CHAPTER 8

Measure of Signals and Systems

and Lyapunov and Riccati Equations

Norm for Signals

The ***norm*** of a signal x(t) has the following properties:

(i) || x || ≥ 0
(ii) || x || = *0* if and only if x=0 ∀ t.
(iii) || cx || = | c | · || x || ∀ $c \in \boldsymbol{R}$.
(iv) || x+y || $\leq$ || x || + || y || for each x, y in ***X***. (triangular inequality).

1-norm:

$$\|x\|_1 := \int_{-\infty}^{\infty} |x(t)| dt$$

$$\|x\|_p = \left(\int_{-\infty}^{\infty} |x(t)|^p \, dt \right)^{1/p}$$

2-norm:

$$\|x\|_2 := \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

∞-norm:

$$\|x\|_\infty := \sup_t |x(t)|$$

$$\sup(1 - e^{-t}) = 1$$

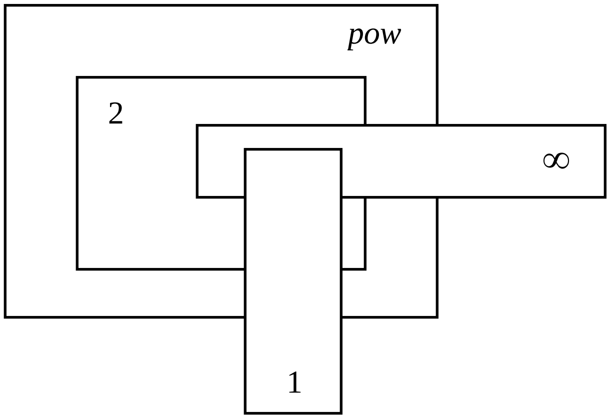
$$\max(1 - e^{-t}) = ?$$

Power Signals

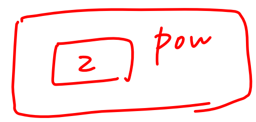
$$pow(x) := \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt}$$

pow is not a norm since it fails (ii). It has properties (i), (iii), and (iv).

Relationship among the sets of 1-norm, 2-norm, ∞-norm, and power signals.



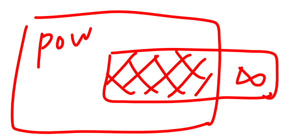
1. If $\|u\|_2 < \infty$, then u in power signal with $pow(u) = 0$.



Pf: $\frac{1}{2T}\int_{-T}^{T} u(t)^2\,dt \le \frac{1}{2T}\|u\|_2^2$

$\Rightarrow pow(u) = 0$ by taking the limit $T \to \infty$

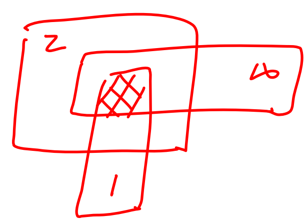
2 If u is a power signal and $\|u\|_\infty < \infty$, then $pow(u) \le \|u\|_\infty$.



Pf: $\frac{1}{2T}\int_{-T}^{T} u(t)^2\,dt \le \|u\|_\infty^2 \frac{1}{2T}\int_{-T}^{T} dt = \|u\|_\infty^2$

3. If $\|u\|_1 < \infty$ and $\|u\|_\infty < \infty$, then

$\|u\|_2 \le \left(\|u\|_\infty \cdot \|u\|_1\right)^{1/2}$ and hence $\|u\|_2 < \infty$.



Pf: $\int_{-\infty}^{\infty} u(t)^2\,dt = \int_{-\infty}^{\infty} |u(t)|\cdot|u(t)|\,dt \le \|u\|_\infty \cdot \|u\|_1$

Norm for Systems

$$\hat{u}(s) \longrightarrow \boxed{\hat{G}(s)} \longrightarrow \hat{y}(s)$$

$$u(t) \qquad G(t) \qquad y(t)$$

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

$$y(t) = G(t) * u(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau$$

convolution

2-norm:

$$\left\|\hat{G}\right\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{G}(j\omega)\right|^2 d\omega}$$

$$\delta(t) \longrightarrow \boxed{\hat{G}(s)} \longrightarrow G(t)$$

If $\hat{G}(s)$ is stable, then

$$G(t) = \mathcal{L}^{-1}[\hat{G}(s)]$$

$$\left\|\hat{G}\right\|_2 = \left\|G\right\|_2 = \sqrt{\int_{-\infty}^{\infty}\left|G(t)\right|^2 dt}$$

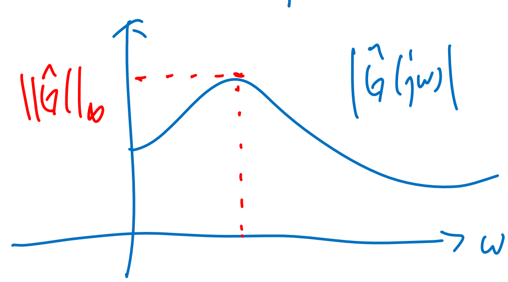
by Parseval's theorem.

