Trapezoids Talk - A paper by Michael Keyton
by Total Gadha - Wednesday, 24 September 2008, 07:25 PM

While I am busy writing the next article for TG.com, here is an offering by another mathematician, Michael Keyton of Illinois Mathematics and Science Academy. In this paper, Mr. Keyton discusses about the properties of a trapezium. Some of the properties are known whereas many of these are unknown. I discovered the paper while searching for one of the properties of the trapezium, namely the sum of the squares of the diagonals of a trapezium. Mr. Keyton has graciously given us the permission to publish the paper on TG. So here is it everyone, the complete compendium of the properties of a trapezium. Include them in your armory of formula.

Trapezoids Talk

In every high school textbook, the trapezoid in included as one of the quadrilaterals to study (or investigate). In almost every one, it is defined as a quadrilateral with exactly one pair of parallel sides. In this talk, I will argue that the definition should be changed and that there is much more to the trapezoid than is given in the books.

Almost every theorem about a trapezoid can be broken into two categories â€" those that are really about parallel sides and those that in some way incorporate a feature of a quadrilateral.

In the history of geometry, many definitions have been changed, some from being exclusive to being inclusive. An exclusive definition is one that separates other objects from being in the same class. For example, in Euclid, an isosceles triangle is defined to be a triangle with exactly two equal sides. Thus, an equilateral triangle is not isosceles. Likewise all his definitions of quadrilaterals are exclusive. During the next 2300 years, most of these have been changed. Now in every high school textbook, equilateral triangles are isosceles, rectangles and rhombi are parallelograms, and squares are rectangles and rhombi. There are two advantages to having inclusive definitions â€" (1) theorems for the more restricted case become corollaries for the more general case and (2) converses do not need to contain an â€œorâ€ conclusion.

By one construction, we can easily see that the parallelogram is a special case of a trapezoid. Take two points on each of two parallel lines. The quadrilateral formed by using these points is a trapezoid (or a parallelogram). Consequently, the definition can be strengthened by including it as such. So why is the definition maintained in the textbooks? I think primarily that many authors do not wish to go against standard terminology, most of the authors have not thought about the inconsistency in terminology, they are not actively engaged in discovering and proving theorems in geometry, and there are no converses for the trapezoid covered in high school geometry.

Thus, for good mathematical reasons letâ€™s change the definition to:

*A **trapezoid** is a quadrilateral with at least one pair of parallel sides.*

In the theorems that follow, some require that a pair of sides be non-parallel, but the parallel case follows as well, usually with little or no additional proof.

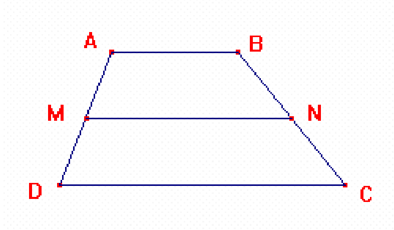
While we are at this change, the same argument applies to the isosceles trapezoid and the rectangle. A simple construction shows that the rectangle is a special case of the isosceles trapezoid. How then can we define the isosceles trapezoid so that the rectangle is a special case. I offer a variety of different definitions.

(1) An **isosceles trapezoid** is a cyclic trapezoid.
(2) An isosceles trapezoid is a trapezoid with a pair of supplementary opposite angles.
(3) An isosceles trapezoid is a trapezoid with the other pair of sides anti-parallel with respect to the parallel sides.
(4) An isosceles trapezoid is a trapezoid with a pair of congruent base angles.
(5) An isosceles trapezoid is a trapezoid with congruent diagonals.

Here are a few other definitions that I offer.

The **diacenter** of a quadrilateral is the intersection point of the diagonals.
A **quord** of a quadrilateral is a segment with endpoints on two sides of a quadrilateral.
A **median** is a quord with endpoints the midpoints on opposite sides.

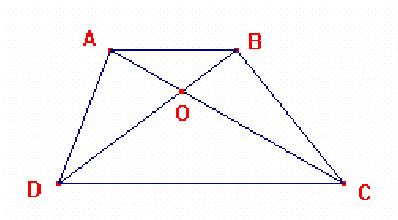
High school geometry theorems about the trapezoid [T]:



1) The median joining the non-parallel sides of a [T] is parallel to the parallel sides.

2) The length of the median joining the non-parallel sides of a [T] is the arithmetic average of the lengths of the parallel sides.

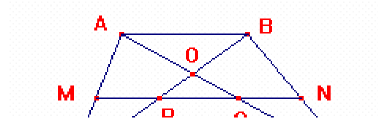
3) The area of a [T] is the height multiplied by the length of the median joining the non-parallel sides. Combine this with the second theorem and the other familiar theorem about a trapezoid is obtained.



4) The diacenter divides the diagonals of a [T] proportionally to each other and to the bases.

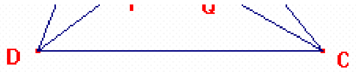
$$\frac{AO}{OC}=\frac{BO}{OD}=\frac{AB}{CD}$$

5) The triangles formed by the diacenter and the non-parallel sides of a [T] are equivalent (equal in area).
Area(ΔAOD) = area(ΔBOC)



6) The median joining the non-parallel sides of a [T] intersect the diagonals at points P and Q. PQ = abs(AB – CD)/2 (The distance between the two points is one-half the absolute value of the difference of the lengths of the

parallel sides of the [T].)

