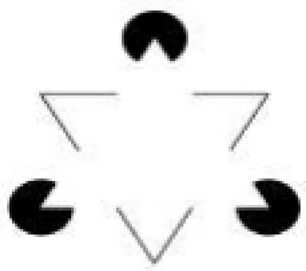
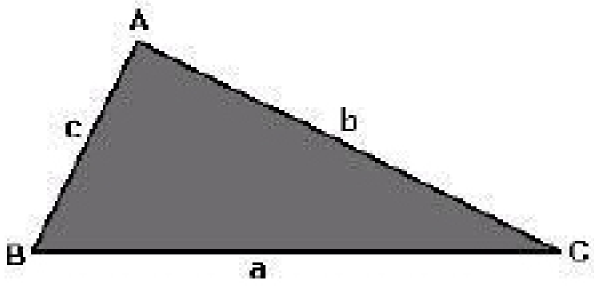
Properties of Triangles
by Total Gadha - Thursday, 5 July 2007, 06:40 AM



It would be a shame if CAT 2007 or CAT 2008 aspirants have still not discovered that geometry is one of the most important areas to prepare in the quant section. They should also know that triangle is the most common figure they will see in the geometry section. Be it a quadrilateral or a hexagon, triangles and their properties will be present in every figure. Although the ideal way to learn these properties is to practice and derive all the theorems related to these properties themselves, the students can go through this small compendium to have a quick recap of those formula. The following properties do not cover the similarity of triangles. Of course that will take an entire article by itself.

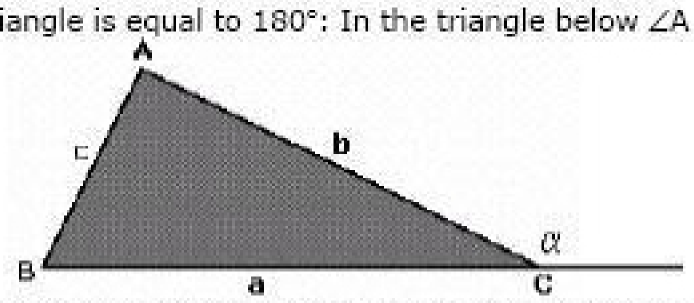


General Properties of Triangles:

1. The sum of the two sides is greater than the third side: $a + b > c$, $a + c > b$, $b + c > a$

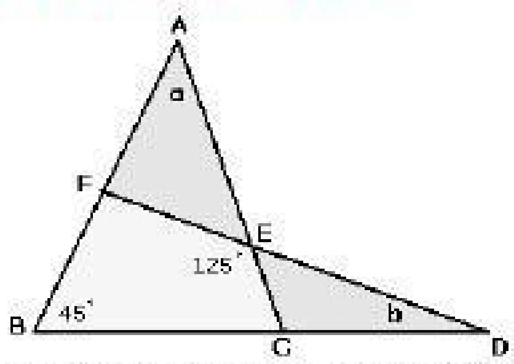
   Problem: The two sides of a triangle are 12 cm and 7 cm. If the third side is an integer, find the sum of all the values of the third side.
   Answer: Let the third side be of x cm. Then, $x + 7 > 12$ or $x > 5$. Therefore, minimum value of x is 6. Also, $x < 12 + 7$ or $x < 19$. Therefore, the highest value of x is 18. The sum of all the integer values from 6 to 18 is equal to 156.

2. The sum of the three angles of a triangle is equal to 180°: In the triangle below $\angle A + \angle B + \angle C = 180°$

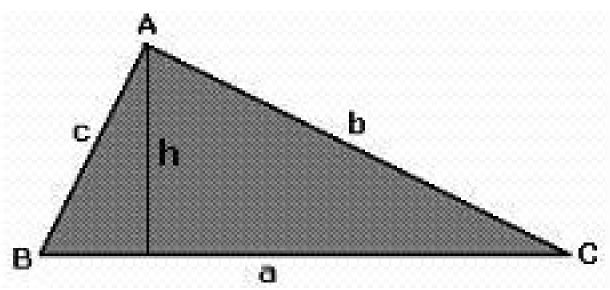


   Also, the exterior angle α is equal to sum the two opposite interior angle A and B, i.e. $\alpha = \angle A + \angle B$.

