II **Limit Superior** : Let $\langle a_n \rangle$ be a real sequence (not necessarily bounded) then limit superior is defined as

$$\overline{\operatorname{Lim}}\, a_n = \inf_n \operatorname{Sup}\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

or limit superior $= \lim_{n\to\infty} \operatorname{Sup} a_n = \overline{\operatorname{Lim}}\, a_n = \inf_A \operatorname{Sup}\{A\}$

I **Limit Inferior** Limit Inferior is defined as

$$\underline{\operatorname{Lim}}\, a_n = \operatorname{Sup}_n \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

or limit inferior $= \lim_{n\to\infty} \inf a_n = \underline{\operatorname{Lim}}\, a_n$

(13)

<u>Example</u>: Let $\{a_n\}$ be a sequence $a_n = (-1)^n \; \forall \; n \geq 1$

find $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$ $\quad | = \langle -1, 1, -1, 1, -1, \ldots \rangle$

<u>Sol</u>: for limit Inferior $\underline{\lim}\, a_n = \sup_n \inf \{a_n, a_{n+1}, a_{n+2}, \ldots\}$

Inferior

Put $n=1 \longrightarrow A_1 = \inf \{a_1, a_2, a_3, \ldots\}$

Put $n=2 \longrightarrow \cancel{A_{2}} = \inf \{-1, 1, -1, 1, -1, 1, \ldots\}$

$= -1$

<u>Put n=2</u> $\quad A_2 = \inf \{a_2, a_3, a_4, \ldots\}$

$= \inf \{1, -1, 1, -1, 1, \ldots\}$

$= -1$

<u>Put n=3</u> $\quad A_3 = \inf \{a_3, a_4, a_5, \ldots\}$

$= \inf \{-1, 1, -1, +1, -1, \ldots\}$

$= -1$

$\ldots$

<u>Now</u> $\underline{\lim}\, a_n = \sup \{A_1, A_2, A_3, \ldots\}$

$= \sup \{-1, -1, -1, \ldots\}$

$= -1$.

for limit Superior $\overline{\lim}\, a_n = \inf_n \sup \{a_n, a_{n+1}, a_{n+2}, \ldots\}$

$B_1 = \sup \{a_1, a_2, a_3, \ldots\}$

$= \sup \{-1, 1, -1, 1, -1, \ldots\} = 1$

$B_2 = \sup \{a_2, a_3, a_4, \ldots\}$

$= \sup \{1, -1, 1, -1, \ldots\} = 1$

$B_3 = \sup \{a_3, a_4, a_5, \ldots\}$

$= \sup \{-1, 1, -1, 1, \ldots\} = 1$

Now. $\overline{\lim}\, a_n = \inf \{B_1, B_2, B_3, \ldots\}$

$= \inf \{1, 1, 1, 1, \ldots\}$

$= 1$