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(See https://cs.stanford.edu/~knuth/programs.html for date.)

1. Intro. This program generates clauses for the transition relation from time t to time t+1 in Conway's Game of Life, assuming that all of the potentially live cells at time t belong to a pattern that's specified in stdin. The pattern is defined by one or more lines representing rows of cells, where each line has '.' in a cell that's guaranteed to be dead at time t, otherwise it has '\*'. The time is specified separately as a command-line parameter.

The Boolean variable for cell (x, y) at time t is named by its so-called "xty code," namely by the decimal value of x, followed by a code letter for t, followed by the decimal value of y. For example, if x = 10 and y = 11 and t = 0, the variable that indicates liveness of the cell is 10a11; and the corresponding variable for t = 1 is 10b11.

Up to 19 auxiliary variables are used together with each xty code, in order to construct clauses that define the successor state. The names of these variables are obtained by appending one of the following two-character combinations to the xty code: A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4, D1, D2, E1, E2, F1, F2, G1, G2. These variables are derived from the Bailleux-Boufkhad method of encoding cardinality constraints: The auxiliary variable Ak stands for the condition "at least k of the eight neighbors are alive." Similarly, Bk stands for "at least k of the first four neighbors are alive," and Ck accounts for the other four neighbors. Codes D, E, F, and C refer to pairs of neighbors. Thus, for instance, 10a11C2 means that at least two of the last four neighbors of cell (10,11) are alive.

Those auxiliary variables receive values by means of up to 77 clauses per cell. For example, if u and v are the neighbors of cell z that correspond to a pairing of type D, there are six clauses

$$\bar{u}d_1$$
,  $\bar{v}d_1$ ,  $\bar{u}\bar{v}d_2$ ,  $uv\bar{d}_1$ ,  $u\bar{d}_2$ ,  $v\bar{d}_2$ .

The sixteen clauses

$$ar{d}_1b_1, \quad ar{e}_1b_1, \quad ar{d}_2b_2, \quad ar{d}_1ar{e}_1b_2, \quad ar{e}_2b_2, \quad ar{d}_2ar{e}_1b_3, \quad ar{d}_1ar{e}_2b_3, \quad ar{d}_2ar{e}_2b_4, \\ d_1e_1ar{b}_1, \quad d_1e_2ar{b}_2, \quad d_2e_1ar{b}_2, \quad d_1ar{b}_3, \quad d_2e_2ar{b}_3, \quad e_1ar{b}_3, \quad d_2ar{b}_4, \quad e_2ar{b}_4$$

define b variables from d's and e's; and another sixteen define c's from f's and g's in the same fashion. A similar set of 21 clauses will define the a's from the b's and c's.

Once the a's are defined, thus essentially counting the live neighbors of cell z, the next state z' is defined by five further clauses

$$\bar{a}_4\bar{z}', \quad a_2\bar{z}', \quad a_3z\bar{z}', \quad \bar{a}_3a_4z', \quad \bar{a}_2a_4\bar{z}z'.$$

For example, the last of these states that z' will be true (i.e., that cell z will be alive at time t+1) if z is alive at time t and has  $\geq 2$  live neighbors but not  $\geq 4$ .

Nearby cells can share auxiliary variables, according to a tricky scheme that is worked out below. In consequence, the actual number of auxiliary variables and clauses per cell is reduced from 19 and 77 + 5 to 13 and 57 + 5, respectively, except at the boundaries.

