(See https://cs.stanford.edu/~knuth/programs.html for date.)

1.* Intro. This program generates clauses for the transition relation from time t to time $t+1$ in Conway's Game of Life, for various values of t, as I'm trying to prove or disprove a certain conjecture.

Namely, it may be possible to set cells (x,y) for $x \le 0$ and $y \le 0$ (i.e., in the lower left quadrant) in such a way that cell (x,y) is reachable in $x+2y$ steps when $0 \le -x \le y$, and in $2y+2x$ steps when $x \ge 0$ and $y \ge 0$. Hopefully by seeing examples for small x and y I will have a handle on that conjecture.

The conjectured bounds agree with lower bounds that are readily proved. Hence the problem is to find matching upper bounds, if possible.

The command line should contain the coordinates x_0 and y_0 being tested.

When the conjectured bound is r, this program uses nested boards of sizes $(2r+1)\times(2r+1)$, $(2r-1)\times(2r-1)$, $\ldots$, 3×3, 1×1, centered on the cell (x,y) that we're trying to turn on. Many of the cells are known to be zero, because of the lower bounds; therefore we don't include them in the computation.

The Boolean variable for cell (x,y) at time t is named by its so-called "xty code," namely by the decimal value of x, followed by a code letter for t, followed by the decimal value of y. For example, if $x=10$ and $y=11$ and $t=0$, the variable that indicates liveness of the cell is `10a11`; and the corresponding variable for $t=1$ is `10b11`.

Up to 19 auxiliary variables are used together with each xty code, in order to construct clauses that define the successor state. The names of these variables are obtained by appending one of the following two-character combinations to the xty code: `A2`, `A3`, `A4`, `B1`, `B2`, `B3`, `B4`, `C1`, `C2`, `C3`, `C4`, `D1`, `D2`, `E1`, `E2`, `F1`, `F2`, `G1`, `G2`. These variables are derived from the Bailleux–Boufkhad method of encoding cardinality constraints: The auxiliary variable `A`k stands for the condition "at least k of the eight neighbors are alive." Similarly, `B`k stands for "at least k of the first four neighbors are alive," and `C`k accounts for the other four neighbors. Codes `D`, `E`, `F`, and `G` refer to pairs of neighbors. Thus, for instance, `10a11C2` means that at least two of the last four neighbors of cell $(10,11)$ are alive.

Those auxiliary variables receive values by means of up to 77 clauses per cell. For example, if u and v are the neighbors of cell z that correspond to a pairing of type `D`, there are six clauses

$$\bar u d_1,\quad \bar v d_1,\quad \bar u\bar v d_2,\quad uv\bar d_1,\quad u\bar d_2,\quad v\bar d_2.$$

The sixteen clauses

$$\begin{gathered}\bar d_1 b_1,\quad \bar e_1 b_1,\quad \bar d_2 b_2,\quad \bar d_1\bar e_1 b_2,\quad \bar e_2 b_2,\quad \bar d_2\bar e_1 b_3,\quad \bar d_1\bar e_2 b_3,\quad \bar d_2\bar e_2 b_4,\\ d_1e_1\bar b_1,\quad d_1e_2\bar b_2,\quad d_2e_1\bar b_2,\quad d_1\bar b_3,\quad d_2e_2\bar b_3,\quad e_1\bar b_3,\quad d_2\bar b_4,\quad e_2\bar b_4\end{gathered}$$

define b variables from d's and e's; and another sixteen define c's from f's and g's in the same fashion. A similar set of 21 clauses will define the a's from the b's and c's.

Once the a's are defined, thus essentially counting the live neighbors of cell z, the next state z' is defined by five further clauses

$$\bar a_4\bar z',\quad a_2\bar z',\quad a_3 z\bar z',\quad \bar a_3 a_4 z',\quad \bar a_2 a_4\bar z z'.$$

For example, the last of these states that z' will be true (i.e., that cell z will be alive at time $t+1$) if z is alive at time t and has ≥ 2 live neighbors but not ≥ 4.

Nearby cells can share auxiliary variables, according to a tricky scheme that is worked out below. In consequence, the actual number of auxiliary variables and clauses per cell is reduced from 19 and $77+5$ to 13 and $57+5$, respectively, except at the boundaries.

