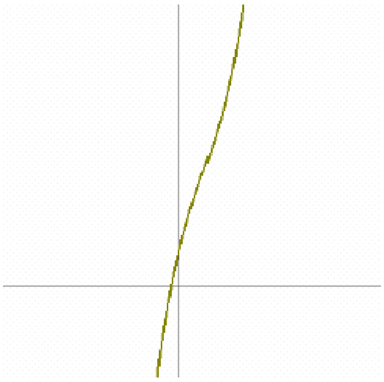
Functions
by Total Gadha - Thursday, 2 July 2009, 10:33 AM

In a CAT classroom or on the internet, there are two kinds of students- those who study and those who believe that they are studying but are actually not. Every time I start taking a fresh batch, it hardly takes one or two classes to identify both types of students. And it has nothing to do with intelligence or background. It has to do with attitude and fighting spirit. Every time I throw a problem in the class or post it in the forums, the students who are studying would be trying hard to solve the problem. They would arrive at an answer, whether right or wrong. On the other hand, the students are who are just '**striking a pose of studying**' would be waiting for me to give the answers and the solutions. They would be acting as if they are thinking hard but in reality would not be thinking at all. On the internet, I frequently see students who waste time in idle chit chat when they should be busy solving problems. They feel satisfied that they have done 'something' for their CAT by looking at problem or talking about CAT to others but in essence they have done nothing and wasted one more day. "Students like these," as I tell my serious students, "are going to fill up the CAT form to give you your 99 percentile." One thing I have learnt from my past experiences is that you can never teach attitude to students. Either they have it or they don't. Whereas winners focus on completing their tasks, whiners have excuses and ways of work avoidance. In TathaGat, we have a strict rule for students to submit one book review per week. And sometimes we see the book review either copied from the net or written after hearing about the content from their friends. We even have strict rule for submission of quant or DI assignments. And many students turn in their assignments half-attempted. The serious students, on the other hand, finish their assignments before time and would frequently come to me with "*Sir aur material mil sakta hai kya?*" There is a world of difference between the two attitudes. Where I constantly hear from the whiners "Sir aap solve kar dijiye", I also constantly hear from the winners "Sir abhi answer mat batayiye, solve karne dijiye." The only sane advice for the instructors in this field, who sometimes feel as frustrated as I do, is- **spend 80% of your time with your winners.** And winners are not the ones who score at the top. They are the ones who are fighting the hardest, whether they are weak or strong in the subject does not matter. And for students on the internet who want to do idle chit chat in the forums or engage in socializing, you would find TotalGadha.com too boring a place. We are serious about studies here.

For our students on TG.com, we would like to share one of our endeavors in CAT CBT Club- **Video lessons**. Dagny and I have been tinkering with our camera and trying to make some good video lessons for our CBT Club students. We have uploaded one such video- Divisors of a number- for our TG junta. Do have a look and give us your feedback. Unfortunately, to watch the rest of the lessons that we are making, you will have to become a member of our exclusive club.

Starting with this new chapter, I would like to ask what would you call an expression such $2x^3 - 3x^2 + 5x + 1$? Your answer would most probably be 'an algebraic expression' and you would be right. What is x here? A variable whose value YOU will have to input. So for x = 1, the value of your algebraic expression would be 5. For x = 2, the value would be 15. If you plot the values against the input value of the variable x, you would get a graph such as the one shown below:



This is a simple graph on the x-y axis which intersects the y axis at y = 1 for x = 0. There are some basic observations that we can make about the graph:

- What values can we put in the expression? Clearly, all the real values, i.e. values which lie on the number line, can be inserted into the expression. To draw the graph in the x-y plane we need to put values which we CAN find on the x-axis. This graph can have all the values from $-\infty$ to ∞.
- For every unique value of x, the expression will give a unique value only, i.e. if we plot the graph completely, at each value of x, we will have only a unique value of y. There cannot be two values for a single value of x.
- What are the values of the expression we can obtain by putting the values of x? Clearly, the values we can obtain will also be from $-\infty$ to ∞.

