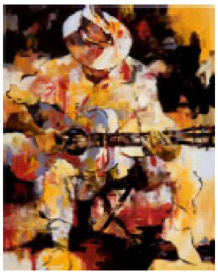
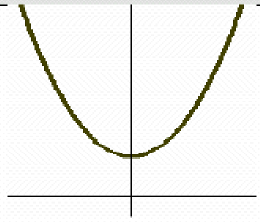
Quadratic Equation
by Total Gadha - Saturday, 5 July 2008, 04:21 AM

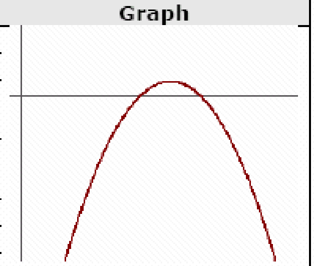


I have not written for a while for my CAT 2008 students, not because I didnâ€™t want to but because the explosive growth of Totalgadha.com and TathaGat kept me busy in multidimensional roles. Gone are the long evening walks, leisurely problem-solving sessions and book readings. Now my time is spent in hurried classroom sessions, editing worksheets, answering incessant phone calls, and managing a rapidly-growing team. The pleasures and pains of entrepreneurship are numerous. Not that I am complaining. It feels good to see that an idea which was once a passing thought has blossomed into something real. It is heartening to see that you can do a lot of good in this world. TG.com was once a thought. Now it is a reality. It would give rise to more realities. Tomorrow, there would be something better than TG.com. There would be better thoughts. And in the end, it would be the students who would benefit the most by the new realities. It is reason enough to rejoice, pain of entrepreneurship notwithstanding.

One more CAT is approaching fast. There is already a tightening in my stomach, a lump in my throat. If I could have my wish, I would want all my students and TGites to clear CAT. But this is not possible, and in the end, life wouldnâ€™t be interesting without a good mix of victories and defeats. No matter what happens on the D-day, I hope that the students donâ€™t stop learning at least. Amen! So here is it ladies and gentlemen, a fruit of painful labour.

An expression of the form $ax^2 + bx + c$, where a, b and c are all real numbers and $a \neq 0$, is called a quadratic expression. Therefore, the expressions $x^2 + 4$, $3x^2 - 4x$, $\sqrt{2}x^2 + 3x - 1$ are all quadratic expressions. As can be seen, the quadratic expressions can be of three types: $\mathbf{ax^2 + c}$, $\mathbf{ax^2 + bx}$ and $\mathbf{ax^2 + bx + c}$. Every quadratic function can be plotted on the x-y plane by finding its value for various values of x. For example, the graphs of $x^2 + 4$ and $-3x^2 + 5x - 2$ have been plotted below for various values of x:

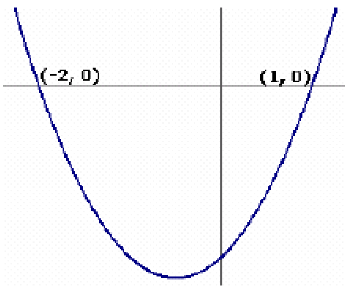
x	$x^2 + 4$	Graph
−3	13	
−2	8	
−1	5	
0	4	
1	5	
2	8	
3	13	

x	$-3x^2 + 5x - 2$	Graph
−1	−10	
0	−2	
$\frac{2}{3}$	0	
$\frac{4}{5}$	0.08	
1	0	
2	−4	
3	−14	

Notice a few things about the two graphs:

- The shape of both the curves is similar, i.e. that of a parabola. Thus, the shape of the curve of every quadratic expression is a parabola.
- The first parabola, in which the coefficient of x^2 is positive, is open upwards whereas the second parabola, in which the coefficient of x^2 is negative, is open downwards.
- The first parabola does not intersect the x-axis. Its value does not become zero or any value of x. The second parabola intersects the x-axis in two points ($\frac{2}{3}$ and 1). Its value becomes zero for these two values of x. We can also have a parabola which only touches the x-axis.
- *If the value of the expression p(x) becomes zero for any value α, it means that $(x - \alpha)$ is a factor of that expression p(x).* As the value of the second expression becomes zero for $x = \frac{2}{3}$ and $x = 1$, $(x - \frac{2}{3})$ and $(x - 1)$ are the factors of $-3x^2 + 5x - 2$. In other words $-3x^2 + 5x - 2 = -3(x - \frac{2}{3})(x - 1)$.
- The value of the first expression always remains positive for all x whereas the value of the second quadratic expression is positive for $\frac{2}{3} < x < 1$, 0 for $x = \frac{2}{3}$ and 1, and negative otherwise.
- From the previous observation, we can see that the value of the quadratic expression changes sign only when the curve crosses the x-axis. As the curve crosses the x-axis at the point where its value becomes zero, the curve changes sign at $x = \alpha$, where $a\alpha^2 + b\alpha + c = 0$. Therefore, the second curve changes sign at $x = \frac{2}{3}$ and $x = 1$.

