Alphametics- The Art of Numbers
by Total Gadha - Friday, 15 February 2008, 03:27 PM

As CAT 2008 preparation starts heating up, I am going to flag the race with an oddball topic: Alphametics. What are alphametics or cryptarithms as they are also known? Alphametics are number puzzles in basic arithmetic operations where digits are represented by alphabets. It is customary that different digits are represented by different alphabets. Why are alphametics relevant? Because they require a good knowledge of properties of numbers and skill in working with them. Solving alphametics helps you quickly identify solution to many problems which might otherwise require a good amount of time. Let me give an example:

Can you solve this problem without any hit or trial method?

The sum of first n natural numbers is a three-digit number, all of whose digits are the same. What is the value of n?

Answer: In 5 seconds, you arrive at the equation $\frac{n(n+1)}{2} = aaa$ (111, 222, etc.). How do you proceed next?

If you think it's hit-and-trial from this point, you are wrong. Here goes the simple logic. It might strike you instantly if you have been working with numbers:

$$\frac{n(n+1)}{2} = aaa = a \times 111 = a \times 3 \times 37$$
$$\Rightarrow n(n+1) = 6a \times 37$$

Look at the L.H.S. of the equation- n(n + 1) is a product of two consecutive natural numbers. Therefore, R.H.S. should also be a product of two consecutive natural numbers. One of the numbers is 37. Therefore, what could the other number, 6a, consecutive to 37 be? It can only be 36, giving a = 6 and n = 36. Therefore, 36 numbers have been summed up and their sum is equal to 666.

Let's start with a very simple alphametic:

S is a six digit number beginning with 1. If the digit 1 is moved from the leftmost place to the rightmost place the number obtained is three times of S. What is the sum of the digits of S?

Answer: The problem might sound abstruse to read but if presented in alphametics form, it is childishly simple:

$$\begin{array}{r} 1ABCDE \\ \times 3 \\ \hline ABCDE1 \end{array}$$

Now can you solve the problem? Yes, you can. What can be the value of E? E × 3 is giving a unit digit of 1. Therefore, E = 7. We get a carryover of 2. Writing the value of E, we get

$$\begin{array}{r} 1ABCD7 \\ \times 3 \\ \hline ABCD71 \end{array}$$

What can be the value of D? Since a carryover of 2 is resulting in 7, we should get a unit digit of 5 when we multiply D y 3. Therefore, D = 5. Proceeding in the same manner, we slowly reach the complete solution in this way:

$$\begin{array}{r} 1ABC57 \\ \times 3 \\ \hline ABC571 \end{array} \;—\; \begin{array}{r} 1AB857 \\ \times 3 \\ \hline AB8571 \end{array} \;—\; \begin{array}{r} 1A2857 \\ \times 3 \\ \hline A28571 \end{array} \;—\; \begin{array}{r} 142857 \\ \times 3 \\ \hline 428571 \end{array}$$

Therefore, the sum of digits of S = 27.

Here is one more:

Find the digits A, B, C and D such that AB × CB = DDD, where AB and CB are two-digit numbers and DDD is a three-digit number.

Answer: AB and CB are two two-digit numbers with the same unit digit. Therefore, R.H.S. should also be a multiplication of two two-digit numbers with the same unit digit.

R.H.S. = DDD = D × 111 = D × 3 × 37. = 3D × 37. Now 37 is a two-digit number with 7 as the unit digit. Therefore, 3D should also be a two-digit number with 7 as the unit digit ⇒ D = 9 and 3D = 27. Therefore, 27 × 37 = 999.

If ABC × CBA = 65125, where A, B and C are single digits, then A + B + C = ?

Answer:

- As the unit digit of the product is 5, therefore, the unit digit of one of the numbers is 5 and the unit digit of the other number is odd. Therefore, AB5 × 5BA = 65125, where A = 1, 3, 5, 7 or 9.
- As the product of two three-digit numbers is a five-digit number, and NOT a six-digit number, A can only be equal to 1. 1B5 × 5B1 = 65125.
- The digit sums of both numbers, 1B5 and 5B1 will be the same. Therefore, the product would give digit sum of a perfect square. The digit sum on the R.H.S. is 1. Therefore, the digit sum of each number can be 1, 8 or 10. Correspondingly B will be 2 or 4 (as digit sum cannot be equal to 1).
- Keeping B = 2, we can see that 125 × 521 = 65125.

Find the four-digit number ABCD such that ABCD × 4 = DCBA.

Answer:

- Any number multiplied by 4 will give us an even number. Hence, the digit D when multiplied by 4 will give us an even number. Since A is the unit digit of the product it is even. Hence, A = 2, 4, 6 or 8 (It cannot be 0. Why?)

- A is also the first digit of the multiplicand and if A = 4, 6 or 8 the product ABCD × 4 will become a 5 digit number. Hence A = 2. Writing the value of A we get **2BCD × 4 = DCB2**.

- What can be the value of D? Looking at the first and last digits of the multiplicand, we can see that 4 × D gives the unit digit of 2 and 4 × 2 gives the first digit of D. Yes, you got it right. D = 8. Writing the multiplication again with the value of D we get **2BC8 × 4 = 8CB2**.

- What can be the value of B? From your repository of formulas associated with 4 recall the one about divisibility of 4. A number is divisible by 4 if the number formed by the last two digits is divisible by 4. Since the number 8CB2 is a multiple of 4, the number **B2** should be divisible by 4. Or, the number B2 = 12, 32, 52, 72 or 92. Hence the original number ABCD is 21C8, 23C8, 25C8, 27C8 or 29C8. But the last 4 numbers when multiplied by 4 will not give you the first digit of 8 in the product! Therefore B = 1 and the original number is 21C8. We write the multiplication again: **21C8 × 4 = 8C12.**

- Can you identify C now? Notice that when you multiply 8, the unit digit of 21C8, by 4 you write 2 in the unit digit of the product and carry 3. The tenths digit of the product is 1. Therefore, 4 × C + 3 (carryover) gives a unit digit of 1. Hence, C is 2 or 7. You can easily check by the hundreds digit in the product (which is C again) that C = 7.

  Therefore, our answer is **2178 × 4 = 8712.**

See how we applied small rules in number theory to decipher the letters. The more we apply these rules, the more understanding we gain about numbers. Let us now see a tough problem to crack. Observe how we apply small rules to break the code:

Find the digits in the multiplication TWO × TWO = THREE, where each alphabet represents a distinct digit.
Answer: In the above multiplication, the square of a three-digit number is a five-digit number.

- A perfect square can end in 00, 1, 4, 5, 6, and 9. Therefore, the last two digits on the R.H.S. can be 00, 11, 44, 55, 66, or 99. It cannot be 00 as the unit digit on the L.H.S., i.e. O is different from the unit digit in the R.H.S., i.e. E. It cannot be 11, 55, 66 or 99 as tens digit of every perfect square is even, except when the square is ending in 6 in which case the tens digit is odd. Therefore, E = 4 and O = 2 or 8. Therefore, the solutions are **TW2 × TW2 = THR44** or **TW8 × TW8 = THR44**.
- The square of a three-digit number is a five-digit number, therefore, T = 1, 2, or 3. Since the leading digit is the same on both sides, i.e. T, therefore, T = 1. Therefore, the possible solutions are **1W2 × 1W2 = 1HR44** or **1W8 × 1W8 = 1HR44.**

- Now the second digit in 1W2 or 1W8 will be 1, 2 or 3 otherwise the leading digit in the product will not be 1 because of the carry over. Therefore, the possible numbers are 102, 108, 112, 118, 122, 128, 132, 138. Out of these, we can eliminate the numbers that do not end in 44 by calculating the last two digits in their square. Therefore, 102, 108, 118, 122, 128, and 132 are out.

- Now 112 × 112 = 12544 which cannot be the number as the second digit of the product and the unit digit of the multiplicand is he same, i.e. 2. Now, **138 × 138 = 19044**, which satisfies our criterion.

Note that many of the processes involved in solving alphametics will be 'mental,' which is good for your brain. Second, although you will use basic mathematical principles, you will use them in the most effective way which help you pinpoint the correct option in a multiple choice questions test very fast.

Last, here are some alphametics puzzles for you to try your hands on:

ABCD × 9 = DCBA

AH × AH = BAH

THREE + THREE + FOUR = ELEVEN

Happy solving!