PROBABILITY AND STATISTICS
(UCS401)

Lecture-9

(Discrete and continuous random variables (Mean Variance))

Random Variables and their Special Distributions(Unit –III & IV)

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Distribution Theory

Random Variable

- Discrete random variable (DRV)
  - e.g. Tossing of coin
- Continuous random variable (CRV)
  - e.g. Heigth of students in a class.

$S = \{H, T\} \rightarrow$ sample space.

$S = \{HH, HT, TH, TT\} \rightarrow$ sample space.

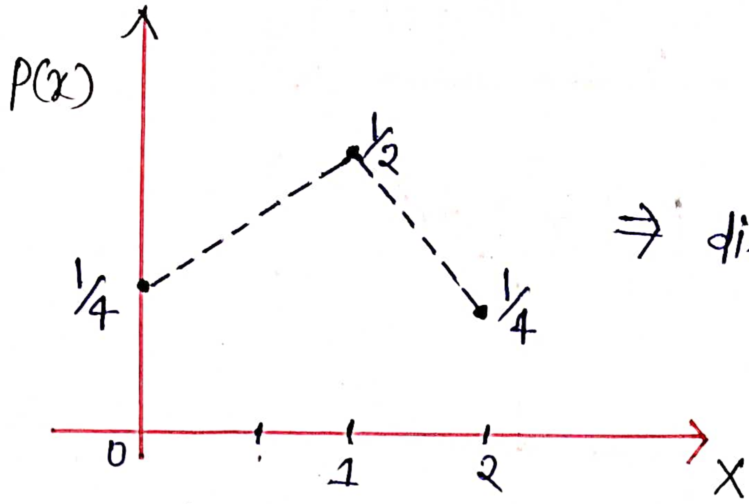
probability distribution $\Rightarrow$

X (No. of Head)	0	1	2
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

This is known as probability distribution.

$p(x) \Rightarrow$ probability mass function (p.m.f)

$p(x) \geq 0$ and $\sum p(x) = 1$ (properties)

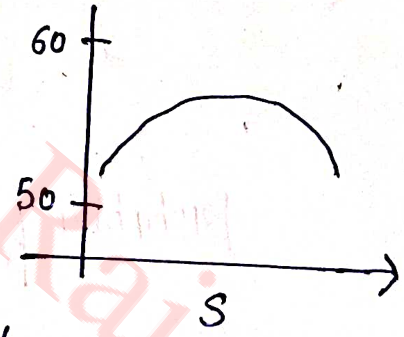


⇒ discreate random variable.

⊛ Such types of Events is known as <u>discreate random variable.</u>

⊛ Continuous random variable ⇒ If sample space is large enough.

e.g. ① Weigth of 1000 students in a college in between 50 to 60 kg ⇒ CRV'



(ii) Plot rate in a particular city.

Random Variable

DRV — CRV

DRV	CRV
$p(x)$ is p.m.f	$f(x)$ is p.d.f.
① $p(x) \geq 0$	① $f(x) \geq 0$
(ii) $\sum p(x) = 1$	(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Distribution

Discreate distribution

1. Bernoli distribution
2. Bionomial distribution
3. Poisson distribution
4. Hypergeometric distribution.

Continuous distribution

(i) Exponential distribution
(ii) Normal distribution
(iii) Chi-square distribution (χ^2-dis)
(iv) Uniform distribution

Mathematical expectation $\Rightarrow$ $E(X) =$ Mean

DRV: $E(X) = \sum x\,p(x)$

CRV: $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$.

Random Variable :-

Random variable is a real valued function which assign a real number to each sample point in sample space.

i.e., Connection of sample space with real number.

Example-: Tossing a fair coin trice then sample space,

$$S = \{HHH, HHT, HTT, HTH, \cancel{HHT}, THH, TTH, THT, TTT\} \Rightarrow \text{sample space.}$$

$$S = \{HHH, HTH, HHT, THH, TTH, THT, HTT, TTT\} \Rightarrow \text{arrenged.}$$

Random variable + probability ⇒ probability distribution

X (No. of Head)	0	1	2	3
$p(x)$ (p.m.f)	$1/8$	$3/8$	$3/8$	$1/8$

$X(s_1) = 3 \qquad X(s_2) = X(s_3) = X(s_4) = 2$

$X(s_5) = X(s_6) = X(s_7) = 1$

$X(s_8) = 0 \Rightarrow$ Random variables

$\{X(s), P\} \Rightarrow$ probability distribution.