What values of x satisfy $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 \leq 0$ ('x' is a real number)? **(CAT 2006)**
Answer: Although we will see later how to answer this question later through properties of quadratic expression, let's examine this equation through the help of a graph. Let $x^{\frac{1}{3}} = y$. Therefore, our expression becomes $y^2 + y - 2$. The graph of this equation is shown below:



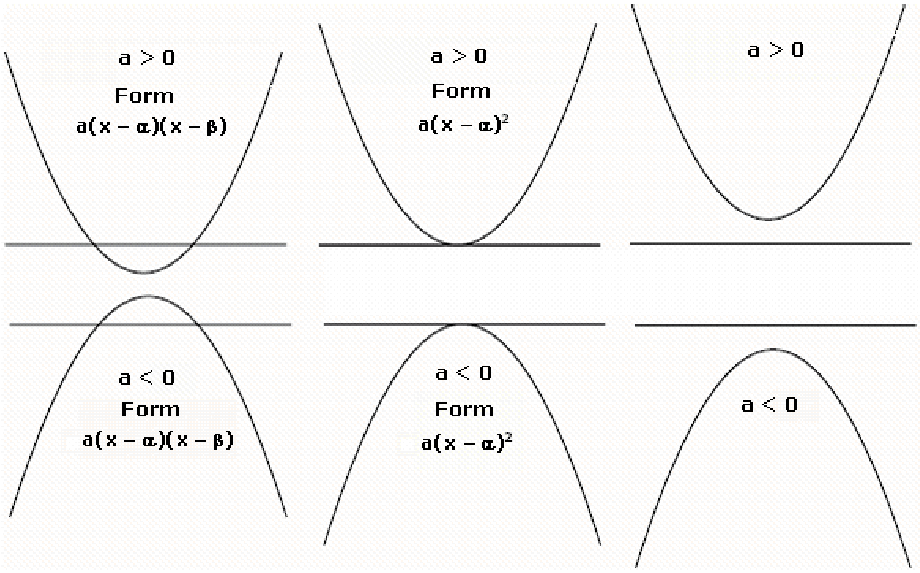
We can see that value of the expression is less than or equal to zero when y varies from −2 to 1. Therefore for $-2 \leq y \leq 1$, $y^2 + y - 2 \leq 0$.

Or for $-2 \leq x^{\frac{1}{3}} \leq 1$, $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 \leq 0$

$\Rightarrow (-2)^3 \leq x \leq 1^3$

$\Rightarrow -8 \leq x \leq 1$

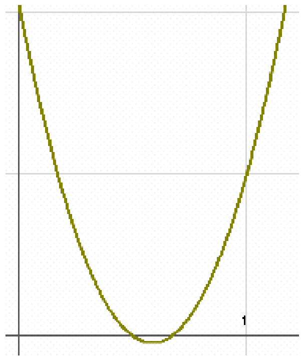
- From the above points we can see that there can be two shapes of the quadratic curve- opening upwards and opening downwards. We can have three possibilities with the quadratic curve- not touching the x- axis, touching the x-axis at a single point, and intersecting the x-axis at two points. Therefore, the curve of a quadratic expression $\mathbf{ax^2 + bx + c}$ can belong to one of the six types:



- In order to form the form of the quadratic curve $ax^2 + bx + c$, we would have to first determine if the coefficient of x^2 (a) is negative or positive. Also, we would have to determine if we could factor the expression in the form $a(x - \alpha)(x - \beta)$.

To factor the polynomial $ax^2 + bx + c$ in the form $a(x - \alpha)(x - \beta)$ we note the following:
If $ax^2 + bx + c = a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta) + a\alpha\beta$
$\Rightarrow \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Factorize $6x^2 - 7x + 2$.



