$$~~l = \sqrt{p+}~~$$

$$l = \sqrt{p+l}$$

$$l^2 - l - p = 0$$

$$l = \frac{+1 \pm \sqrt{1+4p}}{2}$$

$\Rightarrow$ Taking positive limit

$$\lim = \frac{1+\sqrt{1+4p}}{2}$$

CAUCHY SEQUENCE

A sequence $X = (x_n)$ is said to be a ~~th~~ CAUCHY SEQUENCE if for each $\epsilon > 0$ there exists a positive integer $H = H(\epsilon)$ such that

$$|x_n - x_m| < \epsilon \quad \forall\ n, m \geq H$$

$$\forall\ n > m \geq H$$

Eg. $X = (1 + (-1)^n)$

Prove that this sequence is not a Cauchy sequence.

The negation of the definition of Cauchy sequence is that there exists $\epsilon_0 > 0$ such that for every natural number H there exists at least 1 $n > H$ and one $m > H$ such that

$$|x_n - x_m| \geq \epsilon_0$$

Let $\epsilon_0 = 1$

Then, given any natural number H we choose

$$m = 2H \quad \text{and} \quad n = 2H+1$$

then,
$$|x_n - x_m| = |(1 + (-1)^{2H+1}) - (1 + (-1)^{2H})|$$

$$= |-1-1| = 2 > 1$$

A constant sequence is always a CAUCHY SEQUENCE.

Eg. $\{1, 1, \ldots\}$

$H = 1$

THEOREM :

Cauchy convergence criteria / Cauchy's General Principle of Convergence

A sequence of real numbers is convergent iff it is a Cauchy sequence.

$(\Rightarrow)$ Condition is necessary

Let $X = (x_n)$ be a ~~converging~~ convergent sequence of real numbers converging to l (say)

Let $\epsilon > 0$ be any given number

$\Rightarrow$ ~~$X = (x_n) \to l$~~

~~$|x_n - l| < \epsilon \quad \forall \quad n \geq m$~~

$\therefore$ For every ϵ, $\exists$ a natural number H such that

$$|x_n - l| < \epsilon/2 \quad \forall \quad n \geq H$$

Then,

~~$|x_m - l| < \epsilon/2 \quad \forall \quad m \geq H$~~

$$|x_m - l| < \epsilon/2 \quad \forall \quad m \geq H$$

Now,

$$\cancel{|x_n - x_m|}$$

$$|x_n - x_m| = |x_n - l + l - x_m|$$
$$\leq |x_n - l| + |l - x_m|$$
$$= |x_n - l| + |x_m - l|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$
$$\forall \; n, m \geq H$$

$\therefore$ $X = (x_n)$ is a CAUCHY SEQUENCE

Q. Prove that $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

Let $\epsilon > 0$ be any given number

~~Let~~ $\cancel{\epsilon H}$

Let $H \left(> \frac{2}{\epsilon} \right)$ be any natural number (which exists by ARCHIMEDEAN PROPERTY).

Now, $\forall \; n, m \geq H$

$$|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{H} + \frac{1}{H} = \frac{2}{H}$$

$$< \epsilon$$

$*$ $\frac{1}{n} \leq \frac{1}{H} < \frac{\epsilon}{2}$

$$H > \frac{2}{\epsilon}$$

Q. Prove that the sequence (n^2) is ~~not~~ not a Cauchy sequence.

(m and n need not be consecutive)

Let $\epsilon = 1$

Let H be any natural number

If (n^2) is Cauchy, there exists H such that $|x_n - x_m| < \epsilon$ $\forall$ $n, m \geq H$

~~$\exists$ $\epsilon > 0$ such that $n, m \geq H$~~

Let $m = 2H$ and $n = 2H + 1$

$\therefore$ $|x_n - x_m| = |(2H+1)^2 - (2H)^2|$

$= |4H + 1| = 4H+1 > \epsilon = 1$

(and the minimum will be 5.)

