Divisors of a Number
by Total Gadha - Tuesday, 9 October 2007, 01:00 AM

Divisors:
For a natural number N, all the numbers, including 1 and N itself, which divide N completely are called divisors of N.

Example: The number 24 is divisible by 1, 2, 3, 4, 6, 8, 12, and 24. Hence all these numbers are divisors of 24.

How to find the number of divisors of a number:

Let us the find the number of divisors of 60.

$60 = 2^2 \times 3 \times 5$.
Any divisors of 60 will have powers of 2 equal to either 2^0 or 2^1 or 2^2.
Similarly, any divisor of 60 will have powers of 3 equal to either 3^0 or 3^1, and powers of 5 equal to either 5^0 or 5^1.

To make a divisor of 60, we will have to choose a power of 2, a power of 3 and a power of 5. A power of 2 can be chosen in 3 ways out of 2^0 or 2^1, or 2^2. Similarly, a power of 3 can be chosen in 2 ways and a power of 5 can be chosen in 2 ways.

Therefore, the number of divisors $= 3 \times 2 \times 2 = 12$.

Notice that we have added 1 each to powers of 2, 3 and 5 and multiplied.

Now for the formula:

Let N be a composite number such that $N = (x)^a(y)^b(z)^c$.. where x, y, z.. are prime factors. Then, the number of divisors of $N = (a + 1)(b + 1)(c + 1)$..

Find the number of divisors of 21600.
Answer: $21600 = 2^5 \times 3^3 \times 5^2 \Rightarrow$ Number of divisors $= (5 + 1) \times (3 + 1) \times (2 + 1) = 6 \times 4 \times 3 = 72$.

How many divisors of 21600 are odd numbers?
Answer: An odd number does not have a factor of 2 in it. Therefore, we will consider all the divisors having powers of 3 and 5 but not 2. Therefore, ignoring the powers of 2, the number of odd divisors $= (3 + 1) \times (2 + 1) = 4 \times 3 = 12$.

How many divisors of 21600 are even numbers?
Answer: Total number of divisors of 21600 = 72.
Number of odd divisors of 21600 = 12.
$\Rightarrow$ Number of even divisors of 21600 = 72 – 12 = 60.

How many divisors of 360 are not divisors of 540 and how many divisors of 540 are not divisors of 360?
Answer: The best option here is to find the number of common divisors of 360 and 540. For that we find the highest common powers of all the common prime factors in 360 and 540.
Now, $360 = 2^3 \times 3^2 \times 5$ and $540 = 2^2 \times 3^3 \times 5$.

The number of common factors would be made by $2^2 \times 3^2 \times 5$. The number of factors made by this $= 3 \times 3 \times 2 = 18$. Therefore, the two numbers will have 18 factors in common.

Number of factors of $360 = 4 \times 3 \times 2 = 24 \Rightarrow$ Number of factors of 360 which are not factors of 540 = 24 – 18 = 6.
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How many divisors of the number $2^7 \times 3^5 \times 5^4$ have unit digit equal to 5?
Answer: For unit digit equal to 5, the number has to be a multiple of 5 and it should not be a multiple of 2 otherwise the unit digit will be 0. To be a multiple of 5, the powers of 5 that it can have is 5^1, 5^2, 5^3 or 5^4. The powers of 3 can be 3^0, 3^1, 3^2, 3^3, 3^4 or 3^5.

Therefore, the number of divisors which have a unit digit of $5 = 4 \times 6 = 24$.

How many divisors of 36^{36} are perfect cubes?

Answer: $36^{36} = 2^{72}3^{72}$. To find the divisors which are perfect cubes, we need to take those powers of prime factors which are multiples of 3. Therefore, powers of 2 will be 2^0, 2^3, 2^6, 2^9, ... 2^{72} and similarly, powers of 3 will be 3^0, 3^3, 3^6, 3^9, ... 3^{72}. Both are 25 in number. Therefore, number of divisors $= 25 \times 25 = 625$.

Reverse Operations on Divisors:

Find all the numbers less than 100 which have exactly 8 divisors.

Answer: To find the number of divisors of a number, we used to add 1 to powers of all the prime factors and then multiply them together. Now, given the number of divisors, we will express this number as a product and then subtract 1 from every multiplicand to obtain the powers.