Answer: Let $6x^2 - 7x + 2 = 6(x - \alpha)(x - \beta)$

$\Rightarrow \alpha + \beta = \frac{7}{6}$ and $\alpha\beta = \frac{2}{6} = \frac{1}{3}$

$\Rightarrow (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{49}{36} - \frac{4}{3} = \frac{1}{36} \Rightarrow \alpha - \beta = \frac{1}{6}$

$\Rightarrow \alpha = \frac{2}{3} \Rightarrow \beta = \frac{1}{2}$

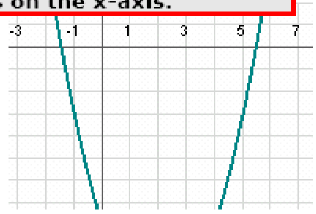
$\Rightarrow 6x^2 - 7x + 2 = 6(x - \frac{2}{3})(x - \frac{1}{2}) = (3x - 2)(2x - 1)$

The graph of the curve $6(x - \frac{2}{3})(x - \frac{1}{2})$ is shown in the adjacent figure. Note that the value of the expression $6(x - \frac{2}{3})(x - \frac{1}{2})$ is zero when $x = \frac{2}{3}$ or $\frac{1}{2}$ and negative when $\frac{1}{2} < x < \frac{2}{3}$. Also, as the curve is symmetrical, the minimum value of the expression occurs at $x = \dfrac{\frac{1}{2} + \frac{2}{3}}{2} = \frac{7}{12}$

If a > 0, the value of $a(x - \alpha)(x - \beta)$
is 0 when $x = \alpha$ or β,
is negative when x lies between α and β,
and positive otherwise.
The **minimum** value of the expression occurs at $x = (\alpha + \beta)/2$

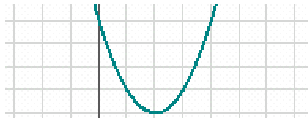
If a < 0, the value of $a(x - \alpha)(x - \beta)$
is 0 when $x = \alpha$ or β,
positive when x lies between α and β,
and negative otherwise.
The **maximum** value of the expression occurs at $x = (\alpha + \beta)/2$

Therefore, the expression $a(x - \alpha)(x - \beta)$ always have the same sign as a except when x lies between α and β, where α and β are real values of intercepts on the x-axis.



Factorize $x^2 - 4x - 8$.

Answer: Let $x^2 - 4x - 8 = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$.
$\Rightarrow \alpha + \beta = 4, \alpha\beta = -8$.



$\Rightarrow \alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \sqrt{16+32} = \sqrt{48} = 4\sqrt{3}$

$\Rightarrow \alpha = 2 + 2\sqrt{3}$ and $\beta = 2 - 2\sqrt{3}$

$\Rightarrow x^2 - 4x - 8 = (x - 2 - 2\sqrt{3})(x - 2 + 2\sqrt{3})$.

As we can see from the graph, the value of the expression is negative for $2 - 2\sqrt{3} < x < 2 + 2\sqrt{3}$. The minimum value of the expression occurs for $x = (2 - 2\sqrt{3} + 2 + 2\sqrt{3})/2 = 2$.

The minimum value of the expression $= 2^2 - 4 \times 2 - 8 = -12$. We can confirm the minimum value from the adjacent graph also.

Looking at the curve of quadratic expressions of the form $ax^2 + bx + c$ we can see that the parabola is either opened upwards or downwards. When the parabola is open upwards ($a > 0$) we have the minimum value of the expression and when the parabola is open downwards ($a < 0$) we have the maximum value of the expression. To find the maximum and minimum values of the quadratic expression, we can use the technique of **completing the square.**

To find the maximum or minimum value of $ax^2 + bx + c$ through **completing the square**, we note that an expression of the form $(x + p)^2$ is a perfect square.

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c = a\left((x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}\right) + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}.$$

Case 1: $a > 0$

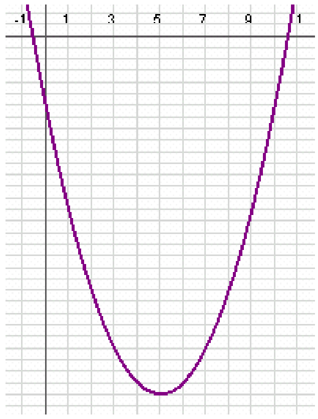
As $a(x + \frac{b}{2a})^2$ is either zero or positive, the minimum value of $ax^2 + bx + c$ occurs when $(x + \frac{b}{2a})^2$ is zero.

Therefore, the minimum value is $c - \frac{b^2}{4a}$.

Case 2: $a < 0$

As $a(x + \frac{b}{2a})^2$ is either zero or negative, the maximum value of $ax^2 + bx + c$ occurs when $(x + \frac{b}{2a})^2$ is zero.

Therefore, the maximum value is $c - \frac{b^2}{4a}$.



Find the minimum value of $x^2 - 10x - 6$.

Answer: Now $x^2 - 10x = x^2 - 10x + 25 - 25 = (x - 5)^2 - 25$.

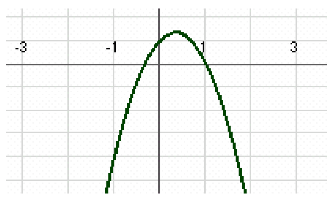
$\Rightarrow x^2 - 10x - 6 = (x - 5)^2 - 31$. As $(x - 5)^2$ is always greater than or equal to zero, the minimum value of the quadratic expression occurs when $(x - 5)^2 = 0$. Therefore, the minimum value $= -31$.

