Results on Burning Number of a Graph

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Certificate

This is to certify that the thesis titled **Results on Burning Number of a Graph** submitted by **Shravan Kumar Singh(16CS60R79)** to the Department of Computer Science and Engineering is a bonafide record of work carried out by him under my supervision and guidance. The thesis has fulfilled all the requirements as per the regulations of the Institute and, in my opinion, has reached the standards needed for submission.

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Abstract

The burning number b(G) of a graph G is a parameter which measures the speed of the spread of contagion in a graph. It was introduced by Bonato, Janssen, and Roshanbin in the paper(How to Burn a Graph?[4]). The most recent result on upper bound of Burning Number is in Max Land and Linyuan Lu's paper[2] (An Upper Bound on Burning Number of Graphs), they have proved $b(G) <= \lceil \frac{-3+\sqrt{24n+33}}{4} \rceil$, which is roughly $\frac{\sqrt{6}}{2}\sqrt{n}$. It is still a conjecture that $b(G) <= \lceil \sqrt{n} \rceil$. In our project, we have arrived at upper bounds, that is dependent on degree of internal nodes of a tree. We have also found the Lower Bound of Burning Number of a tree. We have also explored a new concept of Minimal b-Burnable Tree and provided some results.

Contents

Chapter 1

Introduction

Today, so many social contagions(a meme or a hoax) are in social networking sites. Suppose you want to spread some new contagion in a social network like Facebook or Twitter. For this it is not necessary that all people should be directly connected with each other, many agents spread contagion in social site through friends. They spread it through their friends and followers, and their friends and followers repeat the process. Because of this, whole network gets this contagion as it propagates over time. If our main purpose is : minimum time to spread contagions in the whole network, then which users should be targeted for spreading it, and in what order? So this is a very hot topic in today's time. A deterministic approach which tackles above posed question is called 'Finding the Burning of a Graph'.

We take simple undirected and finite graphs for our discussion. In the burning process either a node is burned or not burned. If a node is burned then it is in burned state during the entire process until the whole network is burned. Time steps are discrete. In this burning process,in the beginning, we choose a node to burn. If a node is burned in time step t then at t+1 step we choose another node to burn, if available, and the neighbors of a node which was chosen in t step are burned also because of spread. So in every step, we choose a node to burn if such a node is available. If all the nodes are burned then the game is over. So burning number b(G) of a graph(G) is minimum no. of time steps required to burn whole graph.

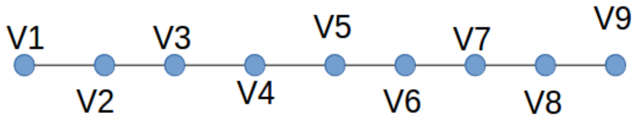
Example. For a complete graph , burning number $b(K_n) = 2$. We burn a node at t=1 and on t=2 every other node in the graph will burn.
For a path it is not that straightforward to compute.

For a path of n vertices, burning number

$$b(P_n) = \lceil \sqrt{n} \rceil \tag{1.1}$$

This eq(1.1) is proved in Bonato's[4] paper.

Example. For the path P_9 and nodes $\{v1, v2, v3, ...v9\}$,the optimal burning sequence will be $\{v3, v7, v9\}$. So b(P_9)=3.



Figure(1): Path P_9

Computing the burning number of a graph is NP-complete problem[5]. Burning process can viewed as spreading of contagion in a social networking site like Facebook or Twitter. If burning number is less, it is easy to spread contagions in the network. Suppose the network is like a Graph G and is G is burned in k steps and vertices that are burned in each steps, are $\{x_1, x_2, x_3, \ldots, x_k\}$ and x_i denotes the node which is burned in ith step where $1 \leq i \leq k$. x_i is called source of fire and $x_1, x_2, x_3, \ldots, x_k$ is called burning sequence for G. So burning number of a graph(G) is length of minimum sequence of a graph (G). Although there can be more than one burning sequence.

Chapter 2

Properties of Burning Number

In this section we will define some properties of burning number of a Graph(G). Some notations and terminologies are used to compute the burning number. First we will define eccentricity.
a). Eccentricity : Eccentricity of v, v is node of a graph(G), is defined as max $\{d(v,u) : u \in V(G)\}$
b). Radius : Radius of a Graph(G) is the minimum eccentricity over the set of all nodes.
c). Center : Center of Graph(G) is collection of those nodes which have minimum eccentricity.
d). K'th Closed neighborhood : For a positive integer k, the K'th closed neighborhood of a particular vertex v is defined as the set
$\{\ u \in V(G) : d(u,v) <= k\}$. It is denoted as $N_k[v]$.

Now suppose for a Graph(G), the burning sequence is $\{x_1, x_2, x_3, \ldots, x_k\}$. Fire started at x_i in the Graph(G) will burn all nodes which are within $k-i$ distance from x_i by the end of the kth step. So every node $v \in V(G)$ of a Graph(G) is either a source of fire or burned from at least one of source of fire(vertices) by the end of kth step. We can say that each node of a Graph(G) must be member of $N_{k-i}[x_i]$, for some $1 <= i <= k$.

For each pair of i and j $\{1 \leq i < j \leq k\}$, we must have $d(x_i, x_j) >= j-i$. If $d(x_i, x_j) >= j-i$ is not true means $d(xi, xj) = l < j-i$, then x_j will be burned at step $l+i < j$ which is contradiction.