$|x_n - x_m| > \epsilon$

$\therefore$ (n^2) is not a Cauchy sequence

(If we have to disprove, choose ϵ)

Q. Prove that the sequence

$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$

is not a Cauchy sequence.

Let if possible, the given sequence be a Cauchy sequence.

Then for $\epsilon = \frac{1}{2} > 0$, $\exists$ a natural number H such that

$|x_n - x_m| <$ $\forall$ $n, m \geq H$

~~$|x_n - x_m|$~~ Now for ~~Let~~ $n = 2H$, $m = H$

$|x_n - x_m| = |x_{2H} - x_H| =$

$|(1 + \frac{1}{2} + \cdots \frac{1}{2H})| -$

$$\left|\left(1+\frac{1}{2}+\cdots\cdots\frac{1}{2H}\right)-\left(1+\frac{1}{2}+\cdots\cdots\frac{1}{H}\right)\right|$$

$$= \frac{1}{H+1}+\frac{1}{H+2}+\cdots\cdots\frac{1}{2H}$$

$$> \frac{1}{H+H}+\frac{1}{H+H}+\cdots\cdots\frac{1}{2H} =$$

$$H\left(\frac{1}{2H}\right)=\frac{1}{2}$$

Q. Prove that $X=(x_n)$ where $x_n=(\log n)$ is not a cauchy sequence.

Let if possible, the given sequence be a cauchy sequence.

Then for $\epsilon=\log 2>0$, $\exists$ a natural number H such that

$$|x_n-x_m|<\epsilon=\log 2 \ \forall \ n, m \geq H$$

Now for $n=3H$, $m=H$

$$|x_n-x_m| = \cancel{\log 3n}$$

$$|x_{3H}-x_H| = |\log 3H-\log H|$$

$$= \left|\log\frac{3H}{H}\right|$$

$$= \log 3 > \epsilon = \log 2$$

Hence this contradicts ~~(i)~~ or (i) and $x_n=(\log n)$ is not a cauchy sequence

Q. Prove that $X = (x_n)$ where $x_n = \left(\frac{n+1}{n}\right)$

$$|x_n - x_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|1 + \frac{1}{n} - 1 - \frac{1}{m}\right|$$

$$\leq \frac{1}{n} + \frac{1}{m}$$

$$\leq \frac{1}{H} + \frac{1}{H} = \frac{2}{H}$$

$$< \epsilon$$

This is a Cauchy sequence

Q. ...... P.

THEOREM : Prove that every Cauchy sequence is bounded

Let $X = (x_n)$ be a Cauchy sequence. Then given any $\epsilon > 0$, $\exists$ a natural number such that $|x_n - x_m| < \epsilon \quad \forall \; n, m \geq H$

Now,

$$|x_n| = |x_n - x_H + x_H|$$

$$\leq |x_n - x_H| + |x_H|$$

$$< \epsilon + |x_H| \quad \forall \; n \geq H$$

Let $M = \max\{x_1, x_2, \ldots \ldots x_{n-1}, \epsilon + |x_n|\}$

Then $|x_n| \leq M \quad \forall \; n \in \mathbb{N}$

$\therefore$ (x_n) is bounded

REMARK : The converse of this theorem is not true.
$[1+(-1)^n]$ is bounded but not Cauchy.

Q. Let $X = (x_n)$

$x_1 = 1$

$x_2 = 2$

$x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n > 2$

Prove that this ~~tt~~ sequence is a Cauchy sequence and find its limit.

$$2x_n = x_{n-2} + x_{n-1}$$
$$2x_n - 2x_{n-1} = x_{n-2} - x_{n-1}$$

$$\Rightarrow \quad |x_n - x_{n-1}| = \frac{1}{2}|x_{n-2} - x_{n-1}|$$
$$= \frac{1}{2}|x_{n-1} - x_{n-2}|$$
$$= \frac{1}{2^2}|x_{n-2} - x_{n-3}|$$
$$\vdots$$
$$= \frac{1}{2^{n-2}}|x_2 - x_1| \quad \forall\, n > 2$$