Sketching the graph of an expression

Let's have a small look at curve sketching here. Let's try and sketch the graph of the expression $x^3 - 4x^2 + 1$. There are two ways we can go about it-

Input Values: Probably the best method for curve sketching for a beginner is to put different values for the variable and check the output values of the expression. The table below shows some of the values we can keep.

Input (x)	Output (y)
−2	−23
−1	−4
0	1
1	−2
2	−7
3	−8
4	1

What do we see? That the value of the expression is increasing as x is increasing from negative to zero, decreases again as x increases from zero to positive, and then increases again. Although, putting values gives us a fairly good idea about the shape of the graph, it is a tedious method, especially so when the increase or decrease of the graph occurs at longer intervals and we do not know how many values to put. To understand the characteristics of the graph accurately we combine this method with our second method.

Finding the derivatives: You would need a small knowledge of calculus here, that of finding the first ($\frac{dy}{dx}$) and second ($\frac{d^2y}{dx^2}$) derivative of the expression. Let us denote first and the second derivative as f′(x) and f″(x), respectively. Observe the following rules:

- If the first derivative *f*′ is positive, then the function *f* is increasing.
- If the first derivative *f*′ is negative, then the function *f* is decreasing.

Let f be a continuous near a.
- If f'(a) = 0 and f''(a) > 0, then f has a local minimum at a.
- f'(a) = 0 and f''(a) < 0, then f has a local maximum at a.

When a graph has a local minimum, the curve is concave upward (and thus lies above the tangent lines) at the minimum. Similarly, the curve is concave downward at a local maximum.

Now let's see the derivatives of the expression $x^3 - 4x^2 + 1$:

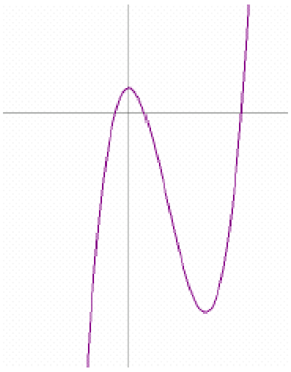
$$\frac{d(x^3 - 4x^2 + 1)}{dx} = 3x^2 - 8x \text{ and } \frac{d^2(x^3 - 4x^2 + 1)}{dx^2} = 6x - 8.$$

Now $f'(x) = 3x^2 - 8x = 0$ at $x = 0$ and $x = \frac{8}{3}$.

$f''(x) = -8$ (<0) at $x = 0$ therefore the graph has a local maximum at $x = 0$

and $f''(x) = 8$ (>0) at $x = \frac{8}{3}$ therefore the graph has a local minimum at $x = \frac{8}{3}$.

Now that we know the graph, we can draw it also. The graph is shown in the adjacent figure.



Combining these two methods, we can sketch the graph of expressions like 2sinx + 5, $2^x + 1$, $6x^2 - x + 1$ and so on.

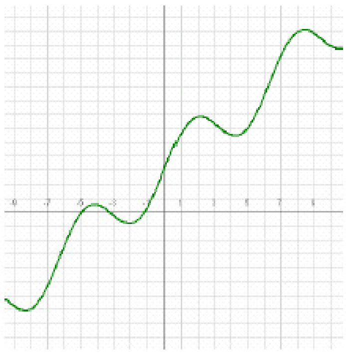
Coming back to my original question, what would you call expression such as 2sinx + 5 or $6x^2 - x + 1$? You would answer 'sinusoidal' expression and 'polynomial,' respectively. You would be right of course. But what would you call an expression such as $2\sin x + 6x^2 + 1$? The expression shares the same characteristics as the ones we have discussed-

- The input values in the expression would be 'real' values, values which lie on the number lines.
- The output values will also be real values.
- For a particular value of x, we get only one value of the expression.
- We can define the range of the real values for the input and output.

