ILFP-21

Recap

- Soundness, Completeness & Decidability of Propositional Logic

- FOL - Syntax $\quad \mathcal{S} = (F, R)$

  1. Terms $\quad t ::= x \in V \mid f(t_1, \ldots, t_n)$
     $\quad f/_n \in F$

  2. Atoms $\quad a ::= P(t_1, \ldots t_n) \mid t = t \mid \bot \mid \top$
     $\quad P/_n \in R$

  3. Formulas
     $$f ::= a \mid \neg f \mid f \underbrace{\circ}_{\text{Propositional Connective}} f \mid \forall x . f \mid \exists x . f$$

Semantics

For signature $S = (F, R)$, an S-model m is a tuple

$$\left(D_m \, , \, \{ f_m : D_m^n \longrightarrow D_m \mid f/_n \in F \}, \, \{ P_m \subseteq D_m^n \mid P/_n \in R \} \right)$$

where

- D_m is the domain of m
- f_m assigns meaning to f under model m

Eg:

$F = \{ i/_0 \}$, $R = \{ P/_1 , Q/_2 \}$

A model m contains concrete elements

D_m – which may be a set of states of a computer program

- Interpretations i_m, P_m, Q_m may be designated as • initial state, • set of final states • state-transition relation

- $D_m = \{a, b, c\}$

$i_m = \{a\}$, $P_m = \{b, c\}$

$Q_m = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$

- Let us check formulas

$$\forall x \exists y : Q(x, y)$$

- States that the model is free from deadlocks, i.e. all states have transitions to some state

Eval: The above pred. is true in our model. $a \to a, b, c$

$b, c \to c$

Eg: $F = \{ e/_0 , \cdot/_2 \}$

$R = \{ \leq /_2 \}$

In infix notation we can write

$(t_1 \cdot t_2) \leq (t \cdot t)$

- Model m : D_m = set of all finite binary strings over the alphabet $\{0,1\}$

e_m = empty word ϵ

$\cdot_m$ = concatenation of words

$\leq_m$ = prefix ordering

Prefix ordering

we say that s_1 is prefix of s_2 if there is s_3 s.t. $s_1 \cdot_m s_3 = s_2$

- Check informally

$$\forall x \exists y \ (y \leq x)$$

i.e. every word has a prefix, which is the case.

Clearly

<u>For variables</u>

A look-up table or environment is defined

$$l : V \longrightarrow D_m$$

we denote $l[x \mapsto a]$ which maps x to a

and any other y to $l(y)$

Satisfaction Relation ($\models$)

$m \models_l \phi$ by structural induction on ϕ

- If $\phi = P(t_1, \ldots, t_n)$, then
  interpret each t_i by look-up into l,
  interpret each $f \in F$ by f_m

Now $m \models_l P(t_1, \ldots, t_n)$ holds iff

$(a_1, \ldots a_n)$ is in the set P_m

where $l[t_i \longmapsto a_i]$

$\forall x$: $m \models_\ell \forall x\, \phi$ holds iff

$m \models_{\ell[x \mapsto a]} \phi$ holds for all

$a \in D_m$

$\exists x$: $m \models_\ell \exists x .\, \phi$ holds iff

$m \models_{\ell[x \mapsto a]} \phi$ holds for some

$a \in D_m$

$\circ$: $m \models_\ell \varphi_1 \circ \varphi_2$ holds iff

$\hookrightarrow \{\vee, \wedge, \rightarrow\}$ $(m \models_\ell \varphi_1) \circ (m \models_\ell \varphi_2)$ holds

$\neg$: $m \models_\ell \neg \varphi$ holds iff

its not the case that

$m \models_\ell \varphi$ holds

we denote $m \not\models_\ell \varphi$ when

$m \models_\ell \varphi$ doesn't hold.

Logical Consequence

Γ = (potentially infinite) set of formulas ϕ_i

s.t.: $\Gamma \models \varphi$ holds iff for all models m and look-up tables l

whenever $m \models_l \phi_i$ holds

$\forall \phi_i \in \Gamma$

then

$m \models_l \varphi$ holds as well.