Now if $n > m$

~~$|x_n$~~

$$|x_n - x_m| = |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots\ldots\ldots (x_{m+1} - x_m)|$$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$

$$= \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^{m-1}}\left[1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right]$$

$$= \frac{1}{2^{m-1}}\left[\frac{1\left(1 - \frac{1}{2^{n-m}}\right)}{1 - 1/2}\right]$$

$$= \frac{1}{2^{m-2}}\left(1 - \frac{1}{2^{n-m}}\right)$$

$$= \frac{1}{2^{m-2}} - \frac{1}{2^{n-2}} < \frac{1}{2^{m-2}} < \epsilon$$

iff $\frac{1}{\epsilon} < 2^{m-2}$

By Archimedean Property,
choose $m \in \mathbb{N}$ such that
$\frac{1}{\epsilon} < 2^{m-2}$

Thus, $\forall\, n \geq m$,
$|x_n - x_m| < \epsilon$
Hence $X = x_n$ is a Cauchy sequence

$$|x_n - x_{n-1}| = \tfrac{1}{2}|x_{n-1} - x_{n-2}|$$

$$x_n - x_{n-1} = \frac{1}{2}(x_{n-2} - x_{n-1})$$

$$x_{n-1} - x_{n-2} = \frac{1}{2}(x_{n-3} - x_{n-2})$$

$$x_{n-2} - x_{n-3} = \frac{1}{2}(x_{n-4} - x_{n-3})$$

$$x_5 - x_4 = \frac{1}{2}(x_3 - x_4)$$

$$x_4 - x_3 = \frac{1}{2}(x_2 - x_3)$$

$$x_3 - x_2 = \frac{1}{2}(x_1 - x_2)$$

$$x_1 = 1 \qquad x_2 = 2$$

$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$$

$$x_n - x_2 = \frac{1}{2}(x_1 - x_{n-1})$$

Taking limit on both sides,

$$\lim_{n\to\infty}(x_n - x_2) = \frac{1}{2}(x_1 - x_{n-1})$$

$$\cancel{l-2 = \frac{1}{2}(1-l)}$$

$$l-2 = \frac{1}{2}(1-l)$$

$$\frac{3l}{2} = \frac{5}{2} \qquad l = \frac{5}{3}$$

Q6.|86| 3.5 Let p be a given natural number. Give an example of a sequence $X = (x_n)$ that is not a Cauchy sequence but that satisfies $\lim_{n\to\infty} |x_{n+p} - x_n| = 0$

Consider the sequence (x_n) given by

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Justify this not a Cauchy sequence

$$0 < |x_{n+p} - x_n| = \frac{1}{n+1}$$

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+p}$$

$$< \frac{1}{n+1} + \frac{1}{n+1} + \cdots \frac{1}{n+1}$$

$$< \frac{p}{n+1} \to 0$$

as $n \to \infty$

Thus, $\lim_{n\to\infty} |x_{n+p} - x_n| = 0$

and the sequence is not a Cauchy sequence.

Q5. If $x_n = \sqrt{n}$, show that (x_n) satisfies ~~limit~~ $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.

$$|x_{n+1} - x_n| = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots \frac{1}{\sqrt{n+n}}$$

$$< 1/\sqrt{n} + \frac{1}{\sqrt{n}} + \cdots \frac{1}{\sqrt{n}}$$

~~$|x_{n+1} - x_n| < \varepsilon$~~ $\frac{n}{\sqrt{n}}$

$$\lim_{n\to\infty} |x_{n+1} - x_n| = \lim_{n\to\infty} |\sqrt{n+1} - \sqrt{n}|$$

$$= \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

Let if possible the given sequence be a Cauchy sequence such that $\varepsilon = 1$

Let $n = 4H$, $m = H$

then, $|x_n - x_m| < 1$

then,

$$|x_n - x_m| = |\sqrt{4H} - \sqrt{H}|$$
$$= |2\sqrt{H} - \sqrt{H}|$$
$$= \sqrt{H} \geq 1$$

$\sqrt{H} \geq 1$ as H is a natural number. which is a contradiction

$\Rightarrow$ The given sequence is not a Cauchy sequence.

