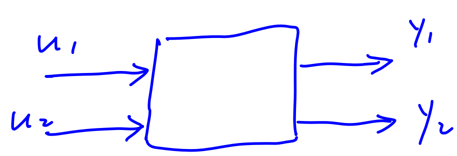
CHAPTER 7

Operations on Linear System and Controller Parametrization

State-space Representation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \text{ and } \{ A, B, C, D \}$$

are used for the same purpose to represent a state-space realization of a system whose transfer function is $C(sI-A)^{-1}B + D$.



$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$G_{11}(s) = \{A, B_1, C_1, D_{11}\}, \quad G_{12}(s) = \{A, B_2, C_1, D_{12}\}$$

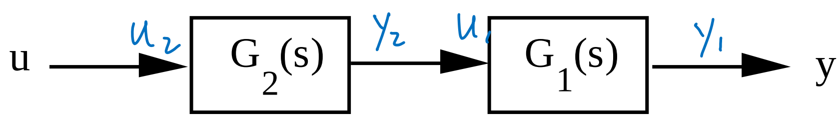
$$G_{21}(s) = \{A, B_1, C_2, D_{21}\}, \quad G_{22}(s) = \{A, B_2, C_2, D_{22}\}$$

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Operations on Linear System

$$y(s) = G_1(s)\,G_2(s)\,u(s)$$

1. Cascade



$$G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \qquad G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

$$G_1(s)\,G_2(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]$$

Proof :

$$G_1(s): \begin{cases} \dot{x}_1 = A_1x_1 + B_1u_1 \\ y_1 = C_1x_1 + D_1u_1 \end{cases}, \qquad G_2(s): \begin{cases} \dot{x}_2 = A_2x_2 + B_2u_2 \\ y_2 = C_2x_2 + D_2u_2 \end{cases}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = y_1$, $u = u_2$, $y_2 = u_1$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1x_1 + B_1C_2x_2 + B_1D_2u \\ A_2x_2 + B_2u \end{bmatrix} = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}u$$

$$y = C_1x_1 + D_1u_1 = C_1x_1 + D_1C_2x_2 + D_1D_2u_2 = \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1D_2u$$

2. Similarity Transformation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]$$

$\hat{A}$ $\hat{B}$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow{\hat{x} = T^{-1}x} \begin{cases} \dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu \\ y = CT\hat{x} + Du \end{cases}$$

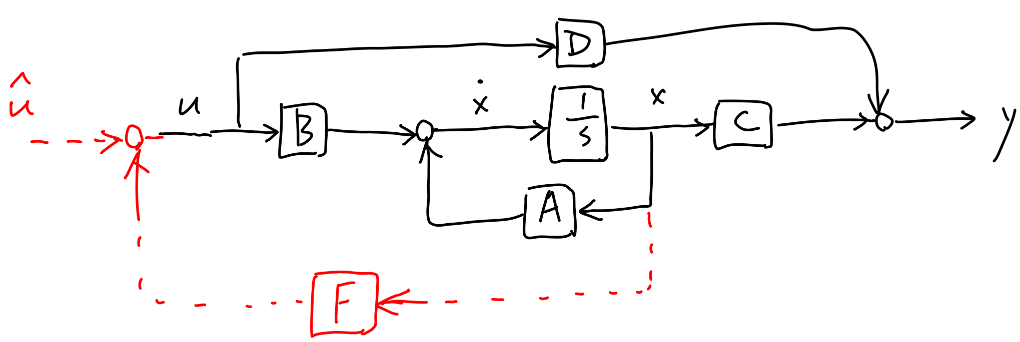
$\hat{C}$

3. State Feedback

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$u = Fx + \hat{u} \quad \rightarrow \quad \begin{cases} \dot{x} = (A+BF)x + B\hat{u} \\ y = (C+DF)x + D\hat{u} \end{cases}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]$$



$$\dot{x} = Ax + B(Fx + \hat{u}) = (A+BF)x + B\hat{u}$$

$$y = Cx + D(Fx + \hat{u}) = (C+DF)x + D\hat{u}$$

4. Output Injection

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \rightarrow \begin{cases} \dot{x} = Ax + Bu + Hy \\ y = Cx + Du \end{cases}$$

=

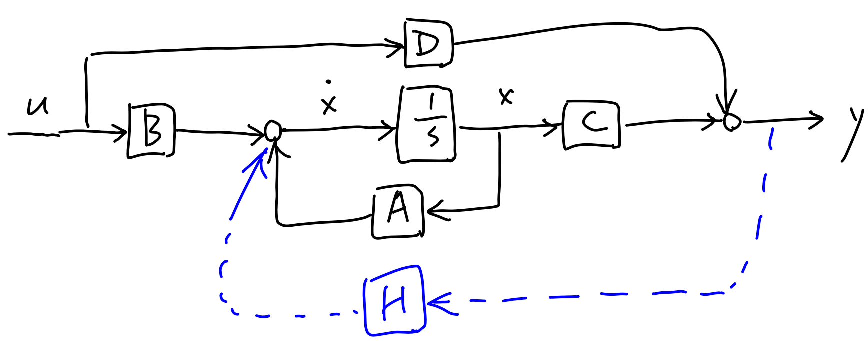
$$\text{i.e.} \begin{cases} \dot{x} = (A+HC)x + (B+HD)u \\ y = Cx + Du \end{cases}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} A+HC & B+HD \\ \hline C & D \end{array}\right]$$ ... output injection

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]$$ ... state feedback

dual

$$\left[\begin{array}{c|c} A^T+C^TH^T & C^T \\ \hline B^T+D^TH^T & D^T \end{array}\right] = \left[\begin{array}{c|c} (A+HC)^T & C^T \\ \hline (B+HD)^T & D^T \end{array}\right]$$