Satisfiability

- φ is SAT iff $\exists m$ and $\exists l$

  s.t. $m \models_l \varphi$

Validity

φ is valid iff $m \models_l \varphi$, $\forall m$, $\forall l$

in which we can check φ

Eg: SAT

Consider $\mathcal{S} = (F, R)$

$F = \{ s/_1, +/_2 \}, \quad R = \{\}$

Consider $\phi = \exists z.\ s(x) + y = s(z)$

Consider a model

$m = (\mathbb{N}, \{succ, +_{\mathbb{N}}\})$

$l = \{x \mapsto 3,\ y \mapsto 2\}$

Under m and l, $s(x) + y$ evaluates to 6

$\exists z.\ z = 5$ so to

$$m_{l[z \mapsto 5]}(s(z)) = 6$$

Therefore,

$$m \models_{l[z \mapsto 5]} s(x) + y = s(z).$$

$$m \models_l \exists z\,(s(x) + y = s(z))$$

Substitutions

$\sigma : V \longrightarrow T_S$, we will write

$t\sigma$ to denote $\sigma(t)$

Substitution for

- Terms
  - $c\sigma \triangleq c$
  - $f(t_1, \ldots, t_n)\sigma \triangleq f(t_1\sigma, \ldots, t_n\sigma)$

    Eg: $\sigma = \{x \mapsto f(x,y),\ y \mapsto f(y,x)\}$

    $f(x,y)\sigma = f(f(x,y), f(y,x))$

    $x\sigma = f(x,y)$

Substitution for

- Atoms

$$\top\sigma \triangleq \top$$

$$\bot\sigma \triangleq \bot$$

$$P(t_1, \dots, t_n) \triangleq P(t_1\sigma, \dots, t_n\sigma)$$

$$(t_1 = t_2)\sigma \triangleq (t_1\sigma) = (t_2\sigma)$$

- Substitution for formulas

  - $(\neg H)\sigma \triangleq \neg (H\sigma)$

  - $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$

  - $(\forall x . G)\sigma \triangleq \forall x\, (G\underbrace{\sigma_x}_{})$; $\forall y \in FV(G)$ s.t. $y \neq x$ then x doesn't occur in $y\sigma$

    Remove the mapping of x from σ

  - $(\exists x . G)\sigma \triangleq \exists x\, (G\sigma_x)$; "

- One can compose substitutions

  σ_1 & σ_2

- Substitution is associative

  ie $\sigma_1(\sigma_2\sigma_3) = (\sigma_1\sigma_2)\sigma_3$

We shall not discuss the syntactic proof rules for FOL;

I am listing them here for your study

In addition to Prop. Logic proof rules we have additional proof rules

Prop. Logic rules

$\wedge$-intro $\dfrac{\Gamma \vdash F, \ \Gamma \vdash G}{\Gamma \vdash F \wedge G}$, $\wedge$-Elim $\dfrac{\Gamma \vdash F \wedge G}{\Gamma \vdash F}$

$\wedge$-Sym $\dfrac{\Gamma \vdash F \wedge G}{\Gamma \vdash G \wedge F}$

$\vee$- Intro $\dfrac{\Gamma \vdash F}{\Gamma \vdash F \vee G}$, $\vee$- Sym $\dfrac{\Gamma \vdash F \vee G}{\Gamma \vdash G \vee F}$

$\neg\neg$- Elim $\dfrac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}$, $\neg$-Intro $\dfrac{\Gamma \vdash \phi}{\Gamma \vdash \neg\neg\phi}$

$\vee$- Elim $\dfrac{\Gamma \vdash F \vee G \, , \quad \Gamma \cup \{F\} \vdash H \, , \quad \Gamma \cup \{G\} \vdash H}{\Gamma \vdash H}$

$\Rightarrow$-intro $\dfrac{\Gamma \cup \{F\} \vdash G}{\Gamma \vdash F \Rightarrow G}$, $\Rightarrow$-Elim $\dfrac{\Gamma \vdash F \Rightarrow G, \ \Gamma \vdash F}{\Gamma \vdash G}$