Discreate random variable-:

A random variable which takes finite or atmost contable number of values is called discreate random variable.

Examples-: (i) No. of Head obtained when two coines are tossed.

(ii) No. of defective items in a lot.

Question:- Four bad oranges are mixed with 16 good oranges, find the probability distribution of no. of bad oranges in a drawn of two oranges.

16G 4B

X (No. of bad oranges)	0	1	2	3	4
P(X)	$\frac{^{16}C_2}{^{20}C_2}$	$\frac{^{16}C_1 \times {^4C_1}}{^{20}C_2}$	$\frac{^{4}C_2}{^{20}C_2}$	0	0

This is the required probability distribution. #

Probability Mass function:-

Let X be a BRV such that $P(X=x)=P_i$, the P_i is said to be probability mass function (p.m.f.) if it satisfy the following conditions.

(i) $\sum P_i = 1$

(ii) $P_i(x) \geq 0$

e.g. Tossing a coin trice,

(sample space) $S = \{HHH, HHT, HTH, THH, THT, TTH, HTT, TTT\}$

Probability distribution

$\Downarrow$

X (No. of Head)	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

$\Rightarrow \quad p_i \geq 0 \quad$ and $\quad \sum p_i = 1/8 + 3/8 + 3/8 + 1/8 = 1$

$\Rightarrow \quad p_i \geq 0 \quad$ and $\quad \sum p_i = 1 \qquad$ #

Distribution function / Cummulative distribution function -:

Let X be a DRV, then its discreate distribution function or cummulative distribution function (c.d.f.) is defined as

$$F(x) = \sum p_i = P(X \leq x_j)$$

$$F(X) = \begin{cases} 1/8 & x \leq 0 \\ 1/8 + 3/8 = 1/2 & x \leq 1 \\ 1/8 + 3/8 + 3/8 = 7/8 & x \leq 2 \\ 1 & x \leq 3 \end{cases}$$

This is cummulative distribution function.

probability distribution.

Question (2)

5 defective pieces of $x y z$ units, which are accidently mixed with 20 good pieces and looking a them. Assume that 4 units are picked up at random. Then Construct probability distribution of R.V.

Solution-:

We have 5 defective pieces

20 good pieces

Total = 25 pieces

you pick = 4 units

Exhaustive Cases = $^{25}C_4$.

X (No. of defective pieces) $= 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$p(X=0) = \frac{{}^{20}C_4}{{}^{25}C_4}$$

$$p(X=1) = \frac{{}^{5}C_1 \times {}^{20}C_3}{{}^{25}C_4}$$

$$p(X=2) = \frac{{}^{5}C_2 \times {}^{20}C_2}{{}^{25}C_4}$$

$$p(X=3) = \frac{{}^{5}C_3 \times {}^{20}C_1}{{}^{25}C_4}$$

$$p(X=4) = \frac{{}^{5}C_4}{{}^{25}C_4}$$

X (No. of defective pieces)	0	1	2	3	4
$p_i = P(X=x_i)$	$\frac{{}^{20}C_4}{{}^{25}C_4} = \alpha_1$	$\frac{{}^{5}C_1 \times {}^{20}C_3}{{}^{25}C_4} = \alpha_2$	$\frac{{}^{5}C_2 \times {}^{20}C_2}{{}^{25}C_4} = \alpha_3$	$\frac{{}^{5}C_3 \times {}^{20}C_1}{{}^{25}C_4} = \alpha_4$	$\frac{{}^{5}C_4}{{}^{25}C_4} = \alpha_5$

$\Downarrow$

probability distribution

If $\alpha_i \geq 0 \quad \forall \; i = 1, 2, 3, 4, 5$

$\sum \alpha_i = 1$ (p.m.f.).

Question ③ A bag contains 6 red and 4 white balls. Three balls are drawn at random. Obtain the probability distribution of number of white ball.