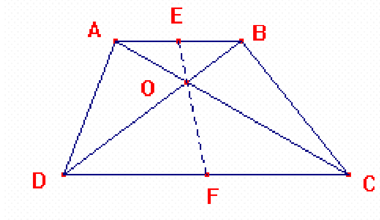


7) Converse of (4): If the diacenter divides the diagonals proportionally, then the quadrilateral is a [T] (or a parallelogram).

A few theorems not seen in high school geometry books.

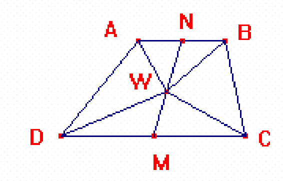
The Other Median

8) The median determined by the midpoints of the parallel sides of a [T] contains the diacenter.

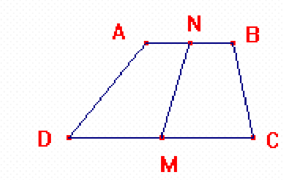


9) In [T] ABCD, S is on $\overline{AD}$, T is on $\overline{BC}$, $\overline{ST} \parallel \overline{AB}$, the locus of the midpoints of $\overline{ST}$ is the median of the parallel sides..

10) If W is on the median $\overline{NM}$ of [T] ABCD with bases $\overline{AB}$ and $\overline{CD}$, then ΔAWD and ΔBWC are equivalent.



11) [T] ABCD with bases $\overline{AB}$ and $\overline{CD}$, M and N midpoints of the parallel sides;

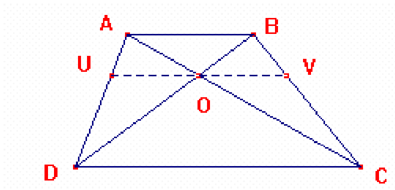


$$NM^2 = \frac{AD^2 + BC^2}{2} - \frac{(AB - CD)^2}{4}$$

$$NM^2 = \frac{AC^2 + Bd^2}{2} - \frac{(AB + CD)^2}{4}$$

12) The area of a [T] is equal to the product of the length of a non-parallel side times the altitude to it from the midpoint of the other non-parallel side.

13a) The diacenter bisects the quord parallel to the parallel sides through the diacenter of a [T].

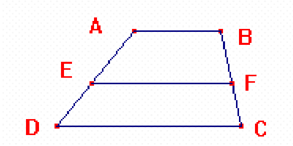


b) A quord between the non-parallel sides of a [T] that is bisected by the diacenter is parallel to the parallel sides.

c) The length of the quord through the diacenter parallel to the parallel sides of a [T] is equal to twice the product of the parallel sides divided by the sum of the lengths of the parallel sides. $UV = \frac{2(AB)(CD)}{AB + CD}$

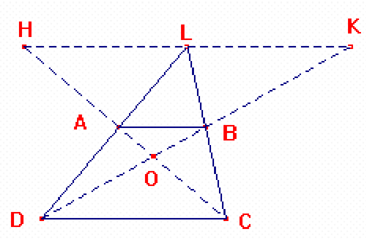
d) $\frac{BV}{BC} = \frac{AB}{AB + CD}$ and $\frac{CV}{BC} = \frac{CD}{AB + CD}$

14) (Generalization) $\overline{EF}$ parallel to the bases of [T] ABCD.



$$EF = \frac{AB \cdot ED + CD \cdot AE}{AE + ED}$$

15) There is another segment connected with a trapezoid which is important. Let the non-parallel sides intersect at L, construct a parallel through L to the parallel sides of the trapezoid. Let the diagonals intersect this parallel at H and K.



a) $\frac{LB}{LA} = \frac{AB}{CD}$ and $\frac{HA}{HC} = \frac{AB}{CD}$

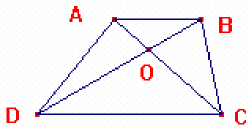
b) The sets {D, O, B, K} and{C, O, A, H} are harmonic sets.

c) $LH = LK = \dfrac{AB \cdot CD}{CD - AB}$

d) L is on the median of the parallel sides of [T].

16) In [T] ABCD with bases $\overline{AB}$ and $\overline{CD}$, the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the non-parallel sides and twice the product of the lengths of the parallel sides

$$AC^2 + BD^2 = AD^2 + BC^2 + 2 \cdot AB \cdot CD$$

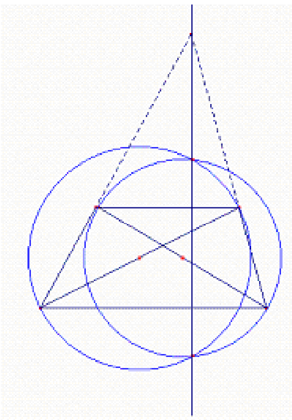


17) If one base is half the other base, then the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the non-parallel sides and the square of the length of the longer base.

If AB = CD/2, then

$$AC^2 + BD^2 = AD^2 + BC^2 + CD^2$$

18) The common chord of circles with diameters the diagonals of a [T] contains the point of intersection of the non-parallel sides.



19) $\overline{GJ} \parallel \overline{AB} \parallel \overline{CD}$, CD = a, AB = b, $\dfrac{CF}{FB} = \dfrac{m}{n}$

$GJ = \dfrac{an + bm}{m - n}$ and $EF = \dfrac{bm - an}{m - n}$