1. Intro. A quick program to output the "domination" or "majorization" relation when it is defined on permutations of multisets instead of on partitions.

Let's say that digits are permuted. Then $x_1 \ldots x_n \succeq y_1 \ldots y_n$ if and only if $\sum_{i=1}^{j}[x_i \geq k] \geq \sum_{i=1}^{j}[y_i \geq k]$ for all j and k.

This relation is self-dual in the sense that $x_1 \ldots x_n \succeq y_1 \ldots y_n$ if and only if $x_n \ldots x_1 \preceq y_n \ldots y_1$. And if the digits consist of equal quantities of the numbers 1 through k, then $x_1 \ldots x_n \succeq y_1 \ldots y_n$ if and only if $\bar{x}_1 \ldots \bar{x}_n \preceq \bar{y}_1 \ldots \bar{y}_n$, where $\bar{x} = k + 1 - x$.

It's emphatically *not* a lattice, in most cases.

Here I just compute the relation and its transitive reduction by brute force. When I learn better algorithms for transitive reduction, I can use this as an interesting example.

(Well, maybe not! In the examples I tried, we seem to have x covers y if and only if x differs from y by a transposition and x has exactly one more inversion than y. Furthermore, it appears that the covering relation on multiset permutations such as $\{1,1,2,2,3\}$ is obtained by taking the relation on set permutations $\{1,1',2,2',3\}$ and removing all cases in which $1'$ occurs before 1 or $2'$ before 2. Thus, some additional theory apparently lurks in the background, making this whole program unnecessary — except as a way to confirm the conjectures in further cases before I go ahead and find proofs.)

```
#define maxn 63    /* this many elements at most */
#define maxp 1000    /* this many perms at most */

#include <stdio.h>
#include <string.h>
  char perm[maxp][maxn + 1];    /* the permutations */
  char work[maxn + 1];    /* where I generate new ones */
  char rel[maxp][maxp];    /* nonzero if x ≺ y */
  char red[maxp][maxp];    /* reduced relation */

  main(int argc, char *argv[])
  {
    register int i, j, k, l, ll, m, n, s, dom;
    ⟨ Set work to the string that is to be permuted, and check it 2 ⟩;
    ⟨ Generate the rest of the permutations 3 ⟩;
    ⟨ Compute the dominance relation 4 ⟩;
    ⟨ Do transitive reduction 5 ⟩;
    ⟨ Print the results 6 ⟩;
  }
```

2. ⟨ Set *work* to the string that is to be permuted, and check it 2 ⟩ ≡

```
if (argc ≠ 2) {
  fprintf(stderr, "Usage:␣%s␣digits_to_permute\n", argv[0]);
  exit(−1);
}
for (j = 0; argv[1][j]; j++) {
  if (j > maxn) {
    fprintf(stderr, "String␣too␣long␣(maxn=%d)!\n", maxn);
    exit(−2);
  }
  if (argv[1][j] < '0' ∨ argv[1][j] > '9') {
    fprintf(stderr, "The␣string␣%s␣should␣contain␣digits␣only!\n", argv[1]);
    exit(−3);
  }
  if (j > 0 ∧ argv[1][j − 1] > argv[1][j]) {
    fprintf(stderr, "The␣string␣%s␣should␣be␣nondecreasing!\n", argv[1]);
    exit(−4);
  }
  work[j + 1] = argv[1][j];
}
n = j;
```

This code is used in section 1.

3. Here I use ye olde Algorithm 7.2.1.2L.

⟨ Generate the rest of the permutations 3 ⟩ ≡

```
  m = 0;
l1: if (m ≡ maxp) {
    fprintf(stderr, "Too␣many␣permutations␣(maxp=%d)!\n", maxp);
    exit(−5);
  }
  for (j = 0; j < n; j++) perm[m][j] = work[j + 1];
  m++;
l2: for (j = n − 1; work[j] ≥ work[j + 1]; j−−) ;
  if (j ≡ 0) goto done;
l3: for (l = n; work[j] ≥ work[l]; l−−) ;
  s = work[j], work[j] = work[l], work[l] = s;
l4: for (k = j + 1, l = n; k < l; k++, l−−) s = work[k], work[k] = work[l], work[l] = s;
  goto l1;
done:
```

This code is used in section 1.

4. We use the fact that dominance is a subset of (reverse) lexicographic order. In other words, if $x_1 \ldots x_n$ is lexicographically less than $y_1 \ldots y_n$ we cannot have $x_1 \ldots x_n \succeq y_1 \ldots y_n$.

⟨ Compute the dominance relation 4 ⟩ ≡

```
for (l = 0; l < m; l++)
  for (ll = l + 1; ll < m; ll++) {
    dom = 0;
    for (k = work[n] + 1; k ≤ work[1]; k++)
      for (j = 0; j < n; j++) {
        for (i = s = 0; i ≤ j; i++) s += (perm[l][i] ≥ k ? 1 : 0) − (perm[ll][i] ≥ k ? 1 : 0);
        if (s > 0) goto fin;
        if (s < 0) dom = 1;
      }
    if (dom) rel[l][ll] = 1;
  fin: continue;
  }
```

This code is used in section 1.

5. Hey, I'm just using brute force today.

⟨ Do transitive reduction 5 ⟩ ≡

```
for (l = 0; l < m; l++)
  for (ll = l + 1; ll < m; ll++) {
    if (rel[l][ll]) {
      for (j = l + 1; j < ll; j++)
        if (rel[l][j] ∧ rel[j][ll]) goto nope;
      red[l][ll] = 1;
    }
  nope: continue;
  }
```

This code is used in section 1.

6. ⟨ Print the results 6 ⟩ ≡

```
for (l = 0; l < m; l++) {
  printf("%s␣<", perm[l]);
  for (ll = l + 1; ll < m; ll++)
    if (red[l][ll]) printf("␣%s", perm[ll]);
  printf("\n");
}
```

This code is used in section 1.

7. Index.

DOMINATION