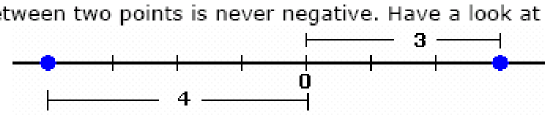
Absolute Value (Modulus)- Basics
by Total Gadha - Tuesday, 18 December 2007, 09:16 AM

For all CAT 2008 aspirants starting their preparations, an introduction to absolute value (modulus) will help them to strengthen their basics. The credit for this chapter goes to my CAT 2007 students. Teaching in a classroom is a learning experience. A teacher learns more about the topic from students than what he learns from the books. This chapter is dedicated to all my CAT 2007 students who gradually, through their inquisitiveness, forced me to find more about the subject than what I already knew.

To understand absolute value function, we study the function from two different points.

Absolute Value as a Distance from a Point

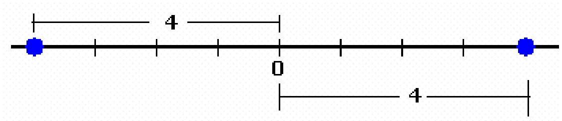
Absolute value $|x|$ of a number x can be understood to be the distance of x from the origin. Absolute value can also be regarded as the distance of a real number from zero on the number line. This is why the absolute value is never negative as the distance between two points is never negative. Have a look at the following figure:



It can be seen that the distance of 3 from zero is 3 units and the distance of –4 from zero is 4 units. Therefore, $|3| = 3$ and $|-4| = 4$.

Which points represent the equation $|x| = 4$?

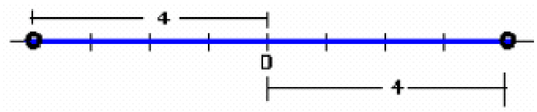
Answer: Since $|x|$ is equal to distance of x from the origin, we are looking for those numbers which are lying at 4 units of distance from zero. We can see from the figure given below that the distance of both the points, 4 and –4, is 4 units from zero.



Therefore, the equation $|x| = 4$ represents the points x = 4 and x = –4.

Which points represent the equation $|x| < 4$?

Answer: From the previous example, we know that the distances of points x = 4 and x = –4 are 4 units from zero. Therefore, $|x| < 4$ means all the points whose distance from zero is less than 4 units. All these points will lie in the blue region between 4 and – 4, as shown below:



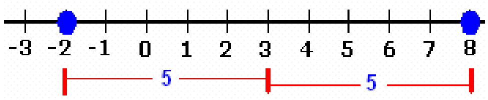
Therefore, the points representing the equation $|x| < 4$ can be represented by the inequality $-4 < x < 4$

What if we want to measure distance from some other number in place of zero?

We use $|x - a|$ to denote distance of number x from the number a. Therefore, $|x - 3|$ denotes distance of number x from 3 and $|x + 5| = |x - (-5)|$ denotes distance of number x from –5.

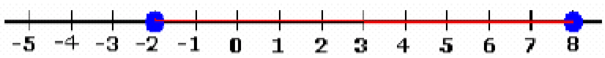
Solve the equality $|x - 3| = 5$.

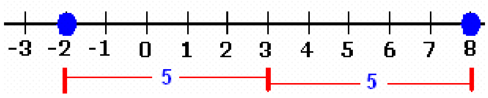
Answer: We know that $|x - 3|$ denotes distance of point x from the number 3. Therefore, we are looking for those numbers x whose distance from 3 is equal to 5 units on the number line. From the figure given below, we can see that there are two such points, 8 and –2, whose distance from 3 is equal to 5 units on the number line.



Which region represents the inequality $|x - 3| > 5$?

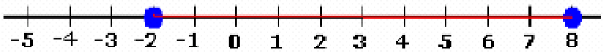
Answer: From the previous question, the blue dots represent the points satisfying the equality $|x - 3| = 5$ and the red line in the figure below represents the region satisfying the points for which $|x - 3| < 5$ because in this region, the distance of the point x – 3 from zero will be less than 5.