5. Transpose (Dual)

$$G(s) \quad \rightarrow \quad G^T(s)$$

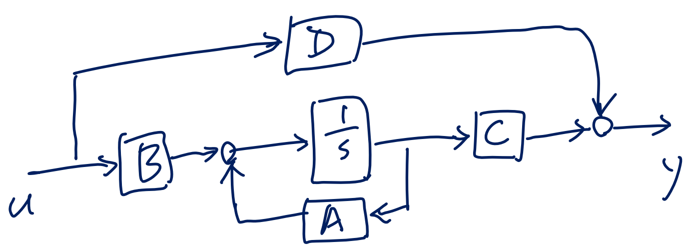
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$$

$$\left[C(sI-A)^{-1}B + D\right]^T$$
$$= B^T(sI-A^T)^{-1}C^T + D^T$$

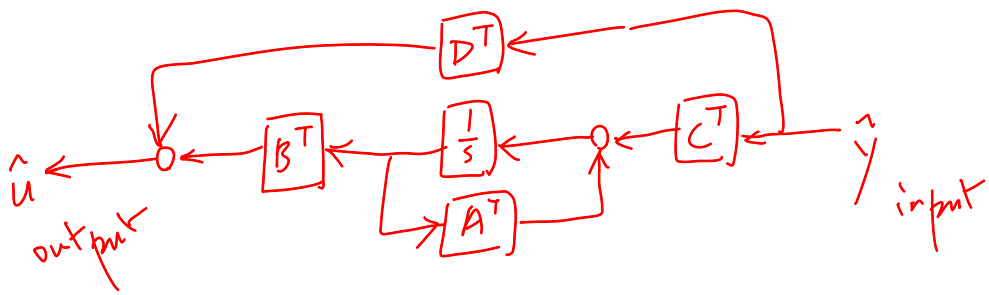
Proof:

$$G(s) = \{A, B, C, D\} = C(sI-A)^{-1}B + D \longrightarrow G^T(s) = B^T(sI-A^T)^{-1}C^T + D^T$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$$



dual $\Rightarrow$



6. Conjugate

$$G(s) \quad \rightarrow \quad G^T(-s)$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]$$

$$G(s) = C(sI-A)^{-1}B + D$$

$$G^T(-s) = B^T(-sI-A^T)^{-1}C^T + D^T$$

$$= -B^T(sI-(-A^T))^{-1}C^T + D^T$$

$$= B^T[sI-(-A^T)]^{-1}(-C^T) + D^T$$

7. Inversion

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

inverse

Suppose $D^\dagger$ is a right (left) inverse of D. Then

$$G^\dagger(s) = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right]$$

is a right (left) inverse of G(s).

Proof: suppose $DD^\dagger = I$, Then

$$G(s)\cdot G^\dagger(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A & BD^\dagger C & BD^\dagger \\ 0 & A - BD^\dagger C & -BD^\dagger \\ \hline C & DD^\dagger C & DD^\dagger \end{array}\right] = \left[\begin{array}{cc|c} A & BD^\dagger C & BD^\dagger \\ 0 & A - BD^\dagger C & -BD^\dagger \\ \hline C & C & I \end{array}\right]$$

Applying similarity transf: $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$, $T^{-1} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$

$$\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} A & BD^\dagger C \\ 0 & A - BD^\dagger C \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} = \cdots = \begin{bmatrix} A & 0 \\ 0 & A - BD^\dagger C \end{bmatrix}$$

$$\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} BD^\dagger \\ -BD^\dagger \end{bmatrix} = \begin{bmatrix} 0 \\ -BD^\dagger \end{bmatrix}, \quad \begin{bmatrix} C & C \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}$$

$$G(s)G^\dagger(s) = \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & A - BD^\dagger C & -BD^\dagger \\ \hline C & 0 & I \end{array}\right] = I$$

$$\left[\begin{array}{cc|c} A_1 & 0 & 0 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & I \end{array}\right] = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} sI - A_1 & 0 \\ 0 & sI - A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + I$$

$$= \begin{bmatrix} C_1(sI - A_1)^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + I = I$$

Controller Parametrization

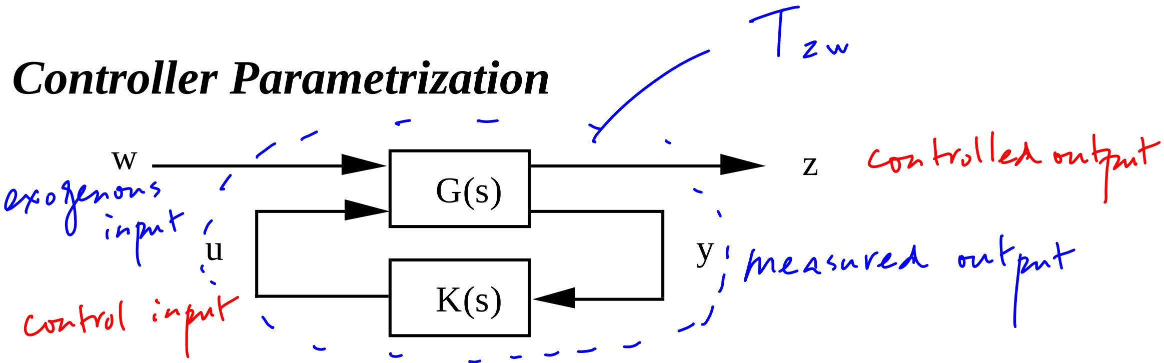


Fig. 1

generalized plant

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$\begin{cases} z = G_{11} w + G_{12} u \\ y = G_{21} w + G_{22} u \end{cases}, \quad u = K y$$

$$y = G_{21} w + G_{22} K y, \quad (I - G_{22}K) y = G_{21} w$$

$$y = (I - G_{22}K)^{-1} G_{21} w, \quad z = G_{11} w + G_{12} K (I - G_{22}K)^{-1} G_{21} w$$

$$= \underbrace{\left[G_{11} + G_{12} K (I - G_{22}K)^{-1} G_{21} \right]}_{T_{zw} = \mathcal{F}_\ell(G, K)} w$$

