PROBABILITY AND STATISTICS (UCS401)

Lecture-4
(Axiomatic definition of Probability and addition rule)
Introduction to Probability (Unit -II)

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Axiomatic definition of probability :-

Sample space :- The set of all possible outcomes of a random experiment is known as sample space. It is denoted by S.

Exp :- Tossing a coin

$$S = \{H, T\}$$

Tossing two coins

$$S = \{HH, TH, HT, TT\}$$

Axiomatic Probability :- Given a sample space S of a random experiment, the probability of occurrence of any event A is defined as a set function P(A) satisfying the following axioms:

Axiom 1 : (Axioms of non-negativity)

$$P(A) \geq 0.$$

Axiom 2 : (Axioms of certainity)

$$P(S) = 1$$

Axiom 3 :- (Axioms of additivity)

If $A_1, A_2, A_3, \ldots, A_n$ is any finite or infinite sequence of disjoint events of S, then

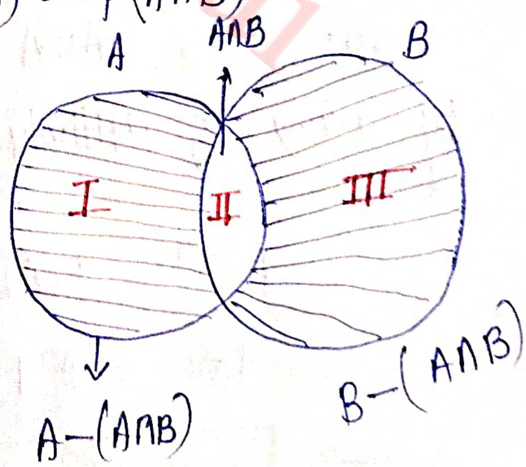
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Addition theorem of probability :-

Theorem :- The probability of occurence of at least one of the two events A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Proof :- For two events A and B

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)}$$

where $n(A \cup B)$ is the number of occurence favourable to the event $A \cup B$.

From figure, we can conclude that

$$n(A\cup B) = [n(A) - n(A\cap B)] + n(A\cap B) + [n(B) - n(A\cap B)]$$

$$= n(A) + n(B) - n(A\cap B)$$

$$P(A\cup B) = \frac{n(A)+n(B)-n(A\cap B)}{n(S)}$$

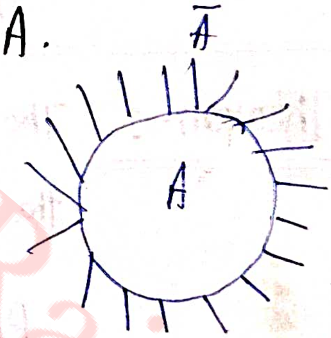
$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A\cap B)}{n(S)}$$

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

Proved

Theorem:- If A is any event, then $P(\bar{A}) = 1 - P(A)$

$\bar{A} \rightarrow$ Complement of event A.



$\bar{A}\cap A = \phi$

$A\cup\bar{A} = S$

Proof: We know that

$$A\cap\bar{A} = \phi$$

also $$S = A\cup\bar{A}$$

By axioms of additivity, we have

$$P(S) = P(A) + P(\bar{A})$$

$$1 = P(A) + P(\bar{A})$$

$$\Rightarrow \boxed{P(\bar{A}) = 1 - P(A)}$$ #

Question-: Given an experiment such that

$$P(A) = 3/8, \quad P(B) = 1/2 \text{ and}$$

$$P(A \cap B) = 1/4. \text{ Find } P(\bar{A} \cap \bar{B})$$

Solution-:

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right]$$

$$= 1 - \left(\frac{3}{8} + \frac{1}{4}\right) = 1 - \frac{5}{8}$$

$$\boxed{P(\bar{A} \cap \bar{B}) = 3/8}$$ Ans

Question-: Two events A and B are such that

$$P(A \cup B) = 3/4, \quad P(A \cap B) = 1/4$$

and $P(\bar{A}) = 2/3$. Find $P(B) = ?$

Solution-: Given that

$$P(\bar{A}) = 2/3 \Rightarrow P(A) = 1 - P(\bar{A})$$

$$\Rightarrow P(A) = 1 - 2/3$$

$$\Rightarrow \boxed{P(A) = 1/3}$$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{3}{4} = \frac{1}{3} + P(B) - \frac{1}{4}$$

$$P(B) = \frac{3}{4} + \frac{1}{4} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$\boxed{P(B) = \frac{2}{3}}$$ Ans

Question [3] The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ and the probability that he will not get an electric contract is $\frac{5}{9}$. If probability of getting one contract is $\frac{4}{5}$, then what is probability that he will get both contracts?