For want of a better word, we call these expressions sharing the same characteristics a **function**. So simply stated, a function is a device or an operator which gives a real number output for a real number input, with one rule- it gives a single output for a single input. The input, denoted by x, is called the independent variable and the output, denoted by y, is called the dependent variable. A function is commonly denoted as f(x). Therefore, in our example above

$f(x) = 2\sin x + x + 3$. The graph of the function is shown in the adjacent figure. The following can be observed about this function-

- You can input all values of x in the range $-\infty$ to ∞. The range of values of the independent variable (x) that we can take is known as the **domain** of the function. Here, the domain of the function is $-\infty$ to ∞.
- The range of values of the dependent variable (y) that we can take is known as the **range** of the function. Here, the range of the function is $-\infty$ to ∞.
- It does not happen that for a single value of x we get two different values of y. **Therefore, if we draw a vertical line for any value of x in the domain it will intersect the curve only once.**



Find the domain and range of the function $f(x) = \sqrt{4 - x^2}$.

Answer- we can see that the expression $4 - x^2$ will become negative if $x > 2$ or $x < -2$. Root of a negative number is not a real number, therefore the **domain** of this function is $-2 \le x \le 2$. We will take only positive values of f(x) for each value of x. The value of f(x) would vary from 0 to 2. Therefore the **range** of this function is [0, 2].

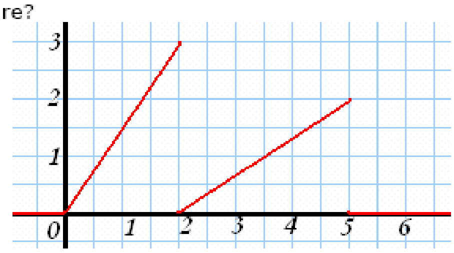
Find the domain and range of the function $f(x) = \frac{1}{1 + \sqrt{x}}$.

Answer- As the roots of a negative number cannot be real, the domain of x would be for $x \ge 0$. The maximum value of f(x) would be 1 for x = 0. Therefore, the range of the function f(x) = [0, 1].

Piecewise- Defined Functions

Piecewise Defined Functions

Till we have only seen functions which are continuous, i.e. functions you can draw without having to raise your pencil from the paper. That is because the functions are defined by a single rule. But how would you define the function given in the following figure?



Let us first see how the function is behaving. We can see that the value of the function is 0 for x < 0 and x > 5. Therefore, one rule is f(x) = 0 for x < 0 and x > 5. From x = 0 to x = 2, the value of f(x) changes from 0 to 3.

Therefore, one more rule is f(x) = $\frac{3}{2}x$ for $0 \le x \le 2$. When x is greater than 2 but less than or equal to 5, the value of f(x) varies from 0 to 2. Therefore, f(x) = $\frac{2x-4}{3}$ for $2 < x \le 5$. Now we can put these rules all in one-

$$fx = \begin{cases} \frac{3}{2}x & \text{when } 0 \le x \le 2 \\ \frac{2x-4}{3}x & \text{when } 2 < x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

There! We have created a rule for a piecewise defined function. Would you like to see some more piecewise-defined functions?

- **Modulus function**- The modulus function $|x|$ can be defined as $|x| = \begin{cases} x & \text{for } x \ge 0 \\ -x & \text{for } x < 0 \end{cases}$. You can read all about the modulus function in out **CAT Quant Lessons.**
- **Greatest Integer Function**- The greatest integer function, for any real number x, gives the greatest integer less than or equal to x. You can read all about the greatest integer function in our **CAT Quant Lessons.**
- **Least integer function**- The least integer function, for any real number x, gives the smallest integer greater than or equal to x.

Find the maximum value of the function $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}$.