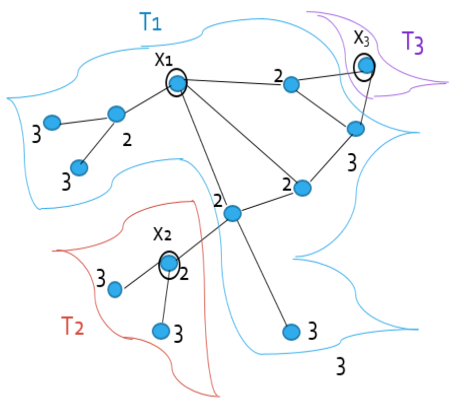
So we can say that $\{x_1, x_2, , , x_k\}$ forms a burning sequence for Graph(G) if and only if, for each pair i and j $\{1 \leq i < j \leq k\}$, $d(x_i, x_j) >= j-i$, and the following set equation holds:

$$N_{k-1}[x_1] \cup N_{k_2}[x_2] \cup \ldots \cup N_0[x_k] = V(G) \tag{2.1}$$

.

For a graph(G), covering of G is a set of subsets of the nodes of G whose union is V (G). In eq(2.1), shows that the burning problem is same as covering problem using closed neighbourhoods with a restriction on their radius.

e). Rooted Tree Partition: Rooted tree means a tree with one vertex r designated as the root. Eccentricity is the height of a tree and leaf vertex have degree 1. Descendant of any vertex v is any vertex which is either the child of v or is(recursively) the descendant of any of the children of v. Subtree of a tree is rooted at v is set of all set of descendant vertex. This can be very useful for our induction steps, while proving the improved bound.
Suppose decomposition of a graph(G) into K rooted subtrees $T_1, T_2, \ldots, T_k$ and for each $1 \leq i \leq k$, T_i is of height at most k-i. We are assuming $x_1, x_2, \ldots, x_k$ are the roots of all subtrees $T_1, T_2, ..., T_k$, respectively. For each pair i and j, with $1 \leq i < j \leq k$, $d(x_i, x_j) \geq j - i$. Our burning sequence is $(x_1, x_2, \ldots, x_k)$, since the distance between any node in subtree T_i and x_i is at most $k - i$. Which means after k steps our graph G will be burned.



Figure(2): Rooted Tree Partition

In figure 2, the burning sequence is (x_1, x_2, x_3). We have done decomposition in figure(2) into subgraphs T_1, T_2 and T_3 based on the burning sequence by drawing dashed curves. In the figure (2) each node is marked with a number corresponding to the step in which it is burned.

The following corollary[4] is very useful to find burning number of graph as it reduces burning number of a graph problem into burning number of a spanning tree of graph G.

For a graph G we have[4] :

$$b(G) = min\{b(T) : \text{T is a spanning subtree of G}\}.$$

So, it is sufficient to consider only burning number b(T) for free T.

Chapter 3

Notations, and the Conjecture

A graph G(V, E) consists of a set of vertices V and a set of edges E. For any positive integer value k, let [k] denotes the set $\{1, 2, 3, \ldots, k\}$. The order of G is denoted as $\|G\|$. For connected graph G, there should be a path between any two vertices of the graph G. Distance between any two vertices d(u, v) is the shortest path from u to v.

From Eq.(2.1) and the definition of a graph G is k-burnable if there is a burning sequence $v_1, ..., v_k$ of vertices such that

$$V \subset \bigcup_{i=1}^{k} N_k - i[v_i] \tag{3.1}$$

$$\forall i, j \in [k] : d(x_i, x_j) \geq j - i \tag{3.2}$$

If G is K-burnable it's mean G contains optimal subsequence $x_1, x_2, x_3, \ldots, x_K$ and b(G) is smallest integer K. It is already proved that that Condition (3.2) is redundant for the definition of burning number b(G). We can rewrite Condition (3.1) by relabeling the vertices in the burning sequence as follows:

$$V \subset \bigcup_{i=1}^{k} N_{i-1}[v_i] \tag{3.3}$$

. From this we will get following generality which will be used for our induction. We will take a set A for k positive integers $\{a_1, a_2, a_3, \ldots, a_k\}$ (it is not necessarily that all are distinct). A graph G is A-burnable,if there exist k vertices $v_1, v_2, \ldots, v_k$ such that $G \subseteq \bigcup_{i=1}^{N} a_{i-1}[v_i]$. For G is [k]-burnable, the burning number b(G) should be least k.

Some theorems on upper bound on the burning number of G which are proved so far:

Conjecture 1: For any connected graph G of order n, $b(G) <= \lceil \sqrt{n} \rceil$.

