DA Assignment 2

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The Mars Equant Model is used to determine the position of Mars at specific observational times (oppositions) by modeling its motion using a circular orbit and comparing these positions with observed values. Below is the detailed mathematical derivation of the model.

Assumptions

a. The Sun is at the origin.
b. Mars's orbit is circular, with the centre at a distance 1 unit from the Sun and at an angle c (degrees) from the Sun-Aries reference line.
c. Mars's orbit has radius r (in units of the Sun-centre distance).
d. The equant is located at (e1, e2) in polar coordinates with centre taken to be the Sun, where e1 is the distance from the Sun and e2 is the angle in degrees with respect to the Sun-Aries reference line.
e. The 'equant 0' angle z (degrees) which is taken as the earliest opposition, also taken as the reference time zero, with respect to the equant-Aries line (a line parallel to the Sun-Aries line since Aries is at infinity).
f. The angular velocity of Mars around the equant is s degrees per day.

1. Orbit Equation

Mars' orbit is assumed to be circular with a center at $(\cos c, \sin c)$ and radius r. The equation of this circle is:

$$(x - \cos c)^2 + (y - \sin c)^2 = r^2. \tag{1}$$

Expanding this equation, we get:

$$x^2 + y^2 - 2x\cos c - 2y\sin c + 1 - r^2 = 0. \tag{2}$$

2. Equation of the Line Representing the i-th Opposition

The line representing the position of Mars at the i-th opposition is defined by the slope $\tan(z + \beta)$ and y-intercept c_1, where:

$$y = \tan(z + \beta) \cdot x + c_1 \tag{3}$$

with:

- z being the initial angular position of Mars,
- s being the angular velocity of Mars around the equant in degrees per day,
- $\beta = \text{timediff}(i) \cdot s$, where timediff($i$) is the time difference in days between the i-th and zeroth oppositions.

3. Equant Coordinates and Line Equation

The equant point $(e_1 \cos e_2, e_1 \sin e_2)$ lies on the line defined by Equation 2. Therefore, substituting the equant coordinates into the line equation gives:

$$e_1 \sin e_2 = \tan(z + \beta) \cdot e_1 \cos e_2 + c_1. \tag{4}$$

Solving for c_1, we get:

$$c_1 = e_1 \sin e_2 - \tan(z + \beta) \cdot e_1 \cos e_2. \tag{5}$$

4. Finding the Intersection of the Line with Mars' Orbit

To find the intersection point P_1 of the line (Equation 2) with Mars' orbit (Equation 1), substitute $y = \tan(z + \beta) \cdot x + c_1$ into the orbit equation:

$$x^2 + (\tan(z+\beta) \cdot x + c_1)^2 - 2x \cos c - 2(\tan(z+\beta) \cdot x + c_1) \sin c + 1 - r^2 = 0. \tag{6}$$

Simplifying, we derive the quadratic equation:

$$ax^2 + bx + c = 0 \tag{7}$$

where:

$$\begin{aligned}
a &= 1 + \tan^2(z + \beta), \\
b &= 2c_1 \tan(z + \beta) - 2 \cos c - 2 \tan(z + \beta) \sin c, \\
c &= c_1^2 - 2c_1 \sin c + 1 - r^2.
\end{aligned}$$

5. Solving the Quadratic Equation

Using the quadratic formula, the solutions for x are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{8}$$

Since a circle and a line can intersect at most at two points, there are two potential roots corresponding to two possible intersection points. The function selects the correct intersection based on the angle $\tan^{-1}\left(\frac{y}{x}\right)$, determined by:

- If $\frac{3\pi}{2} > z + \beta > \frac{\pi}{2}$, use the second root,
- Otherwise, use the first root.

6. Calculating the Longitude and Error

Convert the intersection coordinates (x, y) into a longitude angle:

$$\text{Calculated Longitude} = (\arctan 2(y, x) + 4\pi) \mod 2\pi, \tag{9}$$

and convert it to degrees:

$$\text{Calculated Longitude (degrees)} = \left(\frac{\text{Calculated Longitude}}{\pi} \times 180 \right). \tag{10}$$

The error δ between the calculated longitude and the observed opposition angle is then:

$$\delta = \text{Calculated Longitude (degrees)} - \text{Observed Longitude}. \tag{11}$$

Normalize the error to the range $[-180, 180]$ degrees:

$$\delta = (\delta + 720) \mod 360, \tag{12}$$

adjusting for any value exceeding the bounds by subtracting or adding 360 degrees:

$$\text{If } \delta > 180, \text{ then } \delta = \delta - 360, \quad \text{or if } \delta < -180, \text{ then } \delta = \delta + 360. \tag{13}$$

7. Maximum Error

Finally, the maximum absolute error is calculated from the set of all δ values:

$$\text{Max Error} = \max(|\delta|). \tag{14}$$

If no errors are computed (e.g., due to intersections not occurring), the max error is set to 10000.

Function Descriptions

- **MarsEquantModel**: Calculates predicted positions of Mars using the equant model and computes the error with observed positions.
- **bestOrbitInnerParams**: Optimizes the equant model's key parameters (c, e_1, e_2, z) to minimize the maximum angular error.
- **bestS**: Finds the optimal angular velocity (s) of Mars's orbit by minimizing the maximum error through parameter tuning.
- **bestR**: Determines the best orbital radius (r) for Mars by iterating through possible values and minimizing the maximum error.
- **bestMarsOrbitParams**: Searches for the optimal orbital parameters (r, s, c, e_1, e_2, z) to best fit Mars's observed positions, using a comprehensive grid search and optimization approach.

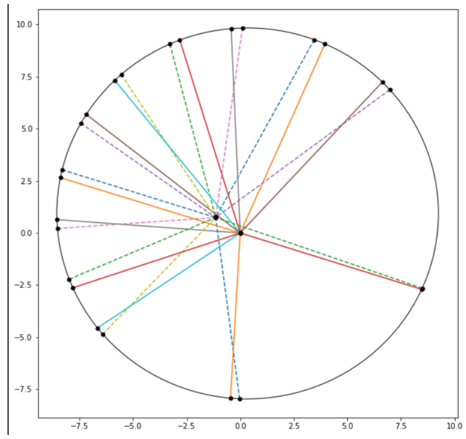

Figure 1: Mars Orbit Predicted using the Model

Results

The Best Parameters Estimated by the model:
$r = 8.9000, s = 0.0091, c = 69.8389, e1 = 1.3658, e2 = 147.0075, z = 61.6588$

Array of errors:
$[3.08599722, 2.97658202, 3.0860075, 2.83507946, 2.75413604, 3.07703146, 0.41606137, -2.9428122, -3.0860154, -2.49497839, -2.51028893, -3.08459698]$
Max Error: **3.086015403943179**