Answer- We can see that f(x) follows the graph of the line y = x for x = 0 to 1 and the graph of y = 2 – x for x = 1 to 2. We can see that the maximum value occurs at the intersection of these two graphs at x = 1. The maximum value of f(x) = f(1) = 1

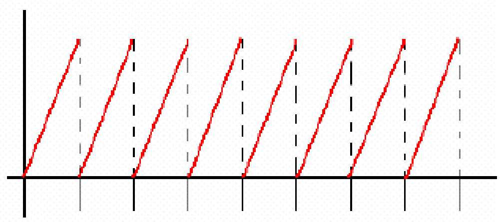
Let f(x) be a function with domain $0 \le x \le 1$ defined as

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ -2x+2 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Find the area between the curve y = f(f(f(x))) and the coordinate axes.

Answer- The graph of the function is drawn as shown below

$$\begin{cases} 0 \le f(f(f(x))) \le 1 & \Leftarrow 0 \le f(f(x)) \le \frac{1}{2} & \Leftarrow 0 \le f(x) \le \frac{1}{4} & \Leftarrow 0 \le x \le \frac{1}{8} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow \frac{1}{2} \le f(f(x)) \le 1 & \Leftarrow \frac{1}{4} \le f(x) \le \frac{1}{2} & \Leftarrow \frac{1}{8} \le x \le \frac{1}{4} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow \frac{1}{2} \le f(f(x)) \le 1 & \Leftarrow \frac{1}{2} \le f(x) \le \frac{3}{4} & \Leftarrow \frac{1}{4} \le x \le \frac{3}{8} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow 0 \le f(f(x)) \le \frac{1}{2} & \Leftarrow \frac{3}{4} \le f(x) \le 1 & \Leftarrow \frac{3}{8} \le x \le \frac{1}{2} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow 0 \le f(f(x)) \le \frac{1}{2} & \Leftarrow \frac{3}{4} \le f(x) \le 1 & \Leftarrow \frac{1}{2} \le x \le \frac{5}{8} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow \frac{1}{2} \le f(f(x)) \le 1 & \Leftarrow \frac{1}{2} \le f(x) \le \frac{3}{4} & \Leftarrow \frac{5}{8} \le x \le \frac{3}{4} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow \frac{1}{2} \le f(f(x)) \le 1 & \Leftarrow \frac{1}{4} \le f(x) \le \frac{1}{2} & \Leftarrow \frac{3}{4} \le x \le \frac{7}{8} \\ 0 \le f(f(f(x))) \le 1 & \Leftarrow 0 \le f(f(x)) \le \frac{1}{2} & \Leftarrow 0 \le f(x) \le \frac{1}{4} & \Leftarrow \frac{7}{8} \le x \le 1 \end{cases}$$



Area is the sum of the area of 8 triangles with base 1/8 and height 1. Total area = 8 × $\frac{1}{2}$ × $\frac{1}{8}$ × 1 = $\frac{1}{2}$

Let f(x) be a function such that

$$f(x) = \begin{cases} \frac{x}{2} & \text{if x is even} \\ x+1 & \text{if x is odd} \end{cases}$$

For how many values of x is f(f(f(x))) = 19?
Answer- f(f(f(x))) = 19 ⇒ f(f(x)) = 38 ⇒ f(x) = 37 or 76 ⇒ x = 74 (for f(x) = 37), 75 or 152 (for f(x) = 76). Therefore, three values of x are possible.

Find the domain of the function $f(x) = \frac{1}{\sqrt{\pi - [x]}}$**, where [x] denotes the greatest integer less than or equal to x.**
Answer- We know that the square root of negative quantities is imaginary. Therefore, for the output (range) to be real, the quantity under the square root should be positive. Therefore, [x] should be less than π = 3.14.. Therefore, as long as x is less than [x] would be 3 or less. Therefore, the domain of the function consists of all values of x less than 4. Therefore, Domain = (– ∞, 4)

Find the domain of the function $f(x) = \frac{1}{[x-2]}$**, where [x] denotes the greatest integer less than or equal to x.**
Answer- The above function will give real values for all values of x except those values where the denominator becomes zero. We cannot divide any number by zero. Therefore when $2 \le x < 3$, $0 \le x - 2 < 1$ and the greatest integer will give zero as the value. Therefore, the domain of the function will contain all values of x except when $2 \le x < 3$. Therefore, Domain = R – [2, 3), where R stands for the set of all real numbers.

