CONTROL STUDIO B (CSB)

STATE FEEDBACK CONTROL

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STATE-SPACE MODEL

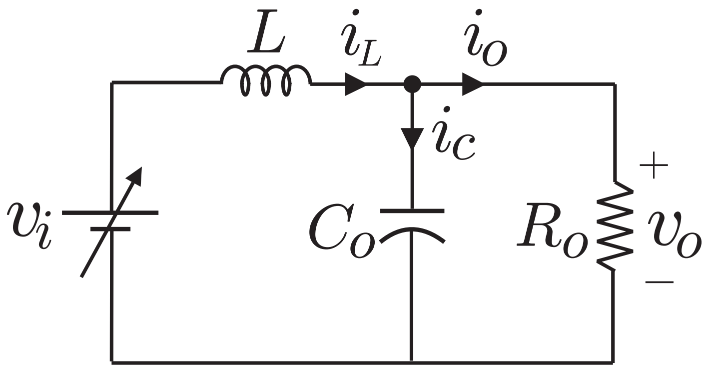
Consider the following Linear Time-Invariant (LTI) dynamic system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

where, $\boldsymbol{x} \in \mathbb{R}^n$ represents for the n-system state; $\boldsymbol{u} \in \mathbb{R}^m$ stands for the m-control input; and $\boldsymbol{y} \in \mathbb{R}^p$ represents the p-system outputs.

STATE-SPACE MODEL

Exercise 1. Obtain the state-space model of the following circuit



DISCRETIZATION

Consider the following continuous-time Linear LTI dynamic system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A_c x}(t) + \boldsymbol{B_c u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C_c x}(t) + \boldsymbol{D_c u}(t)$$

This system can be discretized assuming a zero-order hold for the control input, i.e., the control input remains constant between two consecutive sampling instants. This leads to the following discrete-time LTI system:

$$\boldsymbol{x}(k+1) = \boldsymbol{A_d x}(k) + \boldsymbol{B_d u}(k)$$
$$\boldsymbol{y}(t) = \boldsymbol{C_d x}(k) + \boldsymbol{D_d u}(k)$$

where, $\boldsymbol{A}_d = \mathrm{e}^{\boldsymbol{A}_c T}$

$$\boldsymbol{B}_d = \left(\int_0^T \mathrm{e}^{\boldsymbol{A}_c t} dt\right) \boldsymbol{B}_c \text{ or } \boldsymbol{A}_c^{-1}(\boldsymbol{A}_d - \boldsymbol{I}) \text{ if } \boldsymbol{A}_c \text{ is nonsingular}$$

$$\boldsymbol{C}_d = \boldsymbol{C}_c,\ \boldsymbol{D}_d = \boldsymbol{D}_c$$

STATE-SPACE MODEL

Consider the following Linear Time-Invariant (LTI) dynamic system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

where, $\boldsymbol{x} \in \mathbb{R}^n$ represents for the n-system state; $\boldsymbol{u} \in \mathbb{R}^m$ stands for the m-control input; and $\boldsymbol{y} \in \mathbb{R}^p$ represents the p-system outputs.

Control Target

$$\boldsymbol{y}(t_s) = \boldsymbol{y}^\star, \text{ for a } t_s > 0 \text{ (regulation problem)}$$
$$\boldsymbol{y}(t) = \boldsymbol{y}^\star, \text{ for all } t \geq t_s.$$

STATE-SPACE MODEL

Consider the following Linear Time-Invariant (LTI) dynamic system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

where, $\boldsymbol{x} \in \mathbb{R}^n$ represents for the n-system state; $\boldsymbol{u} \in \mathbb{R}^m$ stands for the m-control input; and $\boldsymbol{y} \in \mathbb{R}^p$ represents the p-system outputs.

Controllability

The linear system (***A*, *B*, *C*, *D***) is completely state controllable if and only if the following controllability matrix, $\mathcal{C}$, is of full rank n:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \cdots A^{n-1}B \end{bmatrix}$$

EQUILIBRIUM POINT (REGULATION)

Exercise 2. Obtain the equilibrium point of the following single-input single-output (SISO) system:

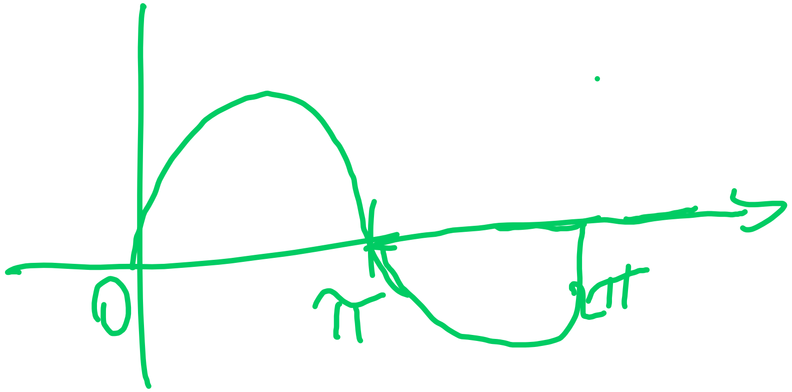
$$\dot{x} = \sin(u)$$

equil point.

