Methods of Summation- Arithmetic Progression, Geometric Progression and Miscellaneous

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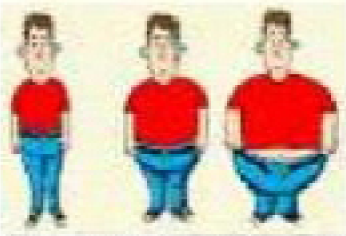

Although the questions on progressions may not come directly in MBA exams, the theory behind progressions is used in every place where we need to sum up numbers. During your CAT 2009 preparations, the formula used in progressions should become second nature to you as they will save a lot of time. Also, the methods of summing up various types of progressions, arithmetic, geometric or otherwise, should be very clear to you so that you are able to instantly spot the type of series you are facing. Knowing the basic methods of progressions also helps you simplify a lot of complex series. So let's start with some basic progressions and their properties:

Arithmetic Progression

Numbers are said to be in Arithmetic Progression (A.P.) when the difference between any two consecutive numbers in the progression is constant i.e. in an Arithmetic Progression the numbers increase or decrease by a constant difference.

Each of the following series forms an Arithmetical Progression:
2, 6, 10, 14...
10, 7, 4, 1, -2...
a, a + d, a + 2d, a + 3d...

If a is the first term and d is the common difference,

- n^{th} term of the series $T_n = a + (n - 1)d$
- Sum of first n terms $S_n = n[2a + (n - 1)d]/ 2$

Example:

1. If the 7^{th} term of an Arithmetical Progression is 23 and 12^{th} term is 38 find the first term and the common difference.
Answer: 7^{th} term = 23 = a + 6d ---- (1)
12^{th} term = 38 = a + 11d ---- (2)
Solving (1) and (2) we get a = 5 and d = 3

2. How many numbers of the series -9, -6, -3 â€¦ should we take so that their sum is equal to 66?
Answer: n[-18 + (n - 1)3]/ 2 = 66
$n^2 - 7n - 44 = 0$
--> n = 11 or -4.
The series is -9, -6, -3, 0, 3, 6, 9, 12, 15, 18, 21..
We can see that the sum of first 7 terms is 0. The sum of next four terms after 7^{th} terms gives us the sum. Otherwise, if we count 4 terms *backward* from -9 we'll get the sum as -66.

3. What is the value of k such that k + 1, 3k - 1, 4k + 1 are in AP?
Answer: If the terms are in AP the difference between two consecutive terms will be the same. Hence,
(3k - 1) - (k + 1) = (4k + 1) - (3k - 1)
2k - 2 = k + 2 --> k = 4.

REMEMBER!

- In an AP, the sum of first term + last term = sum of second term + second last term = the sum of third term + third last term =.. and so on if the number of terms in the AP is even.
  If the number of terms in an AP are odd, the sum of first term + last term = sum of second term + second last term = the sum of third term + third last term = ..= 2 × the middle term.
- Numbers of terms in an AP = $\frac{\text{Last term} - \text{first term}}{\text{Common difference}} + 1$
- Arithmetic Mean between two numbers a and b = (a + b)/2
- If the same quantity is added to, or subtracted from, all the terms of an A.P., the resulting will again with be in A.P. with the same common difference as before.
- If all the terms of an A.P. are multiplied or divided by the same number, the resulting terms will again be in A.P.
- The sum of first n natural numbers = $\frac{n(n+1)}{2}$

TO INSERT ARITHMETIC MEANS BETWEEN TWO NUMBERS

Let n arithmetic means $m_1, m_2, m_3 \dots m_n$ be inserted between two numbers a and b. Therefore, a, $m_1, m_2, m_3, \dots m_n$, b are in arithmetic progression.
Let d be the common difference.
Since b is the $(n + 2)^{th}$ term in the progression, b = a + (n + 1)d
Whence d = (b - a)/(n + 1)
Hence $m_1 = a + (b - a)/(n + 1)$, $m_2 = a + 2(b - a)/(n + 1)$.. and so on.