Following upper bound is due to Bonato-Janssen-Roshanbin[4]

$$b(G) <= 2\lceil \sqrt{n} \rceil - 1 \tag{3.4}$$

The most recent and best upper bound is due to Max Land and Linyuan Lu[2]

$$b(G) <= \left\lceil \frac{-3 + \sqrt{24n + 33}}{4} \right\rceil \tag{3.5}$$

Chapter 4

Bounds Based on Degree of Internal Nodes

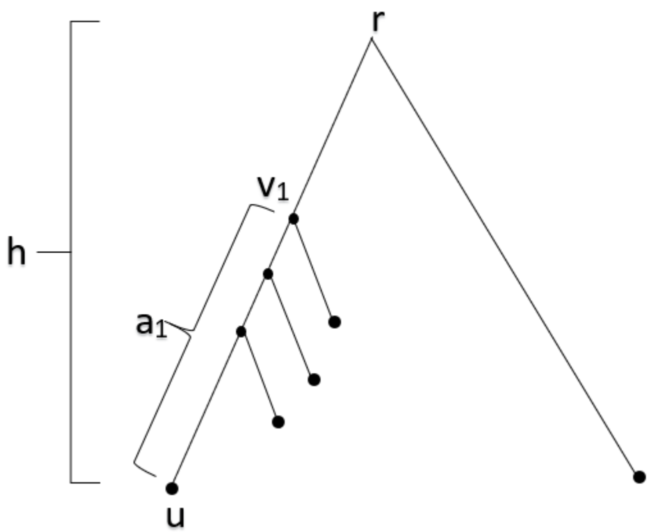
In this project we have arrived to the upper bound that depends on degree of internal nodes of a tree. We have borrowed some part of the technique that Max Land and Linyuan Lu[2] have used to arrive at this result.

A important lemma which is helpful in proving our induction.
Lemma 1. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k non-negative integers. Let T be a tree on n nodes in which every internal(non-leaf node) has degree at least 3. Such a tree T has order at least $2\sum_{i=1}^{k} a_i - k$, and T is A-burnable.

Proof. Suppose $a_1 \leq a_2 \leq \cdots \leq a_k$. We will prove by induction on k. For our initial case: $k = 1, A = a_1$, we need to prove that if tree T has at most $2a_1 - 1$ vertices then T is A-burnable. So we have:

$$r(T) \leq \frac{D(T)+1}{2} \leq \frac{n}{2} \leq a_1 - \frac{1}{2}. \tag{4.1}$$

and the radius r(T) is an integer, so we must have $r(T) \leq a_1 - 1$. Thus T is $\{a_1\}$-burnable since a_1 is greater than $r(T)$.
Now we will assume that this statement holds for $k-1$ integers as our induction hypothesis. Next step is to prove for any A of k integers $0 \leq a_1 \leq a_2 \leq \cdots \leq a_k$ and tree T, in which every internal(non-leaf node) has degree at least 3, has order at least $2a_1 + 2a_2 + ...2a_k - k$ vertices, and T is A-burnable.



Pick an arbitrary vertex r as the root of T . Let h be the height of rooted tree. If $h \leq a_k - 1$, then $V(T) \subseteq N_{a_k-1}(r)$. that is, T is a_k-burnable.

Now suppose $h \geq a_k$. Then we select a leaf vertex u such that $d(r,u) = h$. Let v_1 be the vertex on the ru-path such that the distance $d(u, v_1) = a_1 - 1$. (This is possible since $h \geq a_k \geq a_1 - 1$). Let T_1 be the subtree rooted at v_1, and $T_2 := T \backslash T_1$ be the remaining subtree. Since every node has degree at least 3, number of nodes v_1 will burn is at least 1(first step)+2+2+...+2(at last step). Hence $|T_1| \geq 2a_1 - 1$. Thus we get:

$$|T_2| = |T| - |T_1|$$

$$\leq 2a_1 + 2a_2 + ... + 2a_k - k - (2a_1 - 1)$$
$$= 2a_2 + ... + 2a_k - (k - 1).$$

By induction hypothesis, T_2 is $\{a_2, a_3, \ldots, a_k\}$-burnable. Thus, there exists k - 1 vertices $v_2, v_3, \ldots, v_k$ such that $T2 \subseteq \bigcup_{i=2}^{k} N_{a_i-1}[v_i]$. Also, notice $T_1 \subseteq N_{a_1-1}[v_1]$. Therefore, $T \subseteq \bigcup_{i=1}^{k} N_{a_i-1}[v_i]$. So whole T is burnable.

Corollary 2: For any tree T on n nodes, in which every internal(non-leaf node) has degree at least 3, has $b(G) \leq \lceil \sqrt{n} \rceil$.

<u>Proof</u>. Let $A = \{k, k-1, \ldots, 1\}$. By Lemma 1, any Tree of order $n \leq (2\sum_{i=1}^{k} i) - k = k^2$ is A-burnable. Solving k we get $k \leq \sqrt{n}$. Since burning number is an integer, we can take ceil value. By Corollary 1, the same bound holds true for graphs in which the minimum spanning tree also posses the above mentioned property(every internal node has degree at least 3).

Corollary 3: For any tree T on n nodes, and n' number of internal nodes of degree 2, $b(G) \leq \lceil\sqrt{n+n'}\rceil$.

Proof. We add an extra node to all the internal nodes that have degree 2, thus making every internal node's degree at least 3. Let $A = \{k, k-1, ..., 1\}$. By Lemma 1, such a Tree has order at least $n+n' \leq (2\sum_{i=1}^{k} i) - k = k^2$ and it is A-burnable. Solving k we get $k \leq \sqrt{n+n'}$. Since burning number is an integer, we can take ceil value.