$\mathcal{F}_\ell(G, K)$ Linear fractional transformation

<u>Theorem 1</u>: (Youla's Controller Parametrization)

Consider the system in Fig. 1. Assume that the realization of G(s) shown above is minimal and the subsystem $G_{22}(s)$ is stabilizable and detectable. Let $M_2(s)$, $N_2(s)$, $X_2(s)$, $Y_2(s)$, $M_1(s)$, $N_1(s)$, $X_1(s)$, and $Y_1(s)$ be proper stable rational matrices such that

field ring

$$\begin{bmatrix} M_2(s) & N_2(s) \\ -Y_1(s) & X_1(s) \end{bmatrix}\begin{bmatrix} X_2(s) & -N_1(s) \\ Y_2(s) & M_1(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (1)$$

and

$$M_2(s)^{-1} N_2(s) = G_{22}(s) \qquad (2)$$

Then the set of all proper stabilizing controllers can be described as

$$\left\{ K(s) \left| \begin{array}{l} K(s) = \left[M_1(s)Q(s) + Y_2(s) \right]\left[N_1(s)Q(s) - X_2(s) \right]^{-1} \\ \text{with } Q(s) \text{ proper stable and } \left| N_1(\infty)Q(\infty) - X_2(\infty) \right| \neq 0 \end{array} \right. \right\} \qquad (3)$$

and the closed-loop transfer function matrix $T_{zw}(s)$ from w to z is an affine function of the parameter matrix Q(s),

$$T_{zw}(s) = \overbrace{G_{11}(s) - G_{12}(s)Y_2(s)M_2(s)G_{21}(s)}^{T_{11}} \qquad = T_{11} + T_{12} Q T_{21}$$

$$\underbrace{-\, G_{12}(s)M_1(s)}_{T_{12}}\ Q(s)\ \underbrace{M_2(s)G_{21}(s)}_{T_{21}} \qquad (4)$$

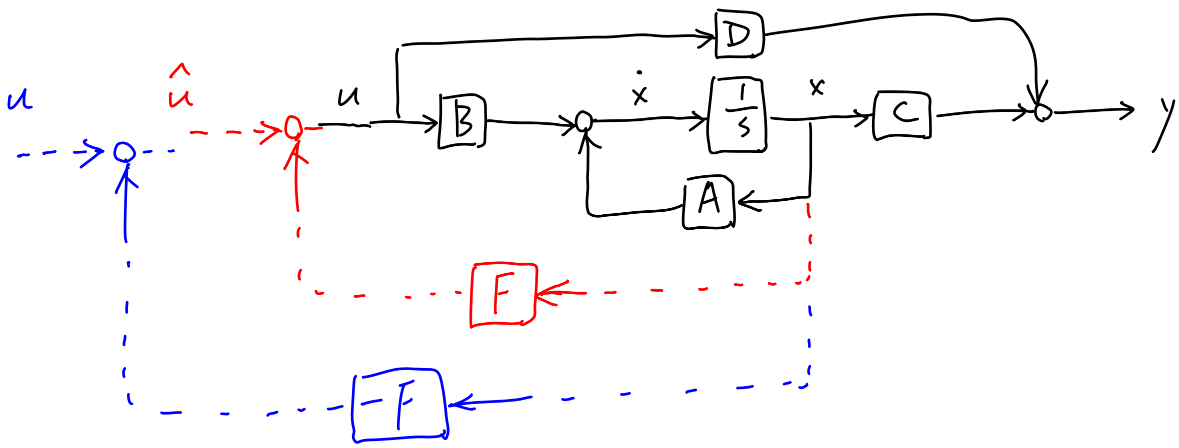
To use the theorem, we need to construct the proper stable rational matrices in (1) and (2). Nett et. al. proposed a convenient state-space approach for this construction. That is, the following realizations

$$\begin{bmatrix} M_2(s) & N_2(s) \\ -Y_1(s) & X_1(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+HC_2 & H & B_2+HD_{22} \\ \hline C_2 & I & D_{22} \\ -F & 0 & I \end{array} \right] \qquad (5a)$$

and

$$\begin{bmatrix} X_2(s) & -N_1(s) \\ Y_2(s) & M_1(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+B_2F & H & B_2 \\ \hline -(C_2+D_{22}F) & I & -D_{22} \\ F & 0 & I \end{array}\right] \qquad (5b)$$

are proper stable and satisfy (1) and (2) where F and H can be arbitrarily chosen such that $A+B_2F$ and $A+HC_2$ are stable.



$$y = G_{22}u \quad , \quad G_{22} = C(sI-A)^{-1}B + D$$

$$y = \{C(sI-A)^{-1}B + D\}\, u$$

$$y = \{\overbrace{(C+DF)[sI-(A+BF)]^{-1}B + D}^{N_1(s)}\}\, \hat{u}$$

$\hat{u} = u - Fx$

$$\hat{u} = u - F[sI-(A+BF)]^{-1}B\,\hat{u}$$

i.e., $\underbrace{\{I + F[sI-(A+BF)]^{-1}B\}}_{M_1(s)}\,\hat{u} = u$

$$y = N_1\hat{u} = N_1 M_1^{-1} u \qquad \begin{bmatrix} -N_1 \\ M_1 \end{bmatrix} = \left[\begin{array}{c|c} A+BF & B \\ \hline -(C+DF) & -D \\ F & I \end{array}\right]$$