   Problem: Find the value of a + b in the figure given below:



   Answer: In the above figure, $\angle CED = 180° - 125° = 55°$. $\angle ACD$ is the exterior angle of ΔABC. Therefore, $\angle ACD = a + 45°$.
   In ΔCED, $a + 45° + 55° + b = 180° \Rightarrow a + b = 80°$



3. **Area of a Triangle:**

   $$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times a \times h$$

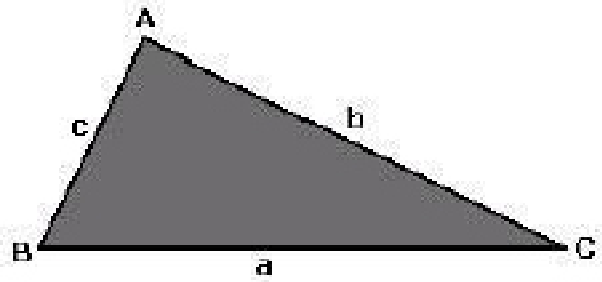
   $$\text{Area of a triangle} = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$

   $$\text{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{a+b+c}{2}$$

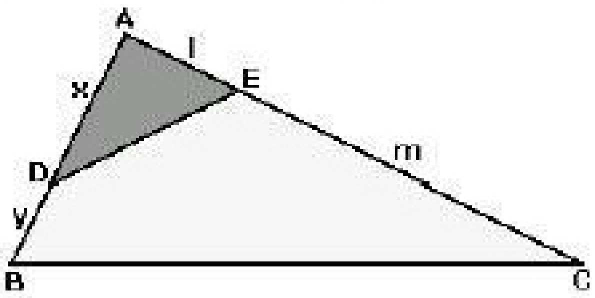
   $$\text{Area of a triangle} = \frac{abc}{4R} \text{ where R = circumradius}$$

   $$\text{Area of a triangle} = r \times s \text{ where r = inradius and } s = \frac{a+b+c}{2}$$

4. **More Rules:**

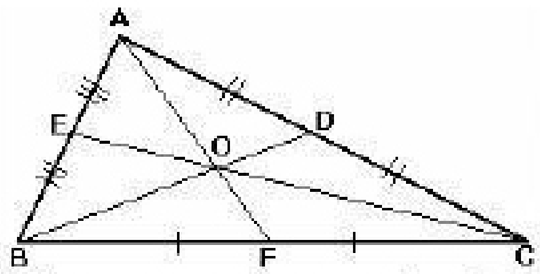


   - Sine Rule: $\dfrac{\text{Sin}A}{a} = \dfrac{\text{Sin}B}{b} = \dfrac{\text{Sin}C}{c}$
   - Cosine Rule: $\text{Cos}A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\text{Cos}B = \dfrac{a^2 + c^2 - b^2}{2ac}$, $\text{Cos}C = \dfrac{b^2 + a^2 - c^2}{2ab}$



Let D and E be on sides AB and AC of triangle ABC such that $\frac{AD}{DB} = \frac{x}{y}$ and $\frac{AE}{EC} = \frac{l}{m}$. Then, area triangle ADE = $\frac{1}{2} lx \sin A$ and area triangle ABC = $\frac{1}{2}(l+m)(x+y)\sin A$. Therefore, $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta ABC} = \frac{lx}{(x+y)(l+m)}$

5. **Medians of a triangle:**

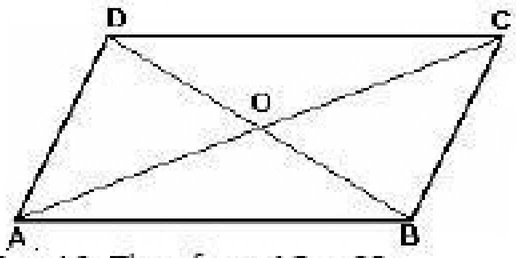


The medians of a triangle are lines joining a vertex to the midpoint of the opposite side. In the figure, AF, BD and CE are medians. The point where the three medians intersect is known as the **centroid**. O is the centroid in the figure.

