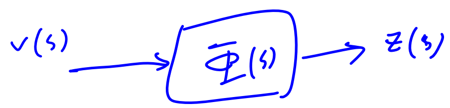
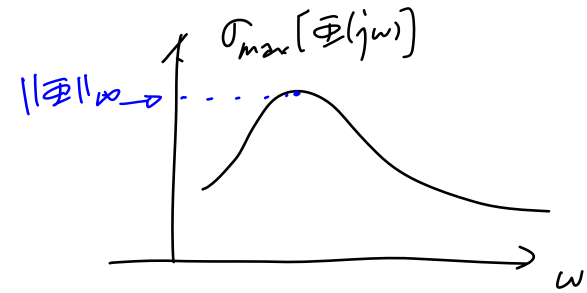
CHAPTER 10

State-space Formulae for Stabilizing Optimal H_∞ Controllers

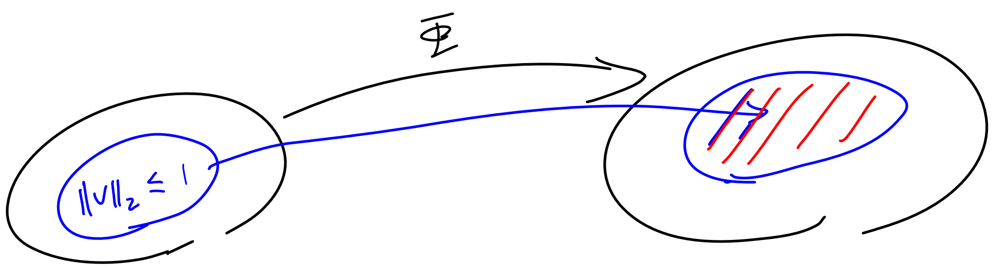
The Standard H_∞ Problem





$$\|\Phi\|_\infty = \sup_{\omega} \sigma_{max}[\Phi(j\omega)]$$

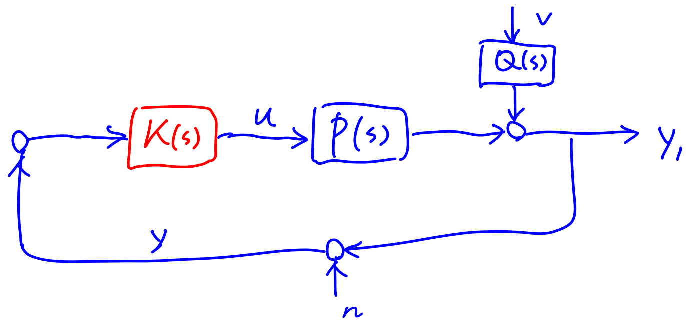
$$= \sup_{\|v\|_2 \le 1} \|z\|_2$$



$$\|\Phi\|_\infty = \sup\left\{ \|z\|_2 : \|v\|_2 \le 1 \right\}$$

H_∞ Control Theory

Ex 1. A disturbance reduction problem



$$P(s) = \frac{1}{s-2}, \quad Q(s) = \frac{1}{s+1}$$

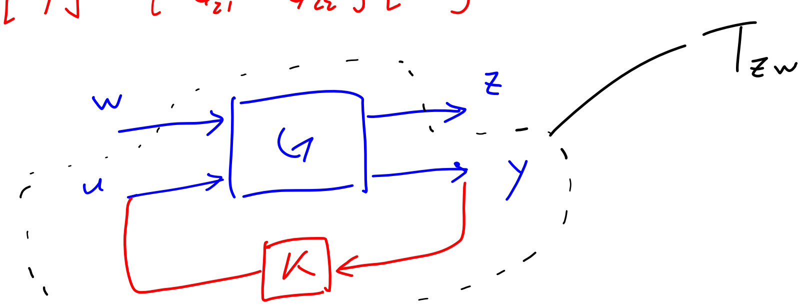
$$\begin{cases} y_1 = \frac{1}{s-2}u + \frac{1}{s+1}v \\ y = n + \frac{1}{s-2}u + \frac{1}{s+1}v \end{cases}$$

$$\begin{bmatrix} y_1 \\ u \\ \hdashline y \end{bmatrix} = \left[\begin{array}{cc:c} \frac{1}{s+1} & 0 & \frac{1}{s-2} \\ 0 & 0 & 1 \\ \hdashline \frac{1}{s+1} & 1 & \frac{1}{s-2} \end{array}\right] \begin{bmatrix} v \\ n \\ \hdashline u \end{bmatrix}$$