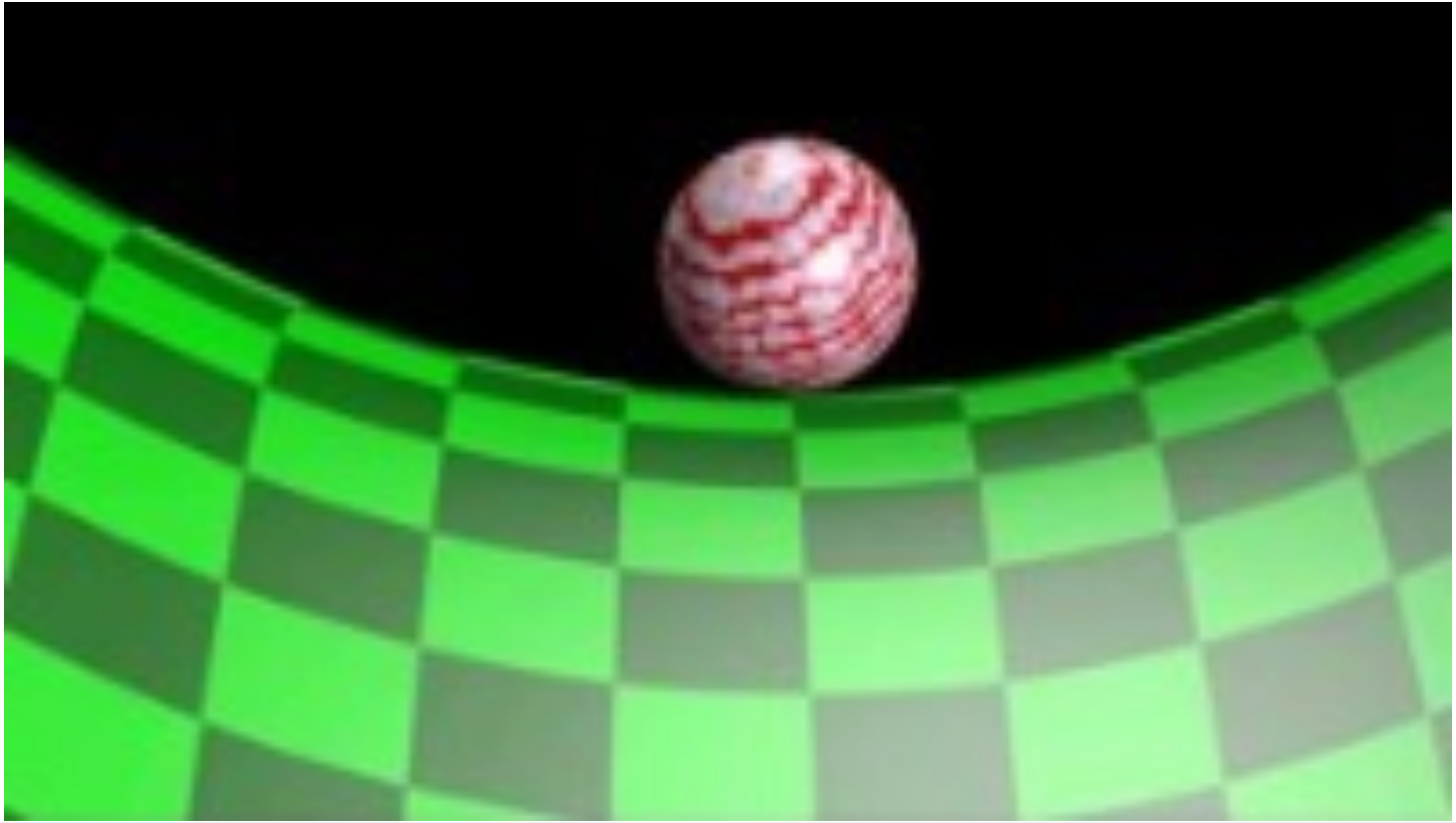
$0 = \sin(u)$

$u = N\pi$

$N \in \mathbb{I}$



EQUILIBRIUM POINT (REGULATION)



EQUILIBRIUM POINT (REGULATION)

Equilibrium Point: $\left(x_{ss}, u_{ss}\right).$

Let's analyze the LTI dynamic system in a steady-date

$$\dot{\boldsymbol{x}}_{ss} = \boldsymbol{A}_c \boldsymbol{x}_{ss} + \boldsymbol{B}_c \boldsymbol{u}_{ss} = \vec{\mathbf{0}}_n$$

zero vector n states.

$$\boldsymbol{y}_{ss} = \boldsymbol{C}\boldsymbol{x}_{ss}$$

If matrix $\boldsymbol{A}_c$ is invertible, it follows that:

$$\boldsymbol{x}_{ss} = -\boldsymbol{A}_c^{-1}\boldsymbol{B}_c \boldsymbol{u}_{ss}$$
$$\boldsymbol{y}_{ss} = \boldsymbol{C}\boldsymbol{x}_{ss}$$

$$\Rightarrow \boldsymbol{y}_{ss} = \underbrace{-\boldsymbol{C}\boldsymbol{A}_c^{-1}\boldsymbol{B}_c}_{\boldsymbol{M}_o} \boldsymbol{u}_{ss} \Rightarrow$$

$$\boldsymbol{u}_{ss} = \boldsymbol{N}_u \cdot \boldsymbol{y}_{ss}$$
$$\boldsymbol{N}_u = \boldsymbol{M}_o^{-1}$$

EQUILIBRIUM POINT (REGULATION)

Equilibrium Point: $\left(x_{ss}, u_{ss}\right).$

Let's analyze the LTI dynamic system in a steady-date

$$\dot{\boldsymbol{x}}_{ss} = \boldsymbol{A}_c\boldsymbol{x}_{ss} + \boldsymbol{B}_c\boldsymbol{u}_{ss} = \vec{\mathbf{0}}_n$$
$$\boldsymbol{y}_{ss} = \boldsymbol{C}\boldsymbol{x}_{ss}$$