We will use the following two result from Max Land and Linyuan Lu[2]:

RESULT 1. For any k-1 distinct positive integers $a_1 < a_2 <...< a_{k-1}$, there exists an a_i such that $2\lfloor\frac{k-1}{3}\rfloor \leq a_i \leq a_{k-1} - \lfloor\frac{k-1}{3}\rfloor$

RESULT 2. For all integer $k \geq 1$

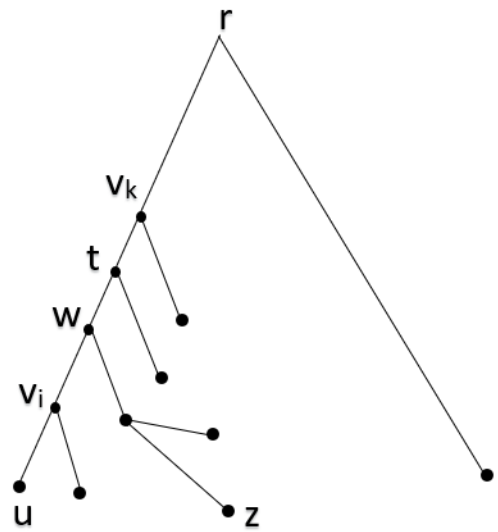
$$\sum_{i=1}^{k}\left\lfloor\frac{i-1}{3}\right\rfloor = \left\lfloor\frac{k^2-3k+2}{6}\right\rfloor \tag{4.2}$$

Theorem 2. Let A = $\{a_1, a_2, ..., a_k\}$ be a set of k non-negative integers $a_1 < a_2 <...< a_k$. Let T be a tree on n nodes in which every internal(non-leaf node) has degree at least 3. Such a tree T has order at least $2\sum_{i=1}^{k} a_i - k + 2\lfloor\frac{k^2-3k+2}{6}\rfloor$, and it is A-burnable.

Proof. We use induction on k. For initial case k=1: A=$\{a_1\}$. Similar to base case of Lemma 1, if a tree has order at most $2a_1$-1 then T is burnable. Let $F(x) = \lfloor\frac{k^2-3k+2}{6}\rfloor$. For k=1, F(x) becomes zero.
Let's assume this statement holds true for any set of k-1 positive integers. Now we have to prove for set A = $\{a_1, a_2, ..., a_k\}$ this statement holds true and at least $2\sum_{i=1}^{k} a_i - k + 2\lfloor\frac{k^2-3k+2}{6}\rfloor$ vertices are burnable in T

Choose an arbitrary root r and view T as a rooted tree. Let u be the leaf vertex which has the farthest distance away from the root r. If d(r, u)$\leq a_k$-1, then $V(T) \subset N_{a_k-1}(r)$; thus T is A-burnable. So we take the other case and assume d(r, u)$\geq a_k$. Let$j = \lfloor\frac{k-1}{3}\rfloor$. By Result 1, there exists a_i that satisfies $2j \leq a_i \leq a_{k-1} - j$. We take three vertices v_i, t, v_{k-1} on the

ru path such that d(u,v_i)=a_i-1, d(u,t)=$a_i - 1 + j$, and d(u,v_{k-1})=a_{k-1}. Let T_1 be the subtree rooted at t. There are two cases that arises when we remove the subtree T_1 from tree T.



Case 1: When $T_1 \subseteq N_{a_i-1}(v_i)$. Case 1 becomes trivial when there are no branches in T_1 that are not fully covered(burned) by v_i. Let T_2 be a subtree such that $T_2 = T - T_1$. We know $|T_1| \geq 2a_i - 1 + 2j$

$$|T_2| = |T| - |T_1|$$

$$\leq 2(a_1 + a_2 + ... + a_k) - k + 2F(k) - (2a_i - 1 + 2j)$$

$$= 2(a_1 + a_2 + ...a_i(missing)... + a_k) - (k-1) + 2(F(k) - j).$$

$$= 2(a_1 + a_2 + ...a_i(missing)... + a_k) - (k-1) + 2F(k-1).$$

By our induction hypothesis T_2 is burnable where A has k-1 integers; T_1 is burnable by v_i;hence whole T is A-burnable.

Case 2: When $T_1 \not\subseteq N_{a_i-1}(v_i)$. Then there is at least a vertex z $\in$ T1 and d(vi,z) $\geq a_i$. Let w be the closest vertex on the path rt to z. Observe that w is not in the subtree rooted at v_i. Thus, w is between v_i and t. So we need another vertex to burn z. We choose v_{k-1} be that vertex. Let

T_3 be the subtree rooted at v_{k-1} and let $T_4 = T - T_3$. We have $|T_3| \geq 2a_{k-1} - 1 + 2d(w, z)$. To estimate value of $d(w, z)$:

$$d(w, z) = d(v_i, z) - d(v_i, w) \geq a_i - d(w, v_i) \geq a_i - d(v_i, t) \geq a_i - j \geq j \quad (4.3)$$

Putting value of $d(w, z)$ back in, we get:

$$|T_3| \geq 2a_{k-1} - 1 + 2d(w, z) \geq 2a_{k-1} - 1 + 2j \quad (4.4)$$

$$\text{We know: } |T_4| = |T| - |T_3|$$

$$\leq 2(a_1 + a_2 + ...a_{k-1} + a_k) - k + 2F(k) - (2a_{k-1} - 1 + 2j)$$

$$= 2(a_1 + a_2 + ... + a_k) - (k - 1) + 2(F(k) - j).$$

$$= 2(a_1 + a_2 + ... + a_k) - (k - 1) + 2F(k - 1).$$

By our induction hypothesis T_4 is burnable where A has k-1 integers; T_3 is burnable by v_{k-1};hence the whole T is A-burnable.

