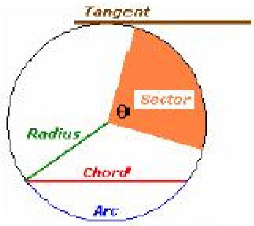
Geometry- Circle and its properties
by Total Gadha - Sunday, 12 August 2007, 01:48 AM

Before all the CAT 2007 aspirants give up on me, here is another big chapter on geometry to whet their appetite for CAT quant. CAT 2008 aspirants are also advised to assimilate the following theorems and results into their system as the present chapter covers one of the most important topics in MBA entrance exams- circles. Time and again, I have been told that students either have a knack for geometry or they donâ€™t. During my three years as a quant instructor, I have seen many instructors and students who could not make head or tail of geometry in the beginning, but who worked hard at memorizing and practicing all the theorems and slowly became very good at it, and developed the required visualization powers. One thing is for sure; if you aspire to do an MBA, you are surely not escaping from geometry. And here comes one more heavy geometry dose- circles.

A circle is a set of all points in a plane that lie at a constant distance from a fixed point. The fixed point is called the center of the circle and the constant distance is known as the radius of the circle.



Arc: An arc is a curved line that is part of the circumference of a circle. A **minor arc** is an arc less than the semicircle and a **major arc** is an arc greater than the semicircle.

Chord: A chord is a line segment within a circle that touches 2 points on the circle.

Diameter: The longest distance from one end of a circle to the other is known as the diameter. It is equal to twice the radius.

Circumference: The perimeter of the circle is called the circumference. The value of the circumference = $2\pi r$, where r is the radius of the circle.

Area of a circle: Area = π x (radius)² = πr^2.

Sector: A sector is like a slice of pie (a circular wedge).

Area of Circle Sector: (with central angle θ) Area = $\frac{\theta}{360} \times \pi \times r^2$

Length of a Circular Arc: (with central angle θ) The length of the arc = $\frac{\theta}{360} \times 2\pi \times r$

Tangent of circle: A line perpendicular to the radius that touches ONLY one point on the circle

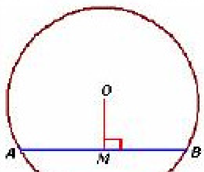
If 45° arc of circle A has the same length as 60° arc of circle B, find the ratio of the areas of circle A and circle B.

Answer: Let the radius of circle A be r_1 and that of circle B be r_2.

$\Rightarrow \frac{45}{360} \times 2\pi \times r_1 = \frac{60}{360} \times 2\pi \times r_2 \Rightarrow \frac{r_1}{r_2} = \frac{4}{3} \Rightarrow$ Ratio of areas $= \frac{\pi r_1^2}{\pi r_2^2} = \frac{16}{9}$.

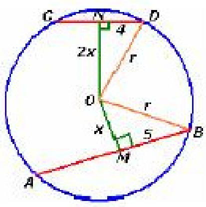
RULE!

The perpendicular from the center of a circle to a chord of the circle bisects the chord. In the figure below, O is the center of the circle and OM ⊥ AB. Then, AM = MB.



Conversely, the line joining the center of the circle and the midpoint of a chord is perpendicular to the chord.

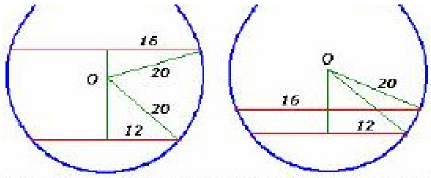
In a circle, a chord of length 8 cm is twice as far from the center as a chord of length 10 cm. Find the circumference of the circle.



Answer: Let AB and CD be two chords of the circle such that AB = 10 and CD = 8. Let O be the center of the circle and M and N be the midpoints of AB and CD. Therefore OM ⊥ AB, ON ⊥ CD, and if ON = 2x then OM = x.
$BM^2 + OM^2 = OB^2$ and $DN^2 + ON^2 = OD^2$.
OB = OD = r → $(2x)^2 + 4^2 = r^2$ and $x^2 + 5^2 = r^2$.
Equating both the equations we get, $4x^2 + 16 = x^2 + 25$
Or $x = \sqrt{3} \rightarrow r = 2\sqrt{7}$. Therefore circumference = $2\pi r = 4\pi\sqrt{7}$.