If matrix $\boldsymbol{A}_c$ is invertible, it follows that:

$$\begin{aligned} \boldsymbol{x}_{ss} &= -\boldsymbol{A}_c^{-1}\boldsymbol{B}_c\boldsymbol{u}_{ss} \\ \boldsymbol{y}_{ss} &= \boldsymbol{C}\boldsymbol{x}_{ss} \end{aligned} \Longrightarrow \boldsymbol{y}_{ss} = \underbrace{-\boldsymbol{C}\boldsymbol{A}_c^{-1}\boldsymbol{B}_c}_{\boldsymbol{M}_o}\boldsymbol{u}_{ss} \Longrightarrow \boxed{\begin{aligned} \boldsymbol{u}_{ss} &= \boldsymbol{N}_u \cdot \boldsymbol{y}_{ss} \\ \boldsymbol{N}_u &= \boldsymbol{M}_o^{-1} \end{aligned}}$$

EQUILIBRIUM POINT (REGULATION)

Equilibrium Point: $\left(x_{ss}, u_{ss}\right).$

Equilibrium Point: $\left(N_x \cdot y_{ss}, N_u \cdot y_{ss}\right).$

We analyze the LTI dynamic system in a steady-date

$$\dot{\boldsymbol{x}}_{ss} = \boldsymbol{A}_c \boldsymbol{x}_{ss} + \boldsymbol{B}_c \boldsymbol{u}_{ss} = \vec{\mathbf{0}}_n$$

$$\boldsymbol{y}_{ss} = \boldsymbol{C} \boldsymbol{x}_{ss}$$

$$\boldsymbol{u}_{ss} = \boldsymbol{N}_u \cdot \boldsymbol{y}_{ss}$$

$$\boldsymbol{N}_u = \boldsymbol{M}_o^{-1}$$

If matrix $\boldsymbol{A}_c$ is invertible, it follows that:

$$\boldsymbol{x}_{ss} = -\boldsymbol{A}_c^{-1} \boldsymbol{B}_c \boldsymbol{u}_{ss}$$

$$\boldsymbol{y}_{ss} = \boldsymbol{C} \boldsymbol{x}_{ss}$$

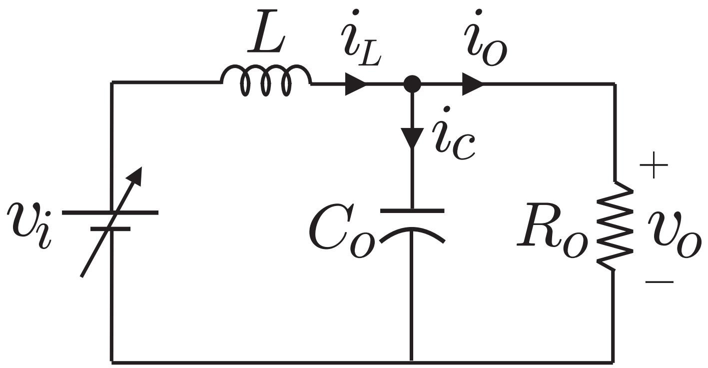
$$\Rightarrow \boldsymbol{x}_{ss} = -\boldsymbol{A}_c^{-1} \boldsymbol{B}_c \boldsymbol{N}_u \boldsymbol{y}_{ss} \Rightarrow$$

$$\boldsymbol{x}_{ss} = \boldsymbol{N}_x \cdot \boldsymbol{y}_{ss}$$

$$\boldsymbol{N}_x = -\boldsymbol{A}_c^{-1} \boldsymbol{B}_c \boldsymbol{N}_u$$

EQUILIBRIUM POINT (REGULATION)

Exercise 3. Obtain the equilibrium point of the following circuit in terms of a desired output voltage reference $v_o^\star$.



KCL

$i_L = i_o + i_c$

$i_L = \frac{v_o}{R_o} + C\frac{dV_c}{dt}$

KVL

$V_i = V_L + \underbrace{V_{co}}_{V_o}$

$V_i = L\frac{di_L}{dt} + V_{co}$

TRACKING ERROR FORM

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The dynamic and steady-state models for the LTI system is given by:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c \boldsymbol{x}(t) + \boldsymbol{B}_c \boldsymbol{u}(t)$$
$$\dot{\boldsymbol{x}}_{ss} = \boldsymbol{A}_c \boldsymbol{x}_{ss} + \boldsymbol{B}_c \boldsymbol{u}_{ss}$$

Now, we introduce the following variable change:

$$\boldsymbol{x}_e(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{ss}$$
$$\boldsymbol{u}_e(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{ss}.$$

$$\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{x}}_{ss} = \boldsymbol{A}_c \boldsymbol{x}(t) - \boldsymbol{A}_c \boldsymbol{x}_{ss} + \boldsymbol{B}_c \boldsymbol{u}(t) - \boldsymbol{B}_c \boldsymbol{u}_{ss}$$

$$\boxed{\dot{\boldsymbol{x}}_e(t) = \boldsymbol{A}_c \boldsymbol{x}_e(t) + \boldsymbol{B}_c \boldsymbol{u}_e(t).}$$

it follows the same dynamics of the linear system.
tracking error form brings error to zero.

