

(See <https://cs.stanford.edu/~knuth/programs.html> for date.)

**1. Introduction.** This short program implements a Françon-inspired bijection between binary trees with Strahler number  $s$  and nested strings with height  $h$ , where  $2^s - 1 \leq h < 2^{s+1} - 1$ . But it uses a direct method that is complementary to his approach. [Reference: Jean Françon, “Sur le nombre de registres nécessaires à l’évaluation d’une expression arithmétique,” *R.A.I.R.O. Informatique théorique* **18** (1984), 355–364.]

```
#define n 17      /* nodes in the tree */
#define nn (n + n)
#include <stdio.h>
int d[nn + 1];    /* the path, a sequence of ±1s */
int l[n + 1], r[n + 1]; /* tree links */
int h[nn + 1], q[n + 1], qm[n + 1]; /* heap and queue structures for decision-making */
int serial;       /* total number of cases checked */
int count[10];    /* individual counts by Strahler number */
<Subroutines 5>
main()
{
    register int i, j, k, jj, kk, m, p, s;
    printf("Checking binary trees with %d nodes...\n", n);
    <Set up the first nested string, d 2>;
    while (1) {
        <Find the tree corresponding to d 7>;
        <Check the Strahler number 4>;
        <Check the inverse bijection 9>;
        <Move to the next nested string, or goto done 3>;
    }
done:
    for (s = 1; count[s]; s++)
        printf("Altogether %d cases with Strahler number %d.\n", count[s], s);
}
```

**2.** Nested strings (aka Dyck words) are conveniently generated by Algorithm 7.2.1.6P of *The Art of Computer Programming*.

```
<Set up the first nested string, d 2> ≡
for (k = 0; k < nn; k += 2) d[k] = +1, d[k + 1] = -1;
d[nn] = -1, i = nn - 2;
```

This code is used in section 1.

**3.** At this point, variable  $i$  is the position of the rightmost ‘+1’ in  $d$ .

```
<Move to the next nested string, or goto done 3> ≡
d[i] = -1;
if (d[i - 1] < 0) d[i - 1] = 1, i--;
else {
    for (j = i - 1, k = nn - 2; d[j] > 0; j--, k -= 2) {
        d[j] = -1, d[k] = +1;
        if (j ≡ 0) goto done;
    }
    d[j] = +1, i = nn - 2;
}
```

This code is used in section 1.

4.  $\langle \text{Check the Strahler number } 4 \rangle \equiv$   
**for** ( $s = j = k = 1$ ;  $k < nn - 1$ ;  $j += d[k], k++$ )  
     **if** ( $j \geq ((1 \ll s) - 1)$ )  $s++$ ;  
      $s--$ ;      /\* now  $s$  is the Strahler number \*/  
      $count[s]++, serial++$ ;  
     **if** ( $strahler(1) \neq s$ ) {  
          $fprintf(stderr, "I\_goofed\_on\_case\_%d.\n", serial)$ ;  
     }

This code is used in section 1.

5.  $\langle \text{Subroutines } 5 \rangle \equiv$   
**int** *strahler*(**int**  $x$ )  
 {  
     **register int**  $sl, sr$ ;  
     **if** ( $l[x]$ )  $sl = strahler(l[x])$ ;  
     **else**  $sl = 0$ ;  
     **if** ( $r[x]$ )  $sr = strahler(r[x])$ ;  
     **else**  $sr = 0$ ;  
     **return** ( $sl > sr ? sl : sl < sr ? sr : sl + 1$ );  
 }

This code is used in section 1.

**6. The main algorithm.** A large family of bijections between nested strings and binary trees was described by Proskurowski in *JACM* **27** (1980), page 1: We build a binary tree by choosing, at each step, some node and some yet-unset link field in that node; then we look at the next element  $d[p]$  of the nested string. The link is set to a new node if  $d[p] > 0$ , and to null if  $d[p] < 0$ . The bijection implemented here is of that type.

To decide what link should be constructed next, we use a heap-like data structure  $h[1], h[2], \dots$ , in which cell  $k$  is the parent of cells  $2k$  and  $2k + 1$ . The cell elements are pointers to nodes in the tree being built, and the nodes recorded in the heap can be embedded as a subtree of that tree. (In other words, if  $h[k]$  and  $h[\lfloor k/2 \rfloor]$  are both nonzero, they point to nodes of the tree in which the first is a descendant of the second. It might be helpful to imagine a set of pebbles on the tree, with the heap cells recording the positions of those pebbles.) When  $h[2k] = 0$ , meaning that heap cell  $2k$  is empty, we also have  $h[2k + 1] = 0$ . The basic idea of the algorithm is to attempt to fill the first empty cell  $k$  in the heap, by setting the links of the tree node pointed to by  $h[k/2]$ .

