PROBABILITY AND STATISTICS
(UCS401)

Lecture-26

(Transformation of two-dimensional random variables)

Random Variables and their Special Distributions(Unit –III & IV)

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Transformation of two-dimensional random variables

Here, we consider the concept and problems of the change of variables in two-dimensional random variables.

* For given random variables X and Y, consider $U = U(x,y)$ & $V = V(x,y)$ are continuously differentiable function.

For given random variables X and Y, if $f(x,y)$ be the joint density function of X and Y, then j.d.f. of U and V denoted by $g(u,v)$ is defined as

$$g(u,v) = f(x,y)\,|J| \qquad \left.\begin{array}{l} x = \phi_1(u,v) \\ y = \phi_2(u,v) \end{array}\right]$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \text{ or } \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

is called Jacobian.

Previously, we have studied for one-dimensional variable.

* Let X be a continuous random variable with p.d.f $f(x)$, then p.d.f. of $Y = v(X)$ is given by

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right| .$$

Working steps

For the transformation

$$U = u(x, y) \quad \& \quad V = v(x, y).$$

Step-I Express X and Y in terms of U and V and hence find Jacobian $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

Step-II Compute

$$g(u, v) = f\big(x(u, v), y(u, v)\big) \, |J| .$$

Step ③: Find the domain of u, v from the domain of x, y.

Question: Let the p.d.f. of the random variable (x, y) be

$$f(x,y) = \begin{cases} \frac{1}{\alpha^2} e^{-\frac{x+y}{\alpha}} & ; \ x, y > 0, \ \alpha > 0 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

Find distribution of $\frac{1}{2}(X-Y)$.

Solution: Let

$u = \frac{1}{2}(x-y)$ and $v = y$ (choose v in such a way so that $|J| \neq 0$)

$u = \frac{1}{2}(x-v)$

$$\Rightarrow \boxed{x = 2u + v \quad ; \quad y = v}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

Thus, joint p.d.f. of u, v is

$$g(u,v) = f(x,y)\,|J|$$

$$g(y,u) = \frac{1}{\alpha^2} e^{-\frac{2y+2u}{\alpha}} \quad (2)$$

$$\boxed{g(y,u) = \frac{2}{\alpha^2} e^{-\frac{2y+2u}{\alpha}}}$$

When $x > 0 \Rightarrow 2y + v > 0$

$\Rightarrow v > -2y$

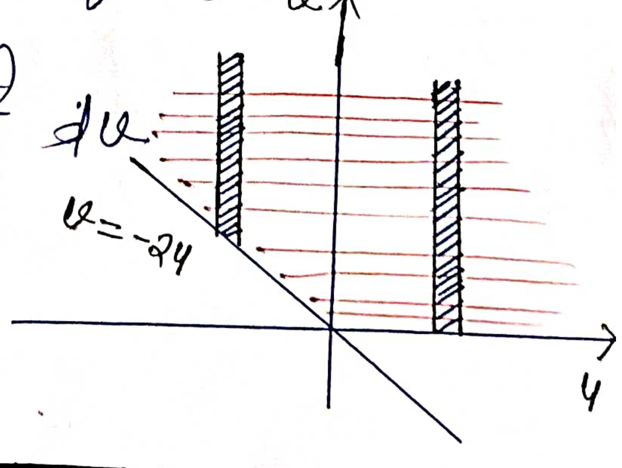
When $y > 0 \Rightarrow v > 0$

Thus, required joint p.d.f. of v and u is

$$g(y,u) = \begin{cases} \frac{2}{\alpha^2} e^{-\frac{2y+2u}{\alpha}} \; ; & -\infty < y < \infty \\ & v > 0, \; 2y + v > 0 \\ 0 \; ; & \text{else where.} \end{cases}$$

