Some Mathematical Curios
by Total Gadha - Tuesday, 31 July 2007, 03:48 AM

Presented below are some of the findings that many CAT 2007 aspirants will find useful and many CAT 2008 asp
them in their armory early on. Unfortunately, I have not been able to provide the proofs for these results as the
attention span of my readers, is at a premium. Perhaps, I will do so in my later chapters. Working as a teacher,
useful results that can prove to be beneficial to oneâ€™s mathematical health but which do not reach the studen
properly documented. Well, hereon, I promise to document every useful result/proof that crosses the eyes and
Feel free to ask any question that arises in your curious mind.

The number of ways a number can be written as sum of two or more consecutive positive integers

Let's start small:
In how many ways can 15 be written as sum of two or more consecutive positive integers?
$15 = 7 + 8$
$15 = 4 + 5 + 6$
$15 = 1 + 2 + 3 + 4 + 5$. Therefore, the number of ways = 3.
Now $15 = 3 \times 5$
$\Rightarrow$ Number of odd divisors = 4.
The number of ways 15 can be written as sum of two or more positive integers = Number of odd divisors – 1.

Let's see it one more time:
In how many ways can 90 be written as sum of two or more consecutive positive integers?
$90 = 29 + 30 + 31$
$90 = 21 + 22 + 23 + 24$
$90 = 16 + 17 + 18 + 19 + 20$
$90 = 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14$
$90 = 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13$. Therefore, the number of ways = 5.
Now $90 = 2 \times 3^2 \times 5$
$\Rightarrow$ Number of odd divisors = 6.
The number of ways 90 can be written as sum of two or more positive integers = Number of odd divisors – 1.

Therefore,

The number of ways a number N can be written as sum of two or more consecutive positive integers = (Number of odd divisors of N) – 1

The number of whole number solutions of the equation ax + by + cz = n, where a, b, and c are constants

Find the number of whole number solutions of the equation be $2x + 3y + 5z = 100$.
The number of solution of the equation = coefficient of x^{100} in $(1 + x^2 + x^4 + x^6 \ldots)(1 + x^3 + x^6 + x^9 + \ldots)(1 + x^5 + x^{10} + x^{15} + \ldots) = 184$

Therefore,

The number of whole number solutions of the equation $ax + by + cz = n$ is equal to the coefficient of x^n in $(1 + x^a + x^{2a} + x^{3a} + \ldots)(1 + x^b + x^{2b} + x^{3b} + \ldots)(1 + x^c + x^{2c} + x^{3c} + \ldots)$

What if x, y or z cannot be equal to zero? Then the term for x^0, y^0, z^0 is absent in the expression.

The number of positive integral solutions of the equation $ax + by + cz = n$ is equal to the coefficient of x^n in $(x^a + x^{2a} + x^{3a} + \ldots)(x^b + x^{2b} + x^{3b} + \ldots)(x^c + x^{2c} + x^{3c} + \ldots)$

Questions for practise:

How many numbers between 100 and 900 have sum of their digits equal to 12?
How many integer solutions exist for the equation $|x| + |y| + |z| = 15$?

To find integer values of x and y in the indeterminate equation ax + by = c, given one pair of values

Find the values (x, y) satisfying the equation $3x + 7y = 200$, where x and y are integers?

In the above example, we saw that some of the values were (62, 2), (55, 5), (48, 8), (41, 11), (34, 14), (27, 17), (20, 20), (13, 23), and (6, 26). Here is one important point: the values of x are in an arithmetic progression with a common difference of 7 and the values of y are in an arithmetic progression with a common difference of 3.

Therefore,

In indeterminate equation $ax + by = c$, the integer values of
- **x** lie in an arithmetic progression with a common difference of **b**
- **y** lie in an arithmetic progression with a common difference of **a**

If *x* and *y* are integers then the equation $5x + 19y = 64$ has **(CAT 2003)**

1. no solution for $x < 300$ and $y < 0$
2. no solution for $x > 250$ and $y > -100$
3. a solution for $250 < x < 300$
4. a solution for $-59 < y < -56$

If we keep $y = 1$ in the equation, we get $x = 9$. Therefore, (9, 1) is a solution. Now, values of x will be in an arithmetic series with a common difference of 19. Therefore, the values of x will be 9, 28, 47, 66.... The values of y will be in an arithmetic progression with a common difference of 5. Therefore, the values of y will be 1, –4, –9, –14...

We can see that there will be a solution for values of x between 250 and 300 whereas no integer value of y falls between –56 and –59. Therefore, C.

Any single digit number N written p – 1 times, where p is a prime number (≠ 2, 3, 5) is divisible by p

For example, 555555 is divisible by 7, $\underbrace{888...888}_{16\text{ times}}$ is divisible by 17 etc.

What is the remainder when $\underbrace{7777....7777}_{37\text{ times}}$ is divided by 19?

Answer: We know that $\underbrace{7777....7777}_{18\text{ times}}$ will be divisible by 19. Similarly, $\underbrace{7777....7777}_{36\text{ times}}$ will be divisible by 19. Therefore, the remainder when $\underbrace{7777....7777}_{37\text{ times}}$ is divided by 19 is 7.

If A is prime to N, the remainder when $A^{p(N)}$ is divided by N, where p(N) is the number of numbers less than and prime to N, is 1

The number of numbers less than and prime to a number N, where N = $(a^x)(b^y)(c^z)$ and a, b, c are prime numbers, $p(N) = N(1-\frac{1}{a})(1-\frac{1}{b})(1-\frac{1}{c})...$

What is the remainder when 35^{100} is divided by 24?

Answer: 24 = $(2^3)(3)$. Therefore, number of numbers less than and prime to 24 = $p(24) = 24(1-\frac{1}{2})(1-\frac{1}{3}) = 8$. Therefore, 35^8 will give a remainder 1 with 24. Now, $35^{100} = (35^8)^{12} \times (35^4)$. $(35^8)^{12}$ will give remainder 1 with 24. $35^4 = (24 + 11)^4 = 11^4 = 121 \times 121 = 1$. Therefore, remainder = 1

Find the remainder when 19^{74} is divided by 13.

Answer: Number of numbers less than and prime to 13 = 12. Therefore, 19^{12} will give remainder 1 with 12. Now, $19^{74} = 19^{72} \times 19^2 \Rightarrow 19^{72}$ will give remainder 1 and $19^2 = 361$ will give remainder 10. Therefore, remainder = 10

Find the remainder when $(11)^{25} + (13)^{25}$ is divided by 36.

Answer: Number of numbers less than and prime to 36 = $p(36) = 36(1-\frac{1}{2})(1-\frac{1}{3}) = 12$. Therefore, 11^{12} and 13^{12} will give a remainder of 1 with 36. Similarly, 11^{24} and 13^{24} will give a remainder of 1 with 36. Therefore, remainder = remainder when 11 + 13 is divided by 36 = 24

Finding the last two digits of a number

To find the last two digits of a number, we find the power that will yield the same last two digits again and again. For example, in case of even numbers, 76 as the last two digits will always give 76 with any power. Similarly, in case of odd numbers, 01 as the last two digits will always give 01 with any power. Therefore, we always try to find a power of the number which will give 76 (if the number is even) or 01 (if the number is odd).