$z = \begin{bmatrix} y_1 \\ u \end{bmatrix}, \quad w = \begin{bmatrix} v \\ n \end{bmatrix}$

i.e.,

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$



Objective : Find $K(s)$ s. t. the closed-loop system is stable and $\|T_{zw}\|_\infty$ is minimized.

$$G(s) = \left[\begin{array}{cc:c} \frac{1}{s+1} & 0 & \frac{1}{s-2} \\ 0 & 0 & 1 \\ \hdashline \frac{1}{s+1} & 1 & \frac{1}{s-2} \end{array}\right]$$

$$= \frac{1}{s+1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \frac{1}{s-2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s+1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \frac{1}{s-2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\left[sI - A\right]^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{array}{c} \begin{array}{cc} B_1 & \quad B_2 \end{array} \\ \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array}, \quad C = \begin{array}{c} \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \\ \hdashline 1 & 1 \end{array}\right] \end{array} \begin{array}{c} C_1 \\ \\ C_2 \end{array}$$

$$D = \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hdashline 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

Assumptions (DGKF):

(i) (A, B_1) stabilizable? no

(C_1, A) detectable? yes

(ii) (A, B_2) stabilizable? yes

(C_2, A) detectable? yes

(iii) $D_{12}^T [C_1 \;\; D_{12}] = [0 \;\; I]$? yes

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$? yes

Define two Hamiltonian matrices :

$$H_\infty(\gamma) = \begin{bmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}$$

$$J_\infty(\gamma) = \begin{bmatrix} A^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}$$

Theorem : There exist a stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions are satisfied.

(i) $H_\infty(\gamma) \in dom(Ric)$ and $X := Ric(H_\infty(\gamma)) \geq 0$

(ii) $J_\infty(\gamma) \in dom(Ric)$ and $Y := Ric(J_\infty(\gamma)) \geq 0$

(iii) $\rho(XY) < \gamma^2$

spectral radius of XY

A suboptimal controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A} & -ZL \\ \hline F & 0 \end{array}\right]$$

where
$$\hat{A} = A + \gamma^{-2} B_1 B_1^T X + B_2 F + ZLC_2$$
$$F = -B_2^T X, \quad L = -YC_2^T, \quad Z = (I - \gamma^{-2} YX)^{-1}$$

Exercise:

Use the data of $G(s)$ in Ex. 1.

Find a suboptimal H_∞ controller such that the closed-loop system is stable and

$\|T_{zw}\|_\infty < \gamma$ with $\gamma - \|T_{zw}\|_\infty < 0.01$.

Computation of $\left\|\hat{G}\right\|_\infty$ in frequency domain:

$$\left\|\hat{G}\right\|_\infty := \sup_\omega \left|\hat{G}(j\omega)\right| = \sup_\omega \bar{\sigma}\left[\hat{G}(j\omega)\right]$$

Computations of H_∞ norm in time domain

Theorem:

For any $\hat{G}(s) \in RH_\infty$, $\left\|\hat{G}\right\|_\infty < \gamma$ if and only if the Hamiltonian matrix

$$H_G = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix}$$

does not have any eigenvalues on the $j\omega$-axis, where γ is a nonnegative number, and $R = \gamma^2 I - D^T D$.

The above theorem actually implies that $\left\|\hat{G}\right\|_\infty = \inf\{\gamma$: H_G does not have $j\omega$-axis eigenvalues$\}$ and that one can compute $\left\|\hat{G}\right\|_\infty$ by an iterative algorithm: choose a positive number γ; calculate the eigenvalues of H and check whether any of them are on the $j\omega$-axis; decrease or increase γ accordingly; repeat, until the infimum is reached within the tolerance.

Computation of H_∞ norm, $\|G\|_\infty$

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad A \text{ is stable}$$

Define a Hamiltonian matrix

$$H(\gamma) = \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

Theorem :

$\|G\|_\infty < \gamma$ if and only if $H(\gamma)$ has no eigenvalues on the $j\omega$-axis.