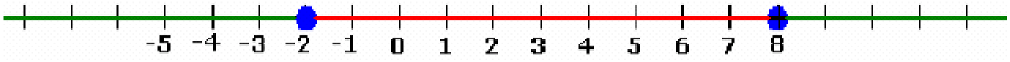


Which region represents the inequality $|x - 3| > 5$?

Answer: From the previous question, the blue dots represent the points satisfying the equality $|x - 3| = 5$ and the red line in the figure below represents the region satisfying the points for which $|x - 3| < 5$ because in this region, the distance of the point $x - 3$ from zero will be less than 5.



Therefore, the rest of the region on the number line will represent the inequality $|x - 3| > 5$, as shown below in green



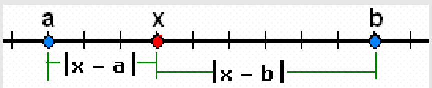
Therefore, the equality is true for $x < -2$ and $x > 8$.

For which value of x is $|2x + 3| = 8$?

Answer: $|2x + 3| = 2\left|x + \frac{3}{2}\right| = 8 \Rightarrow \left|x + \frac{3}{2}\right| = 4$. Therefore, we are looking for those numbers x which lie at a distance of 4 units from -1.5. The points are -5.5 and 2.5.

So what does $|x - a| + |x - b|$ mean?

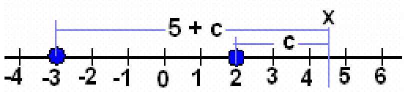
We know that $|x - a|$ and $|x - b|$ are the distances of number x from numbers a and b, respectively. Let's see how this function behaves.



From the figure, we can see that when x is lying between a and b, the sum of the distances of x from a and b is constant and is equal to distance between a and b, i.e. $|a - b|$. When x lies on either side of a and b, the sum $|x - a| + |x - b|$ starts increasing as x starts moving further away.

For which value of x is $|x - 2| + |x + 3| = 8$?

Answer: $|x - 2| + |x + 3| = |x - 2| + |x - (-3)|$. This is the sum of the distances of number x from 2 and -3. We know that if x lies between 2 and -3, the $|x - 2| + |x + 3|$ would be equal to distance between 2 and -3, i.e. 5. Therefore, x cannot lie between 2 and -3 or the equality will not be satisfied. Let x lie on either side of 2 and -3 at distances c and $5 + c$ from two numbers, as shown in the figure below:



$|x - 2| + |x + 3| = 8 \Rightarrow 5 + c + c = 8 \Rightarrow c = 1.5$. Therefore, $x = 3.5$ and -4.5 will both satisfy the above equality.

What is the minimum value of $|x - 2| + |x + 3| + |x - 5|$?

Answer: $|x - 2| + |x + 3| + |x - 5|$ = sum of distances of number x from -3, 2 and 5. The extreme points on the number line out of these three points are -3 and 5. As we have seen already, $|x + 3| + |x - 5|$ would be equal to 8 if x lies between -3 and 5. To keep $|x - 2| + |x + 3| + |x - 5|$ as minimum we need to keep $|x - 2|$ as minimum when x lies between -3 and 5 as $|x + 3| + |x - 5|$ is constant in this range. Therefore, we keep $x = 2$ so that $|x - 2| = 0$. Therefore, the minimum value of $|x - 2| + |x + 3| + |x - 5| = 8$ at $x = 2$.

What is the minimum value of $|x - 1| + |x - 2| + |x - 3| + \dots |x - 10|$?

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also,

$$|x - a| = \begin{cases} x - a & \text{if } x > a \\ 0 & \text{if } x = a \\ -(x - a) & \text{if } x < a \end{cases}$$

In general,

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) = 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

If we need to remove the modulus sign in $|f(x)|$, we need to identify those values of x where f(x) changes its sign, put positive sign in front of f(x) if f(x) is positive and put negative sign in front of f(x) if f(x) is negative. The idea is to make f(x) positive ALWAYS after removing the modulus sign.