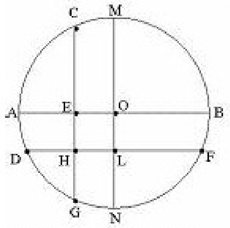
What is the distance in cm between two parallel chords of length 32 cm and 24 cm in a circle of radius 20 cm? (CAT 2005)
1. 1 or 7   2. 2 or 14   3. 3 or 21   4. 4 or 28

Answer: The figures are shown below:



The parallel chords can be on the opposite side or the same side of the centre O. The perpendicular (s) dropped on the chords from the centre bisect (s) the chord into segments of 16 cm and 12 cm, as shown in the figure. From the Pythagoras theorem, the distances of the chords from the centre are $\sqrt{20^2 - 16^2} = 12$ and $\sqrt{20^2 - 12^2} = 16$, respectively. Therefore, the distances between the chords can be 16 + 12 = 28 cm or 16 − 12 = 4 cm.

In the following figure, the diameter of the circle is 3 cm. AB and MN are two diameters such that MN is perpendicular to AB. In addition, CG is perpendicular to AB such that AE:EB = 1:2, and DF is perpendicular to MN such that NL:LM = 1:2. The length of DH in cm is (CAT 2005)



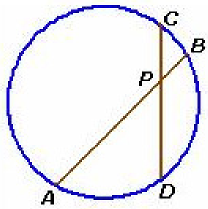
Answer: In the above figure, AB = MN = 3 cm and AE: EB = NL: LM = 1: 2
$\Rightarrow$ AE = NL = 1 cm. Now AO = NO = 1.5 cm $\Rightarrow$ OE = HL = OL = 0.5 cm. Join O and D
$\Rightarrow OD^2 = OL^2 + DL^2 \Rightarrow DL^2 = \sqrt{OD^2 - OL^2} = \sqrt{1.5^2 - 0.5^2} = \sqrt{2} \Rightarrow DH = DL - HL = \sqrt{2} - \frac{1}{2} = \frac{2\sqrt{2} - 1}{2}$

RULE!

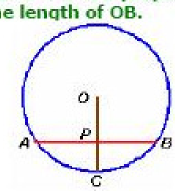
Equal chords are equidistant from the center. Conversely, if two chords are equidistant from the center of a circle, they are equal.

RULE!

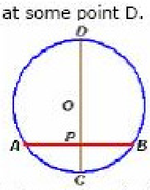
In the following figure, two chords of a circle, AB and CD, intersect at point P. Then, AP × PB = CP × PD.



In the following figure, length of chord AB = 12. O-P-C is a perpendicular drawn to AB from center O and intersecting AB and the circle at P and C respectively. If PC = 2, find the length of OB.



Answer: Let us extend OC till it intersects the circle at some point D.



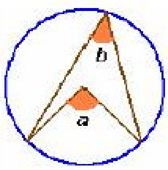
CD is the diameter of the circle. Since OP is perpendicular to AB, P is the midpoint of AB. Hence, AP = PB = 6. Now DP × PC = AP × PB → DP = 18. Therefore, CD = 20 → OC = 10. OB = OC = radius of the circle = 10.

RULE!

In a circle, equal chords subtend equal angles at the center.

RULE!

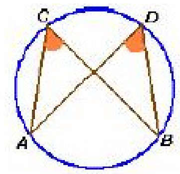
The angle subtended by an arc of a circle at the center is double the angle subtended by it at any point on the remaining part of the circumference.



In the figure shown above, a = 2b.

RULE!

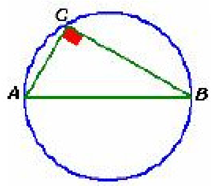
Angles inscribed in the same arc are equal.



In the figure angle ACB = angle ADB.

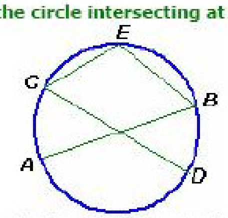
RULE!

An angle inscribed in a semi-circle is a right angle.

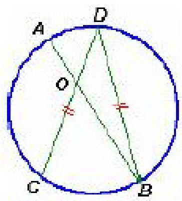


Let angle ACB be inscribed in the semi-circle ACB; that is, let AB be a diameter and let the vertex C lie on the circumference; then angle ACB is a right angle.

