

**1. Intro.** Here's an easy way to calculate the number of graceful labelings that have  $m$  edges and  $n$  nonisolated vertices, for  $0 \leq n \leq m + 1$ , given  $m > 1$ . I subdivide into connected and nonconnected graphs.

The idea is to run through all  $m$ -tuples  $(x_1, \dots, x_m)$  with  $0 \leq x_j \leq m - j$ ; edge  $j$  will go from the vertex labeled  $x_j$  to the vertex labeled  $x_j + j$ .

I consider only the labelings in which  $x_{m-1} = 1$ ; in other words, I assume that edge  $m - 1$  runs from 1 to  $m$ . (These are in one-to-one correspondence with the labelings for which that edge runs from 0 to  $m - 1$ .) But I multiply all the answers by 2; hence the total over all  $n$  is exactly  $m!$ .

I could go through those  $m$ -tuples in some sort of Gray code order, with only one  $x_j$  changing at a time. But I'm not trying to be tricky or extremely efficient. So I simply use reverse colexicographic order. That is, for each choice of  $(x_{j+1}, \dots, x_m)$ , I run through the possibilities for  $x_j$  from  $m - j$  to 0, in decreasing order.

**#define** *maxm* 20 /\* this is plenty big, because 20! is a 61-bit number \*/

**2.** I do, however, want to have some fun with data structures.

Every vertex is represented by its label. Vertex  $v$ , for  $0 \leq v \leq m$ , is isolated if and only if label  $v$  has not been used in any of the edges. (In particular, vertices 0, 1, and  $m$  are never isolated, because of the assumption above.)

It's easy to maintain, for each vertex, a linked list of all its neighbors. These lists are stacks, since they change in first-in-last-out fashion.

It's also easy to maintain a dynamic union-find structure, because of the first-in-last-out behavior of this algorithm.

**3.** OK, let's get going.

**#include** <stdio.h>

**#include** <stdlib.h>

**int** *mm*; /\* command-line parameter \*/

<Global variables 15>;

*main*(**int** *argc*, **char** \**argv*[])

{

**register** *j*, *k*, *l*, *m*;

    <Process the command line 4>;

    <Initialize to  $(m - 1, \dots, 2, 1, 0)$  7>;

**while** (1) {

        <Study the current graph 16>;

        <Move to the next  $m$ -tuple, or **goto** *done* 5>;

    }

*done*: <Print the stats 17>;

}

**4.** <Process the command line 4>  $\equiv$

**if** (*argc*  $\neq$  2  $\vee$  *sscanf*(*argv*[1], "%d", &*mm*)  $\neq$  1) {

*fprintf*(*stderr*, "Usage: %s %m\n", *argv*[0]);

*exit*(-1);

}

*m* = *mm*;

**if** (*m* < 2  $\vee$  *m* > *maxm*) {

*fprintf*(*stderr*, "Sorry, %m must be between 2 and %d!\n", *maxm*);

*exit*(-2);

}

This code is used in section 3.

5.  $\langle \text{Move to the next } m\text{-tuple, or } \mathbf{goto\ done} \text{ } 5 \rangle \equiv$   
**for**  $(j = 1; x[j] \equiv 0; j++)$  {  
 $\langle \text{Delete the edge from } x[j] \text{ to } x[j] + j \text{ } 9 \rangle;$   
}  
**if**  $(j \equiv m - 1)$  **goto done**;  
 $\langle \text{Delete the edge from } x[j] \text{ to } x[j] + j \text{ } 9 \rangle;$   
 $x[j]--;$   
 $\langle \text{Insert an edge from } x[j] \text{ to } x[j] + j \text{ } 8 \rangle;$   
**for**  $(j--; j; j--)$  {  
 $x[j] = m - j;$   
 $\langle \text{Insert an edge from } x[j] \text{ to } x[j] + j \text{ } 8 \rangle;$   
}

This code is used in section 3.

**6. Graceful structures.** An unusual — indeed, somewhat amazing — data structure works well with graceful graphs.

Suppose  $v$  has neighbors  $w_1, \dots, w_t$ . Let  $f_v(w) = w - v$ , if  $w > v$ ;  $f_v(w) = m + v - w$ , if  $w < v$ . Then we set  $\text{arcs}[v] = f(w_1)$ , or 0 if  $t = 0$ ;  $\text{link}[f(w_j)] = f(w_{j+1})$  for  $1 \leq j < t$ ; and  $\text{link}[f(w_t)] = 0$ .

(Think about it. If  $0 < k \leq m$ , we use  $\text{link}[k]$  only for an arc from  $v$  to  $v + k$  for some  $v$ . If  $m < k \leq 2m$ , we use  $\text{link}[k]$  only for an arc from  $v$  to  $v - (k - m)$  for some  $v$ . In either case at most one such arc is present. Thus all of the memory for link storage is preallocated; we don't need a list of available slots.)

**7.** We silently use the facts that  $\text{arcs}[v]$  is initially 0 for all  $v$ , and  $\text{active} = 0$ . But the  $x$  and  $\text{link}$  arrays needn't be initialized (I mean, everything would work fine if they were initially garbage).

```

⟨ Initialize to  $(m - 1, \dots, 2, 1, 0)$  7 ⟩ ≡
  ⟨ Initialize the union/find structures 11 ⟩;
  for  $(j = m; j; j--)$  {
     $x[j] = m - j$ ;
    ⟨ Insert an edge from  $x[j]$  to  $x[j] + j$  8 ⟩;
  }

