$F_X(x) = P(X \le x)$

$$\boxed{P(X=x) = F_X(x) - F_X(x-1)}$$

X

Discrete — **PMF** (Probability Mass Function) — $P_X(X=x)$ —
(i) $P(X=x) \geq 0$
(ii) $\sum_x P(X=x) = 1$

Continuous — **PDF** (Probability Density Function) — $f_X(x)$ —
(i) $f(x) \geq 0$
(ii) $\int_x f(x)\,dx = 1$

$F_X(x) = P(X \le x)$

$= \int_{-\infty}^{x} f(x)\,dx$

$$\boxed{\frac{d}{dx} F(x) = f(x)}$$

Expectations

$E(X)$

$E(X) = \sum_x x\,P(X=x)$

$E(X) = \int_x x\,f(x)\,dx$

$E(g(X)) = \sum_x g(x)\,P(X=x)$

$\int_x g(x)\,f(x)\,dx$

$E(X) = \text{Mean}$

$V(X) = E(X^2) - [E(X)]^2$

$$F_X(x) = P(X \leq x)$$

$$F_X(0) = P(X \leq 0) = \frac{1}{8}$$

$$F_X(1) = P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{4}{8}$$

$$F_X(2) = 7/8$$

$$F_X(3) = 8/8 = 1$$

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$F_X(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8} = 1$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8} & 0 \leq x < 1 \\ 4/8 & 1 \leq x < 2 \\ 7/8 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

$$\boxed{Y = X \pm a} = g(x)$$

$$E_x(y) = \sum_y y\, P(x=y)$$

$$= \sum_x (x \pm a)\, P(x=x)$$

$$= \sum_x x\, P(x=x) \pm a \sum P(x=x)$$

$$\boxed{E(y) = E(x) \pm a}$$

Changing the origin

$$E_y(y) = \int_x y\, f(x)\, dx$$

$$= \int_x (x \pm a)\, f(x)\, dx$$

$$= \int_x x\, f(x)\, dx \pm a \int f(x)$$

$$= E(x) \pm a$$

$Y = aX$, changing the scale

$$E(y) = a\,E(x)$$

Mean is neither independent of change of origin nor scale

Variance $\quad V(x) = E(x^2) - [E(x)]^2$

or

$$E(X - E(x))^2$$

$$E(x^2 + [E(x)]^2 - 2x\,E(x))$$

$$E(x^2) + [E(x)]^2 - 2E(x)E(x)$$

$$E(x^2) - [E(x)]^2$$

Changing The Origin

$$\cancel{E(y) = \sum y \, P(X = x)}$$

$$V(Y) = E(Y - E(Y))^2$$

$$= E(X \pm a - (E(X) \pm a))^2$$

$$= E(X \cancel{\pm a} - E(X) \cancel{\mp a})^2$$

$$= E(X - E(X))^2$$

$$= V(X)$$ —— Independent

Changing The Scale

$$Y = aX$$

$$V(Y) = E(Y - E(Y))^2$$

$$= E(aX - E(aX))^2$$

$$= E(aX - aE(X))^2$$

$$= a^2 E(X - E(X))^2$$

$$= a^2 V(X)$$

Q! $P(X=x) = Kx \quad ; \quad x = 0, 1, 2, 3, 4$

(i) for what value of k, $P(X=x)$ can be seen as a PMF

(ii) find CDF

(iii) find Mean & Variance.

Solⁿ:

$$\sum_{x=0}^{4} P(X=x) = 1 \quad \Rightarrow \quad K \sum_{x=0}^{4} x = 1$$

$$\Rightarrow \quad 10K = 1$$

$$\boxed{K = 1/10}$$

X	0	1	2	3	4
$P(X=x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
$F_X(x)$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{10}{10}$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{10} & 1 \leq x < 2 \\ \frac{3}{10} & 2 \leq x < 3 \\ \frac{6}{10} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

$$F_X(x) = P(X \leq x) = \sum_{x=0}^{x} P(X=x)$$

$$E(x) = \sum_{x=0}^{4} x\, P(X=x)$$

$$= 0 + P(X=1) + 2P(X=2) + 3P(X=3) + 4P(X=4)$$

$$= 0 + \frac{1}{10} + 2\,\frac{2}{10} + 3.\frac{3}{10} + 4.\frac{4}{10}$$

$$= \frac{1+4+9+16}{10}$$

$$= 30/10$$

$$= 3$$

$$V(x) = E(x^2) - [E(x)]^2 \quad \text{or} \quad E(x - E(x))^2$$

Let us find $E(x^2)$

$$E(x^2) = \sum_{x=0}^{4} x^2\, P(X=x)$$

$$= 0 + 1.\frac{1}{10} + 4.\frac{2}{10} + 9.\frac{3}{10} + 16.\frac{4}{10}$$

$$= \frac{1+8+27+64}{10}$$

$$= \frac{100}{10} = 10$$

$$\therefore \; V(x) = 10 - 3^2 = 10 - 9 = 1$$

	Sample	Population
Mean	$\bar{X}$	$E(x) = \mu$
Var	S^2	$V(x) = \sigma^2$

Q(5) $f(x;\beta) = K e^{-x/\beta}, \quad x > 0, \ \beta > 0$

Sol^n:

$$\int_0^\infty f(x)\,dx = 1 \Rightarrow K\int_0^\infty e^{-x/\beta}\,dx = 1$$

$$\Rightarrow K \left.\frac{e^{-x/\beta}}{-1/\beta}\right|_0^\infty = 1$$

$-K\beta\left[e^{-\infty} - e^{-0}\right]$

$\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$

$$\Rightarrow -K\beta(0-1) = 1$$

$$\Rightarrow K\beta = 1$$

$$\boxed{K = 1/\beta}$$

CDF $F_x(x) = \int_0^x f(x)\,dx = \frac{1}{\beta}\int_0^x e^{-x/\beta}\,dx$

$$= \frac{1}{\beta} \left.\frac{e^{-x/\beta}}{-1/\beta}\right|_0^x$$

$$= -e^{-x/\beta} + 1 = \boxed{1 - e^{-x/\beta}}$$

R.v's
X & Y

- Discrete
  - Joint PMF — $P_{X,Y}(X=x, Y=y)$ —
    - (i) $P(X=x, Y=y) \geq 0$
    - (ii) $\sum_{x}\sum_{y} P(X=x, Y=y) = 1$
  - Marginal PMF
    - $P(X=x) = \sum_{y} P(X=x, Y=y)$
    - $P(Y=y) = \sum_{x} P(X=x, Y=y)$
- Continuous
  - Joint PDF — $f(x,y)$ —
    - (i) $f(x,y) \geq 0$
    - (ii) $\int_{x}\int_{y} f(x,y)\, dx\, dy = 1$
  - $f(x) = \int_{y} f(x,y)\, dy$
  - $f(y) = \int_{x} f(x,y)\, dx$

Q1 $P(X=x, Y=y) = k(x+y)$; $x = 0, 1, 2,$ & $y = 1, 2, 3$

$$\sum_{x=0}^{2} \sum_{y=1}^{3} P(X=x, Y=y) = 1$$

Y \ X	0	1	2	Marginal Probabilities for Y
1	K	2K	3K	$6K = P(Y=1) = \sum_{x=0}^{2} P(X=x, Y=1)$
2	2K	3K	4K	$9K = P(Y=2) = \sum_{x=0}^{2} P(X=x, Y=2)$
3	3K	4K	5K	$12K = P(Y=3) = \sum_{x=0}^{2} P(X=x, Y=3)$
Marginal Probs for X →	6K	9K	12K	27K
	$= P(X=0)$	$= P(X=1)$	$= P(X=2)$	

$P(X=0) = \sum_{y=1}^{3} P(X=0, Y=y)$

$27K \rightarrow \sum_{x=0}^{2} \sum_{y=1}^{3} P(X=x, Y=y)$

$27K = 1$

$\Rightarrow \boxed{K = \frac{1}{27}}$

$$P(Y=y) = \sum_{x=0}^{2} P(X=x, Y=y) = \frac{1}{27}\left[0+y+1+y+2+y\right] = \frac{1}{27}\left[3+3y\right] = \frac{y+1}{9}$$

$$P(X=x) = \sum_{y=1}^{3} P(X=x, Y=y) = \frac{1}{27}\left[x+1+x+2+x+3\right] = \frac{1}{27}\left[3x+6\right] = \frac{x+2}{9}$$

Q.2

$$f(x,y) = kxy \;; \quad 0 \le x < 2, \quad 1 < y \le 3$$

$$k\int_0^2\int_1^3 xy\,dy\,dx = 1 \Rightarrow k\int_0^2 x\frac{y^2}{2}\Big|_1^3 dx = 1 \Rightarrow k\int_0^2\left(\frac{9}{2}x - \frac{x}{2}\right)dx = 1$$

$$\Rightarrow k\left(\frac{9}{2}\frac{x^2}{2} - \frac{x^2}{4}\right)\Big|_0^2 = 1 \Rightarrow k\left(9 - \frac{4}{4} - 0 + 0\right) = 1$$

$$\Rightarrow 8k = 1 \quad \Rightarrow \boxed{k = 1/8}$$

$$f(x) = \frac{1}{8}\int_1^3 xy\,dy =$$

$$f(y) = \frac{1}{8}\int_0^2 xy\,dx =$$

$$\boxed{\begin{aligned} P(X=x, Y=y) &= P(X=x)\,P(Y=y) \\ f(x,y) &= f(x)\,f(y) \end{aligned}}$$

