{x} + Bu + H[C\hat{x} - y + Du]$$

$$\dot{\hat{x}} = (A+HC)\hat{x} + (B+HD)u - Hy$$

The stability of the observer is determined by $A+HC$

H can be chosen to make $A+HC$ stable iff (C, A) is detectable.

Reconstruction error

$$\epsilon(t) = x(t) - \hat{x}(t)$$

$$\dot{\epsilon}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Ax + Bu - [A\hat{x} + (B+HD)u + HC\hat{x} - HCx - HDu]$$

$$\dot{\epsilon}(t) = (A+HC)(x - \hat{x}) = (A+HC)\epsilon(t)$$

$$\epsilon(t) = e^{(A+HC)(t-t_0)}\epsilon(t_0)$$

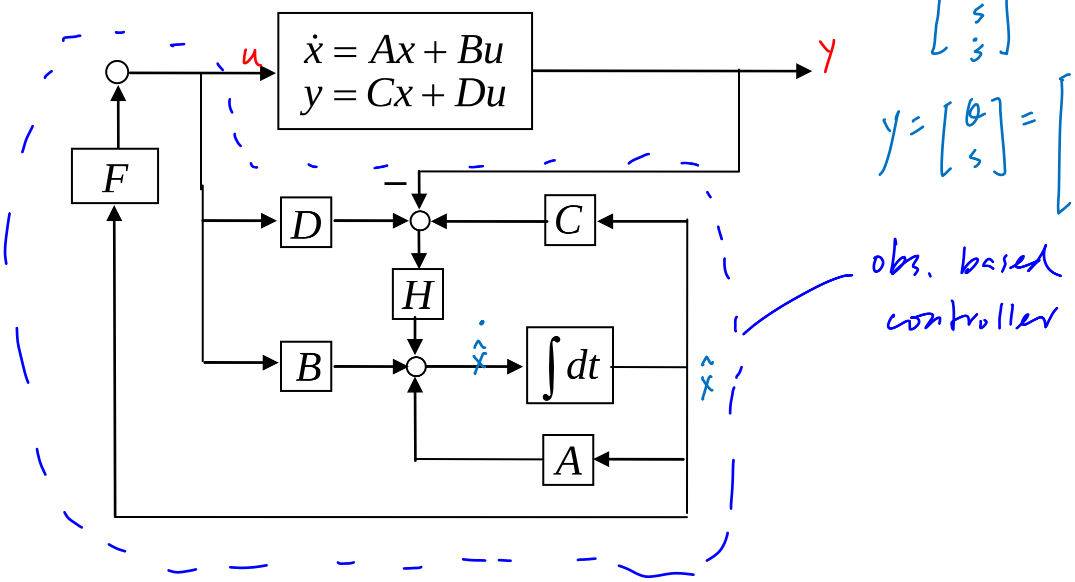
$\epsilon(t) \to 0$ as $t \to \infty$ for all $\epsilon(t_0)$

iff $A+HC$ is stable.

Observer-based controller

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ s \\ \dot{s} \end{bmatrix}$$

$$y = \begin{bmatrix} \theta \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \theta \\ \dot{\theta} \\ s \\ \dot{s} \end{bmatrix}}_{x}$$



observer: $\dot{\hat{x}} = A\hat{x} + Bu + H[C\hat{x} + Du - y]$

state feedback: $u = F\hat{x}$

Then the controller is:

u

$$\dot{\hat{x}} = \underbrace{(A + BF + HDF + HC)}_{A_K}\hat{x} \underbrace{-H}_{B_K} y$$

$$u = \underbrace{F}_{C_K}\hat{x}$$

y

$$\dot{x} = Ax + BC_K x_K$$

$$\dot{x}_K = A_K x_K + B_K C x$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix}$$

Now the closed-loop system is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BF \\ -HC & A+BF+HC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$\dot{x} = Ax + Bu$, $\quad u = F\hat{x}$

$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + HC\hat{x} + H[Du - y]$, $\quad y = Cx + Du$

$$\det \begin{bmatrix} sI - A & -BF \\ HC & sI-(A+BF+HC) \end{bmatrix}$$

$$= \det[sI - (A+BF)] \cdot \det[sI - (A+HC)]$$

The closed-loop system poles are the regulator poles (eigenvalues of $A+BF$) together with the observer poles (eigenvalues of $A+HC$).