Example:
4. If 10 arithmetic means are inserted between 4 and 37, find their sum.
First Method:
Let the means be $m_1, m_2, m_3 \dots m_{10}$. Therefore 4, $m_1, m_2, m_3 \dots m_{10}$, 37 are in AP and 37 is the 12^{th} term in the arithmetic progression. Hence, 37 = 4 + 11d
--> d = 3
Therefore means are 7, 10, 13 ... 34 and their sum is 205.
Second Method:
We know that in an AP-
the sum of first term + last term = sum of second term + second last term = the sum of third term + third last term = .. and so on.
Therefore, $4 + 37 = m_1 + m_{10} = m_2 + m_9 = m_3 + m_8 = m_4 + m_7 = m_5 + m_6 = 41$.
Hence $m_1 + m_2 + m_3$â€¦ $+ m_9 + m_{10} = (m_1 + m_{10}) + (m_2 + m_9) \dots + (m_5 + m_6)$
= 5 x 41 = 205.

- To take three numbers in A.P., take them as a – d, a, a + d.
- To take four numbers in A P., take them as a – 3d, a – d, a + d, a + 3d.

Example:
5. The sum of three numbers in A.P. is 30, and the sum of their squares is 318. Find the numbers.
Answer: Let the numbers be a - d, a, a + d
Hence a - d + a + a + d = 30 or a = 10
The numbers are 10 - d, 10, 10 + d
Therefore, $(10 - d)^2 + 10^2 + (10 + d)^2 = 318$
Or d = 3, therefore the numbers are 7, 10, and 13.

SOME SPECIAL RESULTS FOR ARITHMETIC PROGRESSIONS

- In an A.P. if p^{th} term is q and q^{th} term is p i.e. $T_p = q$ and $T_q = p$, then r^{th} term $T_r = p + q - r$.
- In an A.P. if $T_n = 1/m$, then $T_r = r/mn$ and $T_{mn} = 1$
- In an A.P. if $mT_m = nT_n$ then $(m + n)^{th}$ term i.e. $T_{m+n} = 0$
- In an A.P. if $T_p = q$ and $T_q = p$ then $T_{p+q} = 0$
- In an A.P. if the sum of p terms $S_p = q$ and the sum of q terms $S_q = p$ then sum of $(p + q)^{th}$ terms $S_{p+q} = -(p + q)$
- In an A.P. if the sum of p terms S_p = the sum of q terms S_q, then sum of $(p + q)^{th}$ terms $S_{p+q} = 0$.
- If the ratio $S_p/S_q = p^2/q^2$, then $T_p/T_q = (2p - 1)/(2q - 1)$.
- If the pth, qth, rth terms of an A.P. are a, b, c respectively, then $(q - r)a + (r - p)b + (p - q)c = 0$.

Example:

6. The sum of n terms of two arithmetic series are in the ratio 3n + 2 : 6n + 3. What is the ratio of their 13^{th} term?
Answer: The ratio of the sum of n terms = 3n + 2 : 6n + 3

$$\frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{3n+2}{6n+3} \text{-----}(1)$$

The ratio of 13^{th} terms $= \frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{2a_1 + 24d_2}{2a_2 + 24d_2}$

Keeping n = 25 in (1), we get

$$\frac{2a_1 + 24d_2}{2a_2 + 24d_2} = \frac{77}{152}$$

7. The 288^{th} term of the series a, b, b, c, c, c, d, d, d, d, e, e, e, e, e, f, f, f, f, f, f,... is **(CAT 2003- leaked)**

1. u 2. v 3. w 4. x

Answer: We can see that the nth alphabet is being used n times. Therefore, if we finish writing the nth alphabet (n < 26), the number of terms written down $= \frac{n(n+1)}{2}$. Therefore, we first need to find how many alphabets we will completely finish writing before 288. We need to find the highest value of n for which $\frac{n(n+1)}{2}$ is less than 288. We can see that for n = 23, $\frac{n(n+1)}{2} = 276$. Therefore, we will completely finish writing up to the 23^{rd} alphabet. Then we will start writing the 24^{th} alphabet (x). Therefore, the 288^{th} term will be x.