To find the distribution of y, i.e., Marginal density function of y,

$$g(y) = \int_u \frac{2}{\alpha^2} e^{-\frac{2(y+v)}{\alpha}} \, dv$$



Case-I

when

$-2y < u < \infty$

$$g(y) = \int_{-2y}^{\infty} \frac{2}{\alpha^2} e^{-\frac{2y+2u}{\alpha}} du$$

$$= \frac{2}{\alpha^2} e^{-\frac{2y}{\alpha}} \int_{-2y}^{\infty} e^{-\frac{2u}{\alpha}} du$$

$$= \frac{2}{\alpha^2} e^{-\frac{2y}{\alpha}} \left(\frac{-\alpha}{2}\right) \left(e^{-\frac{2u}{\alpha}}\right)_{-2y}^{\infty}$$

$$= -\frac{1}{\alpha} e^{-\frac{2y}{\alpha}} \left(e^{-\infty} - e^{\frac{4y}{\alpha}}\right)$$

$$= \frac{1}{\alpha} e^{\frac{2y}{\alpha}} ; \quad -\infty < y < 0$$

$$g(y) = \frac{1}{\alpha} e^{\frac{2y}{\alpha}} ; \quad -\infty < y < 0$$

$-2y < u < \infty$

Case-II

when

$0 < u < \infty$

$$g(y) = \int_{0}^{\infty} \frac{2}{\alpha^2} e^{-\frac{2(y+u)}{\alpha}} du$$

$$= \frac{2}{\alpha^2} e^{-\frac{2y}{\alpha}} \int_{0}^{\infty} e^{-\frac{2u}{\alpha}} du$$

$$= \frac{2}{\alpha^2} e^{-\frac{2y}{\alpha}} \left(\frac{-\alpha}{2}\right) \left(e^{-\frac{2u}{\alpha}}\right)_{0}^{\infty}$$

$$= -\frac{1}{\alpha} e^{-\frac{2y}{\alpha}} \left(e^{-\infty} - e^{0}\right)$$

$$= \frac{1}{\alpha} e^{-\frac{2y}{\alpha}} ; \quad 0 < y < \infty$$

$$g(y) = \frac{1}{\alpha} e^{-\frac{2y}{\alpha}} ;$$

$0 < y < \infty$.

Thus, required Marginal density function of U

$$\boxed{g(u) = \frac{1}{\alpha} e^{-\frac{2|u|}{\alpha}} \quad ; \quad -\infty < u < \infty}$$

Question: The joint density of two random variables X_1, X_2 is

$$f(x_1, x_2) = 2e^{-x_1 - x_2} \quad ; \quad 0 < x_1, x_2 < \infty.$$

Consider the transformation

$$U = 2X_1 \quad ; \quad V = X_2 - X_1.$$

Find the joint density of U and V.

Check whether U and V are independent or not?

Solution: Given that

$$u = 2x_1 \quad ; \quad v = x_2 - x_1$$

implies $x_1 = u/2$; $x_2 = v + u/2$

The Jacobian is

$$J = \frac{\partial(x_1, x_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 0 \\ 1/2 & 1 \end{vmatrix} = 1/2$$

Thus, joint p.d.f. of u, v is

$$g(u,v) = f\left(x(u,v), y(u,v)\right) |J|$$

$$= 2e^{-(u+v)} \frac{1}{2}$$

$$g(u,v) = e^{-(u+v)}$$

When

$$0 < x_1 < x_2 < \infty$$

$$\Rightarrow \quad 0 < u/2 < v + u/2 < \infty$$

$$\Rightarrow \quad u > 0 \quad \text{and} \quad 0 < v < \infty$$

Thus, required joint p.d.f. is

$$g(u,v) = \begin{cases} e^{-(u+v)} & ; \quad u > 0, \quad 0 < v < \infty \\ 0 & ; \quad \text{otherwise}. \end{cases}$$

To check whether U and V are independent,
For this we need to check

$$g(u,v) = g(u)\, g(v)$$

$\therefore$ Marginal density of U is

$$g(u) = \int_{v} g(u,v)\, dv$$

$$= \int_{0}^{\infty} e^{-u-v}\, dv$$

$$= e^{-u}\left(-e^{-v}\right)_{0}^{\infty} = e^{-u}\left(-e^{-\infty}+e^{0}\right)$$