Corollary 4: For any tree T on n nodes, in which every internal(non-leaf node) node has degree at least 3, has $b(G) \leq \lceil \frac{3+\sqrt{48n-23}}{8} \rceil$

Proof. Let $A = \{k, k-1, ..., 1\}$. By Theorem 2, any Tree of order $n \leq 2\sum_{i=1}^{k} a_i - k + 2\lfloor \frac{k^2-3k+2}{6} \rfloor$ is A-burnable. Solving for k we get: $k \geq \frac{3+\sqrt{48n-23}}{8}$. Since burning number is an integer, we can take ceil value. Thus $b(T) \leq \frac{3+\sqrt{48n-23}}{8}$. By Corollary 1, the same bound holds true for graphs in which the minimum spanning tree also posses the above mentioned property(every internal node has degree at least 3).

Corollary 5: For any tree T on n nodes, and n' number of internal nodes of degree 3, $b(G) \leq \left\lceil \sqrt{\frac{3(n+n')}{4}} \right\rceil$.

Proof. We add an extra node to all the internal nodes that have degree 2, thus making every internal node's degree at least 3. Let $A = \{k, k-1, ..., 1\}$. By Theorem 2, such a tree T has order at least $2\sum_{i=1}^{k} a_i - k + 2\lfloor \frac{k^2-3k+2}{6} \rfloor$, and it is A-burnable. Solving for k and ignoring constant terms, we get $k \leq \sqrt{\frac{3(n+n')}{4}}$. Since burning number is an integer, we can take ceil value. By Corollary 1, the same bound holds true for graphs.

Chapter 5

Lower Bound of Burning Number

Theorem 3. If T is a binary tree with n nodes, then

$$b(T) \geq \lceil \log_2(\frac{n}{3} + 1 + \frac{2}{3}\log_2(\frac{n}{3}))\rceil \tag{5.1}$$

<u>**Proof**</u>. The idea is to burn maximum possible number of nodes at each time interval. Let say at time t=1, we burn a node v_1, then at time t=2 this fire should spread at maximum possible rate. Since it is a binary tree, fire will spread to at most to 3 nodes at t=2. Similarly we burn another node v_2 at time t=2 and it will spread to another 3 nodes when t becomes 3. Number of nodes that are burnt at t=3 through the fire that was spreaded by v_1 will be 3*2. Let the burning number of binary tree T is b and $\{v_1, v_2, \ldots, v_b\}$ is the sequence of nodes that are burnt at time interval $\{1, 2, \ldots, b\}$.

We add up all these burnt nodes to arrive at:

$$\begin{aligned} n \leq & \overbrace{\left\{1 + 3\times 2^0 + 3\times 2^1 + 3\times 2^2 + \cdots + 3\times 2^{b-2}\right\}}^{v_1} \\ & + \overbrace{\left\{1 + 3\times 2^0 + 3\times 2^1 + 3\times 2^2 + \cdots + 3\times 2^{b-3}\right\}}^{v_2} \\ & + \cdots + \overbrace{\left\{1 + 3\times 2^0\right\}}^{v_{b-1}} + \overbrace{1}^{v_b} \end{aligned} \tag{5.2}$$

The first term represents the number of nodes that are burnt by v_1, second term is for v_2, and last term represents the number of nodes burnt by v_b. By futher solving it, we get:

$$n \leq 3 \times 2^b - 3 - 2 \times b$$

$$b \geq \log_2(n + 3 + 2b) - \log_2 3$$

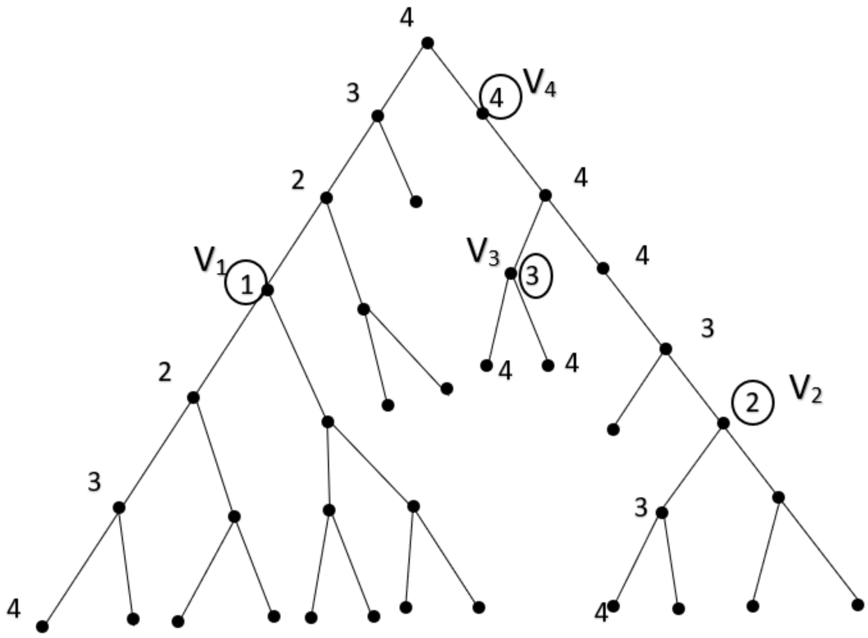
$$\text{since } b \geq \log_2(n + 3 + 2b) - \log_2 3 \geq \log_2 n - \log_2 3$$

$$\text{we replace value of b by } \log_2 n - \log_2 3$$

$$b \geq \log_2(n + 3 + 2(\log_2 n - \log_2 3)) - \log_2 3$$

$$b \geq \log_2(\tfrac{n}{3} + 1 + \tfrac{2}{3}\log_2(\tfrac{n}{3}))$$

Since burning number is an integer we take the ceiling value.Following image shows such a tree where it is possible:



Now we derive the generalized form of the theorem for k-ary tree.
Theorem 4. If T is an k-ary tree with burning number b, then maximum possible nodes it can have is

$$b + \frac{k+1}{(k-1)^2}(k^b - k \times b + b - 1) \tag{5.3}$$

Proof. In the similar fashion as above we derive the expression for maximum possible number of nodes that are burnt in an k-ary tree where burning number is b. Let say at time t=1, we burn a node v_1, then at time t=2 this fire should spread at maximum possible rate. Since it is an k-ary tree, fire will spread to at most to k nodes at t=2. Then we burn another node v_2 at time t=2 and it will spread to another n nodes when t becomes 3. Number of nodes that are burnt at t=3 through the fire that was spreaded by v_1 will be k*(k-1). Let the burning number of k-tree T is b and $\{v_1, v_2, \ldots, v_b\}$ is the sequence of nodes that are burnt at time intervals $\{1, 2, \ldots, b\}$.

We add up all these burnt nodes to arrive at:

$$
\begin{aligned}
n \leq\ & \overbrace{\left\{1+(k+1)\times k^0+(k+1)\times k^1+(k+1)\times k^2+\cdots+(k+1)\times k^{b-2}\right\}}^{v_1} \\
+ & \overbrace{\left\{1+(k+1)\times k^0+(k+1)\times k^1+(k+1)\times k^2+\cdots+(k+1)\times k^{b-3}\right\}}^{v_2} \\
& +\cdots+\overbrace{\left\{1+(k+1)\times k^0\right\}}^{v_{b-1}}+\overbrace{1}^{v_b} \qquad (5.4)
\end{aligned}
$$

The first term represents the number of nodes that are burnt by v_1, second term is for v_2, and last term represents the number of nodes burnt by v_b. By futher solving it, we get:

$$n \leq b + \tfrac{k+1}{(k-1)^2}(k^b - k \times b + b - 1)$$

$$b \geq \log_k\left(\frac{n \times k^2 - 2 \times n \times k + n + k + 1}{k+1} + \frac{2(k+1)}{k+1}\log_k\left(\frac{2 \times k^2 - 2 \times n \times kn + k + 1}{k+1}\right)\right) \qquad (5.5)$$

We can get values of b by plugging different values of k.

Chapter 6

Minimal b-Burnable Tree

A tree T with burning number b is called a minimal b-burnable tree if deleting any leaf node from tree T reduces the burning number to b-1 in resultant tree.

In figure 5.1, tree T is minimal 3-burnable tree. Nodes of tree T can be burned in minimum 3 steps so burning number is 3. If we remove any leaf node {d, e} then burning number will be reduced to 2 in resultant tree. Figure 5.2 illustrates this:

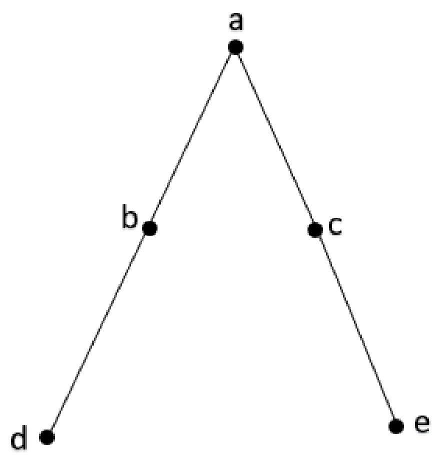


Figure 5.1: **minimal 3-burnable tree T**

In figure 5.2, T1 and T2 are resultant trees after deletion of a leaf node d and e respectively. We can see that burning number of tree T1 and T2 are same and it is 2. So deleting any leaf node from tree T will reduce burning number from 3 to 2 and this implies tree T is minimal 3-burnable tree.

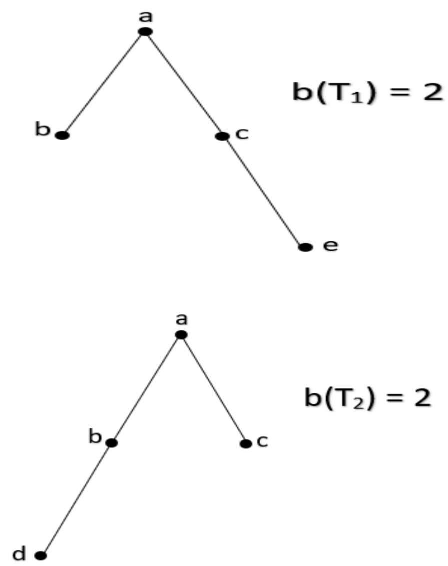


Figure 5.2: **Resultant trees after deletion of a leaf node of tree T**

Now we see another type of a minimal b-burnable tree. In figure 5.3, tree T is minimal 4-burnable tree. If we remove any leaf node {V1,V2, V3 } then burning number will be reduced to 3 in resultant tree, as we can see in figure 5.4.