In the figure AB and CD are two diameters of the circle intersecting at an angle of 48°. E is any point on arc CB. Find angle CEB.



Answer: Join E and D. Since arc BD subtends an angle of 48° at the center, it will subtend half as many degrees on the remaining part of circumference as it subtends at the center. Hence, angle DEB = 24°. Since angle CED is made in a semicircle, it is equal to 90°. Hence, angle CEB = angle CED + angle DEB = 90° + 24° = 114°.



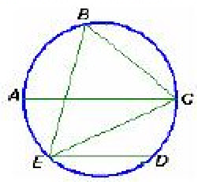
In the above figure, AB is a diameter of the circle and C and D are such points that CD = BD. AB and CD intersect at O. If angle AOD = 45°, find angle ADC.

Answer: Draw AC and CB. CD = BD $\Rightarrow \angle DCB = \angle DBC = \theta$ (say). $\angle ACB = 90° \Rightarrow \angle ACD = 90° - \theta$.

$\angle ABD = \angle ACD = 90° - \theta \Rightarrow \angle ABC = \theta - (90° - \theta) = 2\theta - 90$. In ΔOBC, $45° + 2\theta - 90 + \theta = 180° \Rightarrow 3\theta = 225° \Rightarrow \theta = 75°$.

$\angle ADC = \angle ABC = 2\theta - 90 = 60°$.

In the adjoining figure, chord ED is parallel to the diameter AC of the circle. If angle CBE = 65°, then what is the value of angle DEC? (CAT 2004)

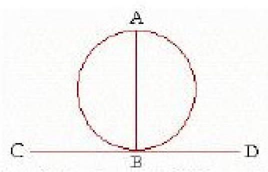


1. 35° 2. 55° 3. 45° 4. 25°

Answer: $\angle ABC = 90° \Rightarrow \angle ABE = 90 - \angle EBC = 25°$. $\angle ABE = \angle ACE = 25°$. $\angle ACE = \angle CED = 25°$ (alternate angles)

RULE!

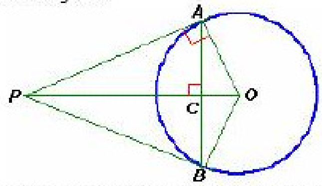
The straight line drawn at right angles to a diameter of a circle from its extremity is tangent to the circle. Conversely, If a straight line is tangent to a circle, then the radius drawn to the point of contact will be perpendicular to the tangent.



Let AB be a diameter of a circle, and let the straight line CD be drawn at right angles to AB from its extremity B; then the straight line CD is tangent to the circle.

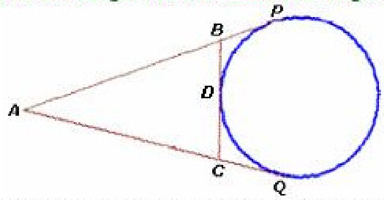
RULE!

If two tangents are drawn to a circle from an exterior point, the length of two tangent segments are equal. Also, the line joining the exterior point to the centre of the circle bisects the angle between the tangents.



In the above figure, two tangents are drawn to a circle from point P and touching the circle at A and B. Then, PA = PB. Also, ∠APO = ∠BPO. Also, the chord AB is perpendicular to OP.

In the following figure, lines AP, AQ and BC are tangent to the circle. The length of AP = 11. Find the perimeter of triangle ABC.

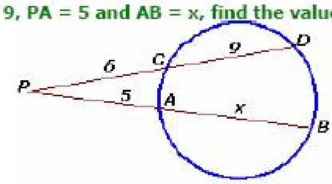


Answer: let AB = x and BP = y. Then, BD = BP because they are tangents drawn from a same point B. Similarly CD = CQ and AP = AQ. Now perimeter of triangle ABC = AB + BC + CA = AB + BD + DC + AC = AB + BP + CQ + AC = AP + AQ = 2AP = 22.

RULE!

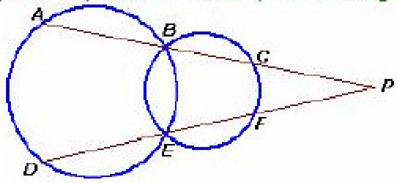
From an external point P, a secant P-A-B, intersecting the circle at A and B, and a tangent PC are drawn. Then, PA × PB = PC^2.

