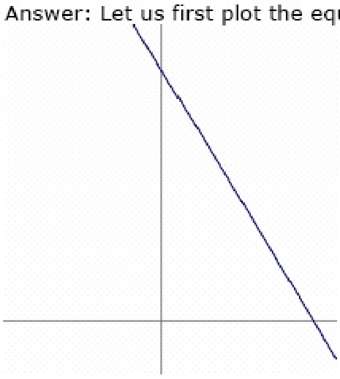
Maxima, Minima and Inequalities- The Basics
by Total Gadha - Monday, 20 October 2008, 10:40 AM

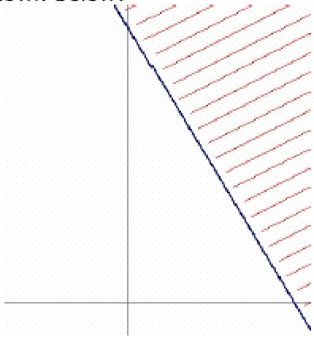
This is a month of distress; students going into depression over their marks, everyone asking for attention, frantic phone calls and emails, long hours of workshops, motivating speeches, exclusive sessions. In short, bullets flying all around and everyday becoming a war zone. Part of the game though. This is the month every instructor in the field tightens his belt and gets ready for the barrage of queries and emotions flying his way. (And I just burnt my tea I left on the burner 10 minutes ago while writing this. Oh well!) I am still amazed how crushing those meaningless percentiles can be to the spirits of the students. I keep on telling students don't take your percentiles seriously. Don't take your percentiles seriously but my exhortations always fall on deaf ears. Students are so much caught in this web that they cannot detect that half their miseries are emanating from something that is not real and cannot supplant the real thing- The CAT. Oh well, I better go and answer those distress calls. For all those students telling me that I have disappeared from TG, here is the new chapter to shush them for a while. Till the mutiny rises again

Plot the region satisfying the inequation $y + 2x > 5$.
Answer: Let us first plot the equation $y = 5 - 2x$. The graph for the equation is shown below:

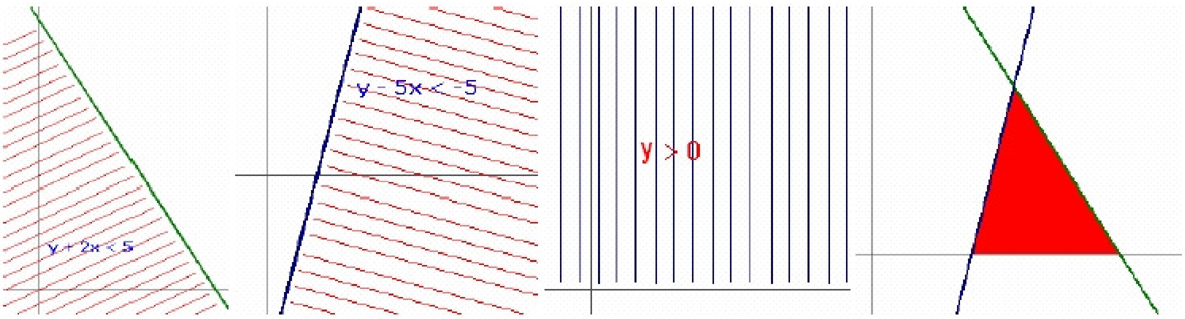


To draw the graph of the inequation $y + 2x > 5$, we note that we need to find the region where $y > 5 - 2x$. This region is shown by red lines in the adjacent figure.

Note that the region $y + 2x < 5$ can be shown by the region on the opposite side of the line. For this region $y < 5 - 2x$.

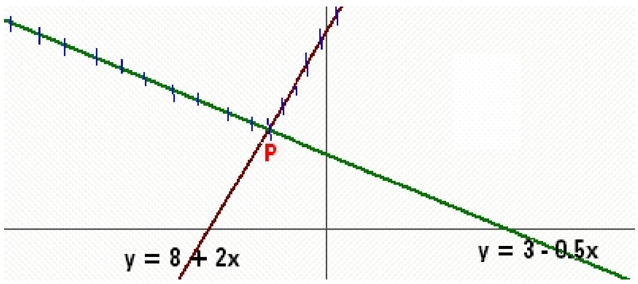


Find the area bounded by the curves $y + 2x < 5$, $y - 5x < -5$ and $y > 0$.
Answer: Let us first see the regions made by the three inequations separately and then together:



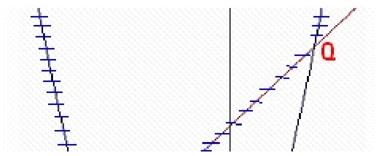
Area of the region bounded by the curves = area of the red triangle = $\frac{1}{2} \times 1.5 \times \frac{15}{7} = \frac{45}{28}$

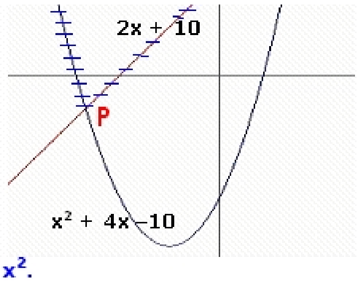
Let $f(x) = \max(8 + 2x, 3 - 0.5x)$ where x is a real number. Then find the minimum value of $f(x)$.
Answer: we graph the lines $y = 8 + 2x$ and $y = 3 - 0.5x$. For every value of x, we will get two corresponding values, one on each line.



Since we have to choose maximum of these two values in every case, the graph of $f(x)$ would be as shown by the crossed lines as shown in the figure. Therefore, the minimum value of $f(x)$ would occur at point P, the intersection of these two lines. Therefore, the minimum value of $f(x)$ occurs at $x = -2$ and the minimum value is equal to 4.

Let $f(x) = \max(2x + 10, x^2 + 4x - 10)$ where x is a real number. Then find the minimum value of $f(x)$.
Answer: Once again, we graph the line $2x + 10$ and the parabola $x^2 + 4x - 10$, as shown in the figure below. For every value of x, we will get two corresponding values, one on each graph. As we have to take the larger of these values, the graph of $f(x)$ can be shown by the crossed curve as shown below.





Find the maximum value of $8 - 2x - x^2$.

Answer: $8 - 2x - x^2 = 9 - (1 + 2x + x^2) = 9 - (1 + x)^2$. $(1 + x)^2$ is always greater than or equal to zero. Therefore, the greatest value of the expression is 9 which occurs at $x = -1$.

> **Remember:**
> - If a quadratic expression achieves a maximum value of b at $x = a$, then the expression can be written as $b - k(x - a)^2$, where k is some constant.
> - If a quadratic expression achieves a minimum value of b at $x = a$, then the expression can be written as $k(x - a)^2 + b$, where k is some constant.

Find the minimum and maximum value of $\frac{x}{x^2 - 5x + 9}$.

Answer: Let $\frac{x}{x^2 - 5x + 9} = y \Rightarrow yx^2 - (5y + 1)x + 9y = 0$. For x to be real, discriminant ≥ 0

$\Rightarrow (5y + 1)^2 - 36y^2 \geq 0$

$\Rightarrow 11y^2 - 10y - 1 \leq 0$ $(11y + 1)(y - 1) \leq 0 \Rightarrow -\frac{1}{11} \leq y \leq 1$.

> **Remember:** If $(x - a)(x - b) \leq 0$, where $b > a$, then $a \leq x \leq b$.

Find the maximum and minimum value of m if $\frac{mx^2 + 3x - 4}{m + 3x - 4x^2}$ can attain all value for real values of x.

Answer: Let $\frac{mx^2 + 3x - 4}{m + 3x - 4x^2} = y \Rightarrow (m + 4y)x^2 + (3 - 3y)x - 4 - my = 0$. For x to be real, discriminant ≥ 0

$\Rightarrow (3 - 3y)^2 + 4(4 + my)(m + 4y) \geq 0$

$\Rightarrow (9 + 16m)y^2 + (46 + 4m^2)y + (16m + 9) \geq 0$.

> **Remember:** The condition that $ax^2 + bx + c \geq 0$ is $a > 0$ and $b^2 - 4ac \leq 0$

$\Rightarrow (46 + 4m^2)^2 - 4(9 + 16m)^2 \leq 0 \Rightarrow m^4 - 41m^2 - 72m + 112 \leq 0$

$\Rightarrow (m - 7)(m^3 + 7m^2 + 8m - 16) \leq 0 \Rightarrow (m - 1)(m - 7)\ (m - 4)^2 \leq 0$.