- The medians divide the triangle into two equal areas. In the figure, area ΔABF = area ΔAFC = area ΔBDC = area ΔBDA = area ΔCBE = area ΔCEA = $\frac{\text{Area } \Delta ABC}{2}$
- The centroid divides a median internally in the ratio 2: 1. In the figure, $\frac{AO}{OF} = \frac{BO}{OD} = \frac{CO}{OE} = 2$
- Apollonius Theorem: $AB^2 + AC^2 = 2(AF^2 + BF^2)$ or $BC^2 + BA^2 = 2(BD^2 + DC^2)$ or $BC^2 + AC^2 = 2(EC^2 + AE^2)$ REMEMBER!

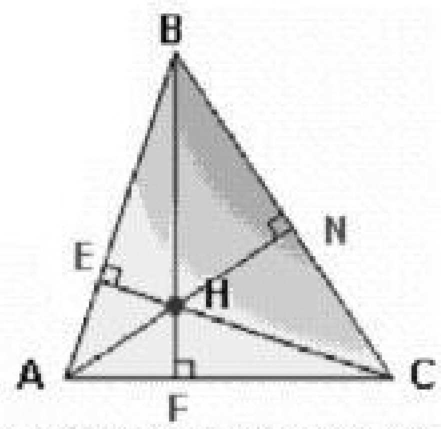
Problem: ABCD is a parallelogram with AB = 21 cm, BC = 13 cm and BD= 14 cm. Find the length of AC.

Answer: The figure is shown below. Let AC and BD intersect at O. O bisects AC and BD. Therefore, OD is the median in triangle ADC.



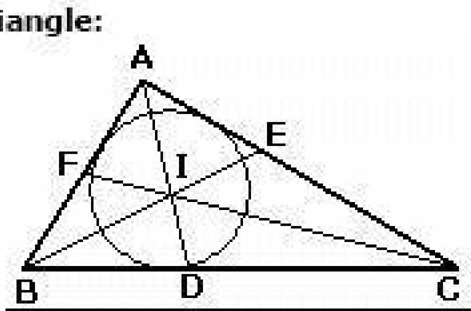
$\Rightarrow AD^2 + CD^2 = 2(AO^2 + DO^2) \Rightarrow AO = 16$. Therefore, AC = 32.

6. **Altitudes of a Triangle:**



The altitudes are the perpendiculars dropped from a vertex to the opposite side. In the figure, AN, BF, and CE are the altitudes, and their point of intersection, H, is known as the orthocenter.
Triangle ACE is a right-angled triangle. Therefore, ∠ECA = 90° – ∠A. Similarly in triangle CAN, ∠CAN = 90° – ∠C. In triangle AHC, ∠CHA = 180° – (∠HAC + ∠HCA) = 180° – (90° – ∠A + 90° – ∠C) = ∠A + ∠C = 180° – ∠B.
Therefore, ∠AHC and ∠B are supplementary angles.

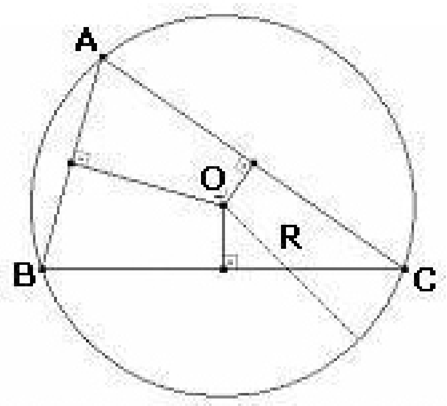
7. **Internal Angle Bisectors of a Triangle:**



In the figure above, AD, BE and CF are the internal angle bisectors of triangle ABC. The point of intersection of these angle bisectors, I, is known as the incentre of the triangle ABC, i.e. centre of the circle touching all the sides of a triangle.