$8 = 2 \times 2 \times 2 = (1 + 1) \times (1 + 1) \times (1 + 1)$. Therefore, the number is of the form $a^1b^1c^1$, where a, b and c are prime.
The numbers can be $2 \times 3 \times 5 =$ **30**, $2 \times 3 \times 7 =$ **42**, $2 \times 3 \times 11 =$ **66**, $2 \times 3 \times 13 =$ **78**, $2 \times 5 \times 7 =$ **70**.

$8 = 4 \times 2 = (3 + 1) \times (1 + 1)$. Therefore, the number is of the form a^3b, where a and b are prime.
The numbers can be $2^3 \times 3 =$ **24**, $2^3 \times 5 =$ **40**, $2^3 \times 7 =$ **56**, $2^3 \times 11 =$ **88**, $3^3 \times 2 =$ **54**.

The number can also be of the form a^7, but there is no such number less than 100.

Find the smallest number with 15 divisors.

Answer: $15 = 3 \times 5 = (2 + 1)(4 + 1) \Rightarrow$ The number is of the form a^2b^4, where a and b are prime. To find the smallest such number, we give the highest power to smallest rime factor, i.e. 2, and the next highest power to next smallest prime number, i.e. 3, and so on. Therefore, the smallest number $= 2^4 \times 3^2 = 144$.

Let N be a composite number such that $N = (2)^a(y)^b(z)^c$.. where y, z.. are prime factors. Then, the number of even divisors of $N = (a)(b + 1)(c + 1)$ and number of odd divisors of $N = (b + 1)(c + 1)$

How many divisors of 21600 are perfect squares?
Answer: In a perfect square, all the prime factors have even powers. For example, $2^5 \times 6^8$ will not be a perfect square as the power of 2 is odd whereas $2^4 \times 6^8$ will be a perfect square because all the prime factors have even powers. $21600 = 2^5 \times 3^3 \times 5^2$ therefore, all the divisors made by even powers of 2, 3 and 5 will be perfect squares. The even powers of 2 are 2^0, 2^2, 2^4, even powers of 3 are 3^0 and 3^2, and even powers of 5 are 5^0 and 5^2. We can select an even power of 2 in 3 ways, even power of 3 in 2 ways, and even power of 5 in 2 ways. Therefore, the number of combinations $= 3 \times 2 \times 2 = 12$.

Let N be a composite number such that $N = (x)^a(y)^b(z)^c$.. where x, y, z.. are prime factors. Then, the sum of divisors of $N = \frac{x^{a+1}-1}{x-1} \times \frac{y^{b+1}-1}{y-1} \times \frac{z^{c+1}-1}{z-1} \ldots$

What is the sum of divisors of 60?

Answer: $60 = 2^2 \times 3 \times 5 \Rightarrow$ Sum of the divisors $= \frac{2^3-1}{2-1} \times \frac{3^2-1}{3-1} \times \frac{5^2-1}{5-1} = 168$

Find the sum of even divisors of $2^5 \times 3^5 \times 5^4$
Answer: All the even divisors of the number will have powers of 2 equal to one of 2, 2^2, 2^3, 2^4, or 2^5. Therefore, sum of even divisors $= (2 + 2^2 + 2^3 + 2^4 + 2^5) \times (1 + 3 + 3^2 + 3^3 + 3^4 + 3^5) \times (1 + 5 + 5^2 + 5^3 + 5^4)$

$$= \frac{2(2^5-1)}{2-1} \times \frac{3^6-1}{3-1} \times \frac{5^5-1}{5-1} = 17625608$$

Let N be a composite number such that $N = (x)^a(y)^b(z)^c$.. where x, y, z.. are prime factors. Then, the product of divisors of $N = (N)^{\frac{(a+1)(b+1)(c+1)}{2}} = \left(x^a y^b z^c\right)^{\frac{(a+1)(b+1)(c+1)}{2}}$

What is the product of divisors of 60?

Answer: $60 = 2^2 \times 3 \times 5 \Rightarrow$ product of divisors of 60 $= (60)^{\frac{3\times2\times2}{2}} = 60^6 = 2^{12} \times 3^6 \times 5^6$

Let A = set of all divisors of 8100 and B = set of all divisors of 21600. What is the product of the elements of AUB?