Proof :

$$G(-s)^T = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & 0 \end{array}\right]$$

$$\gamma^2 I - G(-s)^T G(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -C^T C & -A^T & 0 \\ \hline 0 & -B^T & \gamma^2 I \end{array}\right]$$

$$\left[\gamma^2 I - G(-s)^T G(s)\right]^{-1} = \left[\begin{array}{cc|c} A & \gamma^{-2} B B^T & \gamma^{-2} B \\ -C^T C & -A^T & 0 \\ \hline 0 & \gamma^{-2} B^T & \gamma^{-2} I \end{array}\right]$$

Note that $H(\gamma)$ is the A-matrix of $\left[\gamma^2 I - G^T(-s) G(s)\right]^{-1}$.

Claim : $\left(H(\gamma), \begin{bmatrix} -\gamma^{-2} B \\ 0 \end{bmatrix}\right)$ has no uncontrollable modes on the $j\omega$-axis.

$\left(\begin{bmatrix} 0 & \gamma^{-2} B \end{bmatrix}, H(\gamma)\right)$ has no unobservable modes on the $j\omega$-axis

Proof :

$$\operatorname{rank}\begin{bmatrix} A - j\omega & \gamma^{-2} B B^T & \gamma^{-2} B \\ -C^T C & -A^T - j\omega & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A - j\omega & 0 & \gamma^{-2} B \\ -C^T C & -A - j\omega & 0 \end{bmatrix} = 2n$$

Thus $H(\gamma)$ has no eigenvalues on the $j\omega$-axis if and only if $\left[\gamma^2 I - G(-s)^T G(s)\right]^{-1}$ has no poles on the $j\omega$-axis,

i.e., $\gamma^2 I - G(-s)^T G(s) \in RL_\infty$

Now, we need to prove that

$$\|G\|_\infty < \gamma \iff \gamma^2 I - G^T(-s)G(s) \in RL_\infty$$

If $\|G\|_\infty < \gamma$, then $\gamma^2 I - G(j\omega)^* G(j\omega) > 0 \quad \forall \omega$

and hence $\gamma^2 I - G^T(-s)G(s) \in RL_\infty$

Conversely, if $\|G\|_\infty \geq \gamma$, then $\bar{\sigma}[G(j\omega)] = \gamma$ for some ω

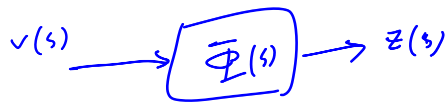
i.e., γ^2 is an eigenvalue of $G^*(j\omega)G(j\omega)$. so

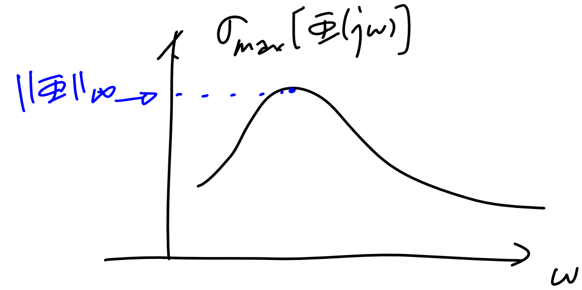
$\gamma^2 I - G^*(j\omega)G(j\omega)$ is singular.

Iterative computation procedure.

1. choose an upper bound γ_u, lower bound γ_ℓ
2. Let $\gamma = (\gamma_u + \gamma_\ell)/2$
3. compute the eigenvalues of $H(\gamma)$
4. If $H(\gamma)$ has no eigenvalues on the $j\omega$-axis, then update γ_u by γ and go to step 6
5. otherwise update γ_ℓ by γ
6. If $\gamma_u - \gamma_\ell < \epsilon$, then $\|G\|_\infty = \gamma_u$ and stop
7. Go to step 2.

