PROBABILITY AND STATISTICS (UCS401)

Lecture-24

(Chi-Square Distribution (Mean, Variance & M.g.f.))

Random Variables and their Special Distributions(Unit –III & IV)

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Chi-square distribution with illustrations

Another very special case of gamma distribution is obtained by letting $\alpha = n/2$ and $\beta = 2$ where n is positive integer. The result is called the chi-square distribution. The distribution has a single parameter n, called degree of freedom.

Chi-square distribution :

A continuous random variable X has a chi-squared distribution with n degree of freedom, if its density function is given by

$$f(x;n) = \begin{cases} \dfrac{e^{-x/2} x^{n/2-1}}{2^{n/2}\,\Gamma(n/2)} ; & x > 0 \\ 0 ; & \text{elsewhere} \end{cases}$$

Where, n is a positive integer.

(i) Mean of x $= E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_0^{\infty} x \frac{e^{-x/2} x^{n/2-1}}{2^{n/2}\sqrt{n/2}}\, dx$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \int_0^{\infty} e^{-x/2} x^{n/2}\, dx$$

We know that

$$\sqrt{n} = \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$$\& \quad \frac{\sqrt{n}}{a^n} = \int_0^{\infty} e^{-ax} x^{n-1}\, dx$$

$$\therefore \quad E(x) = \frac{1}{2^{n/2}\sqrt{n/2}} \int_0^{\infty} e^{-\left(\frac{1}{2}\right)x} x^{\left(\frac{n}{2}+1\right)-1}\, dx$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \frac{\sqrt{n/2+1}}{\left(\frac{1}{2}\right)^{n/2+1}}$$

$$= \frac{2^{\cancel{n/2}+1}}{\cancel{2^{n/2}}\cancel{\sqrt{n/2}}} \frac{n}{2}\cancel{\sqrt{n/2}} = 2 \times \frac{n}{2} = n$$

$$\therefore \quad \boxed{E(x) = n = \text{Mean}}$$

(2) Variance :

The variance is defined as

$$Var(x) = E(x^2) - (E(x))^2$$

Now, $$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{0} x^2 \times 0\, dx + \int_{0}^{\infty} x^2 \frac{e^{-x/2} x^{n/2-1}}{2^{n/2}\sqrt{n/2}}\, dx$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \int_{0}^{\infty} e^{-x/2} \cdot x^{n/2+1}\, dx$$

$$\because \frac{\sqrt{n}}{a^n} = \int_{0}^{\infty} e^{-ax} x^{n-1}\, dx$$

$$\therefore E(x^2) = \frac{1}{2^{n/2}\sqrt{n/2}} \int_{0}^{\infty} e^{-(\frac{1}{2})x} x^{(\frac{n}{2}+2)-1}\, dx$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \frac{\sqrt{\frac{n}{2}+2}}{\left(\frac{1}{2}\right)^{\frac{n}{2}+2}}$$

$$\because \sqrt{n+1} = n\sqrt{n}$$

$$= \frac{2^{\cancel{n/2}+2}}{\cancel{2^{n/2}} \cancel{\sqrt{n/2}}} \left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)\cancel{\sqrt{n/2}}$$

$$= 2^2 \left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right) = n(n+2).$$

$\therefore \quad E(X^2) = n(n+2)$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= n(n+2) - n^2$$

$$\boxed{Var(X) = 2n}$$

(3) Moment-generating function :

If $X \sim \chi^2(n)$ so its p.d.f. is

$$f(x;n) = \begin{cases} \dfrac{e^{-x/2} x^{n/2-1}}{2^{n/2} \Gamma\frac{n}{2}} & ; x > 0 \\ 0 & ; o/w \end{cases}$$

The moment generating function $M_X(t)$ is defined as :

$$M_X(t) = E(e^{tX}) = \int_x e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x/2} x^{n/2-1}}{2^{n/2} \Gamma\frac{n}{2}}\,dx$$

$$= \frac{1}{2^{n/2} \Gamma\frac{n}{2}} \int_0^\infty e^{(t-\frac{1}{2})x} x^{n/2-1}\,dx$$

$$M_X(t) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \int_0^\infty e^{-\frac{1}{2}(1-2t)x}\, x^{\frac{n}{2}-1}\, dx$$

provided $|t| < \frac{1}{2}$.

$$\because \quad \frac{\Gamma n}{a^n} = \int_0^\infty e^{-ax} x^{n-1}\, dx$$

$$\therefore \quad M_X(t) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \quad \frac{\Gamma\frac{n}{2}}{\left(\frac{1-2t}{2}\right)^{n/2}}$$

$$= \frac{\cancel{2^{n/2}}}{\cancel{2^{n/2}}\,\cancel{\Gamma\frac{n}{2}}} \quad \frac{\cancel{\Gamma\frac{n}{2}}}{(1-2t)^{n/2}} \quad ; \quad |t| < \frac{1}{2}$$

$$\boxed{\therefore \quad M_X(t) = (1-2t)^{-n/2} \quad \text{provided } |t| < \frac{1}{2}}$$

"The chi-square distribution is an important component of statistical hypothesis testing and estimation."