SUBSEQUENCE

Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < n_3 \ldots\ldots\ldots < n_k < \ldots$ be a strictly increasing sequence of natural numbers then, the sequence

$$X' = (x_{n_k}) = (x_{n_1}, x_{n_2} \ldots\ldots x_{n_k} \ldots)$$

is ~~called~~ called a subsequence of $X = (x_n)$

eg. $X' = (x_1, x_2, x_3, x_4, x_5 \ldots)$

is a subsequence of

$X = (x_1, x_2, x_3, x_4, x_5, \ldots\ldots)$

$X'' = (x_2, x_3, x_4, \ldots)$

$X''' = (x_3, x_4, x_5, \ldots)$

$X'''' = (x_4, x_6, x_8, x_{10}, \ldots)$

$f : \mathbb{N} \to \mathbb{R}$

$g : \mathbb{N} \to \mathbb{N}$

$g(k) = n_k$

$f \circ g : \mathbb{N} \to \mathbb{R}$

$(f \circ g)(1) = f \circ g(1) = f(n_1)$

~~REMARK~~ REMARK : If X' is a subsequence of X'' and X'' is a subsequence of X' then it is possible that the 2 are not equal.

X' $(0, 1, 0, 1, 0, 1, \ldots)$

X'' $(0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$

THEOREM : ~~Is~~ If a sequence $X = (x_n)$ of real numbers converges to a real number ~~x~~ l then any subsequence $X' = (x_{n_k})$ of X also converges to l.

PROOF : As $X = (x_n) \to l$

$\Rightarrow$ given $\epsilon > 0$, $\exists$ a positive integer K such that

$|x_n - l| < \epsilon \quad \forall \; n \geq K$

$x_{n_1}, x_{n_2}, x_{n_3}, \ldots$

Since $n_1 < n_2 < n_3 < \ldots\ldots\ldots < n_k < \ldots\ldots$
is an increasing sequence of natural numbers
so it is obvious that $n_k \geq k$

$k \geq K$

$n_k \geq k \geq K$

Since n_k

Hence if $k \geq K$
then we also have
$n_k \geq k \geq K$

Hence we have
$|x_{n_k} - l| < \epsilon \quad \forall \quad n_k \geq K$.
$n_k \geq K$

Consequently the subsequence $X' = (x_{n_k})$ also converges to the same limit as the sequence $X = (x_n)$

* $\left(\frac{1}{n}\right) \to 0$

$|x_n - 0| < \epsilon = \frac{1}{10} \quad \forall \quad n \geq 11$

$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10} \ldots\right)$
$x_{n_1} \quad x_{n_2} \quad x_{n_3} \quad x_{n_4} \quad x_{n_5}$

$n_k \geq k \geq K$

MONOTONE SUBSEQUENCE THEOREM

If $X = (x_n)$ is a sequence of real numbers then there is a subsequence of X which is monotone.

To prove this theorem, we define the m^{th} term of the sequence is is a "peak" of the sequence $X = (x_n)$ if

$$x_m \geq x_n \quad \forall \quad n \geq m$$

Eg: $(1, 2, 3, \ldots)$

no peak

$(1, 1/2, 1/3, \ldots \ldots)$

All terms are peaks. Sequence is monotonically decreasing.

$(-1, 1, -1, 1, \ldots \ldots)$

Peak : 1

i.e x_m is never exceeded by any term that follows it in the sequence. Then we say that x_m is a DOMINANT TERM or a PEAK TERM.

Now, 2 cases arise depending on whether $X = (x_n)$ has finitely many or infinitely many peaks.

Not to be done

Case 1: suppose $X = (x_n)$ has infinitely many peaks

Let us list the peaks by increasing subscripts

$x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \dots\dots, x_{n_k}, \dots$

Since each term is a peak we have

$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq x_{n_4} \geq \dots\dots x_{n_k}, \dots$

$\therefore X' = (x_{n_k})$
is a monotonic decreasing subsequence of X.

