COL765
Quiz 4 solutions

Problem 1:

Definition of *multc*

We have

$$x * y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ y + ((x-1) * y), & \text{if } x > 0 \\ -y + ((x+1) * y), & \text{if } x < 0 \end{cases}$$

We also have

$$-x = \begin{cases} 0, & \text{if } x = 0 \\ -(x-1) + (-1), & \text{if } x > 0 \\ -(x+1) + 1, & \text{if } x < 0 \end{cases}$$

Using *plusc*, we define *negc* as:

$$\begin{aligned} (negc\ x) =_\beta\ & \texttt{ITE (IZ } x\texttt{)}, & 0, \\ & \texttt{ITE (GTZ } x\texttt{)}, & (plusc\ (negc\ \texttt{(P } x\texttt{)})\ \texttt{(P Z)}), \\ & & (plusc\ (negc\ \texttt{(S } x\texttt{)})\ \texttt{(S Z)}) \end{aligned}$$

multc can now be defined as:

$$\begin{aligned} (multc\ x\ y) =_\beta\ & \texttt{ITE (OR (IZ } x\texttt{) (IZ } y\texttt{))}, & 0, \\ & \texttt{ITE (GTZ } x\texttt{)}, & (plusc\ y\ (multc\ \texttt{(P } x\texttt{)}\ y)), \\ & & (plusc\ (negc\ y)\ (multc\ \texttt{(S } x\texttt{)}\ y)) \end{aligned}$$

Computation of *multc* (P (P Z)) (S (S Z))

$$multc\ \texttt{(P (P Z)) (S (S Z))} =_{\beta^*} (plusc\ (negc\ \texttt{(S (S Z))})\ (multc\ \texttt{(S (P (P Z))) (S (S Z))})) \tag{1}$$

Reducing *negc* first,

$$\begin{aligned} negc\ \texttt{(S (S Z))} &=_{\beta^*} (plusc\ (negc\ \texttt{(P (S (S Z)))})\ \texttt{(P Z)}) \\ &=_{\beta^*} (plusc\ (negc\ \texttt{(S Z)})\ \texttt{(P Z)}) \\ &=_{\beta^*} (plusc\ (plusc\ (negc\ \texttt{(P (S Z))})\ \texttt{(P Z)})\ \texttt{(P Z)}) \\ &=_{\beta^*} (plusc\ (plusc\ (negc\ \texttt{Z})\ \texttt{(P Z)})\ \texttt{(P Z)}) \\ &=_{\beta^*} (plusc\ (plusc\ \texttt{Z}\ \texttt{(P Z)})\ \texttt{(P Z)}) \\ &=_{\beta^*} (plusc\ \texttt{(P Z) (P Z)}) \\ &=_{\beta^*} (plusc\ \texttt{(S (P Z)) (P (P Z))}) \\ &=_{\beta^*} (plusc\ \texttt{Z (P (P Z))}) \\ negc\ \texttt{(S (S Z))} &=_{\beta^*} \texttt{(P (P Z))} \end{aligned}$$

Using in eq. (1)

$$
\begin{aligned}
\mathit{multc}\ \texttt{(P (P Z)) (S (S Z))} &=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{multc}\ \texttt{(S (P (P Z))) (S (S Z))}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{multc}\ \texttt{(P Z) (S (S Z))}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ (\mathit{negc}\ \texttt{(S (S Z))})\ (\mathit{multc}\ \texttt{(S (P Z)) (S (S Z))})))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{multc}\ \texttt{(S (P Z)) (S (S Z))})))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{multc}\ \texttt{Z (S (S Z))})))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{(P (P Z)) Z}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{(S (P (P Z))) (P Z)}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{(P Z) (P Z)}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{(S (P Z)) (P (P Z))}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z))}\ (\mathit{plusc}\ \texttt{Z (P (P Z))}))\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P (P Z)) (P (P Z))})\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(S (P (P Z))) (P (P (P Z)))})\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(P Z) (P (P (P Z)))})\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{(S (P Z)) (P (P (P (P Z))))})\\
&=_{\beta^*} (\mathit{plusc}\ \texttt{Z (P (P (P (P Z))))})\\
\mathit{multc}\ \texttt{(P (P Z)) (S (S Z))} &=_{\beta^*} \texttt{(P (P (P (P Z))))}
\end{aligned}
$$