STATE FEEDBACK CONTROLLER

For the tracking error form, the state feedback controller is represented by:

$$\boxed{\boldsymbol{u}_e(t) = -\boldsymbol{F}\boldsymbol{x}_e(t).}$$

$$\begin{aligned}\dot{\boldsymbol{x}}_e(t) &= \boldsymbol{A}_c\boldsymbol{x}_e(t) + \boldsymbol{B}_c\boldsymbol{u}_e(t)\\ &= \boldsymbol{A}_c\boldsymbol{x}_e(t) - \boldsymbol{B}_c\boldsymbol{F}\boldsymbol{x}_e(t)\\ &= \underbrace{(\boldsymbol{A}_c - \boldsymbol{B}_c\boldsymbol{F})}_{\boldsymbol{A}_F}\boldsymbol{x}_e(t)\end{aligned}$$

$\Rightarrow$

Closed-Loop System

$$\dot{\boldsymbol{x}}_e(t) = \boldsymbol{A}_F\boldsymbol{x}_e(t)$$

$$\boldsymbol{x}_e(t) = \mathrm{e}^{\boldsymbol{A}_F t} \cdot \boldsymbol{x}_e(0)$$

Stable Control Design

Matrix $\boldsymbol{F}$ must be designed so matrix $\boldsymbol{A}_F$ has all its eigenvalues with negative real part, i.e., $\mathrm{Re}\{\lambda_{Fi}\} < 0, \forall i \in \{1, \ldots, n\}$, where $\mathrm{eig}(A_F) = \{\lambda_{F1}, \lambda_{F2}, \cdots, \lambda_{Fn}\}$.

STATE FEEDBACK CONTROLLER

For the tracking error form, the state feedback controller is represented by:

$$\boxed{\boldsymbol{u}_e(t) = -\boldsymbol{F}\boldsymbol{x}_e(t).}$$

$$\boldsymbol{u}(t) - \boldsymbol{u}_{ss} = -\boldsymbol{F}(\boldsymbol{x}(t) - \boldsymbol{x}_{ss})$$

$$\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{x}(t) + \underbrace{\boldsymbol{F}\boldsymbol{x}_{ss} + \boldsymbol{u}_{ss}}_{\boldsymbol{r}}$$

$$\Rightarrow \quad \boxed{\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{r}}$$

$$\boldsymbol{r} = \boldsymbol{F}\boldsymbol{x}_{ss} + \boldsymbol{u}_{ss}$$

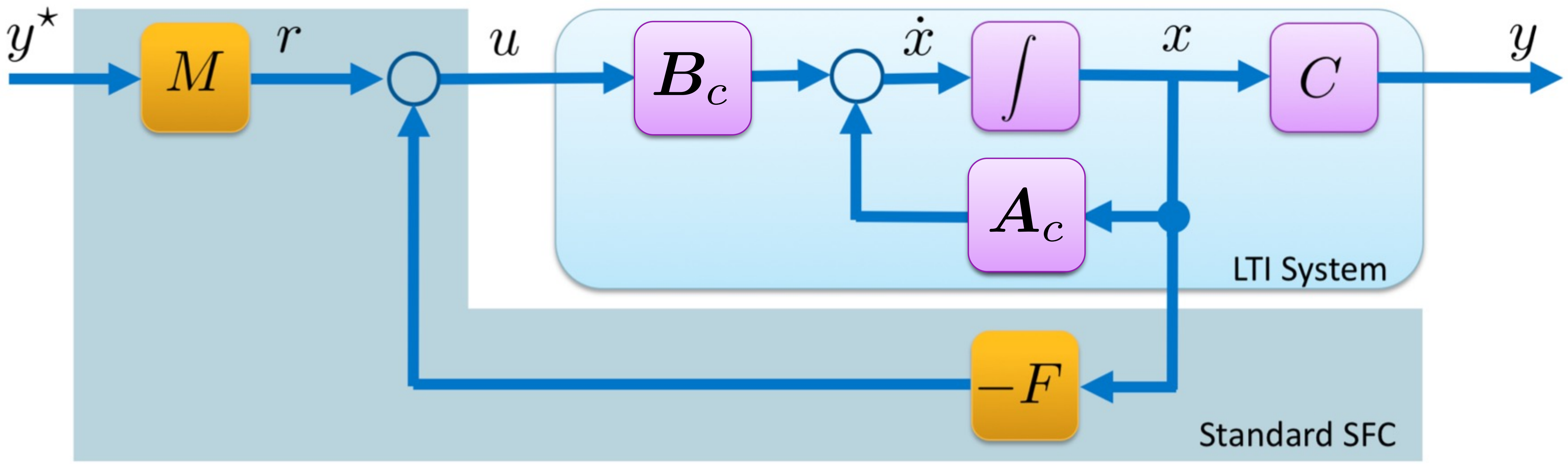
$$r = \boldsymbol{F}\boldsymbol{N}_x\boldsymbol{y}_{ss} + \boldsymbol{N}_u\boldsymbol{y}_{ss}$$

