ILFP - 20

Last Time

- Prop. Logic
  - → Normal forms, Validity checks, Unsat checks
  - → Resolution proofs
  - → Soundness & Completeness of the Prop. Logic deductive system

Resolution

$$\Gamma \models \phi$$

$$\equiv \neg\left(\bigwedge \Gamma \wedge \neg \phi\right) \to \text{Contradiction}$$

1. In CNF form $\to \Delta$

2. Cleaning on Δ
   - $C \subseteq C' \to$ remove C' from Δ
   - $\exists C, (p, \neg p) \in C$, remove C from Δ

3. Δ

$$\frac{\Delta \vdash F \vee G, \quad \Delta \vdash \neg F \vee H}{\Delta \vdash G \vee H}$$

Prop. Logic — Soundness & Completeness

Thms

$\underbrace{\Gamma}\ \underbrace{\vdash}\ \underbrace{\phi}$ iff $\boxed{\Gamma \models \phi}$

Syntactically deduce

semantic Entailment (Logical consequence)

$\Longrightarrow$ Logical implication

$\longrightarrow$ syntactic imp

$\underbrace{\models \left(\bigwedge \Gamma \rightarrow \phi\right)}$

Valid/tautology

Soundness

$$\underbrace{\Gamma \vdash \phi}_{} \Rightarrow \underbrace{\Gamma \models \phi}_{}$$

Proof: We will use all the syntactic identities, transformations & equivalences

Elim:

$$\frac{\Gamma \vdash \phi \wedge \varphi}{\Gamma \vdash \phi, \ \Gamma \vdash \varphi}$$

Consider a model m s.t.

$\underline{m} \models \underline{\Gamma}$ by I.H. $[\phi, \varphi \in \Gamma]$

$m \models \phi \wedge \varphi$

Use the truth table to show that $(\Rightarrow)$

if $m \models \phi \wedge \varphi$ then $m \models \phi$

$\boxed{\phi}$	φ	$m(\phi \wedge \varphi)$
0	0	0
0	1	0
1	0	0
$\boxed{1}$	1	1

$$m \models \phi \wedge \varphi$$

<u>Completeness</u> if $\underbrace{\Gamma \models \phi}$ then $\Gamma \vdash \phi$

Contrapositive

Thm for Completeness

Assumption Γ is a finite set of formulas

$\boxed{\Gamma \text{ is } \underbrace{\text{UNSAT}}}\ , \quad \underbrace{\bot \in \text{Resolvent}(\Gamma)}_{\text{Conclusion}}$

(logical Contradiction)

$\nexists m \models \Gamma$

Use Induction over the number of variables in Γ

Base Case: $\boxed{p \text{ is only variable}}$ in Γ

↳ Assume Γ is unsat

↳ $\{p, \neg p\} \subseteq \Gamma$

↳ $\dfrac{\Gamma \vdash p, \quad \Gamma \vdash \neg p}{\Gamma \vdash \bot}$

$\{\}$

$k < n$, IoHo holds

IoS $\boxed{k \leq n}$ $\longrightarrow$ non-strict inequality

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Observations

- Γ, ϕ $\longrightarrow$ Γ is finite

$\longrightarrow$ $\Gamma \vdash \phi$ $\equiv$

$\underline{\{\}} \in \text{Resolve}(\Gamma')$ where

Γ' is CNF of $(\bigwedge \Gamma \wedge \neg \phi)$

$\equiv$

$\Gamma \models \phi$

Go from finite to infinite

$\underbrace{\Gamma}_{\text{infinite set}} \models \phi$

$\Gamma \vdash \phi$

Because Prop L. is sound

$\Gamma \models \phi$

$\{\} \in \mathrm{Resolve}(\Gamma')$

$(\bigwedge \Gamma \wedge \neg \phi)$ is a contradiction

$(\bigwedge \Gamma \rightarrow \phi)$ is valid

is infinite $\leftarrow$ $\boxed{\Gamma} \models \phi$

Thm (Compactness)

Γ is satisfiable iff every $\boxed{\text{finite}}$ subset of Γ is satisfiable!

↳ potentially infinite

denumerable ⟹ can create an algorithm that compute the above fact!

