ILFP - 22

Recap : FOL Syntax : Substitution, Proof rules

FOL Semantics : Model, Validity, Satisfaction, Logical Consequence, Normal forms

Today :

- Eg: Formal proof using proof rules / derivation rules
- More on Normal forms
- Skolemization
- Unification

FOL CNF

Transformation Steps

1. Rename var. for each quantifier

Thm: if $x, y \notin FV(H(z))$
then $\forall x . H(x) \equiv \forall y . H(y)$

Defn: Every quantifier should be using a different variable

Eg: $\neg(\exists x . \forall y . R(x,y)) \longrightarrow \forall y . \exists x . (R(x,y) \wedge$

$\neg(\exists x . \forall y \ldots \longrightarrow \forall \omega . \exists z \, P(x)))$
$\cdot(R(z,\omega) \ldots) \ldots)$

2. Negation Normal form

Thm: If we have $\Gamma \vdash \neg \exists x . \neg (F(x))$,

we can show

$$\Gamma \vdash \forall x . F(x)$$

Alternatively

if $\Gamma \vdash \neg \forall x . F(x)$, we can show

$\Gamma \vdash \exists x . \neg F(x)$

Defn: $\neg$ is pushed inwards and attaches to only atomic formulas

3. Prenex Form

<u>Thm1</u>: if $x \notin FV(H)$

Then

$$H \equiv \exists x . H \equiv \forall x . H$$

<u>Thm2</u>: if $x \notin FV(G)$, then

$$\exists x . F(x) \wedge G \equiv \exists x . (F(x) \wedge G)$$

<u>Defn</u>: If all quantifiers exist as a prefix

<u>Eg</u>: $(\exists x . \forall y . R(x,y) \wedge \exists z . \forall w . (\neg R(w,z) \vee \neg P(w)))$

4. Skolemization

→ removes $\exists$ quantifiers from prenex form & only $\forall$ quantifiers are left

$(\forall o. \exists a.\ R(o, a))$

Eg: for every orange there is an apple

- to satisfy the above sentence we need to find $f : \text{Orange} \longrightarrow \text{Apple}$

$\forall o.\ R(o, f(m)))$

Skolem function

- Skolemization with F.V.

- Let H be a formula in signature $S = (F, R)$

  $FV(H) = \{x, y_1, \ldots, y_n\}$

  and $f/n \in F$ doesn't occur in H.

  For each model m', there is a model m s.t.

  $$m \models \exists x . H \Rightarrow H(f(y_1, \ldots, y_n))$$

  and m, m' differ only in the interpretation of f.

  Similary it can be shown

  $$\exists m \text{ s.t. } m \models \underset{[x \rightarrow \text{"}]}{H(f(y_1, \ldots, y_n))} \Rightarrow \forall x . H$$

- Skolemization with Quantifiers

  - Let $H(x)$ be a $\mathcal{S}$ formula with $FV = \{x, y_1, \ldots, y_n\}$ and $f/n \in F$ doesn't occur in $H(x)$. Then

    $\forall y_1, \ldots, y_n . \exists x . H(x)$ is SAT

    iff

    $\forall y_1, \ldots, y_n . H(f(y_1, \ldots, y_n))$

- Skolemization of Prenex sentence

→ Apply Skolemization from outside to inside, i.e. remove the outermost $\exists$

Eg: $\exists z. \forall w. \exists x. \forall y.$
$(R(x,y) \wedge (\neg R(w,z) \vee \neg P(w)))$

5. Now convert the body of formula already in that is, skolemized prenex form

$$\forall x_i \circ H$$

$\therefore$ H is quantifier-free, use rule def^n in prop. logic to convert H into CNF form.

<u>Computing CNF</u>

- Rewrite cond^n & bicond^n using equivalences
- Convert to NNF by pushing negation inward
- Convert to CNF by distributing $\vee$ over $\wedge$

Exercise: Give an example to show Skolemization doesn't produce an equivalent formula!

* Exercise (Challenge): The order of quantifiers determines the # of parameters in skolem functions. Give a greedy and efficient strategy for producing prenex formula s.t. the total # of parameters in skolem functions is minimal.

Unification

(with terms and making them equal)

Consider a proof

$$\{\} \vdash (\forall x.(P(x) \vee Q(x)) \Rightarrow \exists x. P(x) \vee \forall x. Q(x))$$

Let us try to show it through FOL CNF