$$r = \boldsymbol{M}\boldsymbol{y}_{ss}$$

$$\boldsymbol{M} = \boldsymbol{F}\boldsymbol{N}_x + \boldsymbol{N}_u$$

STATE FEEDBACK CONTROLLER

Block Diagram: Standard Form

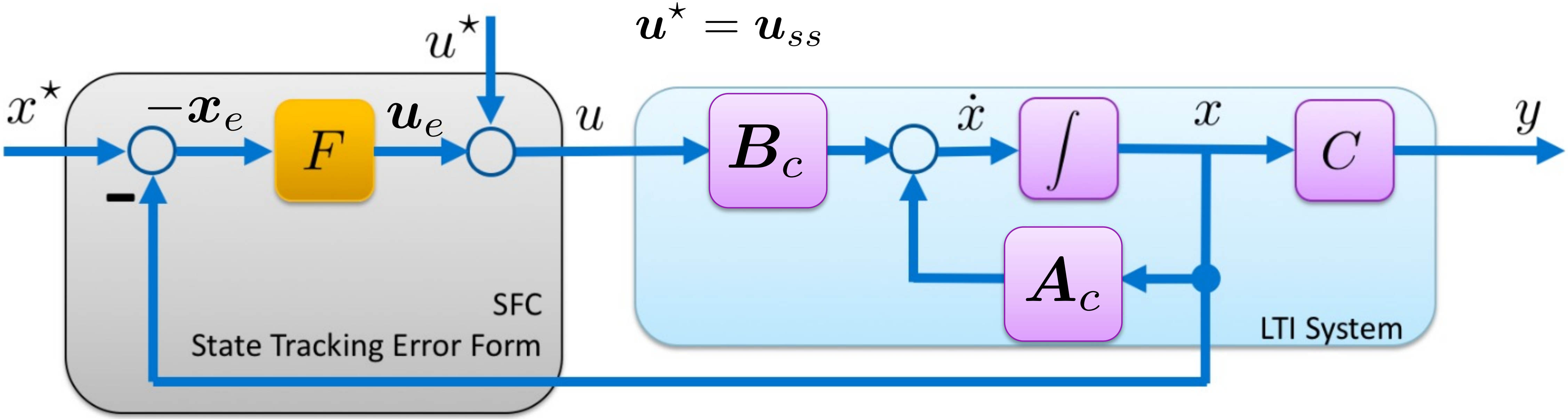


STATE FEEDBACK CONTROLLER

Block Diagram: Tracking Error Form

$$\boldsymbol{x}^\star = \boldsymbol{x}_{ss}$$

$$\boldsymbol{u}^\star = \boldsymbol{u}_{ss}$$



$$\boldsymbol{u}_e(t) = -\boldsymbol{F}\boldsymbol{x}_e(t).$$

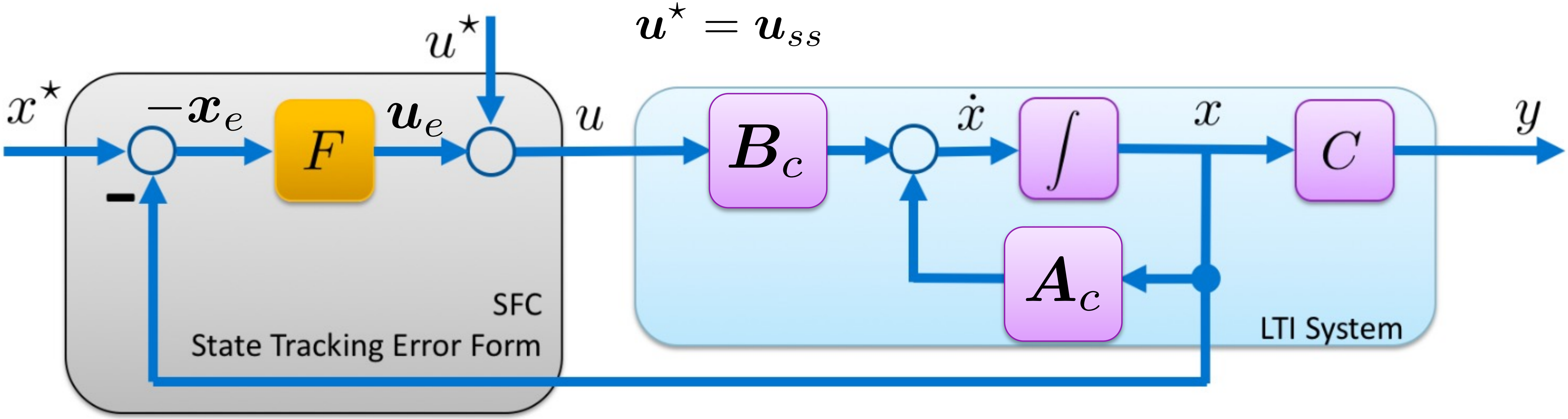
$$\boldsymbol{u}_e(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{ss} \implies \boldsymbol{u}(t) = \boldsymbol{u}_e(t) + \boldsymbol{u}_{ss}$$

STATE FEEDBACK CONTROLLER

Block Diagram: Tracking Error Form

$$\boldsymbol{x}^\star = \boldsymbol{x}_{ss}$$

$$\boldsymbol{u}^\star = \boldsymbol{u}_{ss}$$



$$\boldsymbol{u}_e(t) = -\boldsymbol{F}\boldsymbol{x}_e(t).$$

$$\boldsymbol{u}_e(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{ss} \implies \boldsymbol{u}(t) = \boldsymbol{u}_e(t) + \boldsymbol{u}_{ss}$$

STATE FEEDBACK CONTROLLER

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Block Diagram: Tracking Error Form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}(t)$$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c(-\boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{M}\boldsymbol{y}^\star)$$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_F\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{M}\boldsymbol{y}^\star$$

$$s\boldsymbol{X}(s) = \boldsymbol{A}_F\boldsymbol{X}(s) + \boldsymbol{B}_c\boldsymbol{M}\boldsymbol{Y}^\star(s)$$

$$(s\boldsymbol{I} - \boldsymbol{A}_F)\boldsymbol{X}(s) = \boldsymbol{B}_c\boldsymbol{M}\boldsymbol{Y}^\star(s)$$