$\Rightarrow$-Eq $\dfrac{\Gamma \vdash F \Rightarrow G}{\Gamma \vdash \neg F \vee G}$

Rules for quantifiers

$\exists$-intro .
$$\frac{\Gamma \vdash F(t)}{\Gamma \vdash \exists y . F(y)} \quad y \notin FV(F(z))$$

$\forall$-intro .
$$\frac{\Gamma \vdash F(x)}{\Gamma \vdash \forall y . F(y)} \quad . \; y \in FV(F(z)) \text{ and } x \notin FV(\Gamma \cup \{F(z)\})$$

$\forall$-elim .
$$\frac{\Gamma \vdash \forall x . F(x)}{\Gamma \vdash F(t)}$$

$\forall$ implies $\exists$

i.e. if $\Gamma \vdash \forall x . F(x)$, we can derive $\Gamma \vdash \exists x . F(x)$

$\exists$-Elim .
$$\frac{\Gamma \vdash F(x) \Rightarrow G}{\Gamma \vdash \exists y . F(y) \Rightarrow G} \quad x \notin FV(\Gamma \cup \{G, F(z)\}), \; y \notin FV(F(z))$$

Normal forms

- We can convert FOL sentences into CNF

- Transformation rules

    - Rename : rename vars for each quantifier

    - generate NNF

    - Prenex form : pull quantifiers to front

    - Skolemization : Remove existential quantifiers

    - Turn the quantifier free part of the sentence into CNF

- Remove universal quantifiers : a CNF with free variables

- **Rename**

$$\forall x . F(x) \equiv \forall y . F(y)$$

so long as $x, y \notin FV(F(z))$

Defn: a formula F can be renamed apart if no quantifier in F use a variable that is used by another quantifier or occurs as free variable in F

Eg: $\exists x . \forall y \; R(x,y) \Rightarrow \forall y . \exists x \; (R(x,y) \wedge P(x))$

$\equiv$

$\exists x . \forall y \; R(x,y) \Rightarrow \forall w . \exists z . \; (R(z,w) \wedge P(z))$

<u>Prenex</u>

assume $x \notin FV(F)$

Then, F, $\exists x . F$, $\forall x . F$ are provably Equivalent.

Def^n: Prenex form where all quantifiers of a formula occur as a prefix

$$\exists x \cdot F(x) \wedge \exists x \cdot G$$

(where $x \notin FV(G)$

$$\equiv \exists x \cdot (F(x) \wedge G)$$

Skolemization

- removes $\exists$ from prenex sentences

Eg:

for every orange there is an apple

$$\forall o, \exists a . \; R(o, a)$$

$$\boxed{f : O \longrightarrow A} \longrightarrow \text{Skolem function}$$

$$\forall o . \; R(o, f(o))$$

- Let H be a formula in signature $\mathcal{S} = (F, R)$

  $FV(H) = \{x, y_1, \ldots, y_n\}$

  and $f/_n \in F$ doesn't occur in H.

  For each model m', there is a model m s.t.

  $$m \models \exists x . H \Rightarrow H(f(y_1, \ldots, y_n))$$

  and m, m' differ only in the interpretation of f.

- Let $H(x)$ be a S formula with $FV = \{x, y_1, \ldots, y_n\}$ and $f/_n \in F$ doesn't occur in $H(x)$. Then

$$\forall y_1, \ldots, y_n . \exists x . H(x) \text{ is SAT}$$

iff

$$\forall y_1, \ldots, y_n . H(f(y_1, \ldots, y_n))$$

- Remove outermost $\exists$ first

  ↳ Skolemization is applied from outside to inside

Eg: $\exists z.\ \forall w.\ \exists x.\ \forall y.\ (R(x,y) \wedge (\neg R(w,z) \vee \neg P(w)))$

introduce c/o (why?)

$\longrightarrow$

$\forall w.\ \exists x.\ \forall y$