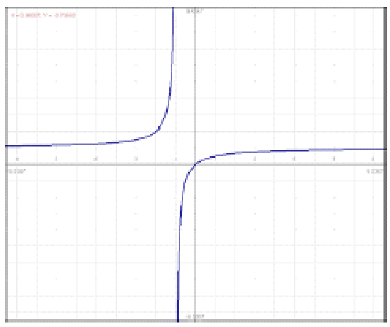
Find the domain of the function $\sqrt{x^2 - [x]^2}$.
Answer- As we only need to avoid having negative numbers inside the square root, we should ask ourselves the question "are there any values of x where the quantity under the square root will become negative?" Now for positive values of x, we know that $x^2 \ge [x]^2$. We only need to consider the negative values. If the negative values are integer, we know that the quantity under the square root will be zero. Let's see non-integer negative values of x. For example, let's take x = – 3.6. $\sqrt{x^2 - [x]^2} = \sqrt{(-3.6)^2 - [-3.6]^2} = \sqrt{(3.6)^2 - 4^2} = \sqrt{\text{negative quantity}}$.
Therefore, whenever we take negative non-integer values of x, the quantity under the root will negative.
Therefore, the domain of x will be all non-negative values of x plus negative integers.
Therefore, Domain = –I ∪ [0, ∞), where –I denotes the set of negative integers.

Methods for Finding Range of a Function

- **Sketch the graph of the function:** This will more often than not give you an idea about the range. Examine the critical points in the domain and see how the curve behaves at those points. Keep some elementary values and check the behaviour.
- **Solve for independent variable:** I would like to thank Deepanshu, a regular TGite, for pointing out this method.
  1. Put f(x) = y
  2. Calculate x in terms of y means solve to obtain x = P(y)
  3. Calculate domain of P(y)
  4. This domain is nothing but range of F(x).

  For example, let $f(x) = \frac{x}{x+1}$. Find the range of f(x).

  Answer- Let $f(x) = y = \frac{x}{x+1} \Rightarrow x = \frac{y}{1-y}$. Now for x to be real, the denominator cannot be zero. Therefore, y cannot be equal to 1. Therefore, y can take all real values except 1. Therefore, the range of f(x) would equal to R – 1, where R stands for the set of all real numbers. You can see the graph of the function in the adjacent figure. Drawing a graph gives a good idea about how a function is going to behave.



Find the range of the function $f(x) = \frac{x}{1+x^2}$

Answer- Let $f(x) = y = \frac{x}{1+x^2}$. $\Rightarrow x^2y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$. Now x would be real for $y \ne 0$ and $1 - 4y^2 \ge 0 \Rightarrow y \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$

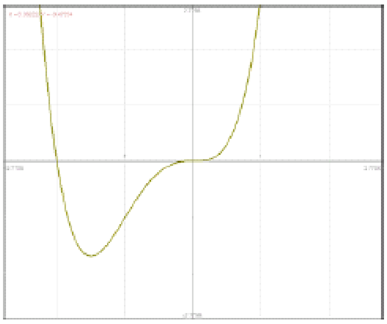
Now that we know the basics of functions, let us see the basic rule for sketching the curve of functions at hand

How to Draw Graph of F(x)

- Find the domain, range and any symmetry f(x) has.
- Calculate. $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
- Determine the critical points of f(x) ($\frac{dy}{dx} = 0$) and use to find if f(x) has local maximum or minimum at these points ($\frac{d^2y}{dx^2} > 0$ for minimum and $\frac{d^2y}{dx^2} < 0$ formaximum).

- Find the inflection points, if any ($\frac{d^2y}{dx^2} = 0$) where the curve changes concavity.
- Determine if the function is increasing or decreasing.
- Identify any asymptotes, i.e. lines which are tangent to the curve at infinity.
- Plot important points such as intercepts on axes, critical points etc.