For example, to remove the modulus sign from $|x^2 - 5x + 6|$, we first identify the points where $x^2 - 5x + 6$ changes its signs. Now, $x^2 - 5x + 6 = (x - 2)(x - 3)$. The expression is positive for $x < 2$ and $x > 3$, 0 for $x = 2$ and 3, and negative for $2 < x < 3$. Therefore, we put positive sign when $x^2 - 5x + 6$ is positive and negative sign when $x^2 - 5x + 6$ is negative.

$$\left|x^2 - 5x + 6\right| = \begin{cases} x^2 - 5x + 6 \text{ if } x < 2 \\ 0 \text{ if } x = 2 \\ -(x^2 - 5x + 6) \text{ if } 2 < x < 3 \\ 0 \text{ if } x = 3 \\ x^2 - 5x + 6 \text{ if } x > 3 \end{cases}$$

For which value of x is $|3x + 4| = 2 + 4x$?

Answer: $|3x + 4| = 3\left|x + \frac{4}{3}\right| = 3\left|x - \left(-\frac{4}{3}\right)\right|$. Therefore, $|3x + 4|$ changes sign at $x = -\frac{4}{3}$.

Case 1: $x > -\frac{4}{3}$

$\Rightarrow 3x + 4 = 2 + 4x \Rightarrow x = 2$. As the value obtained lies in the region $x > -\frac{4}{3}$, the solution is valid.

Case 2: $x < -\frac{4}{3}$

$\Rightarrow -(3x + 4) = 2 + 4x \Rightarrow x = -\frac{2}{7}$. As the value obtained does not lie in the region $x < -\frac{4}{3}$, the solution is NOT valid.

Therefore, only value that satisfies the equality is $x = 2$

Find the values of x for which $|x^2 - 5x + 6| + 4x = 12$.

Answer: As we have seen

$$\left|x^2 - 5x + 6\right| = \begin{cases} x^2 - 5x + 6 \text{ if } x \le 2 \\ -(x^2 - 5x + 6) \text{ if } 2 < x < 3 \\ x^2 - 5x + 6 \text{ if } x \ge 3 \end{cases}$$

Case 1: When $x \le 2$ and $x \ge 3$

$x^2 - 5x + 6 + 4x = 12 \Rightarrow x^2 - x - 6 = 0 \Rightarrow x = -2$ and 3. Since both the values lie in the intervals we have assumed, therefore both the solutions are valid.

Case 2: When $2 < x < 3$

$-(x^2 - 5x + 6) + 4x = 12 \Rightarrow x^2 - 9x + 18 = 0 \Rightarrow x = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = 3, 6$. The first solution ($x = 3$) is valid according to earlier case but the second solution ($x = 6$) does not lie in the interval we have assumed. Therefore, $x = 6$ is not valid.

Hence, $x = -2$ and 3 are the only solutions.

In the X-Y plane, the area of the region bounded by the graph of $|x + y| + |x - y| = 4$ is **(CAT 2005)**

1. 8 2. 12 3. 16 4. 20

Answer: The easy method to solve this question is to take cases for positive/negative values of the functions $(x + y)$ and $(x - y)$, and remove the modulus sign.

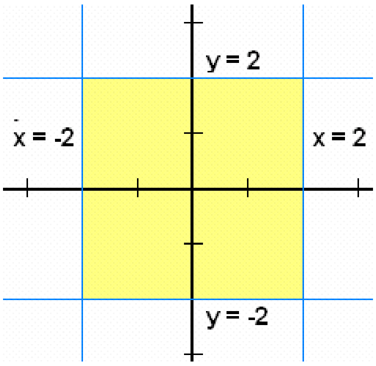
Case 1: Both $(x + y)$ and $(x - y)$ positive:
$\Rightarrow |x + y| + |x - y| = x + y + x - y = 4 \Rightarrow x = 2$.

Case 2: $(x + y)$ positive and $(x - y)$ negative:
$\Rightarrow |x + y| + |x - y| = x + y - (x - y) = 4 \Rightarrow y = 2$

Case 3: $(x + y)$ negative and $(x - y)$ positive:
$\Rightarrow |x + y| + |x - y| = -(x + y) + (x - y) = 4 \Rightarrow y = -2$

Case 4: $(x + y)$ negative and $(x - y)$ negative:
$\Rightarrow |x + y| + |x - y| = -(x + y) - (x - y) = 4 \Rightarrow x = -2$

The four lines are shown in the figure below:



As we can see, the enclosed area = 4 × 4 = 16