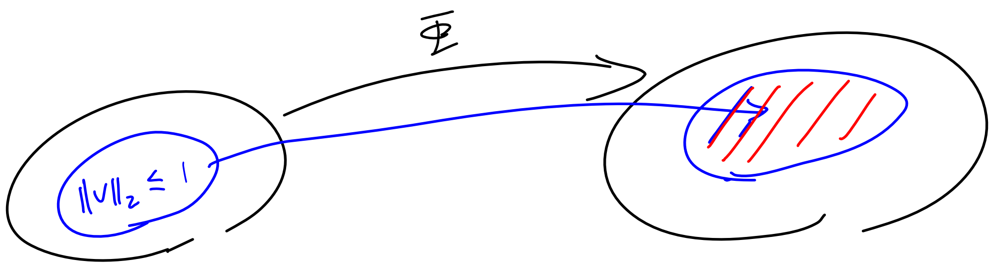
The Standard H_∞ Problem



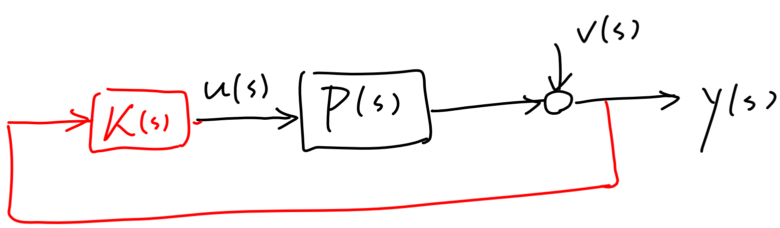


$$\|\Phi\|_\infty = \sup_{\omega} \sigma_{max}[\Phi(j\omega)]$$

$$= \sup_{\|v\|_2 \le 1} \|z\|_2$$

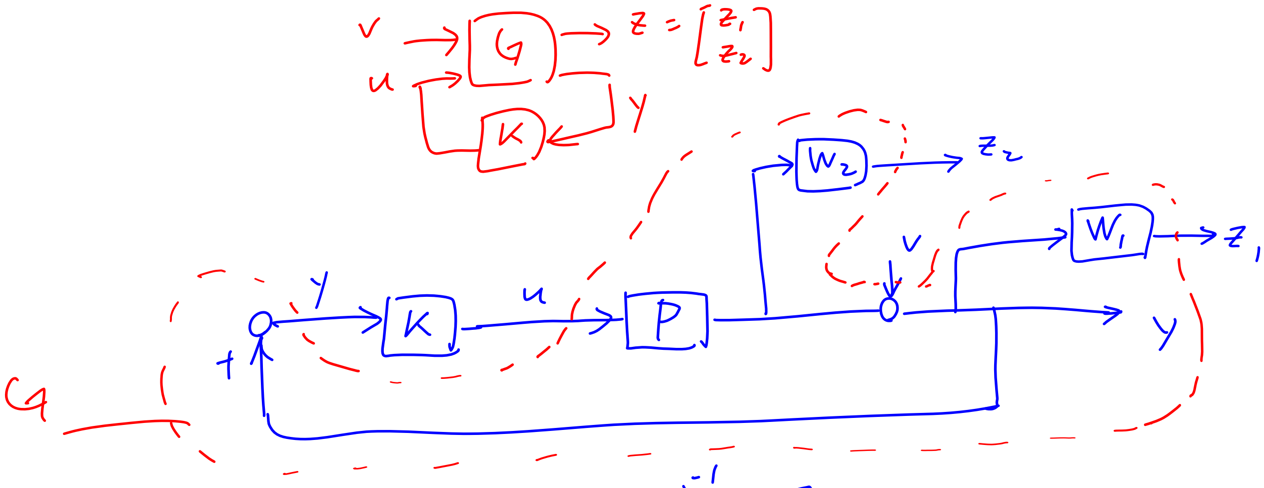


$$\|\Phi\|_\infty = \sup \{ \|z\|_2 : \|v\|_2 \le 1 \}$$



$$y(s) = \underbrace{[I - P(s)K(s)]^{-1}}_{\text{sensitivity function}} v(s)$$

$PK[I - PK]^{-1}$: complementary sensitivity function



$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1 (I - PK)^{-1} \\ W_2 PK (I - PK)^{-1} \end{bmatrix} v = \Phi \cdot v$$

Find K s.t. the closed-loop system is stable and $\|\Phi\|_\infty$ is minimized.

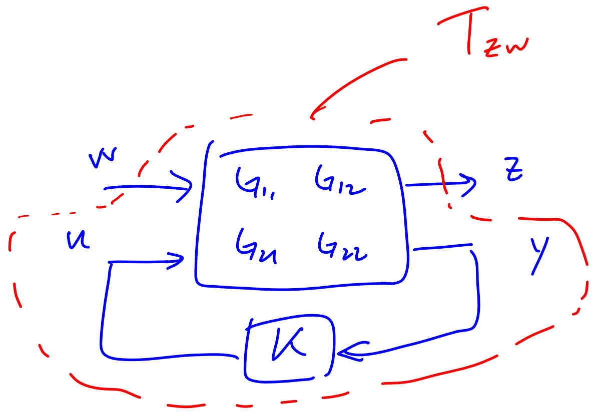
$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = \begin{bmatrix} W_1 (v + Pu) \\ W_2 Pu \\ v + Pu \end{bmatrix} = \left[\begin{array}{c:c} W_1 & W_1 P \\ 0 & W_2 P \\ \hdashline I & P \end{array}\right] \begin{bmatrix} v \\ u \end{bmatrix} = \left[\begin{array}{c:c} G_{11} & G_{12} \\ \hdashline G_{21} & G_{22} \end{array}\right] \begin{bmatrix} v \\ u \end{bmatrix}$$