Solution :- We have

6 red balls
4 white balls

Total = 10 balls — Exhaustive cases $= {}^{10}C_3$

you pick = 3 balls

r.v. : set of all outcomes $\longrightarrow \mathbb{R}$

X (No. of white balls) $= 0, 1, 2, 3$

$$P(X=0) = \frac{{}^4C_0 \times {}^6C_3}{{}^{10}C_3}$$

$$P(X=1) = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3}$$

$$P(X=2) = \frac{{}^4C_2 \times {}^6C_1}{{}^{10}C_3}$$

$$P(X=3) = \frac{{}^4C_3}{{}^{10}C_3}$$

X (No. of white balls)	0	1	2	3
$p_i = P(X=x_i)$	$\frac{{}^4C_0 \times {}^6C_3}{{}^{10}C_3}$	$\frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3}$	$\frac{{}^4C_2 \times {}^6C_1}{{}^{10}C_3}$	$\frac{{}^4C_3}{{}^{10}C_3}$
$p_i = P(X=x_i)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

$\Downarrow$ probability distribution.

Here, $\left.\begin{array}{l} p_i \geq 0 \\ \sum p_i = 1 \end{array}\right\} \Rightarrow$ p.m.f.

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Question :- A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find the value of k

(ii) $P(X<6)$, $P(X\geq 6)$, $P(0<X<5)$

(iii) Distribution function.

(iv) If $P(X\leq c) > \frac{1}{2}$, find the minimum value of c.

(v) find $P\left(\frac{1.5<X<4.5}{X>2}\right)$.

Solution :- We know that if $p(x)$ is p.m.f.

$$\sum p(x) = 1$$

$$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$$

$$\Rightarrow 10k^2+9k = 1$$

$$\Rightarrow 10k^2+9k-1 = 0$$

$$\Rightarrow 10k^2+10k-k-1 = 0$$

$$\Rightarrow 10k(k+1)-1(k+1) = 0$$

$$\Rightarrow (k+1)(10k-1) = 0$$

$$\Rightarrow k=-1 \text{ and } k=\frac{1}{10} \Rightarrow \underline{\text{Ans}}$$

$\times$ $\because p_i \geq 0$

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X	0	1	2	3	4	5	6	7
P(X)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(ii) $P(X<6) = 1 - P(X \geq 6)$

$= 1 - (P(6)+P(7))$

$= 1 - [0.02 + 0.17]$

$= 1 - 0.19$

$\boxed{P(X<6) = 0.81}$ Ans

$P(X \geq 6) = P(6) + P(7)$

$= 0.02 + 0.17$

$\boxed{P(X \geq 6) = 0.19}$ Ans

$P(0<X<5) = P(1)+P(2)+P(3)+P(4)$

$= 0.1 + 0.2 + 0.2 + 0.3$

$\boxed{P(0<X<5) = 0.8}$ Ans

(iii) Distribution function / Cummulative distribution function :-

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 0.1 & x \leq 1 \\ 0.3 & x \leq 2 \\ 0.5 & x \leq 3 \\ 0.8 & x \leq 4 \\ 0.81 & x \leq 5 \\ 0.83 & x \leq 6 \\ 1 & x \leq 7 \end{cases}$$

X	0	1	2	3	4	5	6	7
P(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(iv) If $P(X \leq c) > \frac{1}{2}$ $\Rightarrow$ $P(X \leq 0) = 0$

$P(X \leq 1) = 0.1$

$P(X \leq 2) = 0.3$

$P(X \leq 3) = 0.5$

$P(X \leq 4) = 0.8 > \frac{1}{2}$

$\Rightarrow \boxed{c = 4}$ Ans

(v) 10 We know that

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{P(1.5 < X < 4.5)}{P(X > 2)} = \frac{P(1.5 < X < 4.5 \cap X > 2)}{P(X > 2)}$$

$$= \frac{P((2,3,4) \cap (3,4,5,6,7))}{1 - P(X \leq 2)}$$

$$= \frac{P(3,4)}{1 - (P(0) + P(1) + P(2))}$$

$$= \frac{P(3) + P(4)}{1 - (P(0) + P(1) + P(2))}$$

$$= \frac{0.2 + 0.3}{1 - (0 + 0.1 + 0.2)} = \frac{0.5}{0.7} = \frac{5}{7}$$

$$P\left(\frac{1.5 < X < 4.5}{X > 2}\right) = \frac{5}{7} \quad \text{Ans}$$

Continuous random variable —:

A random variable, which can take infinite no. of values in a interval is known as Continuous random variable.