$\hat{G}(s)$

$\hat{G}(j\omega) = \left|\hat{G}(j\omega)\right| \angle \hat{G}(j\omega)$

∞-norm:

$$\left\|\hat{G}\right\|_\infty := \sup_{\omega}\left|\hat{G}(j\omega)\right|$$



Lemma:

$\left\|\hat{G}\right\|_2$ is finite iff $\hat{G}(s)$ is strictly proper and has no poles on the imaginary axis;

$\left\|\hat{G}\right\|_\infty$ is finite iff $\hat{G}(s)$ is proper and has no poles on the imaginary axis.

Computation of $\left\|\hat{G}\right\|_2$ in frequency domain:

$$\left\|\hat{G}\right\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{G}(j\omega)\right|^2 d\omega}$$

$$= \sqrt{\sum \textit{the residues of } \hat{G}(-s)\hat{G}(s) \textit{ at its poles in the LHP}}$$

Computation of $\left\|\hat{G}\right\|_\infty$ in frequency domain:

$$\left\|\hat{G}\right\|_\infty := \sup_\omega \left|\hat{G}(j\omega)\right|$$

Ex: $\hat{G}(s) = \dfrac{1}{\tau s + 1}$, $\tau > 0$

$\|\hat{G}\|_2^2 = \lim\limits_{s \to \frac{-1}{\tau}} \left(s + \frac{1}{\tau}\right) \cdot \dfrac{1}{-\tau s + 1} \cdot \dfrac{1}{\tau s + 1} = \dfrac{1}{2\tau}$

$\|\hat{G}\|_\infty = \sup\limits_\omega |\hat{G}(j\omega)| = 1$

$|\hat{G}(j\omega)| = \left|\dfrac{1}{j\omega\tau + 1}\right| = \dfrac{1}{\sqrt{\omega^2\tau^2 + 1}}$

$\sin\omega t \longrightarrow \boxed{\hat{G}(s)} \longrightarrow y(t)$

$y_{ss}(t) + y_t(t)$

$\hat{G}(j\omega) = |\hat{G}(j\omega)| \angle \hat{G}(j\omega) = A(\omega) e^{j\theta(\omega)}$

if $u(t) = \sin\omega t$, then $y_{ss}(t) = A\sin(\omega t + \theta)$

$\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} |y(t)|^2 dt} = \infty$

$\|y\|_\infty = \sup |y(t)| = A = |\hat{G}(j\omega)|$

Physical Meaning of $\|\hat{G}\|_2$ and $\|\hat{G}\|_\infty$

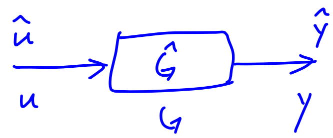


Table 1:

	u(t) = δ(t)	u(t) = sin ωt
$\|y\|_2$	$\|\hat{G}\|_2$	∞
$\|y\|_\infty$	$\|G\|_\infty$	$\lvert\hat{G}(j\omega)\rvert$
$pow(y)$	0	$\frac{1}{\sqrt{2}}\lvert\hat{G}(j\omega)\rvert$

$\|G\|_\infty \neq \|\hat{G}\|_\infty$

$u(t) = \delta(t)$ $\quad \hat{y} = \hat{G}\hat{u} = \hat{G} \Rightarrow \|y\|_2 = \|\hat{y}\|_2 = \|\hat{G}\|_2$ (Parseval)

i.e., $\hat{u}(s) = 1$

$u(t) = \delta(t) \Rightarrow y(t) = \int_0^t G(t-\tau)\delta(\tau)\,d\tau = G(t) \Rightarrow \|y\|_\infty = \|G\|_\infty$

$\hat{u} = 1 \Rightarrow pow(y) = \left(\lim_{T\to\infty} \frac{1}{2T}\cdot\|\hat{G}\|_2^2\right)^{1/2} = 0$, since $\|\hat{y}\|_2$ is finite.

$u(t) = \sin\omega t$, $\quad y(t) = |\hat{G}(j\omega)| \sin(\omega t + \angle\hat{G}(j\omega))$

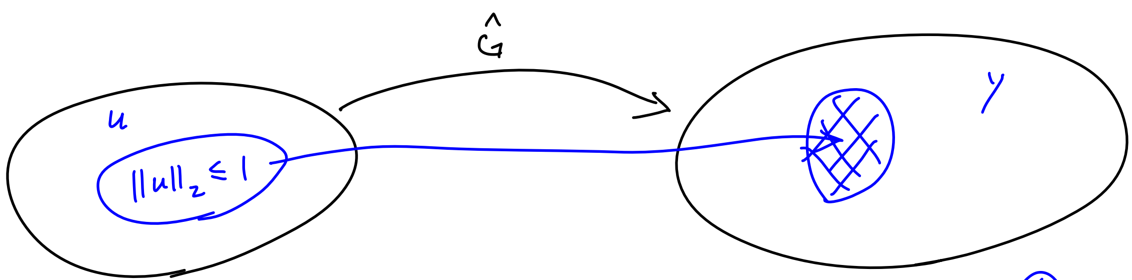
$\Rightarrow \|y\|_2 = \infty$, $\quad \|y\|_\infty = |\hat{G}(j\omega)|$

$pow(y) = \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|y(t)|^2\,dt\right)^{1/2} = \frac{1}{\sqrt{2}}|\hat{G}(j\omega)|$

Table 2: $u(t)$ is not known apriori except $\|u\|_2 \le 1$, ...