Draw the graph of the function $f(x) = x^3(x + 2)$.



Answer- We can see that

- the domain of the function consists of all real values of x.
- the function intersects the x-axis at $x = 0$ and $x = -2$, and it is negative between these two points.
- $\frac{df(x)}{dx} = 2x^2(2x+3)$ which is equal to zero at $x = 0$ and $x = -\frac{3}{2}$. The second derivative is positive at $x = -\frac{3}{2}$ which means that the curve has a local minimum at this point.
- $\frac{d^2f(x)}{dx^2} = 12x(x+1)$ which is equal to zero at $x = 0$ and $x = -1$. Which means that the curve changes concavity at these two points. The graph of the curve is shown in the adjacent figure. See how well our deductions fit with the actual graph of the curve.

Composite Functions

As we said earlier, a function can said to be an operator. For example, the plus sign '+' is an addition operator. In this regard, a function can operate on itself or on other functions. Let us say that there are two functions, f(x) and g(x), such that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

$\Rightarrow f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3.$
$\Rightarrow f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5.$

As we can deduce, we can operate the function as many times as we want.

The operation of function has the same rule as that of real numbers-

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)\,g(x)$$

If $f(x) = \frac{1}{x}$, $x \neq 0$, and $f(g(x)) = g(x)$, then what is the value of $f(g(x)) \times g(x)$?

Answer- As $f(x) = \frac{1}{x}$, $f(g(x)) = \frac{1}{g(x)} \Rightarrow f(g(x)) \times g(x) = 1$

If the functions f(x) and g(x) are defined as follows

$$f(x) = \begin{cases} 2x - 3 & \text{for } x \geq \frac{3}{2} \\ 0 & \text{for } x < \frac{3}{2} \end{cases}$$

$$g(x) = \begin{cases} 3x + 2 & \text{for } 0 \leq x < 2 \\ x^2 + 1 & \text{for } 2 \leq x \leq 5 \end{cases}$$

find the value of $f(g(2)) - g(f(3))$

Answer- $f(g(2)) - g(f(3)) = f(2^2 + 1) - g(2 \times 3 - 3) = f(5) - g(3) = 7 - 10 = -3.$

If $f(x + y) = f(x) + f(y) + f(x) \times f(y)$ and $f(1) = 3$ then find f (3).

1. 63 **2. 48** **3. 53** **4. 36**

Answer- Keeping $x = y = 1$ in the equation, we get
$f(2) = f(1) + f(1) + f(1) \times f(1) \Rightarrow f(2) = 15$. Now keeping $x = 2$, $y = 1$ in the equation
$f(3) = f(2) + f(1) + f(2) \times f(1) \Rightarrow f(3) = 63.$

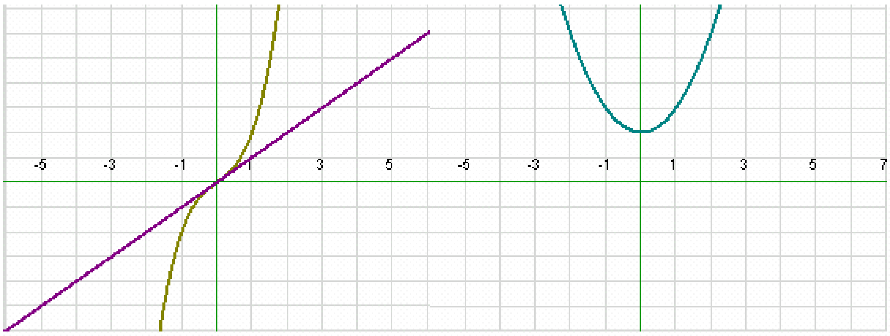
Let f(x) be a function satisfying $f(x)\,f(y) = f(xy)$ for all real x, y. If $f(2) = 4$, then what is the value of $f(\frac{1}{2})$?(CAT 2008)