$$\boxed{g(u) = e^{-u} \;;\quad u>0}$$

and Marginal density function of V is

$$g(v) = \int_{u} g(u,v)\, du = \int_{0}^{\infty} e^{-u-v}\, du$$

$$= e^{-v}\left(-e^{-u}\right)_{0}^{\infty}$$

$$= e^{-v}\left(-e^{-\infty}+e^{0}\right)$$

$$\boxed{g(v) = e^{-v} \;;\quad v>0}$$

Ans

Since

$$g(u,v) = g(u)\, g(v)$$

Thus, U and V are independent. #

Question: The joint density of two random variables X_1, X_2 is

$$f(x_1, x_2) = 8 x_1 x_2 \; ; \quad 0 < x_1 < x_2 < 1.$$

Consider the transformation $U = \frac{X_1}{X_2}$; $V = X_2$. Find the joint density of U and V. Check whether U and V are independent or not?

Solution: Given that

$$u = \frac{x_1}{x_2} \quad \& \quad v = x_2$$

$$\Rightarrow \quad x_2 = v \; ; \quad x_1 = uv$$

The Jacobian is

$$J = \frac{\partial(x_1, x_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Thus, the joint p.d.f of u, v is

$$g(u,v) = f\big(x(u,v), y(u,v)\big)\, |J|$$

$$g(u,v) = 8\,(uv)(v)\; v$$

$$g(u,v) = 84u^3$$

when $0 < x_1 < x_2 < 1$

$$\Rightarrow \quad 0 < uv < v < 1$$

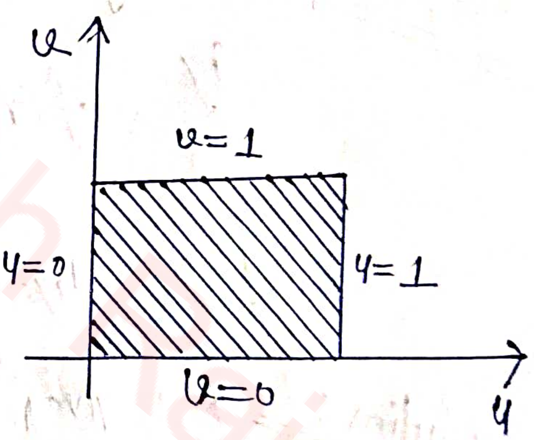
Thus, domain of u, v are

$$0 < u < 1, \qquad 0 < v < 1.$$

Hence, required Joint p.d.f. of U and V is

$$g(u,v) = \begin{cases} 84v^3 & ; \quad 0 < u < 1,\ 0 < v < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

To check whether U and V are independent. For this, we need to check



$$g(u,v) = g(u)\,g(v)$$

Marginal density of U

$$g(u) = \int_v g(u,v)\,dv = \int_0^1 84v^3\,dv$$

$$= 84\left(\frac{v^4}{4}\right)_0^1 = 21$$

$$\boxed{g(u) = 21 \quad ; \quad 0 < u < 1}$$

Marginal density of V

$$g(v) = \int_u g(u,v)\, du = \int_0^1 8uv^3\, du$$

$$= 8v^3 \left(\frac{u^2}{2}\right)_0^1$$

$$\boxed{g(v) = 4v^3 \quad ; \quad 0<v<1}$$

$\because$ $g(u)\, g(v) = (2u)(4v^3) = 8uv^3 = g(u,v)$

Thus, U and V are independent. #

Question: The joint p.d.f. of the variables X and Y is $f(x,y) = \frac{1}{2} x e^{-y}$; $0<x<2$, $y>0$.

Find the distribution of X+Y.

Solution: Given that

$$f(x,y) = \frac{1}{2} x e^{-y} \qquad 0<x<2,\; y>0$$

Let $U = x+y$ & $v = x$ (choose v s.t. $|J| \neq 0$)

which implies that

$$x = v \;;\quad y = u - v \;;$$

The Jacobian is

$$J = \frac{\partial(x,y)}{\partial(v,u)} = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & +1 \\ 1 & -1 \end{vmatrix} = -1$$

Thus, required joint p.d.f. is given by

$$g(v,u) = f\big(x(v,u),\, y(v,u)\big)\, |J|$$

$$= \frac{1}{2} x e^{-y} \Big|_{(x=v,\ y=v-u)} |-1|$$

$$g(v,u) = \frac{1}{2} v e^{-(v-