The number of elements in the heap is always the partial sum  $d[0] + \dots + d[p]$ . If this number is  $2^t - 1$  or more, the Strahler number of the binary tree is at least  $t$ . Conversely, if the Strahler number is  $s$ , one can show without difficulty that the partial sum will indeed reach the value  $2^s - 1$  at some point, with the heap at that time containing the “topmost” complete subtree of size  $2^s - 1$  embedded in the tree.

For validity of this algorithm, we don’t really need to choose the first hole in the heap. Any rule for choosing  $k$  would work, provided only that (a)  $k$  is even; (b)  $h[k/2] \neq 0$ ; and (c)  $k \geq 2^t$  implies  $d[0] + \dots + d[p] \geq 2^t - 1$ . Thus there are many possible bijections, some of which are presumably easier to analyze than others.

**7.** Variable  $m$  represents the number of nodes in the tree; variable  $p$  is our position in the nested string; and variable  $k$  is a lower bound on the location of the least hole in the heap.

```

⟨Find the tree corresponding to  $d$  7⟩ ≡
   $h[1] = m = 1, k = 2, p = 0$ ;
  while (1) {
    while ( $h[k]$ )  $k += 2$ ;    /* find the smallest hole */
     $kk = h[k \gg 1]$ ;    /*  $kk$  is the node pointed to by  $k$ 's parent */
    if ( $d[+p] > 0$ )  $h[k] = l[kk] = ++m$ ; else  $l[kk] = 0$ ;
    if ( $d[+p] > 0$ )  $h[k + 1] = r[kk] = ++m$ ; else  $r[kk] = 0$ ;
    if ( $h[k]$ ) {
      if ( $h[k + 1]$ ) continue;
       $kk = k$ ;
    } else if ( $h[k + 1]$ )  $kk = k + 1$ ;
    else {
       $h[k \gg 1] = 0, kk = (k \gg 1) \oplus 1, k = kk \& -2$ ;
      if ( $k \equiv 0$ ) break;    /* we're done when the heap is empty */
    }
    ⟨Move the subheap rooted at  $kk$  up one level 8⟩;
  }

```

This code is used in section 1.

**8.** Let the binary representation of  $kk$  be  $(b_t \dots b_0)_2$ . We want to set  $h[(b_t \dots b_1 \alpha)_2] \leftarrow h[(b_t \dots b_0 \alpha)_2]$  for all binary strings  $\alpha$ .

(Move the subheap rooted at  $kk$  up one level 8)  $\equiv$

```

 $j = 0, jj = 1, q[0] = kk, qm[0] = 1;$ 
while ( $j < jj$ ) {
     $kk = q[j];$ 
     $h[((kk \gg 1) \& -qm[j]) + (kk \& (qm[j] - 1))] = h[kk];$ 
    if ( $h[kk + kk]$ )  $q[jj] = kk + kk, q[jj + 1] = kk + kk + 1, qm[jj] = qm[jj + 1] = qm[j] \ll 1, jj += 2;$ 
    else  $h[kk] = 0;$ 
     $j++;$ 
}
```

This code is used in sections 7 and 9.

**9. The inverse algorithm.** To reverse the process, we simply look at the tree and build the nested string, instead of vice versa. The same heap-oriented logic applies.

```
#define check(s)
    { if (d[++p] ≠ s) fprintf(stderr, "Rejection at position %d of case %d!\n", p, serial); }

⟨ Check the inverse bijection 9 ⟩ ≡
    h[1] = 1, k = 2, p = 0;
    while (1) {
        while (h[k]) k += 2;    /* find the smallest hole */
        kk = h[k >> 1];    /* kk is the node pointed to by k's parent */
        if (l[kk]) {
            h[k] = l[kk]; check(+1);
        } else check(-1);
        if (r[kk]) {
            h[k + 1] = r[kk]; check(+1);
        } else check(-1);
        if (h[k]) {
            if (h[k + 1]) continue;
            kk = k;
        } else if (h[k + 1]) kk = k + 1;
        else {
            h[k >> 1] = 0, kk = (k >> 1) ⊕ 1, k = kk & -2;
            if (k ≡ 0) break;    /* we're done when the heap is empty */
        }
        ⟨ Move the subheap rooted at kk up one level 8 ⟩;
    }
```

This code is used in section 1.

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# FRANÇON

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