As $(m - 4)^2$ is always positive, $(m - 1)(m - 7) \leq 0 \Rightarrow 1 \leq m \leq 7$. Therefore, the minimum and maximum values of m are 1 and 7, respectively.

Find the minimum value of $|x - 1| + |x - 3| + |x - 10|$.

Answer: Treating $|x - a|$ as the distance of x from a, the value of $|x - a| + |x - b|$ remains constant when x lies between a and b because this is the sum of the distances of x from a and b, and increases otherwise. This value of $|x - a| + |x - b|$ is equal to the distance between a and b, i.e. $|b - a|$ and this is the minimum value.

The value of $|x - 1| + |x - 10|$ would be constant, and equal to 9, when x lies between 1 and 10. Therefore, the minimum value of $|x - 1| + |x - 3| + |x - 10|$ would come when $x = 3$.

Find the minimum value of $|x - 1| + |x - 2| + |x - 3| + \ldots |x - 10|$.

Answer: The value of $|x - 1| + |x - 10|$ would be constant, and equal to 9, when x lies between 1 and 10. The value of $|x - 2| + |x - 9|$ would be constant, and equal to 7, when x lies between 2 and 9. The value of $|x - 3| + |x - 8|$ would be constant, and equal to 5, when x lies between 3 and 8... and so on. Therefore, the minimum value of the expression would be when x lies from 5 to 6, and this minimum value = 9 + 7 + 5 + 3 + 1 = 25.

Inequalities

Some basic properties

If a and b are two positive and real numbers and $a > b$ then

- $a + c > b + c$, where c is a real number
- $a - c > b - c$, where c is a real number
- If $a > b$ and $c > d$, then $ac > bd$ but we **cannot** say that $ad > bc$.
- $\frac{a}{c} > \frac{b}{c}$ if c is positive and real but $\frac{a}{c} < \frac{b}{c}$ if c is negative and real
- $ac > bc$ if c is positive and real but $ac < bc$ if c is negative and real
- $a^n > b^n$, where n is a positive number
- $\frac{1}{a^n} < \frac{1}{b^n}$, where n is a positive number
- Arithmetic mean ≥ geometric mean which means $\frac{a + b}{2} \geq \sqrt{ab}$ and $\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{\frac{1}{n}}$.

  The equality arises when all the quantities involved are equal.

Prove that $(ab + xy)(ax + by) > 4abxy$

Answer: From the relation arithmetic mean > geometric mean for distinct numbers, we get
$\frac{ab+xy}{2} > \sqrt{abxy} \Rightarrow ab + xy > 2\sqrt{abxy}$. Similarly, $ax + by > 2\sqrt{abxy}$
Multiplying the two inequalities we get the desired result.

Prove that ab(a + b) + bc(b + c) + ac(a + c) > 6abc.
Answer: The given expression can be written as

$$abc(\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b})$$

$$\text{Now, } \left(\frac{\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b}}{6}\right) \geq \sqrt{\frac{a}{c}\times\frac{b}{c}\times\frac{b}{a}\times\frac{c}{a}\times\frac{a}{b}\times\frac{c}{b}}$$

And we can prove the desired result.

A fruit-seller has a faulty balance scale, having beams of unequal length. Knowing that the balance is faulty, he tries to give the correct quantity to his customers by the following method: He takes half of the fruits and places it on one pan, with the weights on the other pan. Then he weighs the other half by switching the pans, i.e. by removing the weights and putting the fruits on that pan. Is the quantity sold by the fruit seller is equal to, more than or less than the required quantity?
Answer: Let the lengths of the beams be a and b, respectively. If the fruit-seller places 1 kg in the first pan, he would have to place a weight of a/b kg in the second pan to balance the scale. Similarly, for 1 kg in the second pan, he would have to place b/a kg in the first pan to balance the scale. The total quantity weight would be a/b + b/a kg which would be more than 2 kg (arithmetic mean is greater than geometric mean). Therefore, the quantity the fruit seller is selling is more than the required quantity.