8. If the sum of the first 11 terms of an arithmetic progression equals that of the first 19 terms, then what is the sum of the first 30 terms? **CAT 2004**

Answer: Sum of first 11 terms = sum of first 19 terms $\Rightarrow \frac{11(2a+10d)}{2} = \frac{19(2a+18d)}{2} \Rightarrow 2a = -29d$

Sum of first 30 terms = 15(2a + 29d) = 0

GEOMETRIC PROGRESSION

Numbers are said to be in a Geometric Progression (G.P.) when the ratio between any two consecutive numbers in the progression is th same i.e. in a Geometric Progression the numbers increase or decrease by a common factor.
Each of the following series forms a Geometric Progression:
2, 6, 18, 54...
2, 2/3, 2/9, 2/27...
a, ar, ar^2, ar^3,...

If a is the first term and r is the common ratio of a geometric progression

- The n^{th} term of the series $= ar^{n-1}$.
- The Sum of n terms of a G.P. $S_n = \begin{cases} \frac{a(r^n - 1)}{r-1} & \text{when } r > 1 \\ \frac{a(1-r^n)}{1-r} & \text{when } r < 1 \end{cases}$
- **Note:** The method to calculate the sum of a geometric progression is specially important as the same method is used to calculate the sum of arithmetico-geometric (AGP) series.
  Let $S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$. Multiplying by common ratio r on both sides, we get
  $rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$. Subtracting the first equation from the second, we get
  $(r - 1)S_n = ar^n - a \Rightarrow S_n = \frac{ar^n - a}{r-1}$
- The sum of an infinite number of terms of a G.P. whose common ratio is less than 1 is equal to $\frac{a}{1-r}$.

Example:

9. The sum of an infinite number of terms in G.P. is 6, and the sum of their squares is 18; find the series.
Answer: Let a denote the first term of the series and r the common ratio. Hence, $a + ar + ar^2 + ar^3 + ... = 6$
And $a^2 + a^2r^2 + a^2r^4 + a^2r^6 + ... = 18$
Or $a/(1 - r) = 6$ ---- (1)
$a^2/(1 - r^2) = 18$ ---- (2)
Squaring (1) and dividing it by (2) and noting that $1 - r^2 = (1 + r)(1 - r)$
we get $(1 + r)/(1 - r) = 2$ or $r = 1/3$
Hence, a = 4 and the series is 4, 4/3, 4/9...

10. If 0.234343434.. = p/q, then p + q is equal to
$0.234343434.. = 2/10 + 34/1000 + 34/10^5 + 34/10^7 + ...$
$= 2/10 + 34/1000\ (1 + 1/100 + 1/10000 + ...)$
$= 2/10 + 34/1000\ (100/99)$
$= 2/10 + 34/990$
$= 232/990$
Therefore, p = 232 and q = 990 → p + q = 1222.

11.

Find the sum $5 + 55 + 555 + 5555 + ... + \underbrace{555...555}_{\text{n digits}}$

Answer : $5 + 55 + 555 + 5555 + ... + \underbrace{555...555}_{\text{n digits}} = \frac{5}{9}(9 + 99 + 999 + 9999 + ... + \underbrace{999...999}_{\text{n digits}}$

$= \frac{5}{9}(10 - 1 + 10^2 - 1 + 10^3 - 1 + 10^4 - 1 + ... + 10^n - 1) = \frac{5}{9}(10 + 10^2 + 10^3 + 10^4 + ... + 10^n - n)$

$= \frac{5}{9}\left(\frac{10^{n+1} - 10}{9} - n\right)$

- Geometric Mean between two numbers a and b = $\sqrt{ab}$
- If all the terms of a G.P. are multiplied or divided by the same quantity, the resulting will again be in G.P. with the same common ratio as before.
- If a, b, c, d,.... are in G.P., they are also in continued proportion, i.e. $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = ... = \frac{1}{r}$

TO INSERT GEOMETRIC MEANS BETWEEN TWO NUMBERS

Let n geometric means $m_1, m_2, m_3... m_n$ be inserted between two numbers a and b. Therefore, a, $m_1, m_2, m_3, ...m_n$, b are in geometric progression.
Let r be the common ratio. Since b is the $(n + 2)^{th}$ term in the progression, $b = ar^{n+1}$.
$\therefore r = (b/a)^{1/(n+1)} \Rightarrow m_1 = ar, m_2 = ar^2, m_3 = ar^3$.. and so on, where the value of r is given as above.

Example:

12. Insert three geometric means between 2 and 162.
Answer: Let the means be m_1, m_2, and m_3. Then 2, m_1, m_2, m_3, and 162 are in geometric progression.
If the common ratio is r, then $162 = 2 \times r^4 \rightarrow r = (162/2)^{1/4} \rightarrow r = 3$.
Hence the means are 2 x 3, 2 x 9 and 2 x 27 or 6, 18, and 54.

To take three numbers in G.P. take them as a, ar, and ar^2 **or** a/r, a, and ar

SOME SPECIAL RESULTS FOR GEOMETRIC PROGRESSIONS

- Product of first n terms is $\left(ar^{\frac{n-1}{2}}\right)^n$.
- If the first term is p and the last term is q then the product of all terms is $(pq)^{n/2}$.
- If a, b, and c are three numbers in G.P. and they are greater than zero, then log(a), log(b) and log(c) will be in A.P. (and vice versa also)
- If pth, qth, rth terms of a G.P. are a, b, c respectively
Then $(a)^{q-r}(b)^{r-p}(c)^{p-q} = 1$

SOME OTHER SERIES AND FORMULAS

- The sum of first n natural numbers = $\frac{n(n+1)}{2}$
- The sum of the squares of the 1st n natural numbers = $\frac{n(n+1)(2n+1)}{6}$
- The sum of the cubes of the first n natural numbers = $\left(\frac{n(n+1)}{2}\right)^2$
- The sum of the first n **odd** natural numbers = n^2.
- The sum of the first n **even** natural numbers = n(n + 1)

Example:

13.

Find the sum of the series 2, 6, 12, ...$(n^2 + n)$... for 10 terms

Answer : $\sum n^2 + n = \sum n^2 + \sum n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

To find the sum of 10 terms we keep n = 10

Therefore, sum of the series = 385 + 55 = 440

MISCELLANEOUS EXAMPLES

Method of Differences

In some series, the differences of successive terms (S_n and S_{n-1}) may be in some particular sequence -AP, GP, AGP, etc. In these cases, the sum of series can be found by a technique known as "method of differences".

14. There are 8436 steel balls, each with a radius of 1 centimetre, stacked in a pile, with 1 ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth, and so on. The number of horizontal layers in the pile is **(CAT 2003- leaked)**

1. 34 2. 38 3. 36 4. 32

Answer: Let S_n denote the number of balls in row n.

Therefore, $S_2 - S_1 = 2$
$S_3 - S_2 = 3$
$S_4 - S_3 = 4$
..
..
..
$S_n - S_{n-1} = n$

Adding all the equations we get- $S_n - S_1 = 2 + 3 + 4 + ... + n$.
Since $S_1 = 1$, $S_n = 1 + 2 + 3 + ... + n = n(n + 1)/2$.
Therefore total number of balls = sum of balls in all the row
$= \Sigma n(n + 1)/2 = 1/2\Sigma n^2 + n = 1/2(\Sigma n^2 + \Sigma n) = n(n + 1)(2n + 1)/12 + n(n + 1)/4$
Given that total number of balls = 8436 $\rightarrow n(n + 1)(2n + 1)/12 + n(n + 1)/4 = 8436 \rightarrow n = 36$.
Hence, there are 36 layers in the pile.

Arithmetico-Geometric Series

We employ the same method to find the sum of arithmetico-geometric series as the one we used in finding the sum of the geometric series- we multiply both sides by the common ratio and subtract. We might have to repeat the procedure several times.

15. The infinite sum $1 + \frac{4}{7} + \frac{9}{7^2} + \frac{16}{7^3} + \frac{25}{7^4} + ...$equals **(CAT 2003)**

Answer : Let $S = 1 + \frac{4}{7} + \frac{9}{7^2} + \frac{16}{7^3} + \frac{25}{7^4} + ...$multiplying both sides by the common ratio $(\frac{1}{7})$, we get

$\frac{S}{7} = \frac{1}{7} + \frac{4}{7^2} + \frac{9}{7^3} + \frac{16}{7^4} + \frac{25}{7^5} + ..$ Subtracting the second equation from the first, we obtain

$\frac{6S}{7} = 1 + \frac{3}{7} + \frac{5}{7^2} + \frac{7}{7^3} + \frac{9}{7^4} + ..$ again multiplying both sides by $\frac{1}{7}$ we get

$\frac{6S}{49} = \frac{1}{7} + \frac{3}{7^2} + \frac{5}{7^3} + \frac{7}{7^4} + \frac{9}{7^5} + ...$ subtracting again, we obtain

$\frac{36S}{49} = 1 + \frac{2}{7} + \frac{2}{7^2} + \frac{2}{7^3} + \frac{2}{7^4} + ... = 1 + \frac{2}{7}(1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + ...)$

$\frac{36S}{49} = 1 + \frac{2}{7} \times \frac{7}{6} = \frac{4}{3} \Rightarrow S = \frac{49}{27}$

Arithmetico-Arithmetic Series

16. Find the sum of the series $2 \times 5 + 5 \times 8 + 8 \times 11 + ...$ to 10 terms.

Answer: This series is formed by multiplying the corresponding terms of sequences 2, 5, 8, ... and 5, 8, 11, ... both of which are APs.
Now nth term of first AP = $2 + (n - 1)3 = 3n - 1$
and nth term of second AP = $5 + (n - 1)3 = 3n + 2$
Hence nth term of the given series, $T_n = (3n - 1)(3n + 2) = 9n^2 + 3n - 2$
Thus the required sum, $S_n = \sum T_n = \sum(9n^2 + 3n - 2) = 9\sum n^2 + 3\sum n - 2n$
$= 9n(n + 1)(2n + 1)/6 + 3n(n + 1)/2 - 2n$
$= n(6n^2 + 12n + 2)/2 = n(3n^2 + 6n + 1)$
Hence sum to 10 terms = 3610.

Summation by Method of Partial Fraction

17.
Find the n^{th} term, sum of n terms, and sum to infinity of the series

$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + ...$

Answer : $T_n = \frac{1}{(\text{nth term of } 2, 5, 8...)(\text{nth term of } 5, 8, 11...)} = \frac{1}{(3n-1)(3n+2)} = \frac{1}{3}\left[\frac{1}{3n-1} - \frac{1}{3n+2}\right]$

$T_1 = \frac{1}{3}\left[\frac{1}{2} - \frac{1}{5}\right]$

$T_2 = \frac{1}{3}\left[\frac{1}{5} - \frac{1}{8}\right]$

$T_3 = \frac{1}{3}\left[\frac{1}{8} - \frac{1}{11}\right]$

Adding, $S_n = \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]$

When n becomes large $\frac{1}{3n+2}$ becomes nearly equal to zero.

Hence, $S_\infty = \frac{1}{6}$