State-space representation for $G(s)$:

$$G(s) = \left[\begin{array}{c:c} W_1 & W_1 P \\ 0 & W_2 P \\ \hdashline I & P \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Given
$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \quad W_2 P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_{W_2} & D_{W_2} \end{array}\right], \quad W_1 = \left[\begin{array}{c|c} A_{W_1} & B_{W_1} \\ \hline C_{W_1} & D_{W_1} \end{array}\right]$$

$A, B_1, B_2, C_1, C_2, D_{11}, \ldots$ (3-5) in AIAA paper



$$T_{zw} = \mathcal{F}_\ell[G, K] = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

optimal H_∞ control problem:

Find a stabilizing K s.t. $\|T_{zw}\|_\infty$ is minimized.

suboptimal H_∞ control problem:

Find a stabilizing controller K s.t. $\|T_{zw}\|_\infty < \gamma$. ☆

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{array}{c|c} & \begin{array}{ccc} n & m_1 & m_2 \end{array} \\ \begin{array}{c} n \\ p_1 \\ p_2 \end{array} & \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \end{array}$$

rank $D_{12} = m_2$: D_{12} has full column rank m_2

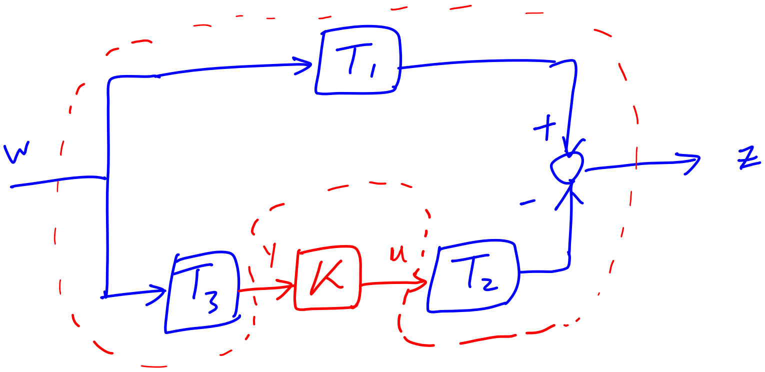
rank $D_{21} = p_2$: D_{21} " " row rank p_2

$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$

Theorem 1 : (Glover & Doyle)

p. 169

Model Matching Problem



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} T_1 & -T_2 \\ T_3 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

Find $K \in RH_\infty$ s.t. $\|T_1 - T_2 K T_3\|_\infty$ is minimized.

Ex: $\left\| \frac{1}{s+1} - \frac{s-1}{s+1} K \cdot 1 \right\|_\infty$

$$\Phi = \frac{1}{s+1} - \frac{s-1}{s+1}K = \frac{+1}{-2}$$

$$= \frac{1 - Ks + K}{s+1} = \pm\frac{1}{2}$$

$$= K+1 - Ks = \pm 0.5(s+1)$$

$$= -0.5 + 1 + 0.5s = 0.5s + 1$$

$$\Rightarrow K = \frac{-1}{2}$$

$$T_1 = \frac{1}{s+1} = \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right]$$

$$-T_2 = \frac{-s+1}{s+1} = -1 + \frac{2}{s+1} = \left[\begin{array}{c|c} -1 & 2 \\ \hline 1 & -1 \end{array}\right]$$

$$T_3 = 1$$

$$\begin{bmatrix} T_1 & -T_2 \\ T_3 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} -1 & 1 & 2 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right]$$

