COL 765 : Lec 20

Last time

Propositional logic
- 1. Tautology checking, Normal forms
- 2. Resolution and UNSAT
- 3. Soundness & Completeness

2. Resolution :

$$\frac{\Sigma \vdash F \vee G \qquad \Sigma \vdash \neg F \vee H}{\Sigma \vdash G \vee H}$$

3. Soundness & Completeness :

$$\Sigma \vdash F \quad \text{iff} \quad \Sigma \models F$$

Soundness

Elim $\dfrac{\Sigma \vdash F \wedge G}{\Sigma \vdash F}$

Consider $m \models \Sigma$

By I.H. $m \models F \wedge G$

Using the Truth table we can show $m \models F \wedge G$
$\Rightarrow m \models F$ $\therefore \Sigma \models F$

Completeness (daunting)

if Σ is UNSAT then $\Sigma \vdash \bot$ via resolution?

Proof: induction over the # of vars in Σ

Decidability:

if Σ is a finite, then $\Sigma \models F$ is decidable
(set of formulas)

Semidecidability when Σ is infinite

Compactness Thm

Consider Γ (possibly infinite)
then Γ is SAT $\boxed{\text{iff}}$ Every finite subset of Γ is SAT.

Observations

$\Gamma \vdash \varphi \rightarrow \Gamma \models \varphi \rightarrow \{\} \in \text{Resolve}(\Gamma') \rightarrow \Gamma \vdash \varphi$

where $\Gamma' = \bigwedge \Gamma \wedge \neg \varphi$

Implication of Compactness Thm on Decidability

- if Γ is finite then $\Gamma \models \varphi$ is decidable
- if Γ is effectively enumerable then $\Gamma \models \varphi$ is semidecidable

if one can enumerate a set using an algorithm

A yes/no decision problem is semi-decidable, if we have an algorithm for only one side of the problem

Proof-Rules in Prop. Logic

1. Assumption
$$\frac{}{\Gamma \vdash F}\ F \in \Gamma$$

2. Monotonic
$$\frac{\Gamma \vdash F}{\Gamma' \vdash \Gamma}\ (\Gamma \subseteq \Gamma')$$

3. DNeg-intro
$$\frac{\Gamma \vdash F}{\Gamma \vdash \neg\neg F}$$

4. $\wedge$-intro
$$\frac{\Gamma \vdash F \quad , \quad \Gamma \vdash G}{\Gamma \vdash F \wedge G}$$

$\wedge$-Elim
$$\frac{\Gamma \vdash F \wedge G}{\Gamma \vdash F}$$

$\wedge$-Sym
$$\frac{\Gamma \vdash F \wedge G}{\Gamma \vdash G \wedge F}$$

Eg:

$$\{p \vee q, r\} \vdash r$$
$$\Big\downarrow \text{DNeg-Intro}$$
$$\{p \vee q, r\} \vdash \neg\neg r$$

So $\vee$-intro
$$\frac{\Gamma \vdash F}{\Gamma \vdash F \vee G}$$

$\vee$-Sym
$$\frac{\Gamma \vdash F \vee G}{\Gamma \vdash G \vee F}$$

$\vee$-Def
$$\frac{\Gamma \vdash F \vee G}{\Gamma \vdash \neg(\neg F \wedge \neg G)}$$

(and the other way too)
$$\frac{\Gamma \vdash \neg(\neg F \wedge \neg G)}{\Gamma \vdash F \vee G}$$

$\vee$-Elim
$$\frac{\Gamma \vdash F \vee G, \quad \Gamma \cup \{F\} \vdash H, \quad \Gamma \cup \{G\} \vdash H}{\Gamma \vdash H}$$

6. $\to$-Intro $\quad \dfrac{\Gamma \cup \{F\} \vdash G}{\Gamma \vdash F \to G}$

$\to$-Elim $\quad \dfrac{\Gamma \vdash F \to G, \quad \Gamma \vdash F}{\Gamma \vdash G}$

$\to$-Def $\quad \dfrac{\Gamma \vdash F \to G}{\Gamma \vdash \neg F \vee G}$

$\to$-Def $\quad \dfrac{\Gamma \vdash \neg F \vee G}{\Gamma \vdash F \to G}$

Eg: Let us prove

$$\{\neg p \vee q, p\} \vdash q$$

1. $\{\neg p \vee q, p\} \vdash p$ [Assumption]

2. $\{\neg p \vee q, p\} \vdash \neg p \vee q$ [Assumption]

3. $\{\neg p \vee q, p\} \vdash p \rightarrow q$ [$\rightarrow$ Def applied to 2]

4. $\{\neg p \vee q, p\} \vdash q$ [$\rightarrow$-Elim applied to 1 and 3]

Intro to Predicate Logic

- There are many arguments which cannot be proven in propositional logic

Eg: All humans are mortal
Socrates is a human
Therefore, Socrates is mortal.

- Requires the notion of quantifiers
- Validity depends on the semantics of "ALL" and certain properties of objects such as "mortal", and the description of information such as membership or other relations, e.g. "is a"

- There is internal structure to sentences that could be parameterized

Eg: All philosophers are mortal
Nagarjuna is a philosopher
Therefore, Nagarjuna is mortal

parameterization - 1

All philosophers are y
x is a philosopher
Therefore, x is y

Parameterization - 2

All z are y
x is a z
Therefore, x is a y

- Parameterized propositions are called predicates
  - → propositions in propositional logic are 0-ary predicates.
- Predicate logic symbols: $\forall$, $\exists$, F, A, V, (,)

Countably infinite set of function symbols

Countably infinite collection of atomic predicate symbols

Countably infinite collection of variable symbols

- FOL = propositional logic + quantifiers over individuals + functions/predicates

- $f \in F$ of arity 0 is called a constant
- $p \in A$ of arity 0 is called a propositional var.

Building FOL

- terms
- atoms
- formulas

- precedence, associativity
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