

(See <https://cs.stanford.edu/~knuth/programs.html> for date.)

1. Intro. A quick program to output the “domination” or “majorization” relation when it is defined on permutations of multisets instead of on partitions.

Let’s say that digits are permuted. Then $x_1 \dots x_n \succeq y_1 \dots y_n$ if and only if $\sum_{i=1}^j [x_i \geq k] \geq \sum_{i=1}^j [y_i \geq k]$ for all j and k .

This relation is self-dual in the sense that $x_1 \dots x_n \succeq y_1 \dots y_n$ if and only if $x_n \dots x_1 \preceq y_n \dots y_1$. And if the digits consist of equal quantities of the numbers 1 through k , then $x_1 \dots x_n \succeq y_1 \dots y_n$ if and only if $\bar{x}_1 \dots \bar{x}_n \preceq \bar{y}_1 \dots \bar{y}_n$, where $\bar{x} = k + 1 - x$.

It’s emphatically *not* a lattice, in most cases.

Here I just compute the relation and its transitive reduction by brute force. When I learn better algorithms for transitive reduction, I can use this as an interesting example.

(Well, maybe not! In the examples I tried, we seem to have x covers y if and only if x differs from y by a transposition and x has exactly one more inversion than y . Furthermore, it appears that the covering relation on multiset permutations such as $\{1, 1, 2, 2, 3\}$ is obtained by taking the relation on set permutations $\{1, 1', 2, 2', 3\}$ and removing all cases in which $1'$ occurs before 1 or $2'$ before 2. Thus, some additional theory apparently lurks in the background, making this whole program unnecessary — except as a way to confirm the conjectures in further cases before I go ahead and find proofs.)

```
#define maxn 63      /* this many elements at most */
#define maxp 1000    /* this many perms at most */
#include <stdio.h>
#include <string.h>
char perm[maxp][maxn + 1]; /* the permutations */
char work[maxn + 1];       /* where I generate new ones */
char rel[maxp][maxp];      /* nonzero if  $x \prec y$  */
char red[maxp][maxp];      /* reduced relation */
main(int argc, char *argv[])
{
    register int i, j, k, l, ll, m, n, s, dom;
    <Set work to the string that is to be permuted, and check it 2>;
    <Generate the rest of the permutations 3>;
    <Compute the dominance relation 4>;
    <Do transitive reduction 5>;
    <Print the results 6>;
}
```

2. \langle Set *work* to the string that is to be permuted, and check it **2** $\rangle \equiv$

```

if (argc  $\neq$  2) {
    fprintf(stderr, "Usage: %s digits_to_permute\n", argv[0]);
    exit(-1);
}
for (j = 0; argv[1][j]; j++) {
    if (j > maxn) {
        fprintf(stderr, "String too long (maxn=%d)!\n", maxn);
        exit(-2);
    }
    if (argv[1][j] < '0'  $\vee$  argv[1][j] > '9') {
        fprintf(stderr, "The string %s should contain digits only!\n", argv[1]);
        exit(-3);
    }
    if (j > 0  $\wedge$  argv[1][j - 1] > argv[1][j]) {
        fprintf(stderr, "The string %s should be nondecreasing!\n", argv[1]);
        exit(-4);
    }
    work[j + 1] = argv[1][j];
}
n = j;

```

This code is used in section 1.

3. Here I use ye olde Algorithm 7.2.1.2L.

\langle Generate the rest of the permutations **3** $\rangle \equiv$

```

m = 0;
l1: if (m  $\equiv$  maxp) {
    fprintf(stderr, "Too many permutations (maxp=%d)!\n", maxp);
    exit(-5);
}
for (j = 0; j < n; j++) perm[m][j] = work[j + 1];
m++;
l2: for (j = n - 1; work[j]  $\geq$  work[j + 1]; j-- ) ;
    if (j  $\equiv$  0) goto done;
l3: for (l = n; work[j]  $\geq$  work[l]; l-- ) ;
    s = work[j], work[j] = work[l], work[l] = s;
l4: for (k = j + 1, l = n; k < l; k++, l-- ) s = work[k], work[k] = work[l], work[l] = s;
goto l1;
done:

```

This code is used in section 1.

4. We use the fact that dominance is a subset of (reverse) lexicographic order. In other words, if $x_1 \dots x_n$ is lexicographically less than $y_1 \dots y_n$ we cannot have $x_1 \dots x_n \succeq y_1 \dots y_n$.

⟨ Compute the dominance relation 4 ⟩ \equiv

```

for ( $l = 0$ ;  $l < m$ ;  $l++$ )
  for ( $ll = l + 1$ ;  $ll < m$ ;  $ll++$ ) {
     $dom = 0$ ;
    for ( $k = work[n] + 1$ ;  $k \leq work[1]$ ;  $k++$ )
      for ( $j = 0$ ;  $j < n$ ;  $j++$ ) {
        for ( $i = s = 0$ ;  $i \leq j$ ;  $i++$ )  $s += (perm[l][i] \geq k ? 1 : 0) - (perm[ll][i] \geq k ? 1 : 0)$ ;
        if ( $s > 0$ ) goto fin;
        if ( $s < 0$ )  $dom = 1$ ;
      }
    if ( $dom$ )  $rel[l][ll] = 1$ ;
  fin: continue;
}

```

This code is used in section 1.

5. Hey, I'm just using brute force today.

⟨ Do transitive reduction 5 ⟩ \equiv

```

for ( $l = 0$ ;  $l < m$ ;  $l++$ )
  for ( $ll = l + 1$ ;  $ll < m$ ;  $ll++$ ) {
    if ( $rel[l][ll]$ ) {
      for ( $j = l + 1$ ;  $j < ll$ ;  $j++$ )
        if ( $rel[l][j] \wedge rel[j][ll]$ ) goto nope;
       $red[l][ll] = 1$ ;
    }
  nope: continue;
}

```

This code is used in section 1.

6. ⟨ Print the results 6 ⟩ \equiv

```

for ( $l = 0$ ;  $l < m$ ;  $l++$ ) {
   $printf("%s\sqcup", perm[l])$ ;
  for ( $ll = l + 1$ ;  $ll < m$ ;  $ll++$ )
    if ( $red[l][ll]$ )  $printf("\sqcup%s", perm[ll])$ ;
   $printf("\n")$ ;
}

```

This code is used in section 1.

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DOMINATION

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