$A = -1$, $B_1 = 1$, $B_2 = 2$

$C_1 = 1$, $D_{11} = 0$, $D_{12} = -1$

$C_2 = 0$, $D_{21} = 1$, $D_{22} = 0$

$$R = \begin{bmatrix} D_{11} \\ D_{12} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \left\{ \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\cdot} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\cdot}^* C_1 & B^* \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \right\} = \begin{bmatrix} 1 & \gamma^{-2} - 4 \\ 0 & -1 \end{bmatrix}$$

$$\tilde{R} = D_{\cdot 1} D_{\cdot 1}^* - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = \left\{ \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \begin{bmatrix} -1 & \gamma^{-2} \\ 0 & 1 \end{bmatrix}$$

$$\left[(-1) I - H \right] e_1 = \begin{bmatrix} -2 & 4 - \gamma^{-2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ \frac{1}{4 - \gamma^{-2}} \end{bmatrix} = 0$$

$$X_\infty = \frac{2}{4 - \gamma^{-2}} \geq 0$$

$$\left[(-1) I - J \right] e_2 = \begin{bmatrix} 0 & -\gamma^{-2} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \quad \Rightarrow \quad Y_\infty = 0$$

$$X_\infty Y_\infty = 0 \quad \Rightarrow \quad \rho\left[X_\infty Y_\infty \right] = 0 < \gamma^2$$

$$4 - \gamma^{-2} \geq 0 \quad \Rightarrow \quad \gamma \geq \frac{1}{2}$$

optimal norm $= \frac{1}{2}$

$$H = [H_{11} \quad H_{12} \quad H_2] = -\left(Y_\infty C^T + B_1 D_{\cdot 1}^T\right)\tilde{R}^{-1}$$

$$= -\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}^T \tilde{R}^{-1} = -[0 \quad 1]\begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= [0 \quad -1] \Longrightarrow H_{12} = 0, \quad H_2 = -1$$

$$[F_{11}^T \quad F_{12}^T \quad F_2^T] = \left(X_\infty B + C_1^T D_{1\cdot}\right) R^{-1} = \left(\frac{2}{4-\gamma^{-2}}[1 \quad 2] + [D_{11} \quad D_{12}]\right) R^{-1}$$

$$= \left(\frac{2}{4-\gamma^{-2}}[1 \quad 2] + [0 \quad -1]\right)\begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left[\frac{2}{4-\gamma^{-2}} \quad \frac{4}{4-\gamma^{-2}} - 1\right]\begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix} = \left[\frac{-2\gamma^{-2}}{4-\gamma^{-2}} \quad \frac{\gamma^{-2}}{4-\gamma^{-2}}\right]$$

$$F_{12} = \frac{-2\gamma^{-2}}{4-\gamma^{-2}}, \quad F_2 = \frac{\gamma^{-2}}{4-\gamma^{-2}}, \quad Z = I, \quad \hat{D} = 0$$

$$\hat{C} = F_2 Z = F_2 = \frac{\gamma^{-2}}{4-\gamma^{-2}}, \quad \hat{B} = -H = 1$$

$$\hat{A} = A + HC + (B_2 + H_{12})\hat{C}$$

$$= -1 + [0 \quad -1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2+0)\frac{\gamma^{-2}}{4-\gamma^{-2}}$$

$$= -1 + \frac{2\gamma^{-2}}{4-\gamma^{-2}} = \frac{-4+\gamma^{-2}+2\gamma^{-2}}{4-\gamma^{-2}} = \frac{-4+3\gamma^{-2}}{4-\gamma^{-2}}$$

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

$$\hat{A} = \frac{-4+3\gamma^{-2}}{4-\gamma^{-2}}, \quad \hat{B} = 1, \quad \hat{C} = \frac{\gamma^{-2}}{4-\gamma^{-2}}$$

$$\begin{cases} \dot{x}_K = \hat{A} x_K + \hat{B} y \\ u = \hat{C} x_K \end{cases}$$

optimal $\gamma_{opt} = \frac{1}{2}$, $\quad 4 - \left(\frac{1}{2}\right)^{-2} = 4 - 4 = 0$

$$K(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} = \frac{\frac{\gamma^{-2}}{4-\gamma^{-2}}}{s - \frac{-4+3\gamma^{-2}}{4-\gamma^{-2}}}$$

$$= \frac{\gamma^{-2}}{(4-\gamma^{-2})s - (-4+3\gamma^{-2})}$$

$\gamma = \frac{1}{2}$

$\gamma^{-1} = 2, \quad \gamma^{-2} = 4$

$$K_{opt} = \frac{4}{0 \cdot s - (-4+12)} = \frac{4}{0-8} = \frac{-1}{2}$$