	$\|u\|_2$	$\|u\|_\infty$	$pow(u)$
$\|y\|_2$	$\|\hat{G}\|_\infty$	∞	∞
$\|y\|_\infty$	$\|\hat{G}\|_2$	$\|G\|_1$	∞
$pow(y)$	0	$\le \|\hat{G}\|_\infty$	$\|\hat{G}\|_\infty$

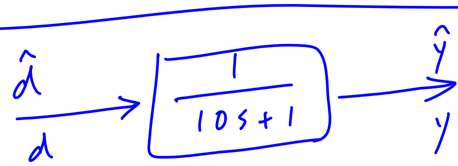
$$\|\hat{G}\|_\infty = \sup_\omega |\hat{G}(j\omega)| = \sup_{\|u\|_2 \le 1} \|y\|_2 = \sup_{pow(u) \le 1} pow(y)$$



$\|\hat{G}\|_\infty$ equals the supremum of $\|y\|_2$ in

$$\|G\|_1 = \sup_{\|u\|_\infty \le 1} \|y\|_\infty = \int_{-\infty}^{\infty} |G(t)|\, dt$$

Ex:



The disturbance input $d(t)$ is known to be energy bounded $\|d\|_2 \le 0.4$. Then the possible largest $\|y\|_\infty$ could be

$$\|y\|_\infty = \|\hat{G}\|_2 \cdot \|d\|_2 = \frac{1}{\sqrt{20}} \cdot 0.4 = \frac{0.4}{\sqrt{20}}$$

Proof of Table 2 :

2 norm / 2 norm induced operator norm

(i) $\|\hat{G}\|_\infty = \sup_\omega |\hat{G}(j\omega)| = \sup_{\|u\|_2 \le 1} \|y\|_2$

Pf: $\|y\|_2^2 = \|\hat{y}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 \cdot |\hat{u}(j\omega)|^2 d\omega$

$$\le \|\hat{G}\|_\infty^2 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega = \|\hat{G}\|_\infty^2 \cdot \|u\|_2^2$$

$\Rightarrow \|\hat{G}\|_\infty^2 \ge \frac{\|y\|_2^2}{\|u\|_2^2}$, i.e., $\|\hat{G}\|_\infty$ is an upper bound of the operator norm

To show $\|\hat{G}\|_\infty$ is the least upper bound, we choose ω_0 where $|\hat{G}(j\omega)|$ is the maximum, i.e.,

$$|\hat{G}(j\omega_0)| = \|\hat{G}\|_\infty$$

Now choose the input u so that

$$|\hat{u}(j\omega)| = \begin{cases} c, & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \end{cases}$$

where ϵ is a small positive number, and c is chosen so that u has unit 2-norm (i.e., $c = \sqrt{\frac{\pi}{2\epsilon}}$). Then

$$\|\hat{y}\|_2^2 = \frac{1}{2\pi}\left[|\hat{G}(-j\omega_0)|^2 \pi + |\hat{G}(j\omega_0)|^2 \pi \right] = |\hat{G}(j\omega_0)|^2 = \|\hat{G}\|_\infty^2 .$$

(ii) $\|\hat{G}\|_2 = \sup_{\|u\|_2 \le 1} \|y\|_\infty$ Cauchy-Schwarz inequality

pf: $|y(t)| = \left| \int_{-\infty}^{\infty} G(t-\tau)\, u(\tau)\, d\tau \right| \le \left(\int_{-\infty}^{\infty} G(t-\tau)^2 d\tau \right)^{1/2} \left(\int_{-\infty}^{\infty} u(\tau)^2 d\tau \right)^{1/2}$

$= \|G\|_2 \cdot \|u\|_2 = \|\hat{G}\|_2 \cdot \|u\|_2$

Hence $\|y\|_\infty \le \|\hat{G}\|_2 \cdot \|u\|_2$

To show $\|\hat{G}\|_2$ is the least upper bound, apply the input $u(t) = \dfrac{G(-t)}{\|G\|_2}$

Then $\|u\|_2 = 1$ and $|y(0)| = \|G\|_2$ so $\|y\|_\infty \ge \|G\|_2$.

(iii) $\|u\|_2 \le 1 \Rightarrow \|y\|_2$ is finite $\Rightarrow pow(y) = 0$

(iv) $u(t) = \sin \omega t$ with $\|u\|_\infty = 1$ $\Rightarrow$ $y(t) = |\hat{G}(j\omega)| \sin\left(\omega t + \angle \hat{G}(j\omega)\right)$ with $\|y\|_2 = \infty$

Power signal analysis :

$u(t)$: a power signal

autocorrelation function :

$$R_u(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)\,u(t+\tau)\,dt$$

i.e., $R_u(\tau)$ is the average value of the product $u(t)\,u(t+\tau)$.