Similarly, we can prove that (using output injection)

$$[M_2 \;\; N_2] = \left[\begin{array}{c|cc} A+HC & H & B+HD \\ \hline C & I & D \end{array}\right]$$

Show that

$$\begin{bmatrix} M_2 & N_2 \\ -Y_1 & X_1 \end{bmatrix} \cdot \begin{bmatrix} X_2 & -N_1 \\ Y_2 & M_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

i.e.,

$$\left[\begin{array}{c|cc} A+HC & H & B+HD \\ \hline C & I & D \\ -F & 0 & I \end{array}\right] \cdot \left[\begin{array}{c|cc} A+BF & H & B \\ \hline -(C+DF) & I & -D \\ F & 0 & I \end{array}\right] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

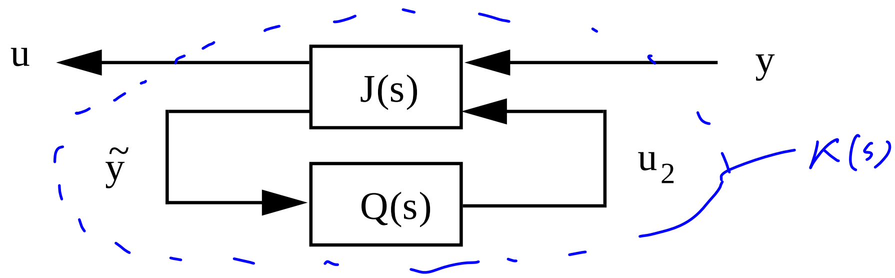
Hint: use similarity transf. $T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$

Doyle et. al. showed that if (5a) and (5b) are used to realize the proper stable rational matrices in (1) and (2) and let

$$J(s) = \begin{bmatrix} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} A+B_2F+HC_2+HD_{22}F & -H & -(B_2+HD_{22}) \\ \hline F & 0 & -I \\ -(C_2+D_{22}F) & I & D_{22} \end{array}\right] \qquad (6)$$

then the set of proper stabilizing controllers described in Theorem 1 will have a structure as that shown in Fig. 2.

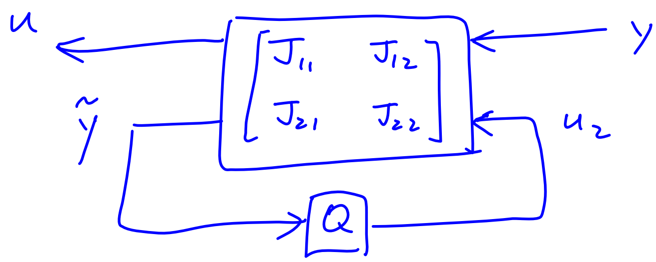


with Q(s) proper stable and I - $D_{22}Q(\infty)$ invertible.

Fig. 2 Structure of stabilizing controller parametrization.

Youla's controller parametrization :

$$\left\{ K(s) \;\middle|\; \begin{array}{l} K(s) = [M_1(s)Q(s) + Y_2(s)][N_1(s)Q(s) - X_2(s)]^{-1} \\ \text{with } Q(s) \text{ proper stable and } |N_1(\infty)Q(\infty) - X_2(\infty)| \neq 0 \end{array} \right\}$$



$$\left\{ \begin{array}{l} \begin{bmatrix} u \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} y \\ u_2 \end{bmatrix} \\ u_2 = Q\tilde{y} \end{array} \right.$$

$$u = \left[J_{11} + J_{12}Q(I - J_{22}Q)^{-1}J_{21} \right] y = \mathcal{F}_\ell(J, Q) \cdot y$$

linear fractional transformation

$$K = \mathcal{F}_\ell(J, Q) = J_{11} + J_{12}Q(I - J_{22}Q)^{-1}J_{21}$$

$$K = (M_1Q + Y_2)(N_1Q - X_2)^{-1}$$
$$= (M_1Q + Y_2)\left[X_2(X_2^{-1}N_1Q - I)\right]^{-1}$$
$$= (M_1Q + Y_2)(X_2^{-1}N_1Q - I)^{-1}X_2^{-1}$$
$$= \left[X_1^{-1}(I - Y_1N_1)Q + Y_2\right](X_2^{-1}N_1Q - I)^{-1}X_2^{-1}$$
$$= \left[X_1^{-1}Q - X_1^{-1}Y_1N_1Q + Y_2\right](X_2^{-1}N_1Q - I)^{-1}X_2^{-1}$$

$$X_1M_1 + Y_1N_1 = I$$
$$M_1 = X_1^{-1}(I - Y_1N_1)$$

$$-Y_1X_2 + X_1Y_2 = 0$$
$$Y_1X_2 = X_1Y_2$$
$$X_1^{-1}Y_1 = Y_2X_2^{-1}$$

$$= X_1^{-1} Q \left(X_2^{-1} N_1 Q - I \right)^{-1} X_2^{-1}$$

$$+ \left(-Y_2 X_2^{-1} N_1 Q + Y_2 \right) \left(X_2^{-1} N_1 Q - I \right)^{-1} X_2^{-1}$$

$$= \underbrace{-Y_2 X_2^{-1}}_{J_{11}} \underbrace{- X_1^{-1}}_{J_{12}} Q \left(I - \underbrace{X_2^{-1} N_1}_{J_{22}} Q \right)^{-1} \underbrace{X_2^{-1}}_{J_{21}}$$

i.e.,

$$J = \begin{bmatrix} -Y_2 X_2^{-1} & -X_1^{-1} \\ X_2^{-1} & X_2^{-1} N_1 \end{bmatrix}$$

$$J_{11} = -Y_2 X_2^{-1}$$

$$= - \left[\begin{array}{c|c} A+BF & H \\ \hline F & 0 \end{array} \right] \cdot \left[\begin{array}{c|c} A+BF & H \\ \hline -(C+DF) & I \end{array} \right]^{-1}$$

