Theory of Equations..
by Total Gadha - Saturday, 3 March 2007, 03:06 PM

GENERAL EQUATION OF N^{th} DEGREE

Let polynomial $f(x) = a_0x^n + a_1x^{n\text{ â€" }1} + a_2x^{n\text{ â€" }2} +$ â€¦ $+ a_n$, where $a_0, a_1, a_2, ..a_n$ are rational numbers and n Â³ 0. Then the values of x for which f(x) reduces to zero are called root of the equation f(x) = 0. The highest whole number power of x is called the degree of the equation.

For example

x^4 â€" $3x^3 + 4x^2 + x + 1 = 0$ is an equation with degree four.

x^5 â€" $6x^4 + 3x^2 + 1 = 0$ is an equation with degree five.

ax + b = 0 is called the linear equation.

$ax^2 + bx + c = 0$ is called the quadratic equation.

$ax^3 + bx^2 + cx + d = 0$ is called the cubic equation.

Properties of equations and their roots

- Every equation of the n^{th} degree has exactly n roots.

For example, the equation $x^3 + 4x^2 + 1 = 0$ has 3 roots,

The equation x^5 â€" $x + 2 = 0$ has 5 roots, and so onâ€¦

- In the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$. where $a_0, a_1, a_2, ..a_n \neq 0$,

  $$\text{Sum of the roots} = -\frac{a_1}{a_0}$$

  $$\text{Sum of the products of the roots taken two at a time} = \frac{a_2}{a_0}$$

  $$\text{Sum of the products of the roots taken three at a time} = -\frac{a_3}{a_0}$$

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  $$\text{Product of the roots} = (-1)^n \frac{a_n}{a_0}$$

- In an equation with real coefficients imaginary roots occur in pairs i.e. if a + ib is a root of the equation f(x) = 0, then a â€" ib will also be a root of the same equation. For example, if 2 + 3i is a root of equation f(x) = 0, 2 â€" 3i is also a root.

- In an equation with rational coefficients, surd roots occur in pairs, i.e. if $a + \sqrt{b}$ is a root of the equation f(x) = 0, then $a - \sqrt{b}$ is also a root of the same equation.
  Hence, if $2 + \sqrt{3}$ is a root of the equation f(x) = 0, $2 - \sqrt{3}$ is also a root.

- If the coefficients of an equation are all positive then the equation has no positive root. Hence, the equation $2x^4 + 3x^2 + 5x + 1 = 0$ has no positive root.
- If the coefficients of even powers of x are all of one sign, and the coefficients of the odd powers are all of opposite sign, then the equation has no negative root. Hence, the equation $6x^4$ â€" $11x^3 + 5x^2$ â€" $2x + 1 = 0$ has no negative root
- If the equation contains **only even** powers of x and the coefficients are all of the same sign, the equation has no real root. Hence, the equation $4x^4 + 5x^2 + 2 = 0$ has no real root.
- If the equation contains **only odd** powers of x, and the coefficients are all of the same sign, the equation has no real root except x = 0. Hence, the equation $5x^5 + 4x^3 + x = 0$ has only one real root at x = 0.
- **Descartesâ€™ Rule of Signs :** An equation f(x) = 0 cannot have more positive roots than there are changes of sign in f(x), and cannot have more negative roots than there changes of sign in f(- x). Thus the equation $x^4 + 7x^3$ â^' $4x^2$ â^' x â€" 7 = 0 has one positive root because there is only change in sign. f(- x) = x^4 â^' $7x^3$ â^' $4x^2$ + x â€" 7 = 0 hence the number of negative real roots will be either 1 or 3.

- If $\frac{a}{b}$ is the rational root of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$, where a_i € I, then a divides a_n, and b divides a_0.