Example —: (i) The weigth of a group of individuals

(ii) Heigh of a group of individuals

(iii) Price of House.

Probability density function -:

A function $f(x)$ is said to be probability density function if

(i) $f(x) \geq 0 \qquad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Question -: If X is a continuous random variable with following probability distribution function

$$f(x) = \begin{cases} \alpha(2x - x^2) & 0 < x < 2 \\ 0 & o/w \end{cases}$$

find the value of

(i) α (ii) $P(X > 1)$

Solution -: We know by definition of p.d.f.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} \alpha(2x - x^2)\,dx + \int_{2}^{\infty} 0\,dx = 1$$

$$\Rightarrow \alpha \int_{0}^{2} (2x - x^2)\,dx = 1$$

$$\Rightarrow \alpha \left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow \alpha\left[4-\frac{8}{3}\right]=1$$

$$\Rightarrow \boxed{\alpha=\frac{3}{4}}$$

Ans

(ii) $$P(X>1)=\int_1^{\infty} f(x)\,dx$$

$$=\int_1^{2} f(x)\,dx+\int_2^{\infty} f(x)\,dx$$

$$=\int_1^{2}\frac{3}{4}(2x-x^2)\,dx+\int_2^{\infty}0\,dx$$

$$=\frac{3}{4}\left[x^2-\frac{x^3}{3}\right]_1^2$$

$$=\frac{3}{4}\left[\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right]$$

$$=\frac{3}{4}\left[\frac{4}{3}-\frac{2}{3}\right]=\frac{3}{4}\times\frac{2}{3}=\frac{1}{2}$$

$$\Rightarrow \boxed{P(X>1)=\frac{1}{2}}$$

Ans

Question–: A random variable X has density function

$$f(x)=\begin{cases} kx^2 & -3<x<3 \\ 0 & o/w \end{cases}$$

find the values of
(i) k
(ii) $P(1\leq X\leq 2)$
(iii) $P(X\leq 2)$
(iv) $P(X>1)$.

Solution:- We know by definition of probability density function (p.d.f.)

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{-3} \cancel{f(x)dx}^{0} + \int_{-3}^{3} f(x)dx + \int_{3}^{\infty} \cancel{f(x)dx}^{0} = 1$$

$$\Rightarrow \int_{-3}^{3} kx^2 dx = 1$$

$$\Rightarrow 2k\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow 2k(9) = 1$$

$$\Rightarrow \boxed{k = 1/18}$$ Ans

$$\therefore f(x) = \begin{cases} x^2/18 & -3 < x < 3 \\ 0 & o/w \end{cases}$$

(i) $$P(1 \leq X \leq 2) = \int_1^2 f(x)dx = \int_1^2 \frac{x^2}{18} dx$$

$$= \frac{1}{18}\left(\frac{x^3}{3}\right)_1^2 = \frac{1}{18}\left(\frac{7}{3}\right) = \frac{7}{54}$$

$$\boxed{P(1 \leq X \leq 2) = \frac{7}{54}}$$ Ans

(ii) $$P(X \leq 2) = \int_{-\infty}^{2} f(x)dx$$

$$= \int_{-\infty}^{-3} 0dx + \int_{-3}^{2} \frac{x^2}{18} dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2} = \frac{1}{18}\frac{(8+27)}{3} = \frac{35}{54}$$

$$\boxed{P(X \leq 2) = \frac{35}{54}}$$ Ans

(iii) $$P(X>1) = \int_1^{\infty} f(x)dx = \int_1^3 \frac{x^2}{18}dx + \int_3^{\infty} 0\,dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{1}{18}\frac{(27-1)}{3} = \frac{26}{54}$$

$$\Rightarrow \boxed{P(X>1) = \frac{13}{27}}$$ Ans

Cummulative/ continuous distribution function (c.d.f.) –:

Let X be continuous random variable having p.d.f. $f(x)$ then $F_X(x)$ will be a continuous distribution function of X if

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

This is also called cummulative distribution function.

Note that –: The relation between distribution function and density function

$$\frac{d}{dx}F(x) = f(x).$$

Question:- The probability density function of random variable X is

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x < 2 \\ 0 & o/w \end{cases}$$

(i) find $P(X \geq 1.5)$

(ii) find cummulative distribution function (c.d.f.).