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A}_F)^{-1}\boldsymbol{B}_c\boldsymbol{M}\boldsymbol{Y}^\star(s)$$

$$\boldsymbol{Y}(s) = C(s\boldsymbol{I} - \boldsymbol{A}_F)^{-1}\boldsymbol{B}_c\boldsymbol{M}\boldsymbol{Y}^\star(s)$$

Multivariable Closed-Loop Transfer Function

$$\boldsymbol{T}(s) = \frac{\boldsymbol{Y}(s)}{\boldsymbol{Y}^\star(s)} = C(s\boldsymbol{I} - \boldsymbol{A}_F)^{-1}\boldsymbol{B}_c\boldsymbol{M}$$

Multivariable Characteristic Equation

$$\boldsymbol{\Delta}(s) = |s\boldsymbol{I} - \boldsymbol{A}_F| = 0$$

STATE FEEDBACK CONTROLLER

$$\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Exercise 5. Consider the following linear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c \boldsymbol{x}(t) + \boldsymbol{B}_c \boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$$

where, $\boldsymbol{A}_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{B}_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

controllable.

a) Design a continuous-time State Feedback Controller (SFC) to obtain a closed-loop response with a time constant $\tau = 0.25s$ and a damping factor of $\zeta = 0.707$.

$\dot{x}_1 = x_2$ $\qquad y = x_1$
$\dot{x}_2 = u$

b) Design a discrete-time SFC to satisfy the same specifications given in a) and considering a sampling time $T_s = 0.05s$.

EQUILIBRIUM POINT (REGULATION)

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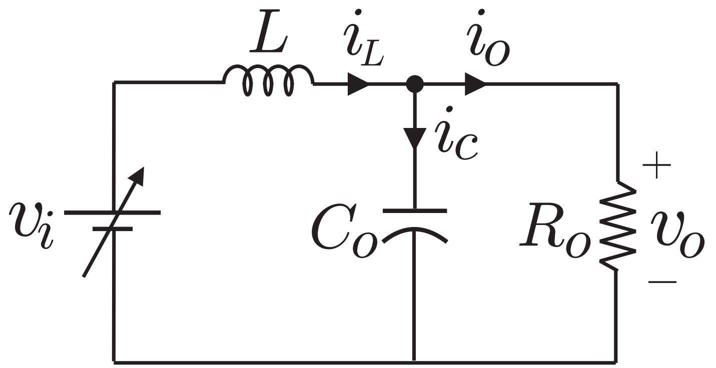
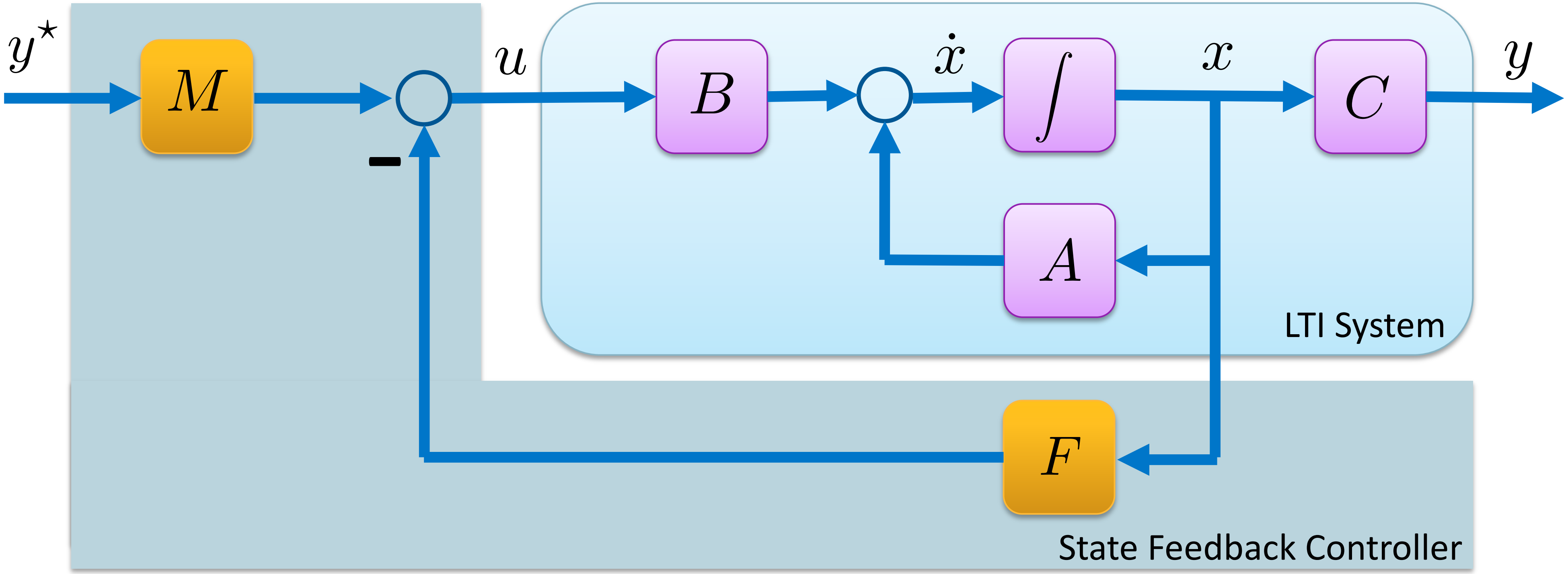
Exercise 6. Design a discrete-time SFC for a buck converter.