Answer- Keeping $x = 1$ and $y = 2$, we get $f(1) \times f(2) = f(2) \Rightarrow f(1) = 1$. Now keeping $x = \frac{1}{2}$ and $y = 2$ in the equation we get $f(\frac{1}{2}) \times f(2) = f(1) \Rightarrow f(\frac{1}{2}) = \frac{1}{4}$

Even and Odd Functions

Both even and odd functions are function which have an axis of symmetry- y-axis for even functions and the line $y = x$ for odd functions. Even and odd can be mathematically defined as follows:

A function f(x) is an
Even function if $f(-x) = f(x)$, for example $f(x) = x^2 + 1$ is an even function.
Odd function if $f(-x) = -f(x)$, for example $f(x) = x^3 + x$ is an odd function.



$f(x) = x^3 + x$ $f(x) = x^2 + 2$

In the figure above are shown graphs of two functions, one odd- $x^3 + x$- and one even- $x^2 + 2$. We can see that the line $y = x$ is the axis of symmetry for the first and the line $x = 0$ is the axis of symmetry for the second.

- If f(x) is even and g(x) is even $\Rightarrow$ f(g(x)) is an even function
- If f(x) is odd and g(x) is odd $\Rightarrow$ f(g(x)) is an odd function
- If f(x) is even and g(x) is odd $\Rightarrow$ f(g(x)) is an even function
- If f(x) is odd and g(x) is even $\Rightarrow$ f(g(x)) is an even function

The function f(x) = $\log(x + \sqrt{x^2 + 1})$ **is**
(a) an even function
(b) an odd function
(c) neither odd nor even
(d) cannot be determined

Answer- Now $f(-x) = \log(-x + \sqrt{x^2+1}) \Rightarrow f(x) + f(-x) = \log(x + \sqrt{x^2+1}) + \log(-x + \sqrt{x^2+1}) = \log(x^2 + 1 - x^2) = \log 1 = 0$. As $f(x) + f(-x) = 0$, the function is an odd function.

Periodic Function
A function f(x) is called a periodic function if there is a positive number p such that $f(x + p) = f(x)$ for every x. The smallest such value of p is known as the period of f(x). For example, the period of trigonometric functions- sinx, cosx, tanx etc. - is 2π.

Let g(x) be a function such that $g(x + 1) + g(x - 1) = g(x)$ for every real x. Then for what value of p is the relation $g(x + p) = g(x)$ necessarily true for every real x? (CAT 2005)
1. 5 **2. 3** **3. 2** **4. 6**
Answer- Let $g(x - 1) = a$ and $g(x) = b \Rightarrow g(x + 1) = b - a$. Now, keeping x + 1 in place of x we get
$\Rightarrow g(x + 2) + g(x) = g(x + 1) \Rightarrow g(x + 2) + b = b - a \Rightarrow g(x + 2) = -a$.
Again, keeping x + 2 in place of x we get
$\Rightarrow g(x + 3) + g(x + 1) = g(x + 2) \Rightarrow g(x + 3) + b - a = -a \Rightarrow g(x + 3) = -b$.
Now we can see that $g(x) = b$ and $g(x + 3) = -b$, therefore, $g(x + 3 + 3) = -(-b) = b \Rightarrow g(x + 6) = b$
Therefore, p = 6.

Inverse of a Function
Simply put, an inverse of a function y = f(x) is a function in the reverse direction $x = f^{-1}y$, i.e. with range as the input and domain as the output.

How to find the inverse of a function? For example, how do I find the inverse of the function $f(x) = y = \frac{2x+3}{x-5}$?