Answer: $8100 = 2^2 \times 3^4 \times 5^2$ and $21600 = 2^5 \times 3^3 \times 5^2$. AUB will have all the divisors of 8100 and 21600 with the common divisors written only once. Therefore, these common divisors will be multiplied only once. The common divisors will come from $2^2 \times 3^3 \times 5^2$ and are 36 in number. Their product will be $(2^2 \times 3^3 \times 5^2)^{18} = 2^{36} \times 3^{54} \times 5^{36}$

Required product$= \frac{\text{product of divisors of 8100} \times \text{product of divisors of 21600}}{\text{product of common divisors}} = \frac{(2^2 \times 3^4 \times 5^2)^{\frac{45}{2}} \times (2^5 \times 3^3 \times 5^2)^{36}}{2^{36} \times 3^{54} \times 5^{36}} = 2^{189} \times 3^{144} \times 5^{81}$

Let N be a composite number such that $N = (x)^a(y)^b(z)^c$.. where x, y, z.. are prime factors.

If N is not a perfect square, then, the number of ways N can be written as a product of two numbers $= \frac{(a+1)(b+1)(c+1)}{2} = \frac{\text{Number of divisors}}{2}$

If N is a perfect square, then, the number of ways N can be written as a product of two numbers $= \frac{(a+1)(b+1)(c+1)+1}{2} = \frac{\text{Number of divisors}+1}{2}$

For example, the divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Now, $60 = 1 \times 60 = 2 \times 30 = 3 \times 20 = 4 \times 15 = 5 \times 12 = 6 \times 10$. Therefore, **divisors occur in pairs for numbers which are not perfect squares**.

The divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18 and 36.
$36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6$. Therefore, **divisors occur in pairs except for the square root for numbers which are perfect squares.**

How many ordered pairs of integers, (x, y) satisfy the equation xy = 110?
Answer: $110 = 2 \times 5 \times 11$. Hence, the number of divisors of 110 is $= 2 \times 2 \times 2 = 8$. Hence, the number of positive ordered pairs of x and y = 8 (as (2, 55) is not same as (55, 2)). Also, since we are asked for integers, the pair consisting of two negative integers will also suffice. Hence the total number of ordered pairs $= 2 \times 8 = 16$.

The number of ways in which a composite number can be resolved into two factors which are prime to each other = 2^{n-1}, where n is the number of different prime factors of the number.

For example, let the number $N = 2^{10} \times 3^7 \times 5^6 \times 7^4$. We have to assign these prime factors and their powers to one of the two factors. As the two factors will be prime to each other, we will have to assign a prime factor with its power (for example 210) completely to one of the factors. For every prime factor, we have two ways of assigning it. Therefore, the total number of ways = $2 \times 2 \times 2 \times 2 = 16$. As we are not looking for ordered pairs, the required number of ways = $\frac{16}{2} = 8$.

Number of numbers less than or prime to a given number:

If N is a natural number such that $N = a^p \times b^q \times c^r$, where a, b, c are different prime factors and p, q, r are positive integers, then the number of positive integers less than and prime to N = $N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)$.

Therefore, $N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)$ numbers have **no** factor in common with N.

The above formula is extremely versatile as it lets us find not only the numbers which do not contain any of the prime factors of N but also the numbers which do not contain some selected prime factors of N. The following examples will make it clear:

How many of the first 1200 natural numbers are not divisible by any of 2, 3 and 5?
Answer: 1200 is a multiple of 2, 3 and 5. Therefore, we need to find the number of numbers which are less than and prime to 1200. Number of numbers = $1200\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 320$

How many of the first 1200 natural numbers are not divisible by any of 2 and 5?
Answer: Unlike the previous problem, this problem only asks for number not divisible by only 2 factors of 1200, i.e. 2 and 5. Therefore, in the formula we remove the part containing the factor of 3 and calculate the numbers of numbers prime to 1200 with respect to prime factors 2 and 5. The required number = $1200\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 480$

How many of the first 1200 natural numbers are either prime to 6 or to 15?
Answer: Number of numbers prime to 6 are $1200\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 400$ and numbers prime to 15 are $1200\left(1-\frac{1}{5}\right)\left(1-\frac{1}{3}\right) = 640$

Out of these numbers, we will have to subtract numbers which are prime to both 6 and 15 (the question asks for either..or). These are 320 in numbers in all (we have already calculated it).
Therefore, the required number = $400 + 640 - 2 \times 320 = 400$

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