Solution:- (i) $P(X \geq 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

$$= \int_{1.5}^{2} (2-x)\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_{1.5}^{2}$$

$$= \left[\left(4 - \frac{4}{2}\right) - \left(3 - \frac{2.25}{2}\right)\right]$$

$$= \left[2 - \frac{15}{8}\right] = \frac{1}{8}$$

$$\boxed{P(X \geq 1.5) = \frac{1}{8}}$$ Ans

(ii) Cummulative distribution function:-

for $x \le 0$

$$P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = 0.$$

$0 \leq x \leq 1$

$$p(X \leq x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$= 0 + \int_{0}^{x} x\,dx = \frac{x^2}{2}$$

$1 < x < 2$ or $x \geq 2$

$$F_X(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^x$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \left(2 - \frac{1}{2}\right)$$

$$= 1 - 2 + 2x - \frac{x^2}{2}$$

$$\boxed{F_X(x) = -1 + 2x - \frac{x^2}{2}}$$

$x > 2$

$$F_X(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx$$

$$F_X(x) = 1$$

$$F_X(x) = P(X \le x) = \begin{cases} 0 & -\infty < x \le 0 \\ x^2/2 & 0 < x \le 1 \\ -1+2x-x^2/2 & 1 \le x \le 2 \\ 1 & o/w \end{cases}$$

$$\frac{d}{dx}F_X(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x \le 2 \\ 0 & o/w \end{cases} = f(x)$$

$F_X(x) = P(X \le x)$

$$P(X \geq 1.5) = 1 - P(X < 1.5)$$

$$P(X \geq 1.5) = 1 - \left(-1 + 2(1.5) - \frac{(1.5)^2}{2}\right)$$

$$= 1 - \left(2 - \frac{2.25}{2}\right)$$

$$\boxed{P(X \geq 1.5) = 0.125}$$

(ii) Cummulative distribution function (for Continuous R.V.)

Let X be a Continuous random variable (CRV), then its Cummulative distribution function is defined as:

$$\boxed{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx}$$

Note that: (i) For DRV is $F(x) = P(X < x)$?? — ①

$\because P(X < x) \neq P(X \leq x)$ in general

$\therefore$ Equation ① does not hold in general for DRV.

(ii) For Continuous random variables

$$P(X \leq x) = P(X < x)$$

$$\therefore \quad F(x) = P(X \leq x) = P(X < x)$$

Always hold in case of Continuous random variables.

Properties of p.d.f.:

For discrete r.v.:

(i) $p_i \geq 0$

(ii) $\sum p_i = 1$

For Continuous r.v.:

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Properties of c.d.f. :-

Let X be a continuous r.v. and $f(x)$ is its probability density function (p.d.f.), then its cummulative distribution function is defined as:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

having following properties:

(i) $0 \leq F(x) \leq 1 \quad \forall x$

(ii) $F(x)$ is non-decreasing function $\left(\because F'(x) = f(x) \geq 0\right)$

if $x < y$, then $F(x) \leq F(y)$

(iii) $F(\infty) = 1$

(iv) $F(-\infty) = \int_{-\infty}^{-\infty} f(x)dx = 0$

(v) $F(x)$ is continuous function of x on the right and they have countable number of discontinuity.

(vi) $F'(x) = f(x)$ at all point of x where $F(x)$ is differentiable.

To Calculate p.d.f. and c.d.f. (for Continuous r.v.)

Question ① If a density function of a continuous random variable is

$$f(x) = \begin{cases} 0 & x < 0 \\ \alpha x & 0 \leq x \leq 2 \\ (4-x)\alpha & 2 \leq x \leq 4 \\ 0 & x \geq 4 \end{cases}$$

then find

(i) value of α

(ii) Cummulative distribution function (c.d.f.)