STATE FEEDBACK CONTROLLER

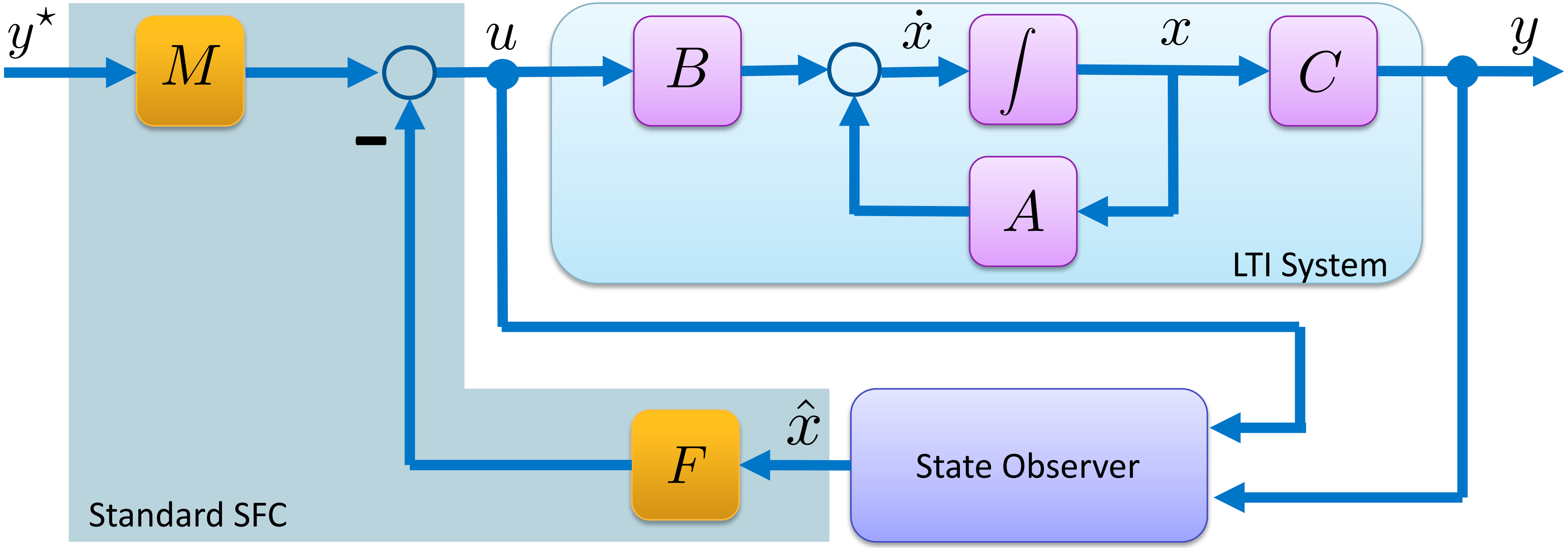
Block Diagram: Standard Form



¿How can we implement an SFC when the full state information is not available?

STATE FEEDBACK CONTROLLER

Block Diagram: Standard Form



¿How can we implement an SFC when the full state information is not available?

STATE-SPACE MODEL

Observability

The linear system ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) is said to be observable if and only if the following observability matrix, $\mathcal{O}$, is of full rank n:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

LUENBERGER OBSERVER

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)) \text{ (continuous-time)}$$
$$\hat{\boldsymbol{y}}(t) = \boldsymbol{C}\hat{\boldsymbol{x}}(t)$$

$\hat{\boldsymbol{x}}(t)$: System State Estimate

$\hat{\boldsymbol{y}}(t)$: System Output Estimate

$\boldsymbol{y}(t)$: System Output (measured)