Note that $R_u(0) = \mathrm{pow}(u)^2 \geq 0$

Theorem : $|R_u(\tau)| \leq R_u(0)$

power spectral density :

$$S_u(j\omega) = \mathcal{F}[R_u(\tau)] = \int_{-\infty}^{\infty} R_u(\tau)\, e^{-j\omega\tau}\, d\tau \quad . \quad \text{Fourier transform}$$

$$R_u(\tau) = \mathcal{F}^{-1}[S_u(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_u(j\omega)\, e^{j\omega\tau}\, d\omega \quad . \quad \text{Inverse Fourier transform}$$

$$\mathrm{pow}(u)^2 = R_u(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_u(j\omega)\, d\omega$$

$$\Rightarrow \frac{S_u(j\omega)}{2\pi} \text{ is power density}$$

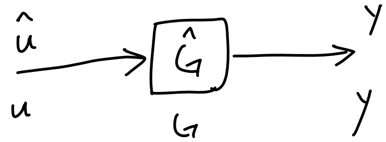
$S_u(j\omega)$ is called the power spectral density of the signal u

Consider two signals, u and v

Their cross-correlation function

$$R_{uv}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)\,v(t+\tau)\,dt$$

$S_{uv}(j\omega) = \mathcal{F}[R_{uv}(\tau)]$: cross power spectral density function



$\hat{G}$: stable & proper

Theorem:

$$R_{uy}(\tau) = \int_{-\infty}^{\infty} G(\alpha)\,R_u(\tau-\alpha)\,d\alpha = G(\tau) * R_u(\tau)$$

Convolution

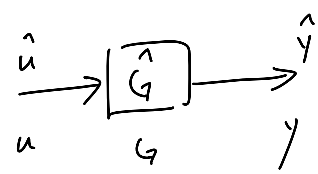
Theorem:

$$R_y(\tau) = G(\tau) * G_{rev}(\tau) * R_u(\tau)$$

where $G_{rev}(t) := G(-t)$.

Theorem:

$$\boxed{S_y(j\omega) = |\hat{G}(j\omega)|^2\, S_u(j\omega)}$$



If u is a power signal, then

$$S_y(j\omega) = |\hat{G}(j\omega)|^2 S_u(j\omega)$$

so

$$pow(y)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 S_u(j\omega)\, d\omega > 0 \quad \text{unless } |\hat{G}(j\omega)|^2 S_u(j\omega) = 0 \text{ for all } \omega$$

Hence $\|y\|_2$ is infinite

$$\|\hat{G}\|_\infty = \sup_{pow(u)\le 1} pow(y)$$

pf:

$$pow(y)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 S_u(j\omega)\, d\omega \qquad (*)$$

$$\le \|\hat{G}\|_\infty^2 \cdot pow(u)^2$$

i.e.,

$$pow(y) \le \|\hat{G}\|_\infty \cdot pow(u)$$

To achieve the equality, suppose that

$$|\hat{G}(j\omega_0)| = \|\hat{G}\|_\infty$$

and let the input be $u(t) = \sqrt{2}\sin(\omega_0 t)$

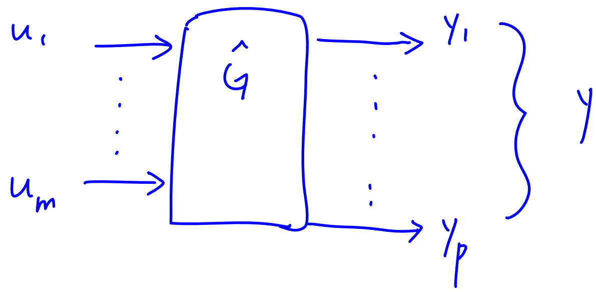
Then $R_u(\tau) = \cos(\omega_0 \tau)$, so

$$pow(u) = R_u(0) = 1 \qquad S_u(j\omega) = \mathcal{F}[R_u(\tau)]$$

Also, $S_u(j\omega) = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

So from $(*)$,

$$pow(y)^2 = \frac{1}{2}|\hat{G}(j\omega_0)|^2 + \frac{1}{2}|\hat{G}(-j\omega_0)|^2 = |\hat{G}(j\omega_0)|^2 = \|\hat{G}\|_\infty^2$$



$\|G\|_2^2 = \|y^1\|_2^2 + \|y^2\|_2^2 + \cdots + \|y^m\|_2^2$

MIMO Case:

$$\|G\|_2 = \sqrt{\sum_i \left\|y^i(t)\right\|_2^2},$$

where $y^i(t)$ is the output due to the input $u(t)$ with $u_i(t) = \delta(t)$, $u_j(t) = 0$ when $j \neq i$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\text{trace } A = a_{11} + a_{22}$

$$\left\|\hat{G}\right\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} trace\left[\hat{G}(j\omega)^* \hat{G}(j\omega)\right] d\omega}$$

$$\left\|\hat{G}\right\|_\infty := \sup_\omega \bar{\sigma}\left[\hat{G}(j\omega)\right] = \sup \frac{\|y\|_2}{\|u\|_2}$$

max. singular value

singular value decomposition

help svd

where

$$\|y\|_2 := \sqrt{\int_{-\infty}^{\infty} y^T(t) y(t) dt}.$$

orthogonal matrix if $U^T U = I$

unitary matrix

Singular value decomposition

$$\underset{m\times n}{X} = \underset{m\times m}{U^*} \underset{m\times n}{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}} \underset{n\times n}{V}$$