(iii) $P(X > 1.5)$

(iv) $P(X > 1)$

(v) $P(1 < X < 3)$

Solution :- (i) For probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} \cancel{f(x)}^{\,0}\,dx + \int_{0}^{2} \cancel{f(x)}^{\,\alpha x}\,dx + \int_{2}^{4} (4-x)\alpha\, dx + \int_{4}^{\infty} 0\, dx = 1$$

$$\Rightarrow \int_{0}^{2} \alpha x\,dx + \int_{2}^{4} (4-x)\alpha\,dx = 1$$

$$\Rightarrow \alpha\left(\frac{x^2}{2}\right)_0^2 + \alpha\left[4x - \frac{x^2}{2}\right]_2^4 = 1$$

$$\Rightarrow \alpha(2) + \alpha\left[\left(16 - \frac{16}{2}\right) - \left(8 - \frac{4}{2}\right)\right] = 1$$

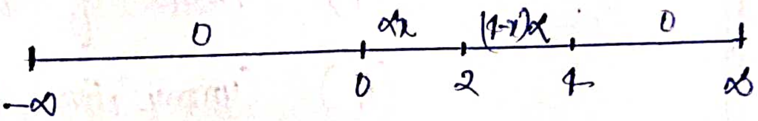
$$\Rightarrow \quad 2\alpha + \alpha[8-6] = 1$$

$$\Rightarrow \quad 4\alpha = 1$$

$$\Rightarrow \quad \boxed{\alpha = \tfrac{1}{4}}$$

$$\therefore \quad f(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{4} & 0 \le x < 2 \\ \frac{(4-x)}{4} & 2 \le x < 4 \\ 0 & x \ge 4 \end{cases}$$

(ii)



The C.d.f. is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

Case-I

when $x < 0$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = 0$$

$$\Rightarrow \quad F(x) = 0 \quad \text{when} \quad x < 0.$$

Case-II

when $0 < x < 2$

$$F(x) = P(X < 0) + P(X \le 2)$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \frac{x}{4}\,dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2}\right)$$

$$\boxed{F(x) = \frac{x^2}{8}} \quad \text{when} \quad 0 < x < 2$$

Case-III When $2 < x < 4$

$$F(x) = P(X \leq 4)$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} \frac{x}{4}\,dx + \frac{1}{4}\int_{2}^{x} (4-x)\,dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2}\right)_0^2 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_2^x$$

$$= \frac{1}{4}\left[2 + 4x - \frac{x^2}{2} - (8-2)\right]$$

$$F(x) = \frac{1}{4}\left[4x - \frac{x^2}{2} - 4\right] \quad \text{when} \quad 2 < x < 4$$

Case-IV When $x > 4$

$$F(x) = \underbrace{\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx}_{\text{Case-(3) at } x=4} + \int_{4}^{x} f(x)\,dx$$

$$F(x) = \frac{1}{4}\left[4x - \frac{x^2}{2} - 4\right]_{\text{at } x=4} + \int_{4}^{x} 0\,dx$$

$$F(x) = \frac{1}{4}\left[16 - 8 - 4\right] = 1$$

$$F(x) = 1 \quad \text{when} \quad x > 4.$$

Thus, required c.d.f. is given by

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{8} & 0 \le x < 2 \\ \frac{1}{4}\left[4x - \frac{x^2}{2} - 4\right] & 2 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

Ans

(iii) To find the value of $P(X > 2.5)$ via p.d.f.

$$P(X > 2.5) = \int_{2.5}^{\infty} f(x)\,dx$$

$$= \int_{2.5}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$$

$$= \frac{1}{4}\int_{2.5}^{4} (4 - x)\,dx + \int_{4}^{\infty} 0\,dx$$

$$= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_{2.5}^{4}$$

$$= \frac{1}{4}\left[\left(16 - \frac{16}{2}\right) - \left(4 \times 2.5 - \frac{6.25}{2}\right)\right]$$

$$= \frac{1}{4}\left[8 - (10 - 3.125)\right]$$

$$= \frac{1}{4}\left[-2 + 3.125\right] = \frac{1.125}{4}$$

$$\boxed{P(X > 2.5) = 0.28125}$$

or

we can find the value of $P(X > 2.5)$ via c.d.f.

$$\because \quad F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\therefore \quad P(X > 2.5) = 1 - P(X \leq 2.5)$$

$$= 1 - F(2.5)$$

$$= 1 - \frac{1}{4}\left[4x - \frac{x^2}{2} - 4\right]_{\text{at } x = 2.5}$$

$$= 1 - \frac{1}{4}\left[10 - \frac{6.25}{2} - 4\right]$$

$$= 1 - \frac{1}{4}\left[6 - 3.125\right]$$

$$= \frac{1}{4}\left[-2 + 3.125\right]$$

$$\boxed{P(X > 2.5) = 0.28125}$$ Ans

(iv) To find the value of $P(X > 1)$ via p.d.f.