```

This code is used in section 3.

```

8.  ⟨ Insert an edge from  $x[j]$  to  $x[j] + j$  8 ⟩ ≡
  {
    register int  $p, u, v, uu, vv$ ;
     $u = x[j]$ ;
     $v = u + j$ ;
    ⟨ Do a union operation  $u \equiv v$  12 ⟩;
     $p = \text{arcs}[u]$ ;
    if  $(\neg p)$   $\text{active}++$ ;
     $\text{link}[j] = p, \text{arcs}[u] = j$ ;
     $p = \text{arcs}[v]$ ;
    if  $(\neg p)$   $\text{active}++$ ;
     $\text{link}[m + j] = p, \text{arcs}[v] = m + j$ ;
  }

```

This code is used in sections 5 and 7.

```

9.  ⟨ Delete the edge from  $x[j]$  to  $x[j] + j$  9 ⟩ ≡
  {
    register int  $p, u, v, uu, vv$ ;
     $u = x[j]$ ;
     $v = u + j$ ;
     $p = \text{link}[m + j]$ ; /* at this point  $\text{arcs}[v] = m + j$  */
     $\text{arcs}[v] = p$ ;
    if  $(\neg p)$   $\text{active}--$ ;
     $p = \text{link}[j]$ ; /* at this point  $\text{arcs}[u] = j$  */
     $\text{arcs}[u] = p$ ;
    if  $(\neg p)$   $\text{active}--$ ;
    ⟨ Undo the union operation  $u \equiv v$  14 ⟩;
  }

```

This code is used in section 5.

**10.** Two vertices are equivalent if they belong to the same component. We use a classic union-find data structure to keep of equivalences: The invariant relations state that  $parent[v] < 0$  and  $size[v] = c$  if  $v$  is the root of an equivalence class of size  $c$ ; otherwise  $parent[v]$  points to an equivalent vertex that is nearer the root. These trees have at most  $\lg m$  levels, because we never merge a tree of size  $c$  into a tree of size  $< c$ .

Variable  $l$  is the current number of edges. It is also, therefore, the number of union operations previously done but not yet undone.

**11.**  $\langle \text{Initialize the union/find structures } 11 \rangle \equiv$   
**for** ( $j = 0$ ;  $j \leq m$ ;  $j++$ )  $parent[j] = -1, size[j] = 1$ ;     $/* \text{ and } l = 0 */$   
 $l = 0$ ;

This code is used in section 7.

**12.**  $\langle \text{Do a union operation } u \equiv v \text{ } 12 \rangle \equiv$   
**for** ( $uu = u$ ;  $parent[uu] \geq 0$ ;  $uu = parent[uu]$ ) ;  
**for** ( $vv = v$ ;  $parent[vv] \geq 0$ ;  $vv = parent[vv]$ ) ;  
**if** ( $uu \equiv vv$ )  $move[l] = -1$ ;  
**else if** ( $size[uu] \leq size[vv]$ )  $parent[uu] = vv, move[l] = uu, size[vv] += size[uu]$ ;  
**else**  $parent[vv] = uu, move[l] = vv, size[uu] += size[vv]$ ;  
 $l++$ ;

This code is used in section 8.

**13.** Dynamic union-find is ridiculously easy because, as observed above, the operations are strictly last-in-first-out. And we didn't clobber the  $size$  information when merging two classes.

**14.**  $\langle \text{Undo the union operation } u \equiv v \text{ } 14 \rangle \equiv$   
 $l--$ ;  
 $uu = move[l]$ ;  
**if** ( $uu \geq 0$ ) {  
 $vv = parent[uu]$ ;     $/* \text{ we have } parent[vv] < 0 */$   
 $size[vv] -= size[uu]$ ;  
 $parent[uu] = -1$ ;  
}

This code is used in section 9.

**15.**  $\langle \text{Global variables } 15 \rangle \equiv$   
**int**  $active$ ;     $/* \text{ this many vertices are currently labeled (not isolated) } */$   
**int**  $parent[maxm + 1], size[maxm + 1], move[maxm]$ ;     $/* \text{ the union-find structures } */$   
**int**  $arcs[maxm + 1]$ ;     $/* \text{ the first neighbor of } v */$   
**int**  $link[2 * maxm + 1]$ ;     $/* \text{ the next element in a list of neighbors } */$   
**int**  $x[maxm + 1]$ ;     $/* \text{ the governing sequence of edge choices } */$

See also section 18.

This code is used in section 3.

**16. Doing it.** Now we're ready to harvest the routines we've built up.

[A puzzle for the reader: Is  $parent[m]$  always negative at this point? Answer: Not if, say,  $m = 7$  and  $(x_1, \dots, x_m) = (5, 4, 3, 2, 0, 1, 0)$ .]

⟨ Study the current graph 16 ⟩  $\equiv$

```

for ( $k = parent[m]$ ;  $parent[k] \geq 0$ ;  $k = parent[k]$ ) ;
if ( $size[k] \equiv active$ )  $connected[active]++$ ;
else  $disconnected[active]++$ ;

```

This code is used in section 3.

**17.** ⟨ Print the stats 17 ⟩  $\equiv$

```

printf("Counts_for_edges:\n", m);
for ( $k = 2$ ;  $k \leq m + 1$ ;  $k++$ )
  if ( $connected[k] + disconnected[k]$ ) {
    printf("on %5d vertices, %lld are connected, %lld not\n", k, 2 * connected[k],
          2 * disconnected[k]);
     $totconnected += 2 * connected[k]$ ,  $totdisconnected += 2 * disconnected[k]$ ;
  }
printf("Altogether %lld connected and %lld not.\n",  $totconnected$ ,  $totdisconnected$ );

```

This code is used in section 3.

**18.** ⟨ Global variables 15 ⟩  $+ \equiv$

```

unsigned long long  $connected[maxm + 2]$ ,  $disconnected[maxm + 2]$ ;
unsigned long long  $totconnected$ ,  $totdisconnected$ ;

```

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