2*. So here's the overall outline of the program.

```
#define maxx 50   /* maximum number of lines in the pattern supplied by stdin */
#define maxy 50   /* maximum number of columns per line in stdin */
#include <stdio.h>
#include <stdlib.h>
  char p[maxx + 2][maxy + 2];   /* is cell (x, y) potentially alive? */
  char have_b[maxx + 2][maxy + 2];   /* did we already generate b(x, y)? */
  char have_d[maxx + 2][maxy + 2];   /* did we already generate d(x, y)? */
  char have_e[maxx + 2][maxy + 4];   /* did we already generate e(x, y)? */
  char have_f[maxx + 4][maxy + 2];   /* did we already generate f(x − 2, y)? */
  int tt;   /* the time being considered */
  int x0, y0;   /* the command-line parameters */
  int r;   /* the conjectured bound */
  int xmax, ymax;   /* the number of rows and columns in the input pattern */
  int xmin = maxx, ymin = maxy;   /* limits in the other direction */
  char timecode[] = "abcdefghijklmnopqrstuvwxyz"
      "ABCDEFGHIJKLMNOPQRSTUVWXYZ"
      "!\"#$%&'()*+,-./:;<=>?@[\\]^_`{|}~";   /* codes for 0 ≤ t ≤ 83 */
  char buf[maxy + 2];   /* input buffer */
  unsigned int clause[4];   /* clauses are assembled here */
  int clauseptr;   /* this many literals are in the current clause */
  ⟨Subroutines 6⟩
  main(int argc, char *argv[])
  {
    register int j, k, x, y;
    ⟨Process the command line 3*⟩;
    for (tt = 0; tt < r; tt++) {
      xmin = ymin = 1 + tt, xmax = ymax = r + r + 1 − tt;
      for (x = xmin + 1; x < xmax; x++)
        for (y = ymin + 1; y < ymax; y++) {
          if (bound(x, y) > tt + 1) continue;
          a(x, y);
          zprime(x, y);
        }
    }
    printf("%d%c%d\n", r + 1, timecode[tt], r + 1);   /* middle variable must be alive */
  }
```

3*. ⟨ Process the command line 3* ⟩ ≡

```
if (argc ≠ 3 ∨ sscanf(argv[1], "%d", &x0) ≠ 1 ∨ sscanf(argv[2], "%d", &y0) ≠ 1) {
  fprintf(stderr, "Usage:␣%s␣x0␣y0\n", argv[0]);
  exit(−1);
}
if (y0 ≤ 0) {
  fprintf(stderr, "The␣value␣of␣y0␣should␣be␣positive!\n");
  exit(−2);
}
if (x0 < −y0) {
  fprintf(stderr, "The␣value␣of␣x0␣should␣be␣at␣least␣-y0!\n");
  exit(−3);
}
r = (x0 > 0 ? 2 ∗ (x0 + y0) : x0 + 2 ∗ y0);
printf("~␣sat-life-upper␣%d␣%d\n", x0, y0);
```

This code is used in section 2*.

4. ⟨ Input the pattern 4 ⟩ ≡

```
for (x = 1; ; x++) {
  if (¬fgets(buf, maxy + 2, stdin)) break;
  if (x > maxx) {
    fprintf(stderr, "Sorry,␣the␣pattern␣should␣have␣at␣most␣%d␣rows!\n", maxx);
    exit(−3);
  }
  for (y = 1; buf[y − 1] ≠ '\n'; y++) {
    if (y > maxy) {
      fprintf(stderr, "Sorry,␣the␣pattern␣should␣have␣at␣most␣%d␣columns!\n", maxy);
      exit(−4);
    }
    if (buf[y − 1] ≡ '*') {
      p[x][y] = 1;
      if (y > ymax) ymax = y;
      if (y < ymin) ymin = y;
      if (x > xmax) xmax = x;
      if (x < xmin) xmin = x;
    } else if (buf[y − 1] ≠ '.') {
      fprintf(stderr, "Unexpected␣character␣`%c'␣found␣in␣the␣pattern!\n", buf[y − 1]);
      exit(−5);
    }
  }
}
```

5. #**define** *pp*(*xx*, *yy*) ((*xx*) ≥ 0 ∧ (*yy*) ≥ 0 ? *p*[*xx*][*yy*] : 0)

⟨ If cell (x, y) is obviously dead at time $t + 1$, **continue** 5 ⟩ ≡

```
if (pp(x − 1, y − 1) + pp(x − 1, y) + pp(x − 1, y + 1) + pp(x, y − 1) + p[x][y] + p[x][y + 1] + pp(x + 1,
      y − 1) + p[x + 1][y] + p[x + 1][y + 1] < 3) continue;
```

6. Clauses are assembled in the *clause* array (surprise), where we put encoded literals.

The code for a literal is an unsigned 32-bit quantity, where the leading bit is 1 if the literal should be complemented. The next three bits specify the type of the literal (0 thru 7 for plain and `A`–`G`); the next three bits specify an integer k; and the next bit is zero. That leaves room for two 12-bit fields, which specify x and y.