$U^* U = I$, $V^* V = I$

$$\underset{r\times r}{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & & \ddots \\ 0 & & & \sigma_r \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ $\quad$ r is $\text{rank}(X)$

$\sigma_1, \sigma_2, \ldots \sigma_r$ are singular values

σ_1 is the max. singular value

Computations of H_2 norm in time domain

Given $\hat{G}(s)$ = $\{A,B,C,0\}$ with A stable. Let L_C and L_O be the controllability and observability grammians respectively. That is,

$$L_c = \int_0^\infty e^{At}BB^T e^{A^T t}dt \quad \text{which satisfies } AL_c + L_c A^T + BB^T = 0$$

$$L_o = \int_0^\infty e^{A^T t}C^T C e^{At}dt \quad \text{which satisfies } A^T L_o + L_o A + C^T C = 0$$

Theorem:

$$\left\|\hat{G}\right\|_2 = \sqrt{trace\left\{CL_c C^T\right\}} = \sqrt{trace\left\{B^T L_o B\right\}}.$$

Computations of H_∞ norm in time domain

$\gamma_l < \|\hat{G}\|_\infty < \gamma_u$

$\gamma = \frac{\gamma_l + \gamma_u}{2}$

Theorem:

For any $\hat{G}(s) \in RH_\infty$, $\left\|\hat{G}\right\|_\infty < \gamma$ if and only if the Hamiltonian matrix

$$H_G = \begin{bmatrix} A + BR^{-1}D^T C & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^T C)^T \end{bmatrix}$$

$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

does not have any eigenvalues on the $j\omega$-axis, where γ is a nonnegative number, and $R = \gamma^2 I - D^T D$.

The above theorem actually implies that $\left\|\hat{G}\right\|_\infty$ = inf{γ: H_G does not have $j\omega$-axis eigenvalues} and that one can compute $\left\|\hat{G}\right\|_\infty$ by an iterative algorithm: choose a positive number γ; calculate the eigenvalues of H and check whether any of them are on the $j\omega$-axis; decrease or increase γ accordingly; repeat, until the infimum is reached within the tolerance.

Lyapunov and Riccati Equations

Lyapunov equation:

$$A^T X + XA + Q = 0$$

Theorem: **(Review)**

If $A \in R^{nxn}$ is stable, and $Q^T = Q \in R^{nxn}$, then the Lyapunov equation has a unique solution,

$$X = \int_0^\infty e^{A^T t} Q e^{At} dt$$

$$\frac{d\,uv}{dt} = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$$

$$\frac{d\,e^{A^T t}}{dt} = A^T \cdot e^{A^T t}$$

$$\frac{d\,e^{At}}{dt} = e^{At} \cdot A$$

$$A^T X + XA = \int_0^\infty \left[A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A \right] dt$$

$$\frac{d}{dt} e^{A^T t} Q e^{At} = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} \cdot A$$

$$A^T X + XA = \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{At} \cdot dt = e^{A^T t} Q e^{At} \Big|_0^\infty = -Q$$

Theorem:

positive semi-definite

A, $X \geq 0$, $Q \geq 0$ satisfy the Lyapunov equation and (Q, A) is detectable.

Then A is stable. **(Review)**

Pf: Suppose A has some eigenvalue λ with $Re\,\lambda \geq 0$.

Let x be a corresponding eigenvector. $Ax = \lambda x$

$$x^*(A^T X + XA + Q)x = 0$$

$$x^* \lambda^* X x + x^* X \lambda x + x^* Q x = 0$$

$$(\lambda^* + \lambda) x^* X x + x^* Q x = 0$$

i.e. $\underbrace{2 Re\,\lambda \cdot x^* X x}_{\geq 0} + \underbrace{x^* Q x}_{\geq 0} = 0 \Rightarrow x^* Q x = 0 \Rightarrow Qx = 0$

Thus $\begin{bmatrix} A-\lambda I \\ Q \end{bmatrix} x = 0$

(Q, A) detectable $\Rightarrow \begin{bmatrix} A-\lambda I \\ Q \end{bmatrix}$ full rank $\Rightarrow x = 0$

A contradiction!

$\Rightarrow$ A must be stable.

$ax + xa + xrx - q = 0$

$rx^2 + 2ax - q = 0$

Algebraic Riccati equation

$$A^T X + XA + XRX - Q = 0 \qquad \text{(ARE)}$$

where A, R and Q are given nxn constant real matrices. Moreover, R and Q are real symmetric.

One way to solve ARE for X is to solve the corresponding Hamiltonian matrix:

the set of eigenvalues of H

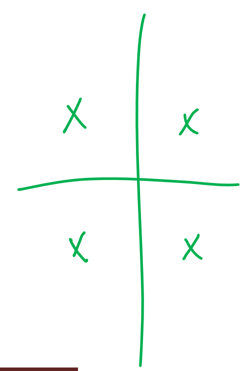
$$H = \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \qquad \text{(Ham)}$$

Claim: $\sigma(H)$ is symmetric about the imaginary axis.