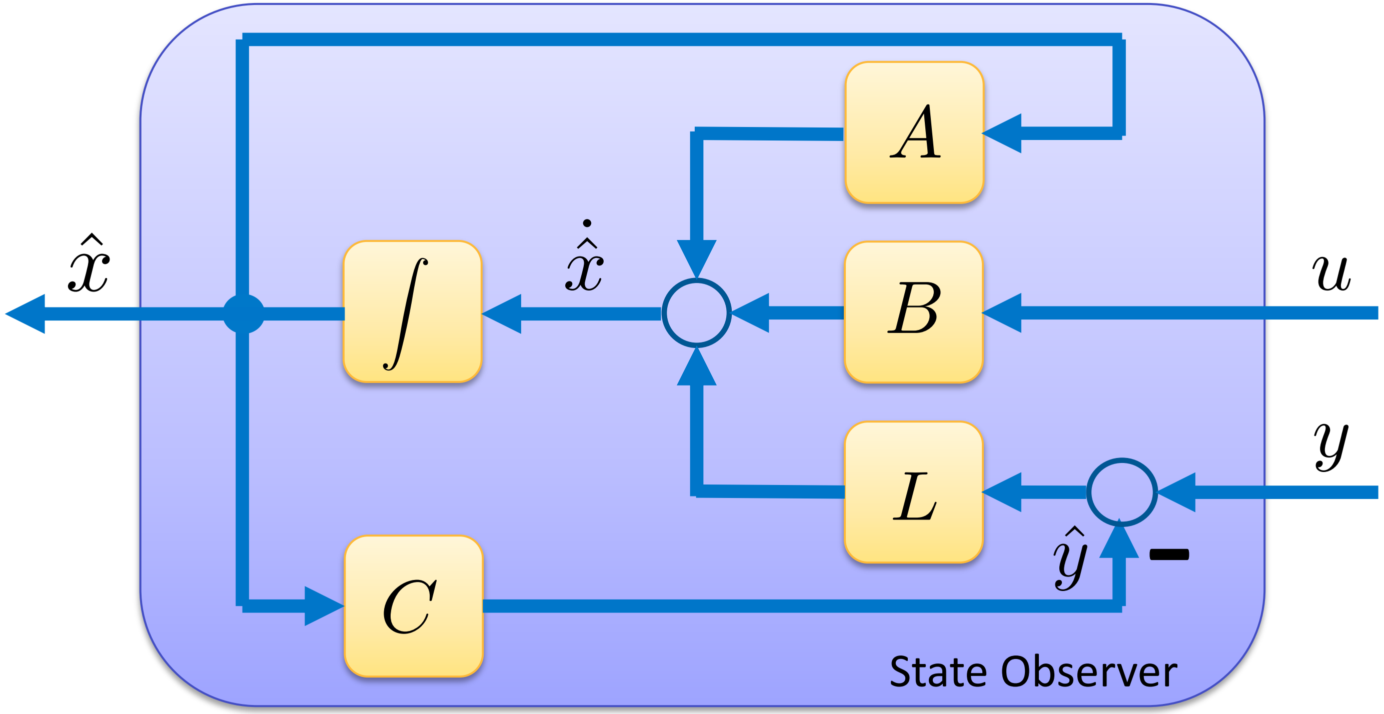
$\boldsymbol{u}(t)$: Control Input (applied by a controller)

$\boldsymbol{L}$: Observer Gain (to be designed by a good controlist)

David G. Luenberger

LUENBERGER OBSERVER

Block Diagram



¿How can we implement an SFC when the full state information is not available?

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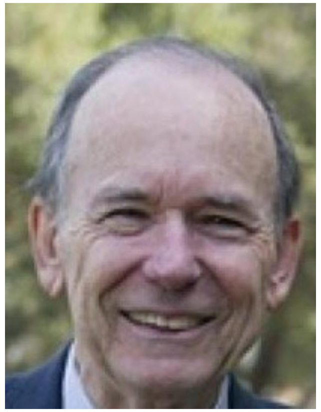
$$
\begin{aligned}
\hat{\boldsymbol{x}}(k+1) &= \boldsymbol{A}\hat{\boldsymbol{x}}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{L}(\boldsymbol{y}(k) - \hat{\boldsymbol{y}}(k)) \quad \text{(discrete-time)} \\
\hat{\boldsymbol{y}}(k) &= \boldsymbol{C}\hat{\boldsymbol{x}}(k)
\end{aligned}
$$



David G. Luenberger

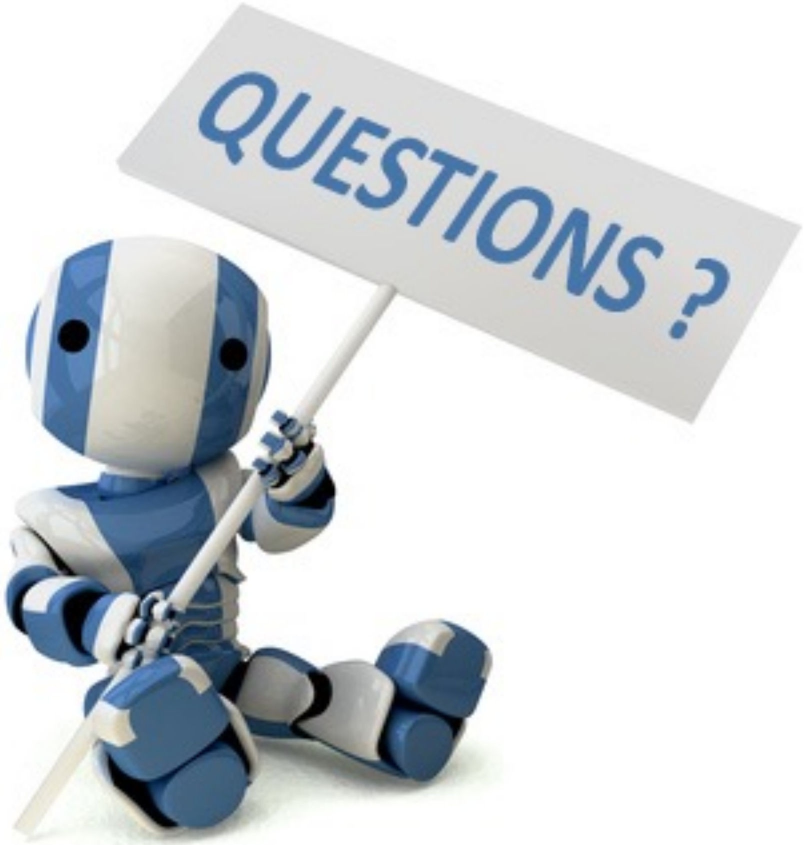
$\hat{\boldsymbol{x}}(t)$: System State Estimate

$\hat{\boldsymbol{y}}(t)$: System Output Estimate

$\boldsymbol{y}(t)$: System Output (measured)

$\boldsymbol{u}(t)$: Control Input (applied by a controller)

$\boldsymbol{L}$: Observer Gain (to be designed by a good controlist)


QUESTIONS ?