Type 0 literals have $k = 0$ for the ordinary xty code. However, the value $k = 1$ indicates that the time code should be for $t + 1$ instead of t. And $k = 2$ denotes a special "tautology" literal, which is always true. If the tautology literal is complemented, we omit it from the clause; otherwise we omit the entire clause. Finally, $k = 7$ denotes an auxiliary literal, used to avoid clauses of length 4.

Here's a subroutine that outputs the current clause and resets the *clause* array.

```
#define taut (2 ≪ 25)
#define sign (1U ≪ 31)
⟨ Subroutines 6 ⟩ ≡
  void outclause(void)
  {
    register int c, k, x, y, p;
    for (p = 0; p < clauseptr; p++)
      if (clause[p] ≡ taut) goto done;
    for (p = 0; p < clauseptr; p++)
      if (clause[p] ≠ taut + sign) {
        if (clause[p] ≫ 31) printf("␣~"); else printf("␣");
        c = (clause[p] ≫ 28) & #7;
        k = (clause[p] ≫ 25) & #7;
        x = (clause[p] ≫ 12) & #fff;
        y = clause[p] & #fff;
        if (c) printf("%d%c%d%c%d", x, timecode[tt], y, c + '@', k);
        else if (k ≡ 7) printf("%d%c%dx", x, timecode[tt], y);
        else printf("%d%c%d", x, timecode[tt + k], y);
      }
    printf("\n");
  done: clauseptr = 0;
  }
```

See also sections 7*, 8, 9, 10, 11, 12, 14, 15, and 16*.

This code is used in section 2*.

7* And here's another, which puts a type-0 literal into *clause*.

```
⟨ Subroutines 6 ⟩ +≡
  void applit(int x, int y, int bar, int k)
  {
    if (k ≡ 0 ∧ bound(x, y) > tt) clause[clauseptr++] = (bar ? 0 : sign) + taut;
    else clause[clauseptr++] = (bar ? sign : 0) + (k ≪ 25) + (x ≪ 12) + y;
  }
```

8. The d and e subroutines are called for only one-fourth of all cell addresses (x, y). Indeed, one can show that x is always odd, and that $y \bmod 4 < 2$.

Therefore we remember if we've seen (x, y) before.

Slight trick: If yy is not in range, we avoid generating the clause $\bar{d}_k$ twice.

```
#define newlit(x, y, c, k)   clause[clauseptr++] = ((c) ≪ 28) + ((k) ≪ 25) + ((x) ≪ 12) + (y)
#define newcomplit(x, y, c, k)   clause[clauseptr++] = sign + ((c) ≪ 28) + ((k) ≪ 25) + ((x) ≪ 12) + (y)
```

⟨ Subroutines 6 ⟩ +≡

```
void d(int x, int y)
{
  register x1 = x − 1, x2 = x, yy = y + 1;
  if (have_d[x][y] ≠ tt + 1) {
    applit(x1, yy, 1, 0), newlit(x, y, 4, 1), outclause();
    applit(x2, yy, 1, 0), newlit(x, y, 4, 1), outclause();
    applit(x1, yy, 1, 0), applit(x2, yy, 1, 0), newlit(x, y, 4, 2), outclause();
    applit(x1, yy, 0, 0), applit(x2, yy, 0, 0), newcomplit(x, y, 4, 1), outclause();
    applit(x1, yy, 0, 0), newcomplit(x, y, 4, 2), outclause();
    if (yy ≥ ymin ∧ yy ≤ ymax) applit(x2, yy, 0, 0), newcomplit(x, y, 4, 2), outclause();
    have_d[x][y] = tt + 1;
  }
}
void e(int x, int y)
{
  register x1 = x − 1, x2 = x, yy = y − 1;
  if (have_e[x][y] ≠ tt + 1) {
    applit(x1, yy, 1, 0), newlit(x, y, 5, 1), outclause();
    applit(x2, yy, 1, 0), newlit(x, y, 5, 1), outclause();
    applit(x1, yy, 1, 0), applit(x2, yy, 1, 0), newlit(x, y, 5, 2), outclause();
    applit(x1, yy, 0, 0), applit(x2, yy, 0, 0), newcomplit(x, y, 5, 1), outclause();
    applit(x1, yy, 0, 0), newcomplit(x, y, 5, 2), outclause();
    if (yy ≥ ymin ∧ yy ≤ ymax) applit(x2, yy, 0, 0), newcomplit(x, y, 5, 2), outclause();
    have_e[x][y] = tt + 1;
  }
}
```