Pf: Let $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ note that $J^2 = -I$

and hence $J^{-1} = -J$

Then $J^{-1} H J = -JHJ = -H^T$



$\Rightarrow \det(\lambda I - H) = 0$ iff $\det(\lambda I + H^T) = 0$

i.e., $\det(-\lambda I - H) = 0$

$\Rightarrow \lambda$ is an eigenvalue of H iff $-\lambda$ is.

Assume H has no eigenvalues on the $j\omega$-axis. Then H has n eigenvalues in $\mathrm{Re}\, s < 0$ and n in $\mathrm{Re}\, s > 0$.

$\mathcal{X}_-(H)$: spectral subspace associated with stable eigenvalues

Let $\mathcal{X}_-(H) = \mathrm{Im} \overset{2n\times n}{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}$ where $X_1, X_2 \in \mathbb{R}^{n\times n}$

$$\begin{bmatrix} X_1 \cdot X_1^{-1} \\ X_2 \cdot X_1^{-1} \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}$$

If X_1 is nonsingular, i.e., if the two subspaces

$$\mathcal{X}_-(H), \quad \mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

$y = f(x)$

$x \mapsto y$

are complementary, we can set $X = X_2 X_1^{-1}$ and hence

$$\mathcal{X}_-(H) = \mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix}$$

Then X is uniquely determined by H, i.e., $\overset{2n\times 2n}{H} \mapsto \overset{n\times n}{X}$ is a function. This function is denoted by Ric and $X = \mathrm{Ric}(H)$.

Summary :

$$\mathrm{Ric} : \mathbb{R}^{2n\times 2n} \longrightarrow \mathbb{R}^{n\times n}, \quad H \mapsto X$$

where $\mathcal{X}_-(H) = \mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix}$

The domain of Ric.

$$\mathrm{dom\,Ric} = \left\{ H \;\middle|\; \begin{array}{l} H \text{ has no eigenvalues on the } j\omega \text{ axis} \\ \text{and the two subspaces } \mathcal{X}_-(H) \text{ and } \mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix} \\ \text{are complementary.} \end{array} \right\}$$

In the following, we denote $\pi(H)$ as the set of all eigenvalues of H; $\pi_n(H)$ as a subset of $\pi(H)$ with n elements; $\pi_{n-}(H)$ ($\pi_{n+}(H)$, respectively) consisting of n eigenvalues of H with negative (positive, respectively) real parts. For any $\pi_n(H)$, one can always find a modal matrix $T \in C^{2nxn}$ which is stacked by n independent eigenvectors (or generalized eigenvectors) of H, corresponding to $\pi_n(H)$. Partition T in such a way that

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

with $T_1, T_2 \in C^{nxn}$. It is not difficult to verify that if T_1 is invertible, then $X = T_2 T_1^{-1}$ is a solution to ARE. Moreover, $\pi_n(H) = \pi(A+RX)$.

It is obvious that the solution to ARE is not unique, since T can be stacked by different set of eigenvectors. However, if X is a solution such that $A+RX$ is stable, then X is called the **stabilizing solution** to ARE. The stabilizing solution can be constructed by $X = T_2 T_1^{-1}$ with T corresponding to $\pi_{n-}(H)$. It is well known that the stabilizing solution is **real**, **symmetric** and **unique**.

Properties of X :

Lemma: Suppose $H \in \text{dom Ric}$, and $X = \text{Ric}(H)$, Then

(i) X is symmetric

(ii) X satisfies the ARE : $A^T X + XA + XRX - Q = 0$.

(iii) $A + RX$ is stable.

pf :

(i) Note that there exists a stable matrix $\Lambda_- \in \mathbb{R}^{n\times n}$ s.t.

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\Lambda_-$$

$$\Rightarrow \quad \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T JH\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T J\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\Lambda_- \qquad \left(J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \right)$$

Since JH is symmetric, the RHS, $(X_2^T X_1 - X_1^T X_2)\Lambda_-$ must be also symmetric

$$(X_2^T X_1 - X_1^T X_2)\Lambda_- = \Lambda_-^T (X_1^T X_2 - X_2^T X_1) = -\Lambda_-^T(-X_1^T X_2 + X_2^T X_1)$$

This is a Lyapunov equation. Since Λ_- is stable, the unique solution is $-X_1^T X_2 + X_2^T X_1 = 0$

i.e., $X_1^T X_2$ is symmetric.

$X X_1 = X_2$

$X_1^T X_2$ symmetric $\longrightarrow$ $X_1^T X X_1 = X_1^T X_2$ is symmetric

$\Rightarrow X$ is symmetric since X_1 is nonsingular.