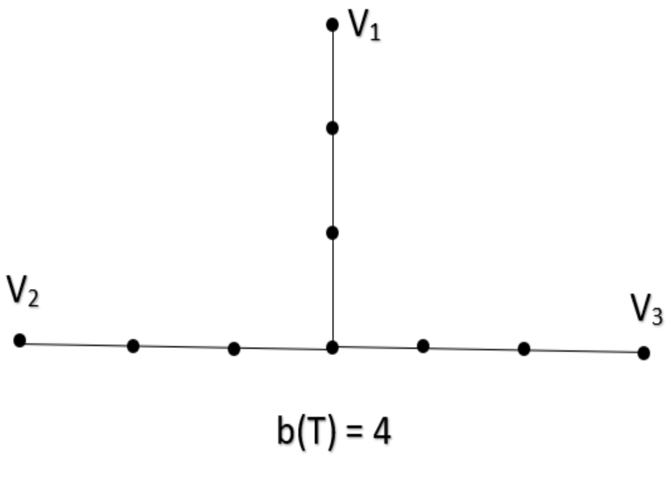


Figure 5.3: **minimal 3-burnable tree T**

In figure 5.4, T1, T2 and T2 are resultant trees after deletion of leaf nodes, V1, V2 and V3 of tree T respectively . We can see that burning number of tree T1, T2 and T3 is 3. So deleting any leaf node from Tree T will reduce burning number from 4 to 3 and this implies tree T is minimal 4-burnable tree.

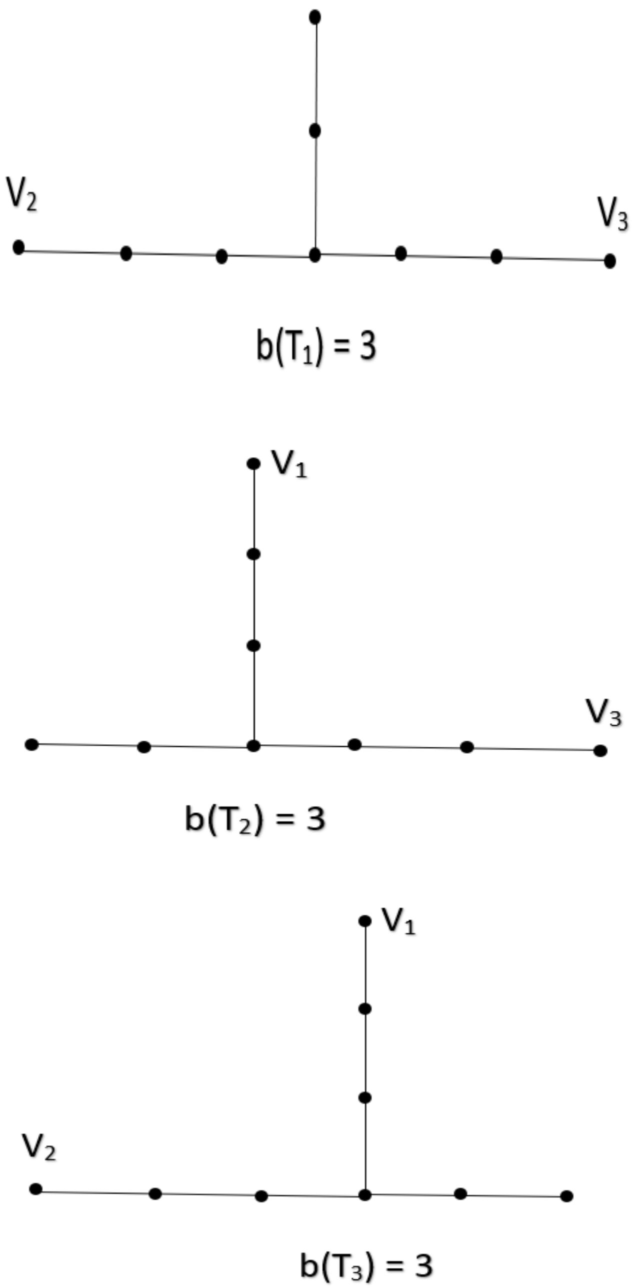


Figure 5.4: **Resultant trees after deletion of a leaf node of tree T**

Theorem 5. A star graph with burning number b is minimal b-burnable, if total branches are $b-1$ and each branch is having $b-1$ length.

<u>Proof</u>. We prove in two steps: 1) Show that such a star graph has burning number b 2) By removing any leaf vertex, the burning number reduces to b-1.

We apply the argument on one branch. By symmetry, it will be true on other branches. Let's assume burning number is $b-1$. If we burn the center of the graph at time $t = 1$, then at $t = b-1$, it will spread to all sides, and burn all vertices except the pendant(leaf) vertex of each branch. Thus, there are $b-1$ leaf vertices remaining to be burnt but, only $b-2$ vertices are remaining that we can burn since we assume that burning number is $b-1$. Leaf vertices that are remaining to be burnt can not burn any other leaf vertices since the distance between two leaf vertices is $2b-2$. Next we try to burn the vertex that is adjacent to the center. It will completely burn one branch, and leave two vertices unburnt in each of the rest of the branches. We have $b-2$ branches left and $b-2$ vertices remaining to be burnt. But, one of the vertex at $t = b-1$ can burn only one vertex, and none of the vertices on one branch can burn vertices of other branches. Thus it will not work either. Similarly, we can show that by moving vertex, that we burn at $t = 1$, farther away from center will not burn the graph. Hence the graph is not $b-1$ burnable. To show it is b burnable we will burn the center vertex at $t = 1$ and it will burn the whole graph at $t = b$, as the length of each branch is $b-1$.