9. The f subroutine can't be shared quite so often. But we do save a factor of 2, because $x + y$ is always even.

⟨ Subroutines 6 ⟩ +≡

```
void f(int x, int y)
{
  register xx = x − 1, y1 = y, y2 = y + 1;
  if (have_f[x][y] ≠ tt + 1) {
    applit(xx, y1, 1, 0), newlit(x, y, 6, 1), outclause();
    applit(xx, y2, 1, 0), newlit(x, y, 6, 1), outclause();
    applit(xx, y1, 1, 0), applit(xx, y2, 1, 0), newlit(x, y, 6, 2), outclause();
    applit(xx, y1, 0, 0), applit(xx, y2, 0, 0), newcomplit(x, y, 6, 1), outclause();
    applit(xx, y1, 0, 0), newcomplit(x, y, 6, 2), outclause();
    if (xx ≥ xmin ∧ xx ≤ xmax) applit(xx, y2, 0, 0), newcomplit(x, y, 6, 2), outclause();
    have_f[x][y] = tt + 1;
  }
}
```

10. The g subroutine cleans up the dregs, by somewhat tediously locating the two neighbors that weren't handled by d, e, or f. No sharing is possible here.

⟨Subroutines 6⟩ +≡

```
void g(int x, int y)
{
  register x1, x2, y1, y2;
  if (x & 1) x1 = x − 1, y1 = y, x2 = x + 1, y2 = y ⊕ 1;
  else x1 = x + 1, y1 = y, x2 = x − 1, y2 = y − 1 + ((y & 1) ≪ 1);
  applit(x1, y1, 1, 0), newlit(x, y, 7, 1), outclause();
  applit(x2, y2, 1, 0), newlit(x, y, 7, 1), outclause();
  applit(x1, y1, 1, 0), applit(x2, y2, 1, 0), newlit(x, y, 7, 2), outclause();
  applit(x1, y1, 0, 0), applit(x2, y2, 0, 0), newcomplit(x, y, 7, 1), outclause();
  applit(x1, y1, 0, 0), newcomplit(x, y, 7, 2), outclause();
  applit(x2, y2, 0, 0), newcomplit(x, y, 7, 2), outclause();
}
```

11. Fortunately the b subroutine *can* be shared (since x is always odd), thus saving half of the sixteen clauses generated.

⟨Subroutines 6⟩ +≡

```
void b(int x, int y)
{
  register j, k, xx = x, y1 = y − (y & 2), y2 = y + (y & 2);
  if (have_b[x][y] ≠ tt + 1) {
    d(xx, y1);
    e(xx, y2);
    for (j = 0; j < 3; j++)
      for (k = 0; k < 3; k++)
        if (j + k) {
          if (j) newcomplit(xx, y1, 4, j);     /* d̄_j */
          if (k) newcomplit(xx, y2, 5, k);     /* ē_k */
          newlit(x, y, 2, j + k);     /* b_{j+k} */
          outclause();
          if (j) newlit(xx, y1, 4, 3 − j);     /* d_{3−j} */
          if (k) newlit(xx, y2, 5, 3 − k);     /* e_{3−k} */
          newcomplit(x, y, 2, 5 − j − k);     /* b̄_{5−j−k} */
          outclause();
        }
    have_b[x][y] = tt + 1;
  }
}
```

12. The (unshared) c subroutine handles the other four neighbors, by working with f and g instead of d and e.

If $y = 0$, the overlap rules set $y1 = -1$, which can be problematic. I've decided to avoid this case by omitting f when it is guaranteed to be zero.

```
⟨Subroutines 6⟩ +≡
  void c(int x, int y)
  {
    register j, k, x1, y1;
    if (x & 1) x1 = x + 2, y1 = (y − 1) | 1;
    else x1 = x, y1 = y & −2;
    g(x, y);
    if (x1 − 1 < xmin ∨ x1 − 1 > xmax ∨ y1 + 1 < ymin ∨ y1 > ymax) ⟨Set c equal to g 13⟩
    else {
      f(x1, y1);
      for (j = 0; j < 3; j++)
        for (k = 0; k < 3; k++)
          if (j + k) {
            if (j) newcomplit(x1, y1, 6, j);    /* f̄_j */
            if (k) newcomplit(x, y, 7, k);    /* ḡ_k */
            newlit(x, y, 3, j + k);    /* c_{j+k} */
            outclause();
            if (j) newlit(x1, y1, 6, 3 − j);    /* f_{3−j} */
            if (k) newlit(x, y, 7, 3 − k);    /* g_{3−k} */
            newcomplit(x, y, 3, 5 − j − k);    /* c̄_{5−j−k} */
            outclause();
          }
    }
  }
```