Here are the steps-

- Write the original function $y = \frac{2x+3}{x-5}$
- Solve for $x \Rightarrow x = \frac{5y+3}{y-2}$
- Switch x and $y \Rightarrow f(x) = y = \frac{5x+3}{x-2}$

Remember that a function and its inverse are mirror images of each other along the line y = x

If the inverse of the function $f(x) = ax + b$ is $f^{-1}(x) = bx + a$, then the values of a and b are
1. 1, 1 **2. −1, 1** **3. 1, -1** **4. −1, -1**
Answer- Let $f(x) = y = ax + b \Rightarrow x = \frac{y}{a} - \frac{b}{a} \Rightarrow f^{-1}(x) = \frac{x}{a} - \frac{b}{a} = bx + a$ (given). Comparing coefficients both sides, we get $\frac{1}{a} = b$ and $-\frac{b}{a} = a$. Solving, we get a = −1 and b = −1.

Function as a Series

Let f(n) be a function such that $f(n+1) = \frac{2f(n)+1}{2}$ **for n = 1, 2, 3, ... and f(1) = 2. Then f(101) is equal to**

Answer- Arranging the equation we get $f(n + 1) - f(n) = \frac{1}{2} \Rightarrow f(2) - f(1) = \frac{1}{2}$, $f(3) - f(2) = \frac{1}{2}$, ... $f(101) - f(100) = \frac{1}{2}$. Adding all the equations we get $f(101) - f(1) = 50 \Rightarrow f(101) = 52$

$= \frac{1}{2}$. Adding all the equations we get f(101) − f(1) = 50 ⇒ f(101) = 52.

A function f(x) satisfies f(1) = 3600, and f(1) + f(2) + … + f(n) = n^2f(n), for all positive integers n > 1. What is the value of f(9)? (CAT 2007)

Answer- Writing the equation for n + 1 instead of n we get f(1) + f(2) + … + f(n + 1) = $(n + 1)^2 f(n + 1)$.
Subtracting the original equation f(1) + f(2) + … + f(n) = n^2f(n) from the above equation we get

$f(n + 1) = (n + 1)^2 f(n + 1) - n^2 f(n) \Rightarrow f(n+1) = \frac{n}{n+2} f(n)$. Therefore,

$$f(9) = \frac{8}{10} f(8) = \frac{8}{10} \times \frac{7}{9} f(7) = \ldots = \frac{8}{10} \times \frac{7}{9} \times \frac{6}{8} \times \frac{5}{7} \times \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} f(1) = 80$$

If $f(x) = \frac{9^x}{9^x + 3}$**, then the value of** $f\left(\frac{1}{882}\right) + f\left(\frac{2}{882}\right) + \ldots + f\left(\frac{881}{882}\right)$ **will be**

Answer- $f(x) = \frac{9^x}{9^x + 3} \Rightarrow f(1 - x) = \frac{9^{1-x}}{9^{1-x} + 3} = \frac{\frac{9}{9^x}}{\frac{9}{9^x} + 3} = \frac{9}{9 + 3 \times 9^x} = \frac{3}{3 + 9^x}$

Therefore, $f(x) + f(1 - x) = \frac{9^x}{9^x + 3} + \frac{3}{3 + 9^x} = 1$

Therefore, $f\left(\frac{1}{882}\right) + f\left(\frac{881}{882}\right) = f\left(\frac{2}{882}\right) + f\left(\frac{880}{882}\right) = \ldots = 1$. There are 440 such pairs and the middle term would be

$f\left(\frac{441}{882}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}$. Therefore, $f\left(\frac{1}{882}\right) + f\left(\frac{2}{882}\right) + \ldots + f\left(\frac{881}{882}\right) = 440 + \frac{1}{2} = 440.5$

For a function f(x), f(x) + f(x – 1) = x^2 and f(19) = 94. Find f(94)

Answer- $f(x) = x^2 - f(x - 1) \Rightarrow f(94) = 94^2 - f(93) = 94^2 - (93^2 - f(92)) = 94^2 - 93^2 + (92^2 - f(91) = \ldots$

$= 94^2 - 93^2 + 92^2 - 91^2 + 90^2 - 89^2 + \ldots + 20^2 - f(19)$

You all can calculate now ☺