X & Y are Independent

$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}}$$

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

if X & Y are Independent Then

$$P(X=x \mid Y=y) = P(X=x)$$

$$P(Y=y \mid X=x) = P(Y=y)$$

$$f(x|y) = f(x)$$

$$f(y|x) = f(y)$$

$$E(\underline{g(x,y)}) = \sum_x \sum_y g(x,y)\, P(X=x, Y=y)$$

$$= \int_x \int_y g(x,y)\, f(x,y)\, dy\, dx$$

$g(x,y) = x$

$$\sum_x \sum_y x\, P(X=x, Y=y)$$

$$\sum_x x \sum_y P(X=x, Y=y)$$

$$\sum_x x\, P(X=x)$$

$$V(X+Y) = ?$$

$$= E\left((X+Y) - E(X+Y)\right)^2$$

or

$$E(X+Y)^2 - \left[E(X+V)\right]^2$$

	Change of origin $x' = x + a$, $y' = y + b$	Chge of Scale $x' = ax$, $y' = by$
Mean	$\mu_{x'} = \mu_x + a$ $\mu_{y'} = \mu_y + b$	$\mu_{x'} = a\mu_x$ $\mu_{y'} = b\mu_y$
Variance	$\sigma^2_{x'} = \sigma^2_x$ $\sigma_{y'} = \sigma^2_y$	$\sigma^2_{x'} = a^2\sigma^2_x$ $\sigma^2_{y'} = b^2\sigma^2_y$
Sd	$\sigma_{x'} = \sigma_x$ $\sigma_{y'} = \sigma_y$	$\sigma_{x'} = \lvert a\rvert\sigma_x$ $\sigma_{y'} = \lvert b\rvert\sigma_y$
Covariance	$Cov(x', y') = Cov(x, y)$	$Cov(x', y') = ab\,Cov(x, y)$
Correlation	$\rho_{x'y'} = \rho_{xy}$	$\rho_{x'y'} = \rho_{xy}$
Regression		

$$V(X \pm Y) = E\left[(X \pm Y) - E(X \pm Y)\right]^2 = E\left[(X \pm Y)^2 - \{E(X) \pm E(Y)\}^2\right]$$

$$= E\left[X^2 + Y^2 \pm 2XY - \left((E(X))^2 + (E(Y))^2 \pm 2E(X)E(Y)\right)\right]$$

$$= E\left[X^2 + Y^2 \pm 2XY - [E(X)]^2 - [E(Y)]^2 \mp 2E(X)E(Y)\right]$$

$$= E(X^2) + E(Y^2) \pm 2E(XY) - [E(X)]^2 - [E(Y)]^2 \mp 2E(X)E(Y)$$

$$= \underbrace{E(X^2) - [E(X)]^2}_{} + \underbrace{E(Y^2) - [E(Y)]^2}_{} \pm 2\underbrace{\left[E(XY) - E(X)E(Y)\right]}_{}$$

$$= V(X) + V(Y) \pm 2\,Cov(X,Y)$$

If X & Y are independent

Then $E(XY) = E(X)E(Y)$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

or

$E\left[(X - E(X))(Y - E(Y))\right]$

$Cov(x, y) = 5$

$Cov(x, y) = 15$

Correlation

$$\rho_{xy} = \frac{Cov(x, y)}{\sigma_x \, \sigma_y}$$

σ_x — $Sd(x)$ σ_y — $Sd(y)$

x	y
5	10
10	15
15	20

$$-1 \leq \rho_{xy} \leq 1$$

To Prove $-1 \le \rho_{xy} \le 1$

Let $V\left(\frac{X}{\sigma_x} \pm \frac{Y}{\sigma_y}\right) = V\left(\frac{X}{\sigma_x}\right) + V\left(\frac{Y}{\sigma_y}\right) \pm 2\,Cov\left(\frac{X}{\sigma_x}, \frac{Y}{\sigma_y}\right)$

Sd(x) Sd(y)

$$= \frac{1}{\sigma_x^2} V(x) + \frac{1}{\sigma_y^2} V(y) \pm 2 \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

$$= 1 + 1 \pm 2\rho_{xy}$$

$$= 2 \pm 2\rho_{xy}$$

We know $V(\cdot) \geq 0$

$\Rightarrow 2 + 2\rho_{xy} \geq 0$	$2 - 2\rho_{xy} \geq 0$
$2 \geq -2\rho_{xy}$	$2 \geq 2\rho_{xy}$
$1 \geq -\rho_{xy}$	$1 \geq \rho_{xy}$
$-1 \le \rho_{xy}$	$\rho_{xy} \le 1$