2 INTRO SAT-LIFE §2

2. So here's the overall outline of the program.

```
#define maxx 50
                         /* maximum number of lines in the pattern supplied by stdin */
#define maxy 50
                        /* maximum number of columns per line in stdin */
#include <stdio.h>
#include <stdlib.h>
  char p[maxx + 2][maxy + 2];
                                      /* is cell (x, y) potentially alive? */
  char have_b[maxx + 2][maxy + 2];
                                           /* did we already generate b(x,y)? */
                                           /* did we already generate d(x, y)? */
  char have_{-}d[maxx + 2][maxy + 2];
  char have_{-}e[maxx + 2][maxy + 4];
                                           /* did we already generate e(x,y)? */
  char have_f[maxx + 4][maxy + 2];
                                          /* did we already generate f(x-2,y)? */
  int tt;
              /* time as given on the command line */
                          /* the number of rows and columns in the input pattern */
  int xmax, ymax;
  int xmin = maxx, ymin = maxy;
                                           /* limits in the other direction */
  char \ timecode[] = "abcdefghijklmnopqrstuvwxyz"
       "ABCDEFGHIJKLMNOPQRSTUVWXYZ"
                                                           /* codes for 0 \le t \le 83 */
       "!\"#$%&'()*+,-./:;<=>?@[\\]^_'{|}~";
                           /* input buffer */
  char buf[maxy + 2];
  unsigned int clause [4];
                                /* clauses are assembled here */
  int clauseptr;
                      /* this many literals are in the current clause */
  (Subroutines 6)
  main(int argc, char *argv[])
    register int j, k, x, y;
    \langle \text{Process the command line } 3 \rangle;
     \langle \text{Input the pattern 4} \rangle;
    for (x = xmin - 1; x \le xmax + 1; x++)
       for (y = ymin - 1; y \le ymax + 1; y++) {
         \langle \text{ If cell } (x,y) \text{ is obviously dead at time } t+1, \text{ continue } 5 \rangle;
         a(x,y);
         zprime(x, y);
       }
  }
3. \langle \text{Process the command line 3} \rangle \equiv
  if (argc \neq 2 \lor sscanf(argv[1], "%d", \&tt) \neq 1) {
    fprintf(stderr, "Usage: \_\%s_t \n", argv[0]);
    exit(-1);
  if (tt < 0 \lor tt > 82) {
    fprintf(stderr, "The_{\sqcup}time_{\sqcup}should_{\sqcup}be_{\sqcup}between_{\sqcup}0_{\sqcup}and_{\sqcup}82_{\sqcup}(not_{\sqcup}%d)! \n", tt);
    exit(-2);
This code is used in section 2.
```

§4 SAT-LIFE INTRO 3

```
4. \langle \text{Input the pattern 4} \rangle \equiv
  for (x = 1; ; x++) {
    if (\neg fgets(buf, maxy + 2, stdin)) break;
    if (x > maxx) {
       fprintf(stderr, "Sorry, \_the\_pattern\_should\_have\_at\_most\_%d\_rows! \n", maxx);
     for (y = 1; buf[y - 1] \neq '\n'; y++)  {
       if (y > maxy) {
         fprintf(stderr, "Sorry, \bot the \bot pattern \bot should \bot have \bot at \bot most \bot %d \bot columns! \n", maxy);
         exit(-4);
       if (buf[y-1] \equiv "") {
         p[x][y] = 1;
         if (y > ymax) ymax = y;
         if (y < ymin) ymin = y;
         if (x > xmax) xmax = x;
         if (x < xmin) \ xmin = x;
       } else if (buf[y-1] \neq '.') {
         fprintf(stderr, "Unexpected_character_'%c'_found_in_the_pattern!\\n", buf[y-1]);
         exit(-5);
This code is used in section 2.
5. #define pp(xx, yy) ((xx) \ge 0 \land (yy) \ge 0 ? p[xx][yy] : 0)
\langle \text{ If cell } (x,y) \text{ is obviously dead at time } t+1, \text{ continue } 5 \rangle \equiv
  if (pp(x-1,y-1) + pp(x-1,y) + pp(x-1,y+1) + pp(x,y-1) + p[x][y] + p[x][y+1] + pp(x+1,y+1)
         y-1)+p[x+1][y]+p[x+1][y+1]<3) continue;
```

This code is used in section 2.

4 INTRO SAT-LIFE §6

**6.** Clauses are assembled in the *clause* array (surprise), where we put encoded literals.

The code for a literal is an unsigned 32-bit quantity, where the leading bit is 1 if the literal should be complemented. The next three bits specify the type of the literal (0 thru 7 for plain and A-G); the next three bits specify an integer k; and the next bit is zero. That leaves room for two 12-bit fields, which specify x and y.

Type 0 literals have k = 0 for the ordinary xty code. However, the value k = 1 indicates that the time code should be for t + 1 instead of t. And k = 2 denotes a special "tautology" literal, which is always true. If the tautology literal is complemented, we omit it from the clause; otherwise we omit the entire clause. Finally, k = 7 denotes an auxiliary literal, used to avoid clauses of length 4.

