Euler's Theorem

If M and N are two numbers coprime to each other, i.e. HCF(M, N) = 1 and $N = a^p b^q c^r \ldots$, $\text{Remainder}[\frac{M^{\phi(N)}}{N}] = 1$, where $\phi(N) = N(1-\frac{1}{a})(1-\frac{1}{b})(1-\frac{1}{c})\ldots$ and is known as Euler's Totient function.. $\phi(N)$ is also the number of numbers less than and prime to N.

Find the remainder when 5^{37} is divided by 63.

Answer: 5 and 63 are coprime to each other, therefore we can apply Euler's theorem here.
$63 = 3^2 \times 7 \Rightarrow \phi(63) = 63(1-\frac{1}{3})(1-\frac{1}{7}) = 18$

Therefore, $\text{Remainder}[\frac{5^{18}}{63}] = 1 \Rightarrow \text{Remainder}[\frac{5^{18} \times 5^{18}}{63}] = \text{Remainder}[\frac{5^{36}}{63}] = 1 \Rightarrow \text{Remainder}[\frac{5^{37}}{63}] = \text{Remainder}[\frac{5^{36} \times 5}{63}] = 5$

Find the last three digits of 57^{802}.

Answer: Many a times (not always), the quicker way to calculate the last three digits is to calculate the remainder by 1 000. We can see that 57 and 1 000 are coprime to each other. Therefore, we can use Euler's theorem here if it's useful.
$1\,000 = 2^3 \times 5^3 \Rightarrow \phi(1000) = 1000(1-\frac{1}{2})(1-\frac{1}{4}) = 400$

Therefore,
$\text{Remainder}[\frac{57^{400}}{1000}] = 1 \Rightarrow \text{Remainder}[\frac{57^{400} \times 57^{400}}{1000}] = \text{Remainder}[\frac{57^{800}}{1000}] = 1$

$\Rightarrow \text{Remainder}[\frac{57^{802}}{1000}] = \text{Remainder}[\frac{57^{800} \times 57^2}{1000}] = 249$

Hence, the last two digits of 57^{802} are 249.

Fermat's Little Theorem

If N in the above Euler's theorem is a prime number, then $\phi(N) = N(1-\frac{1}{N}) = N-1$. Therefore, if M and N are coprime to each other and N is a prime number, $\text{Remainder}[\frac{M^{N-1}}{N}] = 1$

Find the remainder when 52^{60} is divided by 31.

Answer: 31 is a prime number therefore $\phi(N) = 30$. 52 and 31 are prime to each other. Therefore, by Fermat's theorem:
$\text{Remainder}[\frac{52^{30}}{31}] = 1 \Rightarrow \text{Remainder}[\frac{52^{60}}{31}] = 1$

Wilson's Theorem

If P is a prime number then $\text{Remainder}[\frac{(P-1)!+1}{P}] = 0$. In other words, (P – 1)! + 1 is divisible by P if P is a prime number. It also means that the remainder when (P – 1)! Is divided by P is P – 1 when P is prime.

Find the remainder when 40! is divided by 41.

Answer: By Wilson's theorem, we can see that 40! + 1 is divisible by 41 $\Rightarrow \text{Remainder}[\frac{40!}{41}] = 41 - 1 = 40$

Find the remainder when 39! is divided by 41.

Answer: In the above example, we saw that the remainder when 40! is divided by 41 is 40.
$\Rightarrow 40! = 41k + 40 \Rightarrow 40 \times 39! = 41k + 40$. The R.H.S. gives remainder 40 with 41 therefore L.H.S. should also give remainder 40 with 41. L.H.S. = $40 \times 39!$ where 40 gives remainder 40 with 41. Therefore, 39! should give remainder 1 with 41.

Chinese Remainder Theorem

This is a very useful result. It might take a little time to understand and master Chinese remainder theorem completely but once understood, it is an asset.

If a number $N = a \times b$, where a and b are prime to each other, i.e., hcf(a, b) = 1, and M is a number such that $\text{Remainder}[\frac{M}{a}] = r_1$ and $\text{Remainder}[\frac{M}{b}] = r_2$ then $\text{Remainder}[\frac{M}{N}] = ar_2x + br_1y$, where $ax + by = 1$

Confused?

Following example will make it clear.

Find the remainder when 3^{101} is divided by 77.

Answer: 77 = 11 × 7.

By Fermat's little theorem, $\text{Remainder}[\frac{3^6}{7}] = 1$ AND $\text{Remainder}[\frac{3^{10}}{11}] = 1$

$$\text{Remainder}[\frac{3^{101}}{7}] = \text{Remainder}[\frac{3^{96} \times 3^5}{7}] = \text{Remainder}[\frac{(3^6)^{16} \times 3^5}{7}] = \text{Remainder}[\frac{1 \times 3^5}{7}] = 5 = r_1$$
$$\text{Remainder}[\frac{3^{101}}{11}] = \text{Remainder}[\frac{3^{100} \times 3}{11}] = \text{Remainder}[\frac{(3^{10})^{10} \times 3}{11}] = \text{Remainder}[\frac{1 \times 3}{11}] = 3 = r_2$$

Now we will find x and y such that 7x + 11y = 1. By observation we can find out, x = –3 and y = 2.