- If the sum of two or more quantities is constant, their product is maximum when the quantities are equal (or nearly equal). For example, let a and b be positive integers satisfying a + b = 17. The possible pairs satisfying the above equation are (1, 16), (2, 15), (3, 14), (4, 13), (5, 12), (6, 11), (7, 10), and (8, 9). The respective products are 16, 30, 42, 52, 60, 66, 70, and 72, respectively. We see that the maximum product is when the numbers are nearly equal. If we remove the condition of integers, then the maximum product will come when a = b = 8.5. In this case, the maximum value of the product ab = 72.25.

If a + b = 18, find the maximum value of (a – 4)(b + 1).
Answer: many students would make an error here by putting a = b = 9 and calculating the maximum value equal to 5 × 10 = 50. The error is that we are not finding the maximum value of a × b but (a – 4) × (b + 1). Therefore, we need to first find the sum of (a – 4) and (b + 1).
Now (a – 4) + (b + 1) = a + b – 3 = 15 ⇒ a – 4 = b + 1 = 7.5 (for maximum value).
Therefore, the maximum value = 7.5 × 7.5 = 56.25.

If a + b + c = 18, find the maximum value of a^3b^2c.
Answer: Once again, we are not asked the maximum value of the product abc but the product a^3b^2c. To find the maximum value of a^3b^2c, we should know the value of a + a + a + b + b + c, which we don't. Therefore, we try to write the sum a + b + c in the required form. Now a + b + c = 1 $\Rightarrow \frac{a}{3}+\frac{a}{3}+\frac{a}{3}+\frac{b}{2}+\frac{b}{2}+c = 18$. These are 6 numbers whose product is $\frac{a^3b^2c}{108}$. The product will be maximum when a^3b^2c will be maximum which will happen when all the 6 numbers are equal. Therefore, $\frac{a}{3}=\frac{b}{2}=c=\frac{18}{6}=3 \Rightarrow a = 9, b = 6, c = 3$ and the maximum value of a^3b^2c is 78732.

In general, the maximum value of $a^mb^nc^p$... when a + b + c + ... is constant is $m^mn^np^p\left(\frac{a+b+c+\ldots}{m+n+p+\ldots}\right)^{m+n+p+\ldots}$

For which value of x is the product $(1 – x)^5(1 + x)(1 + 2x)^2$ is maximum?
Answer: The sum (1 – x) + (1 – x) + (1 – x) + (1 – x) + (1 – x) + (1 + x) + (1 + 2x) + (1 + 2x) = 8. Therefore, the sum will be maximum when all of them are equal, i.e. (1 – x) = (1 + x) = (1 + 2x) = 1 ⇒ x = 0.

- If the product of two quantities is constant, their sum is the least when the quantities are equal (or nearly equal). For example let a and b be two positive integers such that a × b = 48. The possible solutions for this equation are (1, 48), (2, 24), (3, 16), (4, 12) and (6, 8) and the sum of these pairs are 49, 26, 19, 16 and 14, respectively. We can see that sum is the least when the numbers are nearly equal. If the condition of being an integer is removed, a = b = $\sqrt{48}$ = 6.928 and the sum in this case is 13.856 which is minimum.

- If a, b, c, ..., k are n positive quantities and m is not a proper fraction then
$$\frac{a^m+b^m+c^m+\ldots k^m}{n} > \left(\frac{a+b+c+\ldots+k}{n}\right)^m$$

Prove that $27(a^4 + b^4 + c^4) > (a + b + c)^4$
Answer: From the previous theorem, $\frac{a^4+b^4+c^4}{3} > \left(\frac{a+b+c}{3}\right)^4$ which proves the desired result.

An Important Point:

Many inequalities have letters which are involved symmetrically, i.e. if we interchange any two letters, the inequality would remain the same. For example, $xy + yz + zx$. If we interchange any two variables, the inequality would not change.
In all these cases, the equality arises by putting all the variables equal to each other. Therefore, to find the maximum or minimum value of a symmetrical expression, keep all the variables equal to each other.

Let $a + b + c = 1$, where a, b and c are positive numbers. Find the minimum value of

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$$

Answer: as the given expression is symmetrical, the equality will arise when $a = b = c = 1/3$. Keeping the values in the expression we find the minimum value to be 8.

Find the least value of $(a_1 + a_2 + a_3 + a_4)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right)$ **where all of a_i are positive numbers.**

Answer: Once again, the minimum value will come when $a_1 = a_2 = a_3 = a_4$. And the minimum value is equal to 16.