# MARTINGALES

## INTRODUCTION

Have you ever considered it possible having a gambling technique that you can establish is always a winner, a fair game, and yet might bankrupt you at the same time? The martingale theory and betting strategy essentially demonstrates this.

The martingale probability theory is a mathematical representation of a fair game, in which there is equal chance of winning or losing. It's a process in which today's events serve as the best predictor of what will happen tomorrow. Originally, a martingale was a class of betting strategies. Despite the fact that they are commonly recognized to lead to bankruptcy, they are still widely used.

When the Martingale Strategy is used, the transaction size is doubled every time a loss occurs. A common situation for the technique is to try to trade a result that has a 50% chance of occurring. We also call this strategy a zero-expectation scenario. Here we are doubling our stake each time we lose, so we say our multiplier is 2. You can use 3 or other numbers as well as your multiplier.

Let us consider example of gambling:

Given, (1) You are doubling the stake every time you lose.
(2) You will stop playing the game if you win.
(3) You are starting the game with 1$.

Calculate the profit (or loss) that the gambler will face after $n^{th}$ bet.
Solution:
Case (1): You lose all first n bets, then the loss you will be gone through will be-

$$W = -(1+2+2^2+2^3 \ldots\ldots + 2^{n-1})$$

Case (2): You win at the $n^{th}$ bet-

$$\begin{aligned} W &= -(1+2+2^2+2^3 \ldots\ldots + 2^{n-2}) + 2^{n-1} \\ &= -(2^{n-1}-1) + 2^{n-1} \\ &= 1 \end{aligned}$$

So, you can see in this case, either we are going bankrupt (probability of which is very less) or we are winning 1$. You must have got an intuition that we don't intend to play the game with Martingale strategy from the start but we use Martingale to cover our losses after loss.

## FILTRATION

Now that you have gotten a basic idea of a martingale, the next step is to define it mathematically. To do that, it is crucial to understand the concept of filtration.

A filtration is an increasing sequence of σ - algebras on a measurable probability space. Filtrations represent the information about an experiment that has been revealed up to a certain time t. a σ-algebra (also σ-field) on a set *X* is a collection Σ of subsets of *X*, is closed under complement, and is closed under countable unions and countable intersections.

Looking at an example for a better understanding:

Consider an experiment where a coin is tossed 3 times in 3 seconds.
The sample space for our experiment will be a set of all the possible outcomes -

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

As the sample space is finite, we will take the σ - algebra, F (set of all the events possible), as the power set of Ω -

$$F = 2^{\Omega}$$

Now starting with time t = 0,

The coin has not been tossed yet. So, we have no extra information available to us. All we know is our sample space for the experiment. Given an outcome w, the only events from the σ - algebra that has been revealed to us (i.e., which we know have surely occurred or not) are Ω and φ. This is because we already know whether w will belong in Ω or not.
Making a set of all the events that have been revealed,

$$F_0 = \{\varphi, \Omega\}$$

We can see that F0 itself is a σ - algebra. This will be the first σ - algebra of our filtration.

At time t = 1,

The coin has been tossed once, and we have learnt some more information about the experiment. This extra information helps us reveal two new events (apart from Ω and φ).
The event of the first toss is a head,

$$A_H = \{HHH, HHT, HTH, HTT\}$$

And the event of the first toss is a tail,

$$A_T = \{THH, THT, TTH, TTT\}$$

That is, given an outcome w, we will already know whether the first toss was an H or T, thus revealing the events $A_H$ and $A_T$.
The set of all the events that have been revealed will now be,

$$F_1 = \{\varphi, \Omega, A_H, A_T\}$$

Notice that F1 is a σ - algebra. This will be the next σ - algebra of our filtration.

Similarly, at time t = 2,

The coin has now been tossed twice. The new information will reveal many new events.
Some of them are,
the event of the first two tosses is a head,

$$A_{HH} = \{HHH, HHT\}$$

the event of the first toss is a head and the second is a tail,

$$A_{HT} = \{HTH, HTT\}$$

and so on.
Note that the unions and complements of the revealed events will also be revealed.
The set of all the events that have been revealed will now be,

$$F_2 = \{\varphi, \Omega, A_H, A_T, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A^C{}_{HH}, A^C{}_{HT}, A^C{}_{TH}, A^C{}_{TT},$$

$$A_{HH} \cup A_{HT}, A_{TH} \cup A_{TT}, A_{HT} \cup A_{TH}, A_{HH} \cup A_{TT}\}$$

Notice that F2 is a σ - algebra. This will be the next σ - algebra of our filtration.