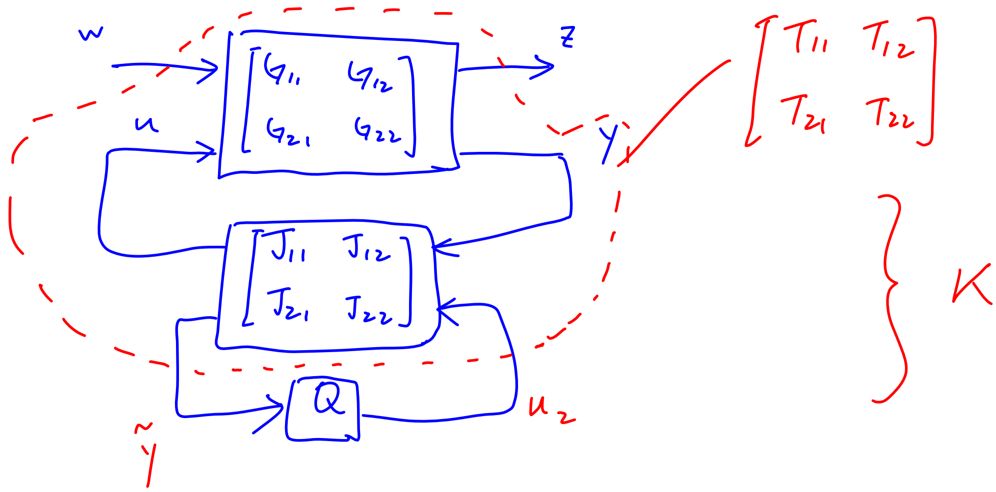
$$= - \left[\begin{array}{c|c} A+BF & H \\ \hline F & 0 \end{array} \right] \cdot \left[\begin{array}{c|c} A+BF+HC+HDF & -H \\ \hline -(C+DF) & I \end{array} \right]$$

$$= - \left[\begin{array}{cc|c} A+BF & -H(C+DF) & H \\ 0 & A+BF+HC+HDF & -H \\ \hline F & 0 & 0 \end{array} \right]$$

applying similarity transf. $\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$

then

$$J_{11}(s) = - \left[\begin{array}{cc|c} A+BF & 0 & 0 \\ 0 & A+BF+HC+HDF & -H \\ \hline F & -F & 0 \end{array} \right] = \left[\begin{array}{c|c} A+BF+HC+HDF & -H \\ \hline F & 0 \end{array} \right]$$



Replace the controller K(s) in Fig. 1 by the structure of Fig. 2, then the closed-loop system can be redrawn as that shown in Fig. 3.

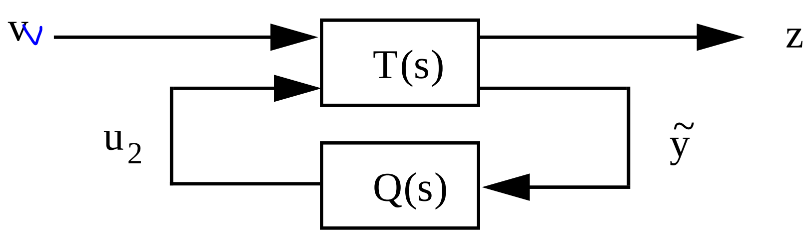


Fig. 3 The closed-loop system characterized in terms of a parameter matrix Q(s).

$$z = \mathcal{F}_\ell(T, Q)\, w = \left[T_{11} + T_{12} Q (I - T_{22} Q)^{-1} T_{21} \right] w$$

$\because T_{22} = 0$

$$\Rightarrow z = \left[T_{11} + T_{12} Q T_{21} \right] w$$ affine function

In Fig. 3, the open-loop transfer function matrix from u_2 to $\tilde{y}$, $T_{22}(s)$, is zero. Therefore, the closed-loop transfer function matrix from w to z, i.e., $T_{zw}(s)$, is a simple affine function of the parameter matrix Q(s). That is,