Here's a subroutine that outputs the current clause and resets the *clause* array.

```
#define taut (2 \ll 25)
#define sign (1_{\rm U} \ll 31)
\langle \text{Subroutines } 6 \rangle \equiv
  void outclause(void)
     register int c, k, x, y, p;
     for (p = 0; p < clauseptr; p++)
       if (clause[p] \equiv taut) goto done;
     for (p = 0; p < clauseptr; p \leftrightarrow)
        if (clause[p] \neq taut + sign) {
          if (clause[p] \gg 31) printf("\square"); else printf("\square");
          c = (clause[p] \gg 28) \& #7;
          k = (clause[p] \gg 25) \& #7;
          x = (clause[p] \gg 12) \& #fff;
          y = clause[p] \& #fff;
          if (c) printf("%d%c%d%c%d", x, timecode[tt], y, c + '@', k);
          else if (k \equiv 7) printf ("%d%c%dx", x, timecode [tt], y);
          else printf("%d%c%d", x, timecode[tt + k], y);
     printf("\n");
  done: clauseptr = 0;
See also sections 7, 8, 9, 10, 11, 12, 14, and 15.
This code is used in section 2.
     And here's another, which puts a type-0 literal into clause.
\langle \text{Subroutines } 6 \rangle + \equiv
  void applit(int x, int y, int bar, int k)
     if (k \equiv 0 \land (x < xmin \lor x > xmax \lor y < ymin \lor y > ymax \lor p[x][y] \equiv 0))
        clause[clauseptr++] = (bar ? 0 : sign) + taut;
     else clause[clauseptr++] = (bar ? sign : 0) + (k \ll 25) + (x \ll 12) + y;
```

§8 SAT-LIFE INTRO 5

8. The d and e subroutines are called for only one-fourth of all cell addresses (x, y). Indeed, one can show that x is always odd, and that  $y \mod 4 < 2$ .

```
Therefore we remember if we've seen (x, y) before.
```

```
Slight trick: If yy is not in range, we avoid generating the clause \bar{d}_k twice.
```

```
#define newlit(x, y, c, k) clause[clauseptr ++] = ((c) \ll 28) + ((k) \ll 25) + ((x) \ll 12) + (y)
#define newcomplit(x, y, c, k) clause[clauseptr ++] = sign + ((c) \ll 28) + ((k) \ll 25) + ((x) \ll 12) + (y)
\langle \text{Subroutines } 6 \rangle + \equiv
  void d(\mathbf{int} \ x, \mathbf{int} \ y)
     register x1 = x - 1, x2 = x, yy = y + 1;
     if (have_{-}d[x][y] \neq tt + 1) {
        applit(x1, yy, 1, 0), newlit(x, y, 4, 1), outclause();
        applit(x2, yy, 1, 0), newlit(x, y, 4, 1), outclause();
        applit(x1, yy, 1, 0), applit(x2, yy, 1, 0), newlit(x, y, 4, 2), outclause();
        applit(x1, yy, 0, 0), applit(x2, yy, 0, 0), newcomplit(x, y, 4, 1), outclause();
        applit(x1, yy, 0, 0), newcomplit(x, y, 4, 2), outclause();
       if (yy \ge ymin \land yy \le ymax) applit (x^2, yy, 0, 0), newcomplit (x, y, 4, 2), outclause ();
        have_{-}d[x][y] = tt + 1;
     }
  }
  void e(\mathbf{int} \ x, \mathbf{int} \ y)
     register x1 = x - 1, x2 = x, yy = y - 1;
     if (have_{-}e[x][y] \neq tt + 1) {
        applit(x1, yy, 1, 0), newlit(x, y, 5, 1), outclause();
        applit(x2, yy, 1, 0), newlit(x, y, 5, 1), outclause();
        applit(x1, yy, 1, 0), applit(x2, yy, 1, 0), newlit(x, y, 5, 2), outclause();
        applit(x1, yy, 0, 0), applit(x2, yy, 0, 0), newcomplit(x, y, 5, 1), outclause();
        applit(x1, yy, 0, 0), newcomplit(x, y, 5, 2), outclause();
       if (yy \ge ymin \land yy \le ymax) applit (x2, yy, 0, 0), newcomplit (x, y, 5, 2), outclause ();
        have_{-}e[x][y] = tt + 1;
  }
```

