Optiver Question: Optimal Dice to Roll

Quant Guild

1 Problem Statement

We seek to determine the number of rolls n of a fair six-sided die that maximizes the probability of observing **exactly one** 6.

2 Probability Formulation

Let X be the random variable representing the number of 6s observed in n independent trials. This follows a binomial distribution with probability of success $p = \frac{1}{6}$ and probability of failure $q = \frac{5}{6}$.

The probability of observing exactly k successes is given by:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

For exactly one success ($k = 1$), we substitute our values:

$$P(X = 1) = \binom{n}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{n-1} \tag{2}$$

Since $\binom{n}{1} = n$, the function simplifies to:

$$f(n) = n \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{n-1} \tag{3}$$

3 Optimization (Calculus Approach)

To find the maximum, we treat n as a continuous variable and take the derivative with respect to n. We use the product rule and the exponential derivative rule $\frac{d}{dn} a^n = a^n \ln(a)$.

$$f'(n) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} + n \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \ln\left(\frac{5}{6}\right) \tag{4}$$

Setting $f'(n) = 0$ to find the critical point:

$$\frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \left[1 + n \ln\left(\frac{5}{6}\right)\right] = 0$$

Since $\frac{1}{6}(\frac{5}{6})^{n-1} \neq 0$, we solve for the term in the brackets:

$$
\begin{aligned}
1 + n \ln\left(\frac{5}{6}\right) &= 0 \\
n &= -\frac{1}{\ln(5/6)} \\
n &\approx 5.48
\end{aligned}
$$

4 Integer Solution

Since the number of rolls n must be an integer, we evaluate the probability function at the integers closest to the critical point 5.48, which are $n = 5$ and $n = 6$.

4.1 Case $n = 5$

Using Equation (3):

$$P(n=5) = 5 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^4 = \frac{5}{6} \cdot \left(\frac{5}{6}\right)^4 = \left(\frac{5}{6}\right)^5 \tag{5}$$

4.2 Case $n = 6$

$$P(n=6) = 6 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^5 = 1 \cdot \left(\frac{5}{6}\right)^5 = \left(\frac{5}{6}\right)^5 \tag{6}$$

5 Conclusion

The probabilities for $n = 5$ and $n = 6$ are identical. Therefore, the optimal strategy is to roll the die either **5 or 6 times**.