13. ⟨Set c equal to g 13⟩ ≡

```
  {
    for (k = 1; k < 3; k++) {
      newcomplit(x, y, 7, k), newlit(x, y, 3, k), outclause();    /* ḡ_k ∨ c_k */
      newlit(x, y, 7, k), newcomplit(x, y, 3, k), outclause();    /* g_k ∨ c̄_k */
    }
    newcomplit(x, y, 3, 3), outclause();    /* c̄_3 */
    newcomplit(x, y, 3, 4), outclause();    /* c̄_4 */
  }
```

This code is used in section 12.

14. Totals over all eight neighbors are then deduced by the a subroutine.

⟨ Subroutines 6 ⟩ +≡

```
void a(int x, int y)
{
  register j, k, xx = x | 1;
  b(xx, y);
  c(x, y);
  for (j = 0; j < 5; j++)
    for (k = 0; k < 5; k++)
      if (j + k > 1 ∧ j + k < 5) {
        if (j) newcomplit(xx, y, 2, j);    /* b̄_j */
        if (k) newcomplit(x, y, 3, k);    /* c̄_k */
        newlit(x, y, 1, j + k);    /* a_{j+k} */
        outclause();
      }
  for (j = 0; j < 5; j++)
    for (k = 0; k < 5; k++)
      if (j + k > 2 ∧ j + k < 6 ∧ j * k) {
        if (j) newlit(xx, y, 2, j);    /* b_j */
        if (k) newlit(x, y, 3, k);    /* c_k */
        newcomplit(x, y, 1, j + k - 1);    /* ā_{j+k-1} */
        outclause();
      }
}
```

15. Finally, as mentioned at the beginning, z' is determined from z, a_2, a_3, and a_4. I actually generate six clauses, not five, in order to stick to 3SAT.

⟨ Subroutines 6 ⟩ +≡

```
void zprime(int x, int y)
{
  newcomplit(x, y, 1, 4), applit(x, y, 1, 1), outclause();    /* ā_4 z̄' */
  newlit(x, y, 1, 2), applit(x, y, 1, 1), outclause();    /* a_2 z̄' */
  newlit(x, y, 1, 3), applit(x, y, 0, 0), applit(x, y, 1, 1), outclause();    /* a_3 z z̄' */
  newcomplit(x, y, 1, 3), newlit(x, y, 1, 4), applit(x, y, 0, 1), outclause();    /* ā_3 a_4 z' */
  applit(x, y, 0, 7), newcomplit(x, y, 1, 2), newlit(x, y, 1, 4), outclause();    /* x ā_2 a_4 */
  applit(x, y, 1, 7), applit(x, y, 1, 0), applit(x, y, 0, 1), outclause();    /* x̄ z̄ z' */
}
```

16* In this variation of the program, I compute the known lower bounds. At time t, only the entries of p that are $\le t$ are considered potentially alive.

I've been thinking "rows and columns" instead of Cartesian coordinates, so the notation is a bit schizophrenic here. An x value in the user interface corresponds to column $x+c$, where $c = 1 + r - x_0$; and a y value corresponds to row $d - y$, where $d = 1 + r + y_0$. (Hence in particular, cell $(0,0)$ corresponds to column c of row d. Since $r \ge 2y_0 - x_0$, we have $c \ge 3$.)

⟨ Subroutines 6 ⟩ +≡

```
int ff(int x, int y)
{
  if (x ≤ 0 ∧ y ≤ 0) return 0;
  if (y < 0) return ff(y, x);
  if (x ≤ −y) return y;
  if (x ≤ 0) return x + y + y;
  return x + x + y + y;
}
int bound(int xx, int yy)
{
  return ff(yy − (1 + r − x0), (1 + r + y0) − xx);
}
```

17*. Index.

The following sections were changed by the change file: 1, 2, 3, 7, 16, 17.

SAT-LIFE-UPPER