$$T_{zw}(s) = T_{11}(s) + T_{12}(s)\,Q(s)\,T_{21}(s) \qquad (7)$$

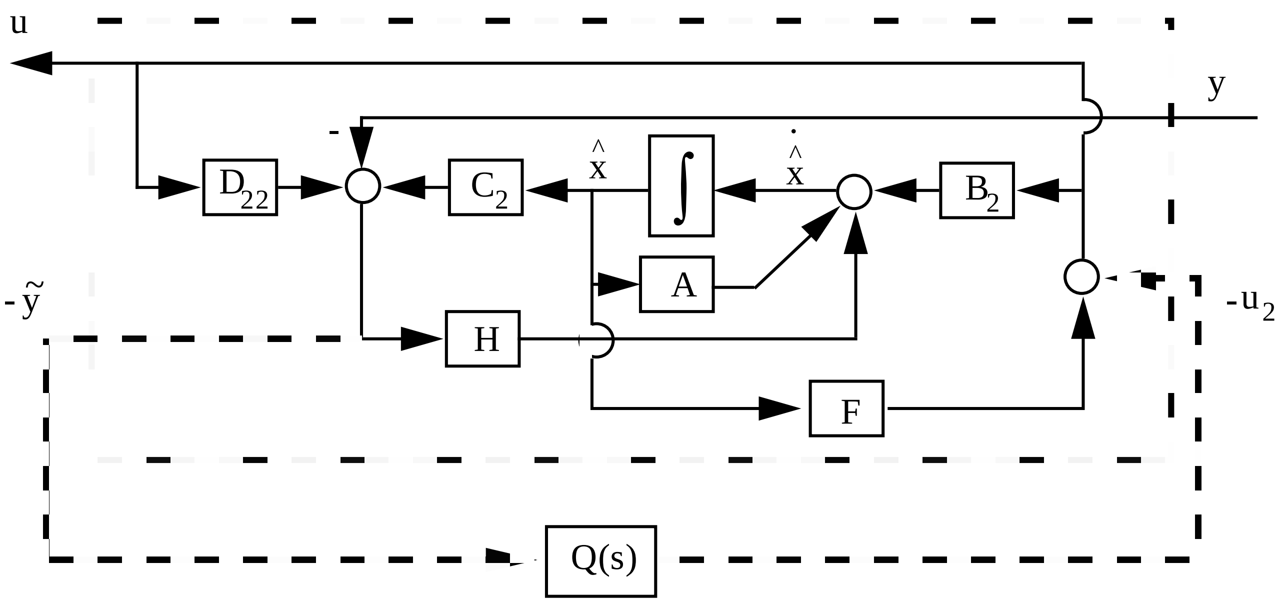
where the realizations of $T_{11}(s)$, $T_{12}(s)$, $T_{21}(s)$ are given by

$$T_{11}(s) = \left[\begin{array}{cc|c} A+B_2F & -B_2F & B_1 \\ 0 & A+HC_2 & B_1+HD_{21} \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} \end{array}\right] \qquad (8a)$$

$$T_{12}(s) = \left[\begin{array}{c|c} A+B_2F & B_2 \\ \hline C_1+D_{12}F & D_{12} \end{array}\right] \qquad (8b)$$

$$T_{21}(s) = \left[\begin{array}{c|c} A+HC_2 & B_1+HD_{21} \\ \hline C_2 & D_{21} \end{array}\right] \qquad (8c)$$

The structure of the stabilizing controller parametrization in Fig. 2 can be realized as an observer-based controller with an added stable dynamics Q(s). The realization is shown in Fig. 4.



with Q(s) proper stable and $I - D_{22}Q(\infty)$ invertible

Fig. 4 The observer-based controller parametrization.

Note that in Fig. 4 the block diagram inside the dotted-line box is the well-known full-order observer-based controller.

The dynamic equations of the observer-based controller in Fig. 4, i.e., the block diagram inside the dotted-line box, can be written as follows,

$$\dot{\hat{x}} = \left(A + B_2F + HC_2 + HD_{22}F\right)\hat{x} + \begin{bmatrix} -H & -(B_2 + HD_{22}) \end{bmatrix} \begin{bmatrix} y \\ u_2 \end{bmatrix} \tag{9a}$$

$$\begin{bmatrix} u \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} F \\ -(C_2 + D_{22}F) \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & -I \\ I & D_{22} \end{bmatrix} \begin{bmatrix} y \\ u_2 \end{bmatrix} \tag{9b}$$

Assume that the added dynamics Q(s) is described by the following <u>minimal</u> realization

$$\begin{aligned} \dot{q} &= \tilde{A}q + \tilde{B}\tilde{y} \\ u_2 &= \tilde{C}q + \tilde{D}\tilde{y} \end{aligned} \tag{10}$$

The controller K(s) is just a combination of (9) and (10). From (9) and (10), we have the dynamic equations of the controller K(s) as follows,

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \hat{x} \\ q \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} y$$

$$u = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} \hat{x} \\ q \end{bmatrix} + \delta y \qquad (11)$$

If $D_{22} = 0$, then
$\beta_1 = -H - B_2 \tilde{D}$

where

$$\beta_1 = -H - (B_2 + HD_{22})(I - \tilde{D}D_{22})^{-1}\tilde{D} \qquad (12a)$$

$$\beta_2 = \tilde{B} + \tilde{B}D_{22}(I - \tilde{D}D_{22})^{-1}\tilde{D} \qquad (12b)$$

$$\gamma_1 = F + (I - \tilde{D}D_{22})^{-1}\tilde{D}(C_2 + D_{22}F) \qquad (12c)$$

$$\gamma_2 = -(I - \tilde{D}D_{22})^{-1}\tilde{C} \qquad (12d)$$

$$\begin{aligned} \alpha_{11} &= A + HC_2 + (B_2 + HD_{22})\gamma_1 \\ &= A + B_2F - \beta_1(C_2 + D_{22}F) \end{aligned} \qquad (12e,f)$$

$$\alpha_{12} = (B_2 + HD_{22})\gamma_2 \qquad (12g)$$

$$\alpha_{21} = -\beta_2(C_2 + D_{22}F) \qquad (12h)$$

$$\alpha_{22} = \tilde{A} - \tilde{B}D_{22}\gamma_2 \qquad (12i)$$

$$\delta = -(I - \tilde{D}D_{22})^{-1}\tilde{D} \qquad (12j)$$

Properties of Observer-based Controller Parametrization

In Fig. 1, the internal stability of the closed-loop system depends only on $G_{22}(s)$ and K(s), i.e., the interconnected system shown in Fig. 5.