(ii) $H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\Lambda_-$, $H\begin{bmatrix} I \\ X \end{bmatrix} X_1 = \begin{bmatrix} I \\ X \end{bmatrix} X_1 \Lambda_-$

$$H\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 \Lambda_- X_1^{-1} \quad \text{- - - -} \quad (*)$$

$$\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X_1 \Lambda_- X_1^{-1} = 0$$

$$\begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} XA - Q & XR + A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix}$$

$$= XA - Q + XRX + A^T X = 0$$

(iii)
$$\begin{bmatrix} I & 0 \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X_1 \Lambda_- X_1^{-1}$$

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = X_1 \Lambda_- X_1^{-1}$$

$$\begin{bmatrix} A & R \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = X_1 \Lambda_- X_1^{-1}$$

$$A + RX = X_1 \Lambda_- X_1^{-1}$$

$$X_1^{-1}(A + RX) X_1 = \Lambda_- \quad \Rightarrow \quad A + RX \text{ is stable.}$$

From the above discussion on the stabilizing solution, we can see that the stabilizing solution exists **if and only if** (i) H has n eigenvalues with negative real parts; (ii) T_1 is invertible. Since Hamiltonian matrix has a symmetric spectrum, then H has n eigenvalues with negative real parts if and only if H has no eigenvalues on $j\omega$-axis. For ease of reference, we use the terminology in DGKF paper: if H has no $j\omega$-axis eigenvalues, we will say that the H has the **stability property**; if H has an invertible T_1 for $\pi_{n-}(H)$, then the H has the **complementary property**. Obviously, complementary property is stronger than the stability property. Since every H, which possesses these two properties, has a one-to-one correspondence with the stabilizing solution X, an operator $Ric: H \mapsto X$ can be defined:

Definition: $X = Ric(H) := T_2 T_1^{-1}$, where T_1 and T_2 are obtained from eq.(Ham) corresponding to $\pi_{n-}(H)$. Moreover, the domain of Ric, denoted as dom(Ric), consists of every H that has the above two properties.

Note that if X is the stabilizing solution, i.e., $X = Ric(H)$, then $A + RX$ is stable. Now a problem arises: under what conditions $H \in$ dom(Ric)? The following lemmas give sufficient conditions.

Lemma: Suppose H has no eigenvalues on $j\omega$-axis, R is signdefinite (i.e. either positive semidefinite or negative semidefinite), and (A, R) is stabilizable. Then $H \in$dom(Ric). Moreover, if, in addition, both R and Q are negative (positive) semidefinite, then $X = Ric(H)$ is positive (negative) semidefinite.

Lemma: If H has the following special form

$$H = \begin{bmatrix} A & -BB' \\ -C'C & -A' \end{bmatrix},$$

then i) (A, B) stabilizable;

ii) (A, C) has no unobservable modes on $j\omega$-axis

if and only if $H \in \mathrm{dom}(Ric)$ and $X:=Ric(H)$ is positive semidefinite. Moreover, if $H \in \mathrm{dom}(Ric)$ then $X:=Ric(H) = 0$ and $\ker(X) = \mathcal{X}$, where $\mathcal{X}$ denotes the stable unobservable subspace.

Theorem :

Suppose H has the form

$$H = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix}$$ with (A, B) stabilizable, (C, A) detectable

Then $H \in \mathrm{dom}\,Ric$, and $Ric(H) \geq 0$

Furthermore, if (C, A) is observable, then $Ric(H) > 0$.

Linear Fractional Transformations

Definition: Suppose M is a complex matrix partitioned as

$$M = \begin{array}{cc} & \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \end{array} & \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \end{array} \in C^{(p1+p2)\times(q1+q2)}$$

and let $\boldsymbol{D}_1 \subset C^{q1\times p2}$ and $\boldsymbol{D}_2 \subset C^{q2\times p1}$, then the Linear Fractional Transformations (LFT) are defined as the maps:

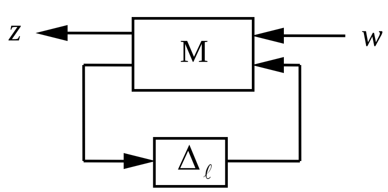
$$F_\ell(M,\bullet): \boldsymbol{D}_2 \to C^{p1\times q1} \qquad F_u(M,\bullet): \boldsymbol{D}_1 \to C^{p2\times q2}$$

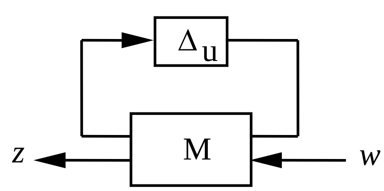
with

$$F_\ell(M,\Delta_\ell) := M_{11} + M_{12}\Delta_\ell(I - M_{22}\Delta_\ell)^{-1}M_{21}$$

$$F_u(M,\Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

The LFT formulae arise naturally when describing feedback systems as shown in the following figures.





The resulting closed-loop transfer functions from w to z are, respectively, $F_\ell(M,\Delta_\ell)$ and $F_u(M,\Delta_u)$. The above M-Δ structure has an exclusive use in robust stability analysis for the systems with uncertainties. By using LFT, one can always separate the nominal plant from the uncertainties. In the following three simple examples are given to show the applications of LFT.