Finally, at time $t = 3$,

The experiment is complete. For any outcome w, we now have information about all tosses.
This means all the events in the σ - algebra have been revealed.
The set of all the events that have been revealed will now be,

$$F_3 = F$$

We can observe that these σ - algebras, $F_0$, $F_1$, $F_2$, and $F_3$ are such that,

$$F_0 \subset F_1 \subset F_2 \subset F_3 = F$$

This family of σ - algebras, $\{F_n\}_{0 \le n \le 3}$, can be called a filtration.

Now that we have a clear understanding of filtrations, let us give a more formal definition:

Given a probability space (Ω, F, P) and some $T > 0$,
For $0 \le t \le T$ there exists a σ - algebra, F(t) such that

$$F(t) \subset F$$

and for some $0 \le s, t \le T$ and $s \le t$, the σ - algebras F(s) and F(t) are such that

$$F(s) \subseteq F(t).$$

Then the family of σ-algebras → $\{F(t)\}$ $0 \le t \le T$ is called a filtration that is associated with the probability space (Ω, F, P).

# MARTINGALES

You may have picked up a bit on Martingales in the introduction part. However, once we've covered filtration, we can move on to the important aspect of Martingales, which includes the basic mathematics underlying it and its applications.

Martingales is a stochastic process (sequence of random variables) in which the conditional expectation of the next value in the sequence, regardless of all previous values, is equal to the current value of the sequence.

To offer a simple illustration, imagine tossing a coin in a sequence: if it comes up head, you get one dollar; if it comes up tail, you lose one dollar. Let's say you have $5 after 100 such tosses. As a result, your expected value after the 101$^{st}$ coin is $5. Isn't it straightforward? This is the core concept of Martingales; we should grasp the basic mathematics underlying it to have delve more into it.

(1) Say, $\{F_n\}_{n\geq 0}$ is an increasing sequence of σ−algebras (filtration) in a probability space (Ω, F, P). Let $X_0, X_1, X_2 \ldots. X_n$ is an adapted sequence ($X_n \in F_n$) of integrable real valued random variable such that $E(X_{n+1}) < \infty$, the sequence $X_0, X_1, X_2 \ldots. X_n$ is said to be a Martingale relative to the filtration $\{F_n\}_{n\geq 0}$ if

$$E\,(X_{n+1}\,|F_n) = X_n \qquad \text{(Martingale condition)}$$

(2) We have, $E(X_{n+1}|F_n) = X_n$.

$$\begin{aligned} E(X_{n+k}|F_n) &= E((X_{n+k}|F_{n+k-1})|F_n) && \text{(Using Tower's property)} \\ &= E(X_{n+k-1}|F_n) && \text{(Using Martingale)} \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &= E(X_{n+1}|F_n) \\ &= X_n \end{aligned}$$

(3) The expectation of next value in sequence to be equal to current value, there are several types of martingales sequences, the most basic one is sums of independent, mean zero random variables. Let $Y_1, Y_2$... be such a series; then with following conditions,

$$E(Y_i) = 0$$

$$F_n = \sigma\,(Y_1, Y_2, Y_3 \ldots. Y_n)$$

the partial sums sequence:

$$X_n = Y_1 + Y_2 + Y_3 \ldots Y_n$$

is a martingale in comparison to the natural filtering caused by the variables $Y_n$. The linearity and stability features, as well as the independence law for conditional expectation, make this easy to verify:

$$\begin{aligned} E\,(X_{n+1}\,|F_n) &= E\,(X_n + Y_{n+1}\,|F_n) \\ &= E\,(X_n\,|F_n) + E\,(Y_{n+1}\,|F_n) && \text{(Linearity)} \end{aligned}$$

$$= X_n + E(Y_{n+1})$$

$$= X_n.$$

Example:

We went through a 3-coin toss example in the filtration part, now let's do the same thing in Martingales. If we apply the random variables and probability terms to the three coin-toss examples, we get $X_n$, which is the total amount won/lost in n bets. The profit/loss on the nth bet is represented by $Y_n$. $F_n$ is a collection of values that may be obtained via the use of n bets. So, if we have been given a $F_n$ that has led us to an $X_n$ value, we may claim that our value X will not change after one more bet since the $Y_{n+1}$ expectation will be zero owing to fair toss.

Now, once we have done the basic mathematics of the martingales we can move towards the applications of Martingales in stock market.

# WHY MARTINGALE IN STOCK MARKET?

(1) According to the Efficient Market Hypothesis, beating the market on a risk-adjusted basis is impossible since the market should only react to fresh information. As a result, trading cannot be done in such a way that you never gain or lose with a positive probability.

(2) In the fundamental theorem of finance, the acceptable probability is known as risk-neutral probability, which is a measure of equivalent Martingale.