**9.** The f subroutine can't be shared quite so often. But we do save a factor of 2, because x + y is always even.

```
 \begin{array}{l} \text{Void } f(\textbf{int } x, \textbf{int } y) \\ \{ \\ \textbf{register } xx = x - 1, \ y1 = y, \ y2 = y + 1; \\ \textbf{if } (have\_f[x][y] \neq tt + 1) \ \{ \\ applit(xx, y1, 1, 0), newlit(x, y, 6, 1), outclause(); \\ applit(xx, y2, 1, 0), newlit(x, y, 6, 1), outclause(); \\ applit(xx, y1, 1, 0), applit(xx, y2, 1, 0), newlit(x, y, 6, 2), outclause(); \\ applit(xx, y1, 0, 0), applit(xx, y2, 0, 0), newcomplit(x, y, 6, 1), outclause(); \\ applit(xx, y1, 0, 0), newcomplit(x, y, 6, 2), outclause(); \\ applit(xx, y1, 0, 0), newcomplit(x, y, 6, 2), outclause(); \\ if (xx \geq xmin \land xx \leq xmax) \ applit(xx, y2, 0, 0), newcomplit(x, y, 6, 2), outclause(); \\ have\_f[x][y] = tt + 1; \\ \} \\ \} \end{array}
```

6 Intro sat-life §10

10. The g subroutine cleans up the dregs, by somewhat tediously locating the two neighbors that weren't handled by d, e, or f. No sharing is possible here.

```
 \begin{array}{l} & \text{ void } g(\textbf{int } x, \textbf{int } y) \\ \{ & \text{ register } x1, \ x2, \ y1, \ y2; \\ & \textbf{ if } (x \& 1) \ x1 = x - 1, y1 = y, x2 = x + 1, y2 = y \oplus 1; \\ & \textbf{ else } \ x1 = x + 1, y1 = y, x2 = x - 1, y2 = y - 1 + ((y \& 1) \ll 1); \\ & applit(x1, y1, 1, 0), newlit(x, y, 7, 1), outclause(); \\ & applit(x2, y2, 1, 0), newlit(x, y, 7, 1), outclause(); \\ & applit(x1, y1, 1, 0), applit(x2, y2, 1, 0), newlit(x, y, 7, 2), outclause(); \\ & applit(x1, y1, 0, 0), applit(x2, y2, 0, 0), newcomplit(x, y, 7, 1), outclause(); \\ & applit(x2, y2, 0, 0), newcomplit(x, y, 7, 2), outclause(); \\ & applit(x2, y2, 0, 0), newcomplit(x, y, 7, 2), outclause(); \\ \} \end{array}
```

11. Fortunately the b subroutine can be shared (since x is always odd), thus saving half of the sixteen clauses generated.

```
 \begin{array}{l} \text{Void } b(\text{int } x, \text{int } y) \\ \{ \\ \text{register } j, \ k, \ xx = x, \ y1 = y - (y \& 2), \ y2 = y + (y \& 2); \\ \text{if } (have\_b[x][y] \neq tt + 1) \ \{ \\ d(xx,y1); \\ e(xx,y2); \\ \text{for } (j=0; \ j < 3; \ j++) \\ \text{for } (k=0; \ k < 3; \ k++) \\ \text{if } (j) \ newcomplit(xx,y1,4,j); \ \ /* \ \bar{d}_j \ */ \\ \text{if } (k) \ newcomplit(xx,y2,5,k); \ \ /* \ \bar{e}_k \ */ \\ newlit(x,y,2,j+k); \ \ /* \ b_{j+k} \ */ \\ outclause(); \\ \text{if } (j) \ newlit(xx,y2,5,3-k); \ \ /* \ d_{3-j} \ */ \\ \text{if } (k) \ newlit(xx,y2,5,3-k); \ \ /* \ \bar{b}_{5-j-k} \ */ \\ outclause(); \\ \} \\ newcomplit(x,y,2,5-j-k); \ \ /* \ \bar{b}_{5-j-k} \ */ \\ outclause(); \\ \} \\ have\_b[x][y] = tt+1; \\ \} \\ \} \end{array}
```