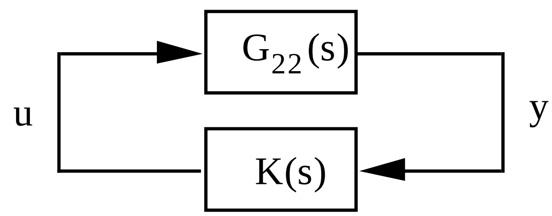


Fig. 5 Equivalent system to Fig. 1 for internal stability.

In the following, the controller K(s) in Fig. 5 is replaced by the block diagram of Fig. 4 which is the observer-based controller with an added dynamics Q(s).

In the following theorem we will show that the poles of the closed-loop system with the observer-based controller parametrization in Fig. 4 are the regulator poles (the eigenvalues of $A+B_2F$), the observer poles (the eigenvalues of $A+HC_2$), together with the poles of the parameter matrix Q(s). In the design of the observer-based controller, F and H are chosen such that the eigenvalues of $A+B_2F$ and $A+HC_2$ are stable. Therefore, the closed-loop system is internally stable if and only if the parameter matrix Q(s) is proper stable. The proof is quite straightforward and is done completely in the state space without referring to the derivations used by Youla et. al., Desoer et. al., and Doyle et. al..

Theorem 2: (Observer-based Controller Parametrization)

Consider the closed-loop system in Fig. 5. Assume that $G_{22}(s) = \{A, B_2, C_2, D_{22}\}$ with order n is stabilizable and detectable and the controller K(s) is replaced by the observer-based controller with an added m-th order dynamics Q(s) as shown in Fig. 4. Then the set of the closed-loop poles is composed of the n eigenvalues of $A+B_2F$, the n eigenvalues of $A+HC_2$, and the m poles of the added dynamics Q(s). That is, the set of the closed-loop poles is

$$\pi_{closed-loop} = \pi_{regulator} + \pi_{observer} + \pi_{Q(s)} \tag{13a}$$

where

$$\pi_{regulator} = \{n \text{ eigenvalues of } A + B_2 F\} \tag{13b}$$

$$\pi_{observer} = \{n \text{ eigenvalues of } A + HC_2\} \tag{13c}$$

and

$$\pi_{Q(s)} = \{m \text{ poles of } Q(s)\} \tag{13d}$$

It is well known that in the observer-based controller design the closed-loop poles are the regulator poles (the eigenvalues of $A+B_2F$) and the observer poles (the eigenvalues of $A+HC_2$). In Theorem 2 we just showed that the above property still remains when we add a dynamics Q(s) to the observer-based controller as shown in Fig. 4. The eigenvalues of $A+B_2F$ and $A+HC_2$ are still parts of the closed-loop poles after we add Q(s) to the controller. Adding Q(s) only introduces additional poles to the closed-loop system and the added closed-loop poles are the poles of Q(s). If F and H have been chosen such that $A+B_2F$ and $A+HC_2$ are stable, then the closed-loop system with the observer-based controller parametrization will be internally stable if and only if the parameter matrix Q(s) is proper stable. From Fig. 4, it is easy to see that the controller K(s) is proper if Q(s) is proper and $I - D_{22}Q(\infty)$ is invertible.

With the observer-based controller parametrization, the closed-loop transfer function matrix from w to z, i.e., $T_{zw}(s)$, is a simple affine function of the parameter matrix Q(s). That is,

$$T_{zw}(s) = T_{11}(s) + T_{12}(s)\,Q(s)\,T_{21}(s) \qquad (14)$$

$Q(s) = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B} + \tilde{D}$

where $T_{11}(s)$, $T_{12}(s)$, and $T_{21}(s)$ are given by (8). The added dynamics Q(s) is a proper stable rational matrix to be chosen such that $I-D_{22}Q(\infty)$ is invertible and $T_{zw}(s)$ has some desired performance. No matter which Q(s) is to be selected, we always have clear idea that the closed-loop poles will be the eigenvalues of $A+B_2F$ and $A+HC_2$ together with the poles of the added dynamics Q(s) if the controller is realized as that shown in Fig. 4. $\tilde{D}$

Assume that the orders of the plant and the parameter matrix Q(s) are n and m respectively. If the controller is realized as that shown in Fig. 4, then the order of the controller is n+m and the closed-loop system has 2n+m poles described by the set $\pi_{\text{closed-loop}}$ in (13). The realization of the controller in Fig. 4 may not be minimal. Suppose it is not and there are r poles in the controller either uncontrollable or unobservable, then the controller can be realized by a minimal realization with order n+m-r and the number of closed-loop poles will be reduced to 2n+m-r.

In the following theorem, we will show that the uncontrollable or unobservable poles of the controller of Fig. 4 must be the eigenvalues of $A+B_2F$ or $A+HC_2$ and the closed-loop poles will always include all m poles of the added stable dynamics Q(s). If r pole-zero cancellations occur in the controller, then the closed-loop poles will include m poles of Q(s), and 2n-r eigenvalues out of the set $\pi_{regulator}+\pi_{observer}$ which was defined in (9).