$$P(X > 1) = \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$$

$$= \int_1^2 \frac{x}{4}\,dx + \int_2^4 \frac{1}{4}(4-x)\,dx + \int_4^\infty 0\,dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2}\right)_1^2 + \frac{1}{4}\left(4x - \frac{x^2}{2}\right)_2^4$$

$$= \frac{1}{4}\left[\left(\frac{4}{2}-\frac{1}{2}\right) + \left(\left(16-\frac{16}{2}\right) - \left(8-\frac{4}{2}\right)\right)\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + (8-6)\right] = \frac{1}{4}\left[\frac{3}{2}+2\right] = \frac{1}{4}\left[\frac{7}{2}\right]$$

$$\boxed{P(X>1) = \frac{7}{8}}$$ Ans

or

we can find the value of $P(X>1)$ via using c.d.f.

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\therefore \quad P(X>1) = 1 - P(X \leq 1)$$

$$= 1 - F(1)$$

$$= 1 - \left(\frac{x^2}{8}\right)_{at\ x=1} = 1 - \frac{1}{8} = \frac{7}{8}$$

$$\boxed{P(X>1) = \frac{7}{8}}$$

(v) To find the value of $P(1<X<3)$ via using p.d.f.

$$P(1<X<3) = \int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \frac{1}{4}\int_1^2 x\,dx + \frac{1}{4}\int_2^3 (4-x)\,dx$$

$$= \frac{1}{4}\left[\left(\frac{x^2}{2}\right)_1^2 + \left(4x-\frac{x^2}{2}\right)_2^3\right]$$

$$= \frac{1}{4}\left[\left(\frac{4}{2}-\frac{1}{2}\right) + \left(12-\frac{9}{2}\right) - \left(8-\frac{4}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + 4 - \frac{9}{2} + \frac{4}{2}\right]$$

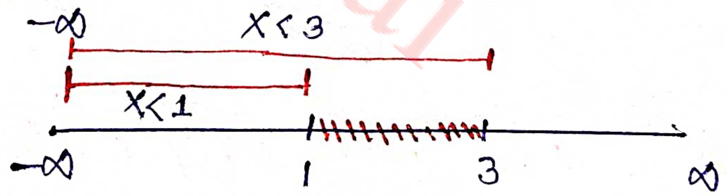
$$= \frac{1}{4}\left[4-1\right] = \frac{3}{4}$$

$$\boxed{P(1<X<3) = \frac{3}{4}}$$ Ans

Or

we can find the value of $P(1<X<3)$ via using c.d.f.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$



$$P(1<X<3) = P(X<3) - P(X<1)$$

$$= P(X\le 3) - P(X \le 1)$$

$$= F(3) - F(1)$$

$$= \frac{1}{4}\left[4x - \frac{x^2}{2} - 4\right]_{\text{at } x=3} - \left[\frac{x^2}{8}\right]_{\text{at } x=1}$$

$$P(1<X<3) = \frac{3}{4}$$ Ans

Mathematical Expectation —:

Let X be any random variable and $\phi(x)$ is any function of X. Then expectation of $\phi(x)$ is denoted by $E(\phi(x))$ & is defined as

$$E(\phi(x))$$

DRV

$$\sum_{x} \phi(x)\, P(x)$$

CRV

$$\int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx$$

If $\phi(x) = x$

Mean $\Rightarrow$ $E(x)$

DRV: $E(x) = \sum_{x} x\,P(x) = \bar{x}$

CRV: $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$.