Case 2: ~~Suppose~~ suppose $X = (x_n)$ has only a finite number of peaks (which includes the case of 0 peaks)

Let these peaks be listed by increasing subscripts

$x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \dots\dots x_{n_r}$

Let $s_1 = n_{r+1}$
be the first index beyond the last peak

Since x_{s_1} is not a peak, there exists
$s_2 > s_1$
such that $x_{s_1} < x_{s_2}$

Again, x_{s_2} is not a peak so there exists $s_3 > s_2$

such that

$x_{n_3} > x_{n_2}$

Continuing this way we obtain monotonic increasing subsequence

$x_{n_1} < x_{n_2} < x_{n_3} < x_{n_4} \dots$

BOLZANO WEIERSTRASS THEOREM

A bounded sequence of real numbers has a convergent subsequence.
(Every bounded sequence has a limit point)
Equivalently, it has some convergent subsequence

Let $X = (x_n)$ be a bounded sequence
Then, by monotone subsequence theorem, it has a monotone subsequence
say $X' = (x_{n_k})$
Since the sequence X is bounded, the subsequence X' is also bounded.

Thus X' is a bounded monotone subsequence and hence by MONOTONE CONVERGENCE theorem, it is convergent.

CAUCHY CONVERGENCE CRITERION (Converse)

Every Cauchy sequence is convergent.

Let $X = (x_n)$ be a Cauchy sequence.
Then it is bounded (Every Cauchy sequence bounded).

So, by the BOLZANO WEIERSTRASS THEOREM it ~~…~~ has a convergent ~~…~~ subsequence say

$X' = (x_{nk})$

and let it ~~conv~~ converges to l.

Now, we claim that the sequence $X = (x_{nk})$ $X = (x_n)$ also converges to l.

Let $\epsilon > 0$ be any given number. Then since ~~…~~ $X = (x_n)$ is a Cauchy sequence so for given $\epsilon > 0$, there exists a natural number such that

$|x_n - x_m| < \epsilon/2 \quad \forall \; n, m \geq H \quad (i)$

~~So~~ since the subsequence $X' = (x_{nk})$ converges to l, there exists a natural number $K \geq H$ belonging to the set $\{n_1, n_2, n_3, n_4, \ldots\}$ such that

$|x_k - l| < \epsilon/2$

Since $K \geq H$, it follows from (i) with $m = K$ that

$|x_n - x_k| < \epsilon/2 \quad \forall \; n \geq H$

Thus, $|x_n - l| = |x_n - x_k + x_k - l|$

for

Thus, $|x_n - l| = $ ~~Lim~~

$$|x_n - x_k + x_k - l|$$
$$\leq |x_n - x_k| + |x_k - l|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$
$$\forall \ n \geq H$$

Thus, ~~the limit~~ $X = (x_n)$ converges ~~conv~~ [illegible] to l.

DIVERGENCE CRITERIA

If a sequence $X = (x_n)$ of real numbers has either of the following properties then it is divergent :

(i) X has 2 convergent subsequences
$X' = (x_{n_k})$
and $X'' = (x_{n_k})$
whose limits are not equal

(ii) X is ~~unbou~~ ~~unbounded~~ unbounded

eg 1. $X = ((-1)^n)$
is divergent (not convergent)
(not diverging to ∞)

The subsequence $X' = ((-1)^{2n}) = $ ~~(1, 1, 1, 1, ...)~~
$(1, 1, 1, 1, \ldots)$
$X'' = ((-1)^{2n-1})$
$= (-1, -1, -1, -1, \ldots)$

$\therefore$ by the (i)st ~~statement~~ criterion ~~criteria~~ of the above divergence ~~criterion~~ criteria, the sequence is divergent.