Solution-:

$$P(\text{plumbing}) = \frac{2}{3}$$

$$P(\overline{\text{electric}}) = \frac{5}{9}$$

probability of getting one contract is $\frac{4}{5}$

plumbing or electric

or $\rightarrow \cup$

and $\rightarrow \cap$

Given

$$P(\text{plumbing} \cup \text{electric}) = \frac{4}{5}$$

$$P(\text{plumbing} \cap \text{electric}) = ?$$

Let

A : Contractor will get plumbing contract.

B : Contractor will get electric contract.

Given, $P(A) = \frac{2}{3}$ $\quad P(\overline{B}) = \frac{5}{9}$

$$\Downarrow$$

$$P(B) = 1 - \frac{5}{9} = \frac{4}{9}$$

$P(A \cup B) = \frac{4}{5}$; $P(A \cap B) = ?$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P(A \cap B)$$

$$P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{30+20-36}{45} = \frac{14}{45}.$$

$$P(A \cap B) = \frac{14}{45}$$

i.e., probability that contractor will get both the contract is $\frac{14}{45}$. Ans

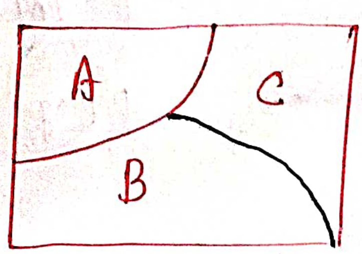
Question:- If A, B and C are three mutually exclusive (disjoint) and exhaustive events (total) associated with a random experiment. Find $P(A)$ given that

$$P(B) = \frac{3}{2} P(A) ;$$

$$P(C) = \frac{1}{2} B(A).$$

disjoint and exhaustive

Solution:- $S = A \cup B \cup C$.



S

$A \cap B \cap C = \phi$

$$P(S) = P(A) + P(B) + P(C)$$

$$\Rightarrow \quad P(A) + P(B) + P(C) = 1$$

$$\Rightarrow \quad P(A) + \frac{3}{2}P(A) + \frac{1}{2}P(B) = 1$$

$$\Rightarrow \quad \frac{5}{2}P(A) + \frac{1}{2}\left(\frac{3}{2}P(A)\right) = 1$$

$$\Rightarrow \quad \frac{5}{2}P(A) + \frac{3}{4}P(A) = 1$$

$$\Rightarrow \quad \left(\frac{5}{2} + \frac{3}{4}\right)P(A) = 1$$

$$\Rightarrow \quad \frac{13}{4}P(A) = 1$$

$$\Rightarrow \quad \boxed{P(A) = \frac{4}{13}} \qquad \text{Ans}$$

Question–: The probability that a student passes a physics test is $2/3$ and the probability that he passes both a physics test and an English test is $\frac{14}{45}$. The probability that he passes at least one test is $4/5$. What is the probability that he passes the english test.

Solution–:

$$P(\text{Physics}) = 2/3$$

$$P(\text{Phy} \cap \text{Eng}) = \frac{14}{25}$$

$$P(\text{Phy} \cup \text{Eng}) = \frac{4}{5}$$

$$P(\text{Eng}) = ??$$

Let A : students pass the physics test

B : students pass the english test.

$$P(A) = 2/3 \quad P(A \cap B) = \frac{14}{15}, \quad P(A \cup B) = \frac{4}{5}$$

$$P(B) = ??$$

we Know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{4}{5} = \frac{2}{3} + P(B) - \frac{14}{45}$$

$$\boxed{P(B) = \frac{4}{9}}$$

Question :- Three newspapers A, B and C are published in a certain city. It is estimated from survey that of the adult population, 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A & C, 4% read both B and C, 2% read all three. Find the percentage who read at least one of the papers?

Solution :-

$$P(A) = 20\%, \quad P(B) = 16\% \quad P(C) = 5\%$$

$$P(A \cap B) = 8\%, \quad P(A \cap C) = 5\% \quad P(B \cap C) = 4\%$$

$$P(A \cap B \cap C) = 2\%$$

$$P(A \cup B \cup C) = ?$$

We know that

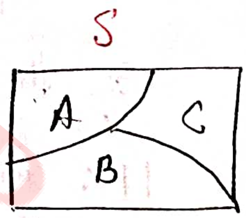
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= 20\% + 16\% + 5\% - 8\% - 4\% - 5\% + 2\%$$

$$\boxed{P(A \cup B \cup C) = 35\%}$$ Ans

Practice sheets

Question ① Three horses A, B and C are in a derby race. The chance of A winning the race is twice of that of B and the chance of B is winning the race is twice that of C. Find the respective chances of winning the race.