Implication of this thm

Decidability

if Γ is a finite set then $\boxed{\Gamma \models \phi}$ is decidable

if Γ is $\boxed{\text{Effectively Enumerable}}$ then $\Gamma \models \phi$ is $\boxed{\text{semi-decidable}}$

Proof: Comes from compactness result

finite subset ← $\Gamma' \subseteq \Gamma$ such that $\boxed{\Gamma' \models \phi}$

Intro to predicate logic

- Arg. can't be proven in Prop Logic

Eg: All humans are mortal
Socrates is a human → relationship
therefore, Socrates is mortal

X

- Needs quantifiers
- Validity depends on the semantics of "All", prop. of objects such as "mortals"
- Describe information such as membership, relations

Eg:

All philosophers are mortal

Nagarjuna is a philosopher

Therefore, Nagarjuna is a mortal

Parameterisation - 1

All ~~philosophers~~ z are ~~y~~ y

x is a ~~philosopher~~ z

Therefore, x is y

- Parameterized propositions are essentially called predicates

  → prop. in Prop. Logic are 0-ary predicates

- Predicate logic:

  $\boxed{\forall, \exists,}$ Quantifiers, $\underline{\underline{F}} \rightarrow$ Countably infinite set of function symbols

  (,) $\underline{\underline{R}} \rightarrow$ Countably inf. set of predicate symbols

  $V \rightarrow$ Countably inf. set of variables, FOL := Prop. Logic + Quant. over individuals

+ functions / Pred.

$$\underline{\underline{f/_n}} \in \underline{\underline{F}} \qquad +/_2 \in F$$

$$P/_K \in \underline{R} \qquad -/_2 \in F$$

$$\underline{\leq}/_2 \in R$$

Syntax of FOL

Signature : $S = (\underline{F}, \underline{R})$

In Prop. Logic

$$\begin{cases} F = \emptyset \\ R = \{ p_1/_0, \; p_2/_0, \; \dots \; p_n/_0 \} \end{cases}$$

FOL signature : $\left\{ \begin{array}{l} \text{defines a} \\ \text{set of FOL} \\ \text{formulas} \end{array} \right.$

- terms
- atoms
- formulas

Terms

$$S = (F, R)$$

S-terms T_S given by the foll. grammar

$$t ::= x \in V \mid f(t_1, \ldots, t_n)$$

$$f/_n \in F$$

Eg:

Consider $\underline{\underline{F}} = \{ c/_0 , f/_1 , g/_2 \}$

S-terms

- $\underline{f(x_1)}$, $x_1 \in V$
- $g(\underline{f}(\underline{c}), g(x_1, x_2))$, $x_1, x_2 \in V$

Atoms

S-atoms , A_S

$$a ::= P(t_1, \ldots t_n) \mid t = t \mid \bot \mid \top$$

where $P/n \in \underline{\underline{R}}$

Eg:

Consider

$F = \{ s/0 \}$, $R = \{ H/1, M/1 \}$

- $H(x)$, $M(s)$, $H(M(s))$

$x \in V$

Formulas

$\mathcal{S}$- formulas , $P_{\mathcal{S}}$

$$\mathcal{F} ::= a \mid \neg \mathcal{F} \mid \mathcal{F} \circ \mathcal{F} \mid \forall x.\, \mathcal{F} \mid \exists x.\, \mathcal{F}$$

$$\{\underbrace{\wedge, \vee, \rightarrow, \leftrightarrow, \ldots}_{\text{Prop. connectives}}\}$$