Now we can say that $\text{Remainder}[\frac{3^{101}}{77}] = 7 \times 3 \times -3 + 11 \times 5 \times 2 = 47$

Friends we can also solve this problem by Euler's theorem and this is the method I follow most of the time. No confusion remains thereby.

Find the remainder when 3^{101} is divided by 77.

Answer: $\phi(77) = 77(1 - \frac{1}{7})(1 - \frac{1}{11}) = 60$

$$\text{Remainder}[\frac{3^{60}}{77}] = 1 \Rightarrow \text{Remainder}[\frac{3^{101}}{77}] = \text{Remainder}[\frac{3^{60} \times 3^{41}}{77}] = \text{Remainder}[\frac{1 \times 3^{41}}{77}] = \text{Remainder}[\frac{3^{41}}{77}]$$

$$\text{Remainder}[\frac{3^4}{77}] = \text{Remainder}[\frac{81}{77}] = 4$$

$$\Rightarrow \text{Remainder}[\frac{3^{41}}{77}] = \text{Remainder}[\frac{(3^4)^{10} \times 3}{77}]\text{Remainder}[\frac{4^{10} \times 3}{77}] = \text{Remainder}[\frac{4^4 \times 4^4 \times 4^2 \times 3}{77}] = \text{Remainder}[\frac{256 \times 256 \times 48}{77}]$$

$$= \text{Remainder}[\frac{25 \times 25 \times 48}{77}] = \text{Remainder}[\frac{9 \times 48}{77}] = 47$$

Solved Examples:

Find the remainder when $32^{32^{32}}$ is divided by 9.

Answer: Notice that 32 and 9 are coprime. $\phi(9) = 9(1 - \frac{1}{3}) = 6$

Hence by Euler's theorem, $\text{Remainder}[\frac{32^6}{9}] = 1$. Since the power is 32^{32}, we will have to simplify this power in terms of 6k + r. Therefore, we need to find the remainder when 32^{32} is divided by 6.

$$\text{Remainder}[\frac{32^{32}}{6}] = \text{Remainder}[\frac{2^{32}}{6}] = \text{Remainder}[\frac{(2^8)^4}{6}] = \text{Remainder}[\frac{256 \times 256 \times 256 \times 256}{6}] = \text{Remainder}[\frac{256}{6}] = 4$$

Therefore, $32^{32^{32}} = 32^{6k+4} = (32^6)^k \times 32^4$

$$\Rightarrow \text{Remainder}[\frac{(32^6)^k \times 32^4}{9}] = \text{Remainder}[\frac{32^4}{9}] = \text{Remainder}[\frac{5 \times 5 \times 5 \times 5}{9}] = \text{Remainder}[\frac{625}{9}] = 4$$

What will be the remainder when N = $10^{10} + 10^{100} + 10^{1000} + \ldots\ldots\ldots + 10^{10000000000}$ is divided by 7?

Answer: By Fermat's Little Theorem 10^6 will give remainder as 1 with 7.

$$\text{Remainder}[\frac{10^{10}}{7}] = \text{Remainder}[\frac{10^6 \times 10^4}{7}] = \text{Remainder}[\frac{10^4}{7}] = \text{Remainder}[\frac{3^4}{7}] = 4$$

Similarly, all the other terms give remainder of 4 with 7. Therefore, total remainder = 4 + 4 + 4… (10 times) = 40.

Remainder of 40 with 7 = 5

What is the remainder when N = $2222^{5555} + 5555^{2222}$ is divided by 7?

2222^6 will give remainder 1 when divided by 7.

$5555 = 6K+5 \Rightarrow 2222^{5555} = 2222^{6k+5} \Rightarrow$ Remainder$[\frac{2222^{5555}}{7}]$ = Remainder$[\frac{2222^5}{7}]$ = Remainder$[\frac{3^5}{7}] = 5$

Also 55556 will give remainder 1 when divided by 7.

$5555^{2222} = 5555^{6k+2} \Rightarrow$ Remainder$[\frac{5555^{2222}}{7}]$ = Remainder$[\frac{5555^2}{7}]$ = Remainder$[\frac{4^2}{7}] = 2$

So final remainder is (5 + 2) divided by 7 = 0

Find the remainder when 8^{643} is divided by 132.

Answer: Note that here 8 and 132 are not coprime as HCF (8, 132) = 4 and not 1. Therefore, we cannot apply Euler's theorem directly.

Remainder$[\frac{8^{643}}{132}]$ = Remainder$[\frac{2^{1929}}{132}] = 4 \times$ Remainder$[\frac{2^{1927}}{33}]$. Now we can apply Euler's theorem.

$\phi(33) = 33(1-\frac{1}{3})(1-\frac{1}{11}) = 20 \Rightarrow$ Remainder$[\frac{2^{20}}{33}] = 1 \Rightarrow$ Remainder$[\frac{2^{1927}}{33}]$ = Remainder$[\frac{2^7}{33}] = 29$

$\Rightarrow$ Real remainder $= 4 \times 29 = 116$

I tried to cover the concepts by giving the examples. Hope this article will be of some help to you guys. I welcome more problems so we can cover these concepts completely.

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