Example:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx = [1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2 \end{cases} \qquad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 1]$$

Choose F s.t. $A+BF$ is stable, Let $F = [f_1 \quad f_2]$

$$A+BF = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[f_1 \quad f_2] = \begin{bmatrix} f_1 & f_2 \\ 1 & 0 \end{bmatrix}$$

$$|\lambda I - (A+BF)| = \begin{vmatrix} \lambda - f_1 & -f_2 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - f_1\lambda - f_2 = (\lambda+1)(\lambda+2)$$
$$= \lambda^2 + 3\lambda + 2$$

$$\Rightarrow \quad f_1 = -3, \quad f_2 = -2, \quad \text{i.e.,} \quad F = [-3 \quad -2]$$

choose H s.t. $(A+HC)$ is stable.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}[1 \quad 1] = \begin{bmatrix} h_1 & h_1 \\ 1+h_2 & h_2 \end{bmatrix}$$

$$|\lambda I - (A+HC)| = \begin{vmatrix} \lambda - h_1 & -h_1 \\ -1-h_2 & \lambda - h_2 \end{vmatrix} = \lambda^2 - (h_1+h_2)\lambda + h_1(2h_2 - 1)$$
$$= (\lambda+2)^2 = \lambda^2 + 4\lambda + 4$$

$$\Rightarrow \quad h_1 = -4, \quad h_2 = 0, \quad \text{i.e.,} \quad H = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

Computation of the State Feedback Gain *F* Using Linear Quadratic Regulator Design

The state feedback gain matrix F can be computed as follows

$$F = -R^{-1}B^T X$$

where X is the positive semi-definite stabilizing solution of the following algebraic Riccati equation

$$A^T X + XA - XB\, R^{-1}B^T X + Q = 0$$

Note that the eigenvalues of $A + BF$, the regulator poles, are identical to the stable eigenvalues of the following Hamiltonian matrix

$$H_{reg} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

Algebraic Riccati equation

$$A^T X + XA + XRX - Q = 0 \qquad \text{(ARE)}$$

where A, R and Q are given nxn constant real matrices. Moreover, R and Q are real symmetric.

One way to solve ARE for X is to solve the corresponding Hamiltonian matrix:

$$H = \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \qquad \text{(Ham)}$$

In the following, we denote $\pi(H)$ as the set of all eigenvalues of H; $\pi_n(H)$ as a subset of $\pi(H)$ with n elements; $\pi_{n-}(H)$ ($\pi_{n+}(H)$, respectively) consisting of n eigenvalues of H with negative (positive, respectively) real parts. For any $\pi_n(H)$, one can always find a modal matrix $T \in C^{2nxn}$ which is stacked by n independent eigenvectors (or generalized eigenvectors) of H, corresponding to $\pi_n(H)$. Partition T in such a way that

$$T = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

with $X_1, X_2 \in C^{nxn}$. It is not difficult to verify that if X_1 is invertible, then $X = X_2 X_1^{-1}$ is a solution to ARE. Moreover, $\pi_n(H) = \pi(A+RX)$.

4×4

H: 2 stable eigenvalues, 2 unstable eigenvalues

λ_1, λ_2

$X = X_2 X_1^{-1}$

$H\lambda_1 = \lambda_1 e^1, \quad H\lambda_2 = \lambda_2 e^2$

$$\mathcal{X}_-(H) = Im\,[\underset{4\times1}{e^1}\ \underset{4\times1}{e^2}] = Im \begin{bmatrix} \boxed{\begin{matrix} x & x \\ x & x \end{matrix}} \\ \boxed{\begin{matrix} x & x \\ x & x \end{matrix}} \end{bmatrix} \begin{matrix} X_1 \\ \\ X_2 \end{matrix} = Im \begin{bmatrix} I \\ X \end{bmatrix}$$

It is obvious that the solution to ARE is not unique, since T can be stacked by different set of eigenvectors. However, if X is a solution such that $A+RX$ is stable, then X is called the **stabilizing solution** to ARE. The stabilizing solution can be constructed by $X = X_2 X_1^{-1}$ with T corresponding to $\pi_{n-}(H)$. It is well known that the stabilizing solution is **real**, **symmetric** and **unique**.

Properties of X:

Lemma: Suppose $H \in \text{dom}\,Ric$, and $X = Ric(H)$, Then

(i) X is symmetric

(ii) X satisfies the ARE: $A^T X + XA + XRX - Q = 0$.

(iii) $A + RX$ is stable.