Linear Algebra for Engineers

A Commentary/Notes on course EE5007

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Preface

Linear algebra is a part of "Abstract" algebra. Hence some understanding of abstraction, jargon, and mathematical maturity are prerequisites to this usually semester long bachelor's degree level course. A math enthusiast, or a student from B.Tech background should not feel that this is a tough nut to crack. The goals of this notes are:

0. To endure writing a self-contained technical article.
1. To introduce abstraction.
2. To introduce mathematical notation.
3. To illustrate some common proofs and the thought process behind them.
4. To elaborate on some topics.
5. To add some problems.

This is not a formal paper, the motto here is to educate. This is my first attempt. The tone is conversational as if I was explaining it all to you. It has words like ain't, kind of, etc., etc. Almost all of it is my interpretation, and I might be wrong. All constructive criticism is welcome. If you feel even a little wiser w.r.t mathematical thinking after reading this document, I feel I've served my purpose. I haven't blatantly copy-pasted any material from any source, however all credit goes to every math-circler out there, who has taught me any infinitesimal thing. You can use this under creative commons share alike noncommercial license.

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Introduction - Algebraic Structures

An **algebraic structure** is a set along with some operations. This is similar to data structures in computer science which consist of a storage space and rules/procedures defined on them; something like a stack, which has some memory, say an array, and operations like push and pop. If S, R are sets, and *, +, % are some operations, the common **notation** of a structure looks like (S, *) or <R, +, %>.

An **operator** is a relation that maps some elements of set to other elements of the set. Firstly, an operator can have a one or more inputs or arguments. A particular example we shall be using commonly is the **binary operator,** one which takes an ordered pair and maps it to another element of the set. A ternary operator takes three inputs, … Secondly, a structure can have one or more operations. Some examples are groups, which is a set with one binary operation, a ring – a set with two binary operations, and a field with four binary operations (some say two are enough. Ok!). To introduce some maths notations, we define * as a binary operator on a set S if: * : S x S -> S.

Now, can you just blend any set and arbitrarily operate on them? You may, but it may not be of utility. The reason we call these structures are because they always satisfy some rules, and they are abstract as they make generic restrictions on all members of the family. This is where a vector stops being a pointy line and its course becomes handwaving.

Some useful sets are subsets of natural numbers (N), real numbers(R), complex numbers(C). Some operators are addition (+), multiplication (*, sometimes x), remainder or modulo. So, how do these sets and operators bind together? What are the rules they have to obey? Let S be a set, and * be a binary operator that assigns every ordered pair (a, b) of elements of S an element in S denoted by ab (= a*b, * is sometimes skipped). The operation if unknown, is usually called multiplication.

Some of the commonly used properties of operations are:

1. **Closure**: The operator should map the input n-tuple from set S to an element of the same set S. Note that this property is covered in the definition of the operator itself. If a set obeys closure property under an operation, we say it's closed w.r.t that operation.

   S is closed under * if for all a, b in S, a*b is in S.

   Is the set of negative numbers closed under multiplication? Is the set of natural numbers closed under subtraction?

2. **Associativity:** The "grouping" of the operands should be immaterial. This property deals with the case when we perform the operation more than once. In a nutshell, reordering the brackets should not change the result.

   *** is associative in S if for all a, b, c in S, (a*b)*c = a*(b*c).**

3. **Existence of identity:** Identity is about "do nothing" operation on an input. An element e in S is called the identity if **for all a in S, a*e = e*a = a.**

   The identity element of the structure is one with which the operation has no effect for all elements of the set. Adding zero to any number gives the same number. 1 is the identity w.r.t multiplication on the set of complex numbers. It is common to represent additive identity by **0** and multiplicative identity by **1.**

Is 0 the identity of subtraction? Is 1 the identity of division operation?(see point c below).

Points to note are:

a. The above property defines identity, it doesn't guarantee the existence of one. Identity needs to be the same set.

b. Identity has to work for all elements of the set, not for some elements. Identity is for the whole set, not for individual element.

c. Operation on an input with identity should commutatively return the input.

4. **Existence of inverses:** Inverses talk about "undo" operation. **b in S is an inverse of a (in S) if a*b = b*a = e**. A real-life example would be the undo-redo pair of operations in a software like MS paint. If taking a step forward is the operation, then its inverse is taking a step back. The negative of a real number is its additive inverse. Reciprocals are multiplicative inverses.

   Points to note are:

   a. The existence of inverse in the same set is not guaranteed. Does 0 have a multiplicative inverse?

   b. An inverse (if it exists) is for an element, not for the whole set. So, we use 'a/an' inverse but "the" identity.

   c. An inverse (if it exists), should be commutative. Otherwise they are called left-inverse or right-inverse depending on which side the operation leads to identity. This is generally observed in case of matrix multiplication.

   d. Common notation for additive inverse of a is -a and for multiplicative inverse of a is a^{-1}.

5. **Commutativity:** This property is about the "order" of the inputs i.e which operand comes first. * is commutative **if for all a, b in S, a*b = b*a**. A well-known case is multiplication of matrices A, B where AB ≠ BA. Sometimes the multiplication is not even defined due to incompatible dimensions of matrices. Is subtraction commutative?

When the set is real numbers R, addition obeys all the above properties. (C, +) obeys all of them. (Z, +) obeys all of them. But we didn't have to create such a system if we just talk about numbers – real or complex. What's the abstraction in that?

Consider the set of all symmetric matrices of order n with all real entries, S under the operation addition, +. Let A, B, C ∈ S. You can check that (S, +) follows the above rules.

- The sum of symmetric matrices is still symmetric.
- A + (B + C) = (A + B) + C. This carries over from addition of real numbers.
- The zero matrix of order n is symmetric. Identity is in S.
- The negative of symmetric matrix is symmetric. Inverse(additive) for each matrix is in S.
- A + B = B + A for all A and B.

Therefore, the set of symmetric matrices under addition is "kind of" similar to addition of real numbers. The set of square matrices of order n with complex entries also satisfies above properties under addition. Variation of sets and operations give rise to various complicated systems which are difficult to analyze individually. Instead we form one structure that obeys certain properties. The most fundamental structure has one set and one binary operation.

A **group** is a set G, with a binary operator * such that:

a. (G, *) is closed.
b. * is associative in G.
c. (G, *) has an identity.
d. All elements in G have inverses under *.

The set of symmetric matrices, real numbers, complex numbers and integers are groups under addition. Check if set of natural numbers is a group under addition. Is (Z, *) a group?

In addition, if a group satisfies the commutative property, then it is called a **commutative group** or an **Abelian group**. (Z, +) is an abelian group.

Few additional points:

- Inverses are important for "cancellation". Cancellation is operating with the inverse on one particular side of the expression (on both sides of an equation) so that the result is a simplified equation.
- All the properties above dealt with one operand.

6. **Distributivity:** When there are two operators, it is of interest to know how they act together. An operator * distributes over + in a set S if **for all a, b, c, in S**

   a*(b+c) = (a*b) + (a*c) (left distributivity), and

   (a+b)*c = (a*c) + (b*c) (right distributivity).

   For example, multiplication distributes over addition in Z. The above definition simplifies to 'product of sums is equal to sum of products'. Both the conditions have to be satisfied to say "distributive" in general.

Fields

A field is simply an algebraic structure with compatible addition, subtraction, multiplication, and division operations defined.

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