Q. If $X = (x_n)$ and $Y = (y_n)$ are Cauchy sequences then prove that $X+Y$ is also a Cauchy sequence

Q. Prove that $X \cdot Y$ is a Cauchy sequence if $X = (x_n)$ and $Y = (y_n)$ are Cauchy sequences.

Q. Prove that the following sequence is a Cauchy sequence:

$$X = (x_n) = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \frac{1}{n+1!}$$

Eg 2. $\left(1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \dots\right)$

$$x_n = \begin{cases} n & \text{if } n = \text{odd} \\ 1/n & \text{if } n \text{ is even} \end{cases}$$

This sequence is unbounded hence it is not convergent.

Eg 3. $X = (\sin n)$ (Not to be done)

$$\sin \frac{\pi}{6} = \frac{1}{2} = \sin \frac{5\pi}{6}$$

and $\sin x > 1/2$ for $x \in I_1 = \left(\pi/6, \frac{5\pi}{6}\right)$

$$\frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$\approx \frac{2}{3} \times \frac{22}{7}$$

$$= \frac{44}{21}$$

Since the length of the interval is > 2, there are at least 2 natural numbers lying inside I_k.

We let n_1 be the first such natural number.

Similarly, similarly for each k let n_k, $k \in \mathbb{N}$

since $\sin x > 1/2$, $x \in I_k = \left(\frac{\pi}{6} + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1) \right)$

since the length of $I_k > 2$, there are at least 2 natural numbers inside I_k.

we let n_k be the ~~fir~~ first such number

Then, the subsequence

$$x' = (\sin n_k)$$

of X obtained in this way has the property ~~such~~ that all of its values lie in the interval $\left[\frac{1}{2}, 1\right]$

~~similarly~~ Similarly if $k \in \mathbb{N}$ and

$$J_k = \left(\frac{7\pi}{6} + 2\pi(k-1), \frac{11\pi}{6} + 2\pi(k-1) \right)$$

Then, $\sin x < -1/2$ $\forall$ $x \in J_k$

and ~~length~~ length of $J_k > 2$.

let m_k be the first natural number lying in

Thus, we obtain a subsequence

$X'' = (\sin n_k)$

of X having the property that all of its values lie in the interval $[-1, -\frac{1}{2}]$

Now, given any real ~~number~~ number c we observe that at least one of the subsequences X' and X'' lies entirely outside the ½ neighbourhood of c.

So, c cannot be a ~~limit~~ limit of X. c ∈ ℝ is arbitrary, we conclude that the sequence. $X = (\sin n)$ is divergent.

Q. Prove that ~~X = co~~ $X = (\cos n)$ is divergent.

THEOREM

Let $X = (a_n)$ be a ~~sequence of~~ sequence of real numbers. Then, the following are equivalent :

(i) The sequence $X = (a_n)$ does not converge to $l \in \mathbb{R}$

(iii) $\exists$ an $\epsilon_0 > 0$ such that for any $k \in \mathbb{N}$, $\exists$ $n_k \in \mathbb{N}$ such that

$$n_k \geq k \text{ and } |x_{n_k} - l| \geq \epsilon_0$$

(iv) $\exists$ an $\epsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that

$$|x_{n_k} - l| \geq \epsilon_0 \quad \forall \; k \in \mathbb{N}$$

(i) $\Rightarrow$ (ii)

If the sequence $X = (x_n)$ does not converge to l then for some $\epsilon_0 > 0$, it is impossible to find a natural number k such that

$$|x_n - l| < \epsilon_0 \quad \forall \; n \geq k$$

i.e.

for each $k \in \mathbb{N}$, it is not so $\exists$ a natural number $n_k \geq k$ such that

$$|x_{n_k} - l| \geq \epsilon_0$$

Hence, (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (iii)

Let $\epsilon_0 > 0$ be chosen as in part (ii) and let $n_1 \geq 1$ such that

$$|x_{n_1} - l| \geq \epsilon_0$$

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then let $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $|x_{n_2} - l| \geq \epsilon_0$

Further let $n_3 \in \mathbb{N}$ such that $n_3 > n_2$ and $|x_{n_3} - l| \geq \epsilon_0$