We show by deleting any of the leaf vertex from the graph, we can burn the residual graph in $b-1$ time. After deletion a leaf vertex, the residual graph has $b-2$ branches of $b-1$ length and one branch of $b-2$ length. At t=1, we burn the center. At $t = b-1$, it will spread to all sides, and burn $b-2$ vertices on each branch. Thus we are left with one vertex on each of b-2 branches that are unburnt. We burn one unburnt vertex on time steps $\{2, 3, \ldots, b-2, b-1\}$. This burning sequence will burn the residual graph in $b-1$ time.

Theorem 6. If a path P has length $l = (b-1)^2 + 1$ then it is minimal b-burnable.

<u>Proof</u>. From equation 1.1, we know any path of n length has $b(P) = \lceil \sqrt{n} \rceil$. Hence the burning number of path of length $(b-1)^2 + 1$ is b. There are two leaf vertices in a path, and to prove that it is minimal b-burnable we remove them one at a time and check whether the burning number remains the same or reduced by 1. First, remove the left leaf vertex of the path. After removal, total number of nodes that are remaining in the path is $(b-1)^2$. By equation 1.1, we know we can burn this path in $n-1$ steps. Right leaf vertex case is also similarly handled. Thus by removal of any of these two nodes the burning number drops by 1. Hence path P is minimal b-burnable.

Theorem 7. Let center of a tree is v_c and $L = \{l_1, l_2, \dots, l_k\}$ be the leaf nodes in a tree, and radius of a tree is $\max(d(v_c, l_i))$ where $l_i \in L$. If T is any tree with n nodes and its burning number is b, then number of leaf nodes that are burned at $t = b$ is equal to number of leaf nodes whose $d(v_c, l_i) = radius(T)$ where $l_i \in L$.

Proof. We prove by contradiction. Let's say there is a node l_1 whose $d(v_c, l_1) = radius(T)$ but it is burning at time $t \lneq b$, say $t = t'$. Then there must be a vertex v_k, burnt at time t=t_k, through which fire is spread and reached to l_1, and distance $d(v_k, l_1) = t' - t_k$. Now we propose to change the vertex v_k to one which falls on a path from l_1 to center v_c of the tree, such that the distance between this new vertex v'_k and l_1 becomes $b - t_k$, and then l_1 will burn at t=b. Let say by choosing such a vertex v'_k there is a node v_x that was previously being burned but not after the change. Then the $d(v_x, v'_k) \gneq d(l_1, v'_k)$ which is not possible since l_1 is one of the farthest node from center of tree and v'_k falls on the path from l_1 to v_c. If such a vertex v_x exists then it will change the radius of the tree, which clearly contradicts our assumption. Thus choosing such a vertex v'_k is always possible.

A tree that has two centers(c_1, c_2) is similar to above case. We divide the leaves of tree into parts: 1) $L_1 = \{l_i : d(l_i, c_1) = radius(T)\}$ 2) $L_2 = \{l_i : d(l_i, c_2) = radius(T)\}$. Important thing to note is, all the shortest paths from c_1 to vertices in L_1 passes through c_2 and similarly all the shortest paths from c_2 to vertices in L_2 passes through c_1. When dealing with L_1 we use center as c_1, and apply the same argument as we did in single center. Similarly use c_2 for set L_2.

Corollary 6. If T is any tree with n nodes and its burning number is b, then there will always be at least two nodes that are burned at time t=b.

Proof. Opposite ends of the diameter of a tree are two such vertices that are burnt at t=b as they satisfy equation $d(v_c, l_i) = radius(T)$ where $l_i \in L$, from theorem 6. Case for bicenter can be similarly proven as in theorem 6.

Conjecture 2. A tree T with burning number b is not a minimal b-burnable tree, if every internal node is having more than equal to 2 leaf nodes.

Chapter 7

Conclusion and Future Work

In Max Land and Linyuan Lu's paper (An Upper Bound on Burning Number of Graphs), they have proved $b(G) <= \lceil \frac{-3+\sqrt{24n+33}}{4} \rceil$, which is roughly $\frac{\sqrt{6}}{2}\sqrt{n}$, where n is the number of vertices in G. We are trying to prove the conjecture $b(n) \leq \sqrt{n}$. We have achieved it when every internal node of a tree has at least degree 3. We show that $b(G) \leq \left\lceil \sqrt{\frac{3(n+n')}{4}} \right\rceil$, where n' is the number of internal nodes with degree 2. In chapter 6, we study minimal b-burnable tress. A tree T is a minimal b-burnable, if by removing any of the leaves, the burning number reduces to $b-1$. We observed that if we prove the burning number conjecture on minimally b-burnable trees then we have proved it for general graphs. With this in mind. in Chapter 6, we study several properties of a minimal b-burnable tree.

Chapter 8

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