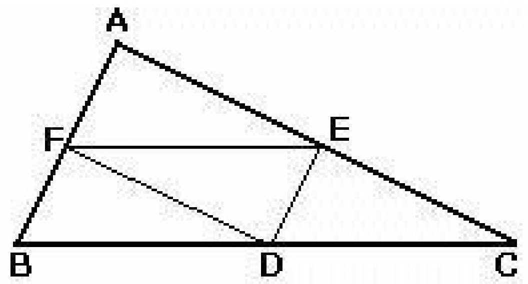
- $\angle BIC = 180° - (\angle IBC + \angle ICB) = 180 - \left(\frac{B}{2} + \frac{C}{2}\right) = 180 - \left(\frac{B+C}{2}\right) = 180 - \left(\frac{180 - A}{2}\right) = 90 + \frac{A}{2}$
- $\frac{AB}{AC} = \frac{BD}{CD}$ (internal bisector theorem) REMEMBER!

8. **Perpendicular Side Bisectors of a Triangle:**



In the figure above, the perpendicular bisectors of the sides AB, BC and CA of triangle ABC meet at O, the circumcentre (centre of the circle passing through the three vertices) of triangle ABC. In figure above, O is the centre of the circle and BC is a chord. Therefore, the angle subtended at the centre by BC will be twice the angle subtended anywhere else in the same segment. Therefore, ∠BOC = 2∠BAC.

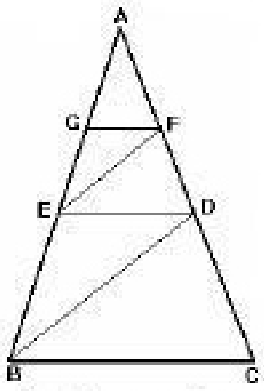
9. **Line Joining the Midpoints:**



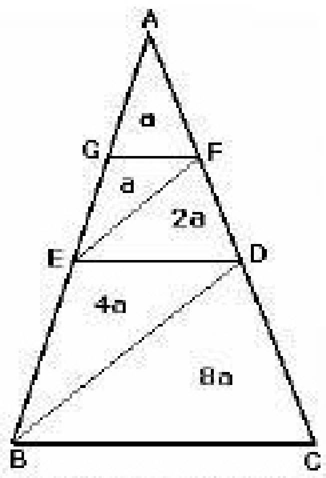
In the figure above, D, E and F are midpoints of the sides of triangle ABC. It can be proved that:

- FE // BC, DE // AB and DF // AC.
- $FE = \frac{BC}{2}$, $DE = \frac{AB}{2}$, $FD = \frac{AC}{2}$
- Area ΔDEF = Area ΔAFE = Area ΔBDF = Area $\Delta DEC = \frac{\text{Area } \Delta ABC}{4}$
- **Corollary:** If a line is parallel to the base and passes through midpoint of one side, it will pass through the midpoint of the other side also.

Problem: In the figure given below: AG = GE and GF // ED, EF //BD and ED // BC. Find the ratio of the area of triangle EFG to trapezium BCDE.



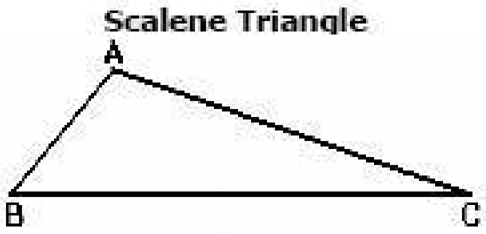
Answer: We know that line parallel to the base and passing through one midpoint passes through another midpoint also. Using this principle, we can see that G, F, E and D are midpoints of AE, AD, AB, and AC respectively. **Therefore, GF, EF, ED, and BD are medians in triangles AFE, AED, ADB and ABC.**



We know that medians divide the triangle into two equal areas. Let the area of triangle AGF = a. Therefore, the areas of the rest of the figures are as shown above.
The required ratio = a/12a = 1/12.

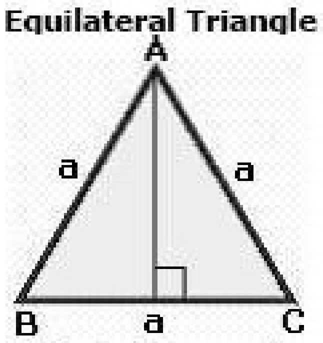
TYPES OF TRIANGLES

Scalene Triangle



No side equal.
All the general properties of triangle apply

Equilateral Triangle



Each side equal
Each angle = 60°

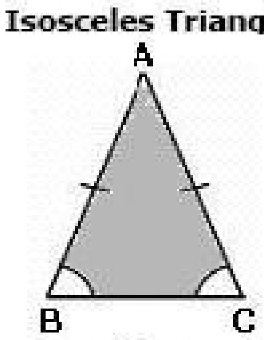
Length of altitude $= \frac{\sqrt{3}}{2}a$