Varience $\Rightarrow$

$$Var(X) = E(X-\bar{x})^2 = E\left(X^2 - 2X\bar{x} + (\bar{x})^2\right)$$

$$= E(X^2) - 2\bar{x}\,E(X) + \bar{x}^2$$

$$= E(X^2) - 2(E(x))^2 + (E(x))^2$$

$$= E(X^2) - (E(X))^2$$

$$\boxed{Var(X) = E(X^2) - (E(X))^2}$$

$$Var(X) = E(X - E(X))^2 = E(X^2) - (E(X))^2. \quad \#$$

Question:- Find mean and varience of the probability distribution given by the following table.

x	1	2	3	4	5
$P(x)$	0.2	0.35	0.25	0.15	0.05

Solution:-

$$\text{Mean} = E(X) = \sum x\,P(x)$$

$$= 1\times 0.2 + 2\times 0.35 + 3\times 0.25 + 4\times 0.15 + 5\times 0.05.$$

$$= 0.2 + 0.7 + 0.75 + 0.60 + 0.25$$

$$\boxed{\text{Mean} = E(X) = 2.5}$$ Ans

$$Var(X) = \text{Varience} = E(X - E(X))^2 = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum x^2 p(x)$$

$$= 1 \times 0.2 + 4 \times 0.35 + 9 \times 0.25 + 16 \times 0.15 + 25 \times 0.05$$

$$= 0.2 + 1.4 + 2.25 + 2.4 + 1.25$$

$$\boxed{E(X^2) = 7.5}$$

$$Var(X) = \sigma_X^2 = E(X^2) - (E(X))^2$$

$$= 7.5 - (2.5)^2$$

$$= 7.5 - (6.25)$$

$$\boxed{Var(X) = \sigma_X^2 = 1.25}$$ #

Question:- Thirteen Cards are drawn simultaneously from a pack of 52 Cards if ace count 1 and face Card 10 and other according to their denomination. Find the expectation of total score in 13 Cards.

										Q	B	B	J
X	1	2	3	4	5	6	7	8	9	10	10	10	10	10
$P(X)$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$	$4/52$
	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$	$1/13$

$$\text{Mean} = E(X) = \sum x.P(x)$$

$$= 1\times\frac{1}{13} + 2\times\frac{1}{13} + 3\times\frac{1}{13} + 4\times\frac{1}{13} + 5\times\frac{1}{13}$$
$$+ 6\times\frac{1}{13} + 7\times\frac{1}{13} + 8\times\frac{1}{13} + 9\times\frac{1}{13} + 10\times\frac{1}{13}$$
$$+ 10\times\frac{1}{13} + 10\times\frac{1}{13} + 10\times\frac{1}{13} + 10\times\frac{1}{13}$$

$$= \frac{1}{13}(1+2+3+4+5+6+7+8+9+10+10+10+10+10)$$

$$\boxed{E(X) = \frac{85}{13}}$$ **Ans**

In case of continuous random variable-:

Question-: A continuous random variable X has density function given by

$$f(x) = \begin{cases} 2e^{-2x} & x>0 \\ 0 & o/w \end{cases}$$

Find the expected value and varience of X.

solution-:

$$\text{Mean} = \text{Expected value} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{0} x\times 0\, dx + \int_{0}^{\infty} x.2e^{-2x} dx$$

$$= 2\left[\left(x \cdot \frac{-1}{2}e^{-2x}\right)_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x}dx\right]$$

$$= 2\,\frac{1}{2}\left[\frac{-e^{-2x}}{2}\right]_0^\infty = \frac{1}{2}\left(-e^{-\infty}+e^{0}\right) = \frac{1}{2}$$

$$\Rightarrow \boxed{E(X) = \frac{1}{2}}$$ Ans

and $\operatorname{Var}(X) = E(X-E(X))^2 = E(X^2) - (E(X))^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{0} x^2 \times 0\,dx + \int_{0}^{\infty} x^2\, 2e^{-2x}\,dx$$

$$= 2\int_0^\infty x^2 e^{-2x}\,dx$$

We know that

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \qquad \Gamma(n+1) = n\,\Gamma(n)$$

$$\frac{\Gamma(n)}{a^n} = \int_0^\infty e^{-ax} x^{n-1}\,dx \qquad \Gamma(n+1) = n!$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$= 2\int_0^\infty x^{3-1} e^{-2x}\,dx = 2\,\frac{\Gamma(3)}{2^3} = \frac{2\times 2}{2^3} = \frac{1}{2}$$

$$\Rightarrow \boxed{E(X^2) = \frac{1}{2}}$$

$$\operatorname{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\boxed{\operatorname{Var}(X) = 1/4}$$ Ans