This is bit hard to understand if you are new to the stock world, but there is very simple meaning behind this i.e., This suggests that, regardless of how much money a firm is losing at the time, an investor should not give up, but rather keep investing in the hope that perseverance will eventually pay off. Before using the martingales in real world one should be aware of its risk. The majority of investors do not employ martingales as their primary investment strategy, but rather combine it with other momentum techniques and then use it.

# APPLICATIONS OF MARTINGALES IN STOCK MARKET

(1) Most investors do not strictly follow the Martingale method since it is not the greatest way to invest, but every investor indirectly utilises the Martingale approach to lower the average price of the stocks he owns down if he learns that the stock price is going to rise in a short period of time.

(2) Martingale trading is a common forex trading method. The fact that currencies, unlike equities, seldom fall to zero, is one of the reasons why the martingale technique is so popular in the currency market. Although businesses may readily go bankrupt, most countries do so on their own volition. A currency's value will fluctuate

from time to time. Even in extreme circumstances of depreciation, the currency's value seldom falls to zero.

# IMPLEMENTATION OF MARTINGALES

Let us see what we have,

Initially the investor will buy one stock. The primary premise behind using Martingales to generate an idea for this stock is that if the current day's stock price is less than 1.5% lower than the previous day's, the investor should double his investment at the end of the day. Investment will rise, and investors might be able to profit from the market even if there are fewer victories.

Dataset:

Apple data containing for date and price (closing) for the time interval 6/3/2019 to 31/12/2019

Simple Martingale strategy:

Signal is a indication of action to the investor if he has buy, hold. If current day's price is 1.5% greater than previous day's close then +1, if current day's price is 1.5% less than previous day's close then -1, Otherwise 0.

```
# Signal is a indication of action to the investor if he has buy, hold.
# If current day’s price is 1.5% greater than previous day’s close then +1,
# if current day’s price is 1.5% less than previous day’s close then -1, Otherwise 0.

df = copy
signal = []
for i in range(len(df)):
  if i ==0:
    signal.append(0)
  elif df['price'][i] > (1.015)*df['previous_close'][i]:
    signal.append(1)
  elif df['price'][i] < (0.985)*df['previous_close'][i]:
    signal.append(-1)
  else:
    signal.append(0)
```

```
copy['signal'] = signal    #signal is for next day
```

Quantity is number of shares the investor own on the day.

```
#quantity is number of shares the investor own on the day.
df = copy
quantity = [1,1]
for i in range(2, len(df)):
  if df['signal'][i-1]==-1:
    quantity.append(quantity[i-1]*2)
  else:
    quantity.append(quantity[i-1])

copy['quantity']=quantity
```

invested money is total money invested from the starting date to till date.

```
# invested money is total money invested from the starting date to till date.
df = copy
invested = ['NAN',df['price'][0]]
for i in range(2, len(df)):
  if df['quantity'][i]==df['quantity'][i-1]:
    invested.append(invested[i-1])
  else:
    invested.append(invested[i-1]+(df['previous_close'][i]*(df['quantity'][i]-df['quantity'][i-1])))
print(invested)
copy['invested_money'] = invested
```

Returns are returns on the given date and returns_in_per are returns on the given date in percentage

```
# Returns are returns on the given date and returns_in_per are returns on the given date in percentage
df = copy
returns =[]
returns_per = []
for i in range(len(df)):
  if i == 0:
    returns.append('NAN')
    returns_per.append('NAN')
  else:
    returns.append((df['price'][i]-df['previous_close'][i])*df['quantity'][i])
    returns_per.append(returns[i]/(df['previous_close'][i]*df['quantity'][i]))
copy['returns'] = returns
copy['returns_in_per'] = returns_per
```

cumulative returns is net returns from the starting date to till date, cumulative returns in percentage is total returns from starting date to till date in percentage.

```
# cumulative returns is net returns from the starting date to till date,
# cumulative returns in percentage is total returns from starting date to till date in percentage.
df = copy
cumulative_returns = ['NAN']
cumulative_returns_per = ['NAN']
net_returns = 0
for i in range(1, len(df)):
  net_returns+=df['returns'][i]
  cumulative_returns.append(net_returns)
  cumulative_returns_per.append((net_returns*100)/df['invested_money'][i])


copy['cumulative_returns'] = cumulative_returns
copy['cumulative_returns_in_per'] = cumulative_returns_per
```

In the final file given below you can see, the cumulative returns in percentage for the period of 6 months is 9.57 for total invested money 312360.7804$.

<u>Code and dataset:</u>

Dataset: [File]

Code: [Code]

Final results: [Final file]