In the following figure, if PC = 6, CD = 9, PA = 5 and AB = x, find the value of x



Answer: Let a tangent PQ be drawn from P on the circle. Hence, PC × PD = PQ^2 = PA × PB → 6 × 15 = 5 × (5 + x) → x = 13

In the following figure, PC = 9, PB = 12, PA = 18, and PF = 8. Then, find the length of DE.

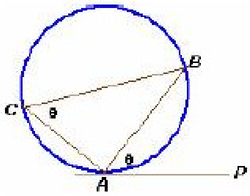


Answer: In the smaller circle PC × PB = PF × PE → PE = $12 \times \frac{9}{8} = \frac{27}{2}$. In the larger circle, PB × PA = PE × PD → PD = $12 \times 18 \times \frac{2}{27}$ = 16.

Therefore, DE = PD – PE = 16 – 13.5 = 2.5

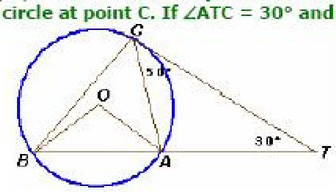
RULE!

The angle that a tangent to a circle makes with a chord drawn from the point of contact is equal to the angle subtended by that chord in the alternate segment of the circle.



In the figure above, PA is the tangent at point A of the circle and AB is the chord at point A. Hence, angle BAP = angle ACB.

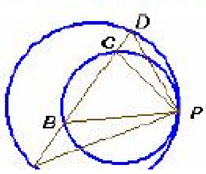
In the figure given below (not drawn to scale), A, B and C are three points on a circle with centre O. The chord BA is extended to a point T such that CT becomes a tangent to the circle at point C. If ∠ATC = 30° and ∠ACT = 50°, then the angle ∠BOA is (CAT 2003)



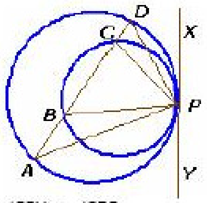
1. 100°   2. 150°   3. 80°   4. not possible to determine

Answer: Tangent TC makes an angle of 50° with chord AC. Therefore, ∠TBC = 50°. In triangle TBC, ∠BCT = 180° – (30° + 50°) = 100°. Therefore, ∠BCA = ∠BCT – ∠ACT = 100° – 50° = 50°. ∠BOA = 2∠BCA = 100°.

Two circles touch internally at P. The common chord AD of the larger circle intersects the smaller circle in B and C, as shown in the figure. Show that , ∠APB = ∠CPD.



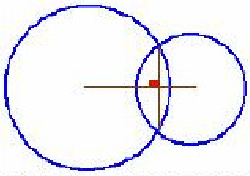
Answer: Draw the common tangent XPY at point P.



Now, for chord DP, ∠DPX = ∠DAP, and for chord PC, ∠CPX = ∠CBP
⇒ ∠CPD = ∠CPX − ∠DPX = ∠CBP − ∠DAP. In triangle APB, ∠CBP is the exterior angle
⇒ ∠CBP = ∠CAP + ∠APB
⇒ ∠CBP − ∠CAP = ∠APB
⇒ ∠CPD = ∠CPX − ∠DPX = ∠CBP − ∠DAP = ∠APB

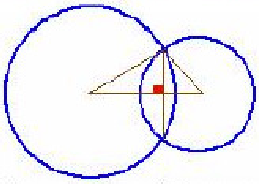
RULE!

When two circles intersect each other, the line joining the centers bisects the common chord and is perpendicular to the common chord.



In the figure given above, the line joining the centers divides the common chord in two equal parts and is also perpendicular to it.

Two circles, with diameters 68 cm and 40 cm, intersect each other and the length of their common chord is 32 cm. Find the distance between their centers.



Answer: In the figure given above, the radii of the circles are 34 cm and 20 cm, respectively. The line joining the centers bisects the common chord. Hence, we get two right triangles: one with hypotenuse equal to 34 cm and height equal to 16 cm, and the other with hypotenuse equal to 20 cm and height equal to 16 cm. Using Pythagoras theorem, we get the bases of the two right triangles equal to 30 cm and 12 cm. Hence, the distance between the centers = 30 + 12 = 42 cm.