Also, $(x - 5)^2 - 31 = (x - 5)^2 - (\sqrt{31})^2$

$= (x - 5 - \sqrt{31})(x - 5 + \sqrt{31})$ which is in the form $(x - \alpha)(x - \beta)$. Therefore,

- The value of the expression becomes zero at $x = 5 - \sqrt{31}$ and $5 + \sqrt{31}$.
- The value of the expression is negative when x is between $5 - \sqrt{31}$ and $5 + \sqrt{31}$.
- The minimum value of the expression occurs at $x = \frac{\alpha + \beta}{2}$ or at

  $x = \frac{5 - \sqrt{31} + 5 + \sqrt{31}}{2} = 5$.



Find the maximum value of $1 + 2x - 3x^2$.

Answer: $1 + 2x - 3x^2 = 1 - 3(x^2 - \frac{2x}{3}) = 1 - 3(x^2 - \frac{2x}{3} + \frac{1}{9} - \frac{1}{9})$

$= 1 - 3[(x - \frac{1}{3})^2 - \frac{1}{9}] = \frac{4}{3} - 3[(x - \frac{1}{3})^2]$. As $3(x - \frac{1}{3})^2$ is always greater than or equal to zero, the maximum value of the expression $= \frac{4}{3}$ and it occurs at $x = \frac{1}{3}$.

What does root of the quadratic equation $ax^2 + bx + c = 0$ mean?

In short, the roots are those values of x at which the value(s) of the expression $\mathbf{ax^2 + bx + c}$ become equal to zero. As we have seen, the values of the expression becomes zero at those points where the curve of the expression intersects the x-axis. We can see that if $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, then the curve intersects the x-axis at $x = \alpha$ and β.

We have seen that

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c = a\left[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}\right] + c = a[(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}]$$

$$= a[(x + \frac{b}{2a})^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2] = a(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a})(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a})$$

Comparing this with $a(x - \alpha)(x - \beta)$

$\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

Therefore, roots of the quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

We have already seen that the sum of the roots $\alpha + \beta = -\frac{b}{a}$ and product of roots $\alpha\beta = \frac{c}{a}$.

Therefore, the equation $ax^2 + bx + c = 0$ can be written as $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

The quantity under the radical $b^2 - 4ac$ is known as the discriminant. Depending on the value of the discriminant, the roots can be of different types:

- $b^2 - 4ac > 0$: The roots are real and unequal. It means that curve intersects the x-axis in two distinct points.
- $b^2 - 4ac = 0$: The roots are real and equal. It means that curve touches the x-axis at one point.
- $b^2 - 4ac < 0$: The roots are imaginary. It means that the curve does not intersect the x-axis at all.
- $b^2 - 4ac$ is a perfect square: The roots are rational and unequal.

If the equation $x^2 + 2(p + 2)x + 9p = 0$ has equal roots, find the value of p.

Answer: Here, $a = 1$, $b = 2(p + 2)$ and $c = 9p$. For equal roots, $b^2 - 4ac = 0$
$\Rightarrow 4(p + 2)^2 - 36p = 0 \Rightarrow p^2 - 5p + 4 = 0 \Rightarrow (p - 4)(p - 1) = 0 \Rightarrow p = 1$ or 4.

Here are some more conditions on the type of roots.

For the equation $ax^2 + bx + c = 0$

- If $b = 0$, the roots are equal in magnitude but opposite in sign.
- If $a = c$, the roots are reciprocals of each other.
- If a and c have the same sign but b has the opposite sign, the roots are positive.
- If a and b have the same sign but c has the opposite sign, the roots have opposite signs.

Find the maximum and minimum value of $\frac{x^2 - x + 1}{x^2 + x + 1}$.

Answer: Let $\frac{x^2 - x + 1}{x^2 + x + 1} = y$

$\Rightarrow (1 - y)x^2 - (1 + y)x + 1 - y = 0$

If x is real $b^2 - 4ac \geq 0 \Rightarrow (1 + y)^2 \geq 4(1 - y)^2 \Rightarrow 3y^2 - 10y + 3 \leq 0 \Rightarrow (y - \frac{1}{3})(y - 3) \leq 0$.

Remember that if $a > 0$, $a(x - \alpha)(x - \beta) < 0$ when x lies between α and β. Therefore, $(y - \frac{1}{3})(y - 3) \leq 0$ when $\frac{1}{3} \leq y \leq 3$. Therefore, the minimum and maximum values of y are $\frac{1}{3}$ and 3.