Theorem 3:

Consider the closed-loop system in Fig. 5. Assume that $G_{22}(s) = \{A, B_2, C_2, D_{22}\}$ with order n is stabilizable and detectable and the controller K(s) is replaced by the observer-based controller with an added m-th order dynamics Q(s) as shown in Fig. 4. Define $\pi_{regulator}$, $\pi_{observer}$ and $\pi_{Q(s)}$ by (13b), (13c), and (13d) respectively and let

$$\pi_{removal} = \{\text{the controller poles which are either uncontrollable or unobservable}\} \tag{15}$$

Then

$$\pi_{removal} \subset \pi_{regulator} + \pi_{observer} \tag{16}$$

and the closed-loop system with the minimal order controller will have a set of poles described by the following

$$\begin{aligned} &\pi_{closed\text{-}loop\ with\ \min\ controller} \\ &= \pi_{Q(s)} + (\pi_{regulator} + \pi_{observer} - \pi_{removal}) \end{aligned} \tag{17}$$

AN ILLUSTRATIVE EXAMPLE

Although the results presented in the previous section are shown in multivariable case, a simple scalar-case example is given in the following for simplicity. Consider Fig. 5 where

$$G_{22}(s) = \frac{s+1}{s^2}$$

with realization { A, B_2, C_2, D_{22}} as

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_{22} = 0$$

The structure of the controller K(s) is shown in Fig. 4. F and H can be arbitrarily chosen such that $A+B_2F$ and $A+HC_2$ are stable. Let

$$F = \begin{bmatrix} -4 & -3 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

Then the regulator poles (i.e., the eigenvalues of $A+B_2F$) are { -1, -3 } and the observer poles (i.e., the eigenvalues of $A+HC_2$) are { -2, -2 }. That is,

$$\pi_{regulator} = \{-1, -3\}, \qquad \pi_{observer} = \{-2, -2\}$$

From Theorem 1 or Theorems 2 and 3, we know that the closed-loop system is internally stable if and only if the parameter matrix Q(s) is stable. In the following, we will verify the results of Theorems 2 and 3.

(i) Choose $Q(s)=\dfrac{1}{s+5}$ whose realization $\left\{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\right\}$ is { -5, 1, 1, 0 } and therefore

$$\pi_{Q(s)}=\{-5\}.$$

From (11) we have a state-space representation for the controller K(s) as follows.

$$K(s) = \left[\begin{array}{ccc|c} -8 & -7 & -1 & 4 \\ 1 & 0 & 0 & 0 \\ -1 & -1 & -5 & 1 \\ \hline -4 & -3 & -1 & 0 \end{array}\right]$$

which is a minimal realization and its transfer function is

$$K(s) = \frac{-(17s^2+92s+60)}{(s+1)(s^2+12s+34)}$$

The closed-loop characteristic polynomial is

$$(s+1)(s+3)(s+2)(s+2)(s+5)$$

and therefore the set of the closed-loop poles are the regulator poles {-1, -3}, the observer poles {-2, -2}, and the pole of Q(s) {-5}. That is,

$$\pi_{closed-loop}=\pi_{regulator}+\pi_{observer}+\pi_{Q(s)}$$

(ii) Choose $Q(s)=\dfrac{20}{s+6}$ whose realization $\left\{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\right\}$ is {-6, 1, 20, 0}, and therefore

$$\pi_{Q(s)}=\left\{-6\right\}.$$

From (11) we have a state-space representation for the controller K(s) as follows.

$$K(s) = \left[\begin{array}{ccc|c} -8 & -7 & -20 & 4 \\ 1 & 0 & 0 & 0 \\ -1 & -1 & -6 & 1 \\ \hline -4 & -3 & -20 & 0 \end{array}\right]$$

which has an uncontrollable eigenvalue at -1 and an unobservable eigenvalue at -2. Its transfer function is

$$K(s) = \frac{\text{-36 (s+1) (s+2)}}{\text{(s+11) (s+1) (s+2)}} = \frac{\text{-36}}{\text{s+11}}$$

The set of controller poles which are either uncontrollable or unobservable is

$$\pi_{removal}=\left\{-1,-2\right\}$$

which is a subset of $\pi_{regulator}+\pi_{observer}$. The characteristic polynomial of the closed-loop system with the minimal order controller $K(s)=\dfrac{-36}{s+11}$ is (s+2) (s+3) (s+6)

and therefore

$$\pi_{closed\text{-}loop\ with\ \min\ controller} = \{-2, -3, -6\}$$

which is equal to

$$\pi_{Q(s)} + (\pi_{regulator} + \pi_{observer} - \pi_{removal})$$

(iii) Choose $Q(s) = \dfrac{2s^2 + 2}{s^2 + 2s + 2}$ whose realization

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \quad is \quad \left[\begin{array}{cc|c} -2 & -2 & 1 \\ 1 & 0 & 0 \\ \hline -4 & -2 & 2 \end{array}\right]$$

and therefore

$$\pi_{Q(s)} = \{-1 + j, -1 - j\}$$

From (11) we have a state-space representation for the controller K(s) as follows.

$$\left[\begin{array}{cccc|c} -6 & -5 & 4 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -2 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ \hline -2 & -1 & 4 & 2 & -2 \end{array}\right]$$

whose uncontrollable eigenvalues are { -1, -3 } and unobservable eigenvalues are { -2, -2 }. Its transfer function is

$$K(s) = \frac{-2\,(s+1)\,(s+3)\,(s+2)\,(s+2)}{(s+1)\,(s+3)\,(s+2)\,(s+2)} = -2$$

The set of controller poles which are either uncontrollable or unobservable is

$$\pi_{removal} = \{-1, -3, -2, -2\}$$

which is equal to $\pi_{regulator} + \pi_{observer}$. The characteristic polynomial of the closed-loop system with the minimal order controller K(s) = -2 is

$$s^2 + 2s + 2$$

and therefore

$$\pi_{closed\text{-}loop\ with\ \min\ controller} = \{-1+j, -1-j\}$$

which is equal to

$$\pi_{Q(s)} + (\pi_{regulator} + \pi_{observer} - \pi_{removal}) = \pi_{Q(s)}$$