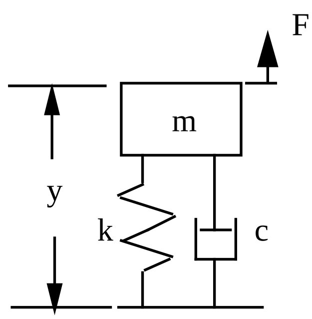
Example:

This example is used to show the basic idea of using LFT to represent a system with uncertainties. Consider the following spring-mass-damper system

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = \frac{F}{m}$$

with the transfer function

$$y = \frac{1}{ms^2 + cs + k}F$$



Assume that $m = \overline{m}(1+\omega_m\delta_m)$, $\tilde{c} = \frac{c}{m} = \overline{c}(1+\omega_c\delta_c)$, and $k = \overline{k}(1+\omega_k\delta_k)$. Then defining $x_1 = y$ and $x_2 = m\dot{y}$ we can write the differential equation in state space form as:

$$\begin{aligned}\dot{x}_1 &= \frac{1}{m}x_2 = \overline{m}^{-1}(1+\omega_m\delta_m)^{-1}x_2 = \overline{m}^{-1}\left[1-\omega_m\delta_m(1+\omega_m\delta_m)^{-1}\right]x_2 \\ &= \overline{m}^{-1}x_2 - \omega_m q_m\end{aligned}$$

$$\dot{x}_2 = -\overline{k}((1+\omega_k\delta_k)x_1 - \overline{c}(1+\omega_c\delta_c)x_2 + F = -\overline{k}x_1 - \overline{c}x_2 + F + \omega_k q_k + \omega_c q_c$$
$$q_m = \delta_m\overline{m}^{-1}(1+\omega_m\delta_m)^{-1}x_2 = \delta_m p_m$$
$$p_m = \overline{m}^{-1}(1+\omega_m\delta_m)^{-1}x_2 = \overline{m}^{-1}x_2 - \omega_m q_m$$

$$q_k = -\delta_k \bar{k} x_1 = \delta_k p_k$$
$$p_k = -\bar{k} x_1$$
$$q_c = -\delta_c \bar{c} x_2 = \delta_c p_c$$
$$p_c = -\bar{c} x_2$$
$$y = x_1$$

Then we have

$$y = F_u(M, \Delta)\, F,$$

with

$$\begin{array}{c} \\ y_1 \\ \\ \\ \\ y_2 \end{array}
\begin{bmatrix} x_1 \\ \dot{x}_2 \\ p_m \\ p_k \\ p_c \\ y \end{bmatrix} =
\begin{array}{c} \begin{array}{cc} M_{11} & M_{12} \end{array} \\
\left[\begin{array}{ccccc|c}
0 & \bar{m}^{-1} & -\omega_m & 0 & 0 & 0 \\
-\bar{k} & -\bar{c} & 0 & \omega_k & \omega_c & 1 \\
0 & \bar{m}^{-1} & -\omega_m & 0 & 0 & 0 \\
-\bar{k} & 0 & 0 & 0 & 0 & 0 \\
0 & -\bar{c} & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{cc} M_{21} & M_{22} \end{array} \end{array}
\begin{bmatrix} x_1 \\ x_2 \\ q_m \\ q_k \\ q_c \\ F \end{bmatrix}
\begin{array}{c} u_1 \\ \\ \\ \\ \\ u_2 \end{array}$$

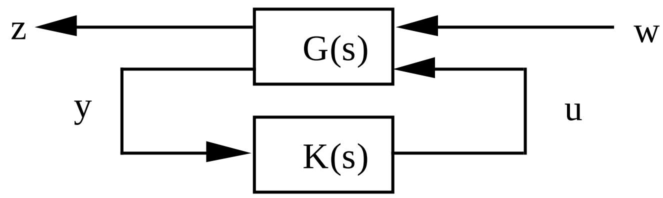
and $\Delta = \mathrm{diag}\,(I_2 \frac{1}{s}, \delta_m, \delta_k, \delta_c).$ $= \Delta_u$

Hence, the uncertainties have been separated from the nominal plant.

Example:

A closed loop system can be expressed by LFT. Consider that G(s) is the generalized plant and K(s) is controller. Then the transfer function of the closed loop system can written as:

$$T_{zw}(s) = F_\ell(G,K) = G_{11}(s) + G_{12}(s)K(s)\,[\,I - G_{22}(s)K(s)]^{-1}\,G_{21}(s).$$



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \qquad u = Ky$$

Example:

The state space realization of a transfer function can also be represented by a LFT:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = F_u\left(\begin{array}{cc} M_{11} & M_{12} \\ \left[\begin{matrix} A & B \\ C & D \end{matrix}\right] & \\ M_{21} & M_{22} \end{array}, \overset{\Delta_u}{\frac{1}{s}I}\right) = D + C(sI - A)^{-1}B$$

$$= M_{22} + M_{21}\cdot\Delta_u\,(I - M_{11}\Delta_u)^{-1} M_{12}$$

$$= D + C\cdot\frac{1}{s}I\left(I - A\cdot\frac{1}{s}I\right)^{-1} M_{12}$$

$$= D + C\left[s\left(I - A\cdot\frac{1}{s}\right)\right]^{-1} B$$

$$= D + C\,[sI - A]^{-1}\,B$$