§12 SAT-LIFE INTRO 7

12. The (unshared) c subroutine handles the other four neighbors, by working with f and g instead of d and e.

If y = 0, the overlap rules set y1 = -1, which can be problematic. I've decided to avoid this case by omitting f when it is guaranteed to be zero.

```
\langle \text{Subroutines } 6 \rangle + \equiv
  void c(\mathbf{int} \ x, \mathbf{int} \ y)
  {
     register j, k, x1, y1;
     if (x \& 1) x1 = x + 2, y1 = (y - 1) | 1;
     else x1 = x, y1 = y \& -2;
     if (x1-1 < xmin \lor x1-1 > xmax \lor y1+1 < ymin \lor y1 > ymax) (Set c equal to g 13)
     else {
        f(x1, y1);
        for (j = 0; j < 3; j ++)
          for (k = 0; k < 3; k ++)
             if (j+k) {
               if (j) newcomplit (x1, y1, 6, j); /* \bar{f}_j */
               if (k) newcomplit(x, y, 7, k); /* \bar{g}_k */
                newlit(x, y, 3, j + k); /* c_{j+k} */
                outclause();
               if (j) newlit(x1, y1, 6, 3 - j); /* f_{3-j} */
               if (k) newlit(x, y, 7, 3 - k); /* g_{3-k} */
                newcomplit(x, y, 3, 5 - j - k); /* \bar{c}_{5-j-k} */
                outclause();
             }
     }
  }
13. \langle \text{Set } c \text{ equal to } g \text{ 13} \rangle \equiv
     for (k = 1; k < 3; k ++) {
        newcomplit(x, y, 7, k), newlit(x, y, 3, k), outclause();
                                                                        /* \bar{g}_k \vee c_k */
        newlit(x, y, 7, k), newcomplit(x, y, 3, k), outclause();
                                                                         /* g_k \vee \bar{c}_k */
     newcomplit(x, y, 3, 3), outclause();
                                                  /* \bar{c}_3 */
     newcomplit(x, y, 3, 4), outclause();
                                                  /* \bar{c}_4 */
This code is used in section 12.
```

8 INTRO SAT-LIFE §14

14. Totals over all eight neighbors are then deduced by the a subroutine.

```
\langle \text{Subroutines } 6 \rangle + \equiv
   void a(int x, int y)
       register j, k, xx = x \mid 1;
       b(xx,y);
       c(x,y);
       for (j = 0; j < 5; j++)
          for (k = 0; k < 5; k++)
              if (j + k > 1 \land j + k < 5) {
                  \begin{array}{lll} \textbf{if} \ (j) \ \ newcomplit(xx,y,2,j); & /* \ \bar{b}_j \ */ \\ \textbf{if} \ (k) \ \ newcomplit(x,y,3,k); & /* \ \bar{c}_k \ */ \end{array}
                  newlit(x, y, 1, j + k); /* a_{j+k} */
                  outclause();
       for (j = 0; j < 5; j++)
          for (k = 0; k < 5; k++)
              if (j+k > 2 \land j+k < 6 \land j*k) {
                  \begin{array}{lll} \textbf{if} \ (j) \ \ newlit(xx,y,2,j); & /* \ b_j \ */ \\ \textbf{if} \ (k) \ \ newlit(x,y,3,k); & /* \ c_k \ */ \end{array}
                  newcomplit(x, y, 1, j + k - 1); /* \bar{a}_{j+k-1} */
                  outclause();
              }
   }
```