Solution:-

$S = A \cup B \cup C$

$P(S) = P(A) + P(B) + P(C)$

$\Rightarrow \quad P(A) + P(B) + P(C) = 1 \quad$ —①

Let the probability of winning horse C is x

then $P(B) = 2x$ & $P(A) = 4x$

$4x + 2x + x = 1$

$\Rightarrow \quad 7x = 1 \quad \Rightarrow \quad x = 1/7$

Hence, probability of winning horses A, B and C are

$4/7$, $2/7$ and $\frac{1}{7}$, resp.

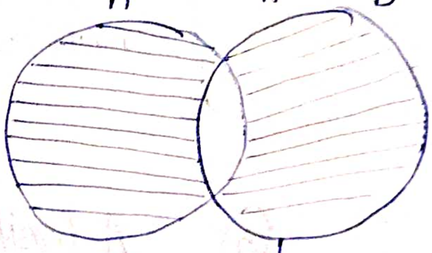
Question (2) Events A and B are such that

$$P(A\cup B)=\frac{1}{2};\quad P(\bar{A}\cap B)=\frac{1}{5};\quad P(\bar{B})=\frac{3}{4}$$

Find $P(A)$ and $P(A\cap\bar{B})=?$

Solution:- We know that

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$



Given $P(\bar{B})=1-P(B)$

$$P(B)=1-P(\bar{B})$$

$$P(B)=1-\frac{3}{4}=\frac{1}{4}$$

$$\boxed{P(B)=\frac{1}{4}}$$

$$P(\bar{A}\cap B)=P(B)-P(A\cap B)$$

$$\frac{1}{5}=\frac{1}{4}-P(A\cap B)$$

$$\Rightarrow P(A\cap B)=\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$$

$$\Rightarrow \boxed{P(A\cap B)=\frac{1}{20}}$$

$$P(A\cap\bar{B})=P(A)-P(A\cap B)$$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$P(A\cup B)=P(A)+P(\bar{A}\cap B)$$

$$\frac{1}{2}=P(A)+\frac{1}{5}$$

$$P(A)=\frac{1}{2}-\frac{1}{5}=\frac{3}{10}\quad\Rightarrow\quad\boxed{P(A)=\frac{3}{10}}$$ Ans

Now

$$P(\bar{A} \cap B) =$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$P(A \cap \bar{B}) = \frac{3}{10} - \frac{1}{20}$$

$$P(A \cap \bar{B}) = \frac{6-1}{20} = \frac{5}{20}$$

$$\boxed{P(A \cap \bar{B}) = \frac{1}{4}}$$ Ans

Question - 3 A positive integer from one to six is to be chosen by casting a die. Thus element of the sample space are 1, 2, 3, 4, 5, 6. Let $C_1 = \{1, 2, 3, 4\}$ and $C_2 = \{3, 4, 5, 6\}$. If the probability set function P assigns a probability of $\frac{1}{6}$ to each elements of S, Compute $P(C_1)$, $P(C_2)$, $P(C_1 \cap C_2)$ and $P(C_1 \cup C_2)$.

Question - 4 A box contains 4 blue and 6 red balls. Two balls are drawn together. Find the probability that both balls have the same color.

Solution 3

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$C_1 = \{1, 2, 3, 4\}$$

$$\& \; C_2 = \{3, 4, 5, 6\}$$

$P(G_1) = \frac{4}{6} = 2/3$

$$\boxed{P(G_1) = 2/3}$$

$P(G_2) = \frac{4}{6} = 2/3$

$G_1 \cap G_2 = \{3, 4\}$ $\qquad P(G_1 \cap G_2) = 2/6 = 1/3$

$G_1 \cup G_2 = \{1, 2, 3, 4, 5, 6\}$

$$\boxed{P(G_1 \cup G_2) = 1}$$

(ii) A box contain — 4 blue, 6 red

Total = 10 balls. $\qquad$ Exhaustive cases $= {}^{10}C_2$

you pic = 2

both have same color — 2B or 2R

Favourable cases $= {}^4C_2 + {}^6C_2$

Thus, required probability

$$P(E) = \frac{{}^4C_2 + {}^6C_2}{{}^{10}C_2}$$

Ans