Eg: $x \in V$

$\mathcal{S} = (F, R)$

$F = \{ s/0 \}$, $R = \{ H/1 , M/1 \}$

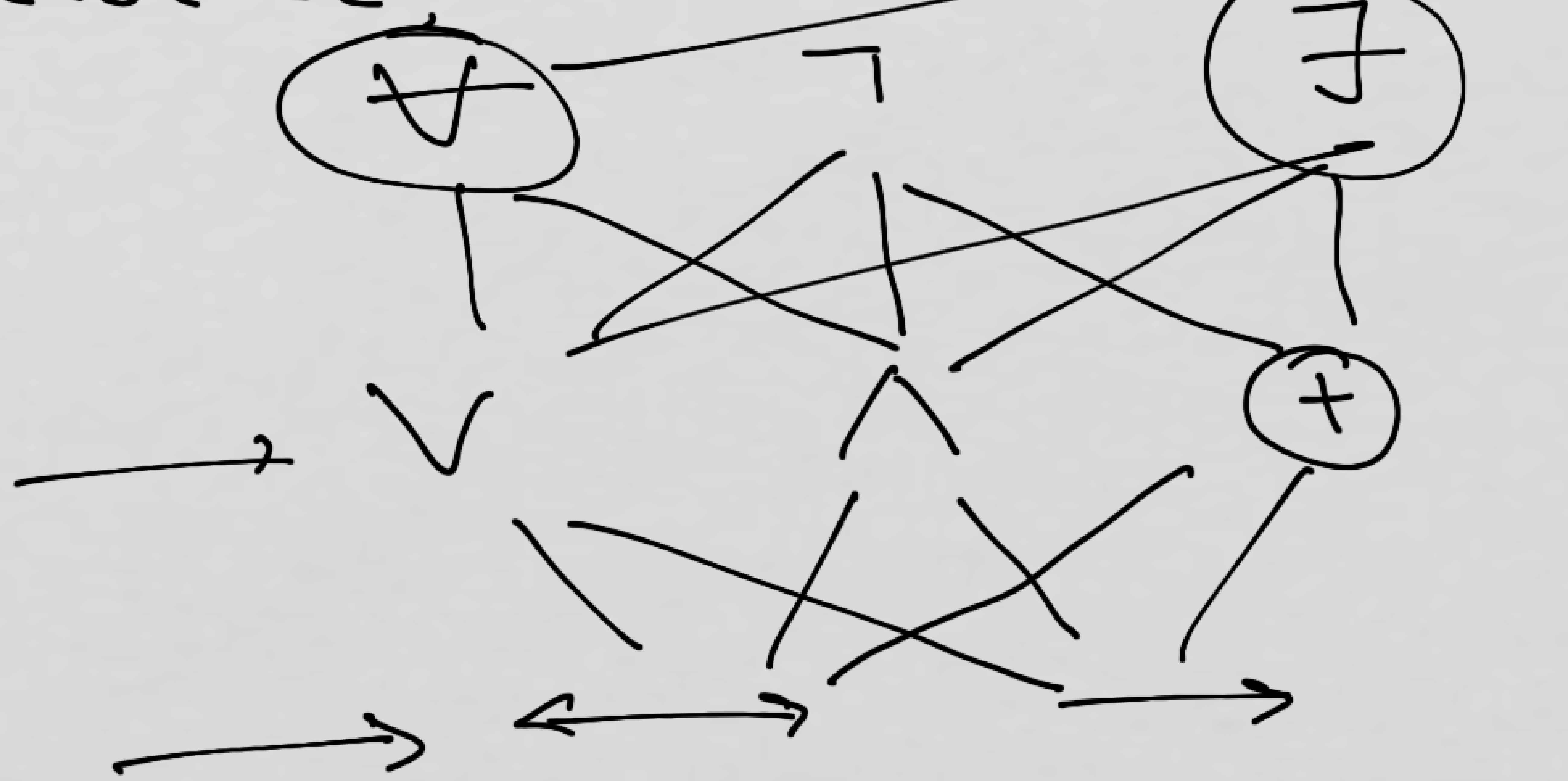
$\mathcal{S}$. formulas

$$\forall x.\, (H(x) \Rightarrow M(x)) \wedge (H(s) \Rightarrow M(s))$$

Parsing

Precedence, Assoc.

Right to left



FOL - semantics

$\underline{\underline{f}} \in F$

$\mathcal{S} = (F, R)$

$\mathcal{S}$-model m is defined as

$$\Big(\underbrace{D_m}_{\text{Nonempty set}},\ \underbrace{\{f_m : D_m^n \to D_m \mid f/n \in F\}}_{\text{Assigns meaning to } f/n \text{ under a model } m},\ \{P_m \subseteq D_m^n \mid P/n \in R\}\Big)$$

FOL Semantics

- Recap
- Eg: Assignment of vars
  Assignment of values to fncs