Area $= \frac{\sqrt{3}}{4}a^2$

Inradius $= \frac{a}{2\sqrt{3}}$

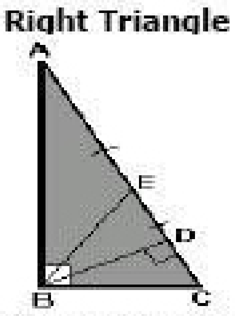
Circumradius $= \frac{a}{\sqrt{3}}$

Isosceles Triangle



Two sides equal.
The angles opposite to the opposite sides are equal.

Right Triangle



One of the angles is a right angle, i.e. 90°.

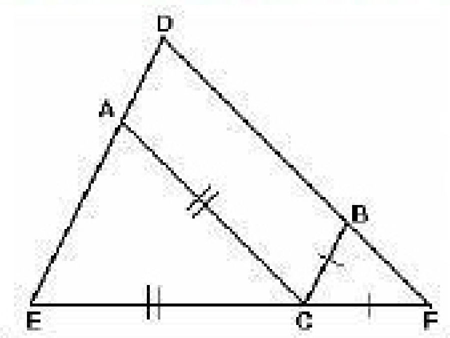
Area $= \frac{1}{2} AB \times BC$

$AC^2 = AB^2 + BC^2$

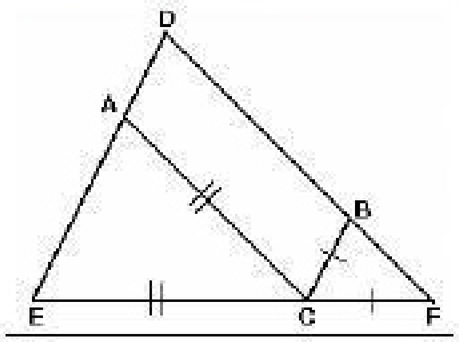
Altitude BD $= \frac{AB \times BC}{AC}$

The midpoint of the hypotenuse is equidistant from all the three vertices, i.e. EA = EB = EC

Problem: In triangle DEF shown below, points A, B, and C are taken on DE, DF and EF respectively such that EC = AC and CF = BC. If angle D = 40° then what is angle ACB in degrees? (CAT 2001)



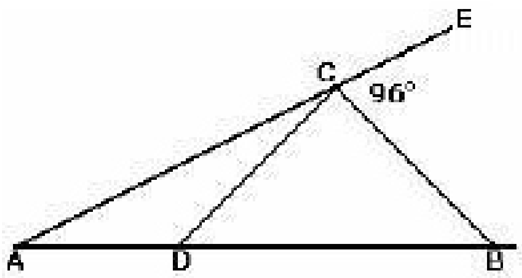
1. 140 2. 70 3. 100 4. None of these



1. 140 2. 70 3. 100 4. None of these

Answer: Let ∠AEC = ∠EAC = α and ∠CBF = ∠CFB = β. We know that $\alpha + \beta = 180° - \angle D = 140°$. $\angle ACB = 180° - (\angle ECA + \angle BCF) = 180° - (180° - 2\alpha + 180° - 2\beta) = 100°$.

Problem: In the figure (not drawn to scale) given below, if AD = CD = BC, and ∠BCE = 96°, how much is ∠DBC? (CAT 2003)



1. 32° 2. 84° 3. 64° 4. Cannot be determined.

Answer: Let ∠DAC = ∠ACD = α and ∠CDB = ∠CBD = β. As ∠CDB is the exterior angle of triangle ACD, $\beta = 2\alpha$.
Now $\angle ACD + \angle DCB + 96° = 180° \Rightarrow \alpha + 180° - 2\beta + 96° = 180° \Rightarrow 3\alpha = 96° \Rightarrow \alpha = 32° \Rightarrow \beta = 64°$

In the next chapter, I will cover similarity of triangles and its uses.

Total Gadha