15. Finally, as mentioned at the beginning, z' is determined from z,  $a_2$ ,  $a_3$ , and  $a_4$ . I actually generate six clauses, not five, in order to stick to 3SAT.

```
 \begin{array}{l} \langle \, {\rm Subroutines} \,\, 6 \, \rangle \, + \equiv \\ & {\bf void} \,\, zprime({\bf int} \,\, x, {\bf int} \,\, y) \\ \{ \\ & \,\, newcomplit(x,y,1,4), applit(x,y,1,1), outclause(\,); \quad / * \,\, \bar{a}_4 \bar{z}' \,\, */\\ & \,\, newlit(x,y,1,2), applit(x,y,1,1), outclause(\,); \quad / * \,\, a_2 \bar{z}' \,\, */\\ & \,\, newlit(x,y,1,3), applit(x,y,0,0), applit(x,y,1,1), outclause(\,); \quad / * \,\, \bar{a}_3 a_4 z' \,\, */\\ & \,\, newcomplit(x,y,1,3), newlit(x,y,1,4), applit(x,y,0,1), outclause(\,); \quad / * \,\, \bar{a}_3 a_4 z' \,\, */\\ & \,\, applit(x,y,0,7), newcomplit(x,y,1,2), newlit(x,y,1,4), outclause(\,); \quad / * \,\, x\bar{a}_2 a_4 \,\, */\\ & \,\, applit(x,y,1,7), applit(x,y,1,0), applit(x,y,0,1), outclause(\,); \quad / * \,\, x\bar{z}z' \,\, */\\ \} \end{array}
```

§16 SAT-LIFE INDEX 9

## 16. Index.

```
y2: \ \underline{9}, \ \underline{10}, \ \underline{11}.
a: <u>14</u>.
applit: 7, 8, 9, 10, 15.
                                                                                                    zprime: 2, \underline{15}.
argc: \underline{2}, 3.
argv: \underline{2}, \underline{3}.
b: \underline{11}.
bar: \underline{7}.
buf: \underline{2}, 4.
c: \ \underline{6}, \ \underline{12}.
clause: \underline{2}, 6, 7, 8.
clauseptr: \underline{2}, 6, 7, 8.
d: 8.
done: \underline{6}.
e: <u>8</u>.
exit: 3, 4.
f: \underline{9}.
fgets: 4.
fprintf: 3, 4.
g: \underline{10}.
have_b: \underline{2}, \underline{11}.
have_{-}d: 2, 8.
have_e: \underline{2}, 8.
have_{-}f: \underline{2}, \underline{9}.
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k: 2, 6, 7, 11, 12, 14.
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maxx: \underline{2}, 4.
maxy: \underline{2}, 4.
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newlit: 8, 9, 10, 11, 12, 13, 14, 15.
outclause: <u>6</u>, 8, 9, 10, 11, 12, 13, 14, 15.
p: 2, 6.
pp: \underline{5}.
printf: 6.
sign: \underline{6}, 7, 8.
sscanf: 3.
stderr: 3, 4.
stdin: 1, 2, 4.
taut: \underline{6}, 7.
timecode: 2, 6.
tt: 2, 3, 6, 8, 9, 11.
x: \quad \underline{2}, \ \underline{6}, \ \underline{7}, \ \underline{8}, \ \underline{9}, \ \underline{10}, \ \underline{11}, \ \underline{12}, \ \underline{14}, \ \underline{15}.
xmax: \ \underline{2}, \ 4, \ 7, \ 9, \ 12.
xmin: \ \underline{2}, \ 4, \ 7, \ 9, \ 12.
xx: 5, \underline{9}, \underline{11}, \underline{14}.
x1: \ \ \underline{8}, \ \underline{10}, \ \underline{12}.
x2: 8, 10.
y: 2, 6, 7, 8, 9, 10, 11, 12, 14, 15.
ymax: 2, 4, 7, 8, 12.
ymin: 2, 4, 7, 8, 12.
yy: 5, 8.
y1: \ \underline{9}, \ \underline{10}, \ \underline{11}, \ \underline{12}.
```

10 NAMES OF THE SECTIONS SAT-LIFE

```
 \begin{array}{lll} \left\langle \text{If cell } (x,y) \text{ is obviously dead at time } t+1, \, \textbf{continue} \, \, 5 \right\rangle & \text{Used in section 2.} \\ \left\langle \text{Input the pattern 4} \right\rangle & \text{Used in section 2.} \\ \left\langle \text{Process the command line 3} \right\rangle & \text{Used in section 2.} \\ \left\langle \text{Set } c \text{ equal to } g \, \, 13 \right\rangle & \text{Used in section 12.} \\ \left\langle \text{Subroutines 6, 7, 8, 9, 10, 11, 12, 14, 15} \right\rangle & \text{Used in section 2.} \\ \end{array}
```

## SAT-LIFE

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