EXAMPLES:

1. Solve the equation $9x^3 - 54x^2 + 92x - 40 = 0$, given that the roots are in arithmetical progression.

   Answer: Let the roots be a − d, a, a + d. We know that

   $$\text{Sum of roots} = -\left(\frac{-54}{9}\right)$$

   $$\text{Product of roots} = -\left(\frac{-40}{9}\right)$$

   --> a − d + a + a + d = 6 or a = 2 and

$(a - d)(a)(a + d) = \left(\frac{40}{9}\right)(2 - d)(2 + d) = \left(\frac{20}{9}\right)$ or $4 - d^2 = \left(\frac{20}{9}\right)$

or $d = \frac{4}{3}$. Hence the roots are $\frac{2}{3}$, 2, and $\frac{10}{3}$.

2. Find the sum of the squares and of the cubes of the roots of the equation $x^3 - 2x^2 + x - 3 = 0$

Answer: Let the roots be denoted α, β, and γ. We know that
$\alpha + \beta + \gamma = 2$ and $\alpha\beta + \beta\gamma + \alpha\gamma = 1$
Therefore $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma) = 4 - 2 = 2$
Substituting α, β, and γ in the original equation and adding
$(\alpha^3 + \beta^3 + \gamma^3) - 2(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta + \gamma) - 9 = 0$
or $(\alpha^3 + \beta^3 + \gamma^3) - 2 \times 2 + 2 - 9 = 0 \rightarrow (\alpha^3 + \beta^3 + \gamma^3) = 11$

3. Solve the equation $2x^4 - 5x^3 - 9x^2 - x + 1 = 0$, having given that one root is $2-\sqrt{3}$.

Answer : Since $2-\sqrt{3}$ is root, $2+\sqrt{3}$ is also a root of the equation. Hence, the polynomial $[x-(2-\sqrt{3})][x-(2+\sqrt{3})] = x^2 - 4x + 1$ is a factor of the equation. Dividing the polynomial $2x^4 - 5x^3 - 9x^2 - x + 1$ by $x^2 - 4x + 1$, we get $2x^2 + 3x + 1$. Hence, $2x^4 - 5x^3 - 9x^2 - x + 1 = (x^2 - 4x + 1)(2x^2 + 3x + 1)$. Therefore, other roots of the equation can be found by the equation $2x^2 + 3x + 1 = 0 \rightarrow (2x + 1)(x + 1) = 0 \rightarrow x = \frac{-1}{2}$ or $x = -1$.

Hence roots are $2-\sqrt{3}$, $2+\sqrt{3}$, $\frac{-1}{2}$, -1.

4. Which of the following is **NOT** the root of the equation $6x^3 + 5x^2 + 2x + 2 = 0$?
(a) 2/3 (b) 1/6 (c) 1/3 (d) none of these

Answer: (d)

5. If two roots of the equation $x^3 - 3x^2 + 5x + k = 0$ are equal but opposite in sign, then what is the value of k?
Answer: Let the roots be α, −α, and β. Then,
$\alpha + (-\alpha) + \beta = 3, \Rightarrow \beta = 3$
$\alpha \times (-\alpha) + \alpha \times \beta + (-\alpha) \times \beta = 5, \Rightarrow -\alpha^2 = 5$
and $\alpha \times (-\alpha) \times \beta = -k \Rightarrow \alpha^2\beta = k$
$\Rightarrow k = -15$.

6. If the roots of the equation $x^{10} - 1 = 0$ are 1, α, β, γ, .. then what is the value of $(1 - \alpha)(1 - \beta)(1 - \gamma)..$?
Answer: $x^{10} - 1 = (x - 1)(x^9 + x^8 + x^7 + ... + 1)$
Also $x^{10} - 1 = (x - 1)(x - \alpha)(x - \beta)(x - \gamma)...$
$\Rightarrow (x - 1)(x - \alpha)(x - \beta)(x - \gamma)... = (x - 1)(x^9 + x^8 + x^7 + ... + 1)$
Canceling $(x - 1)$ on both sides and keeping $x = 1$ we get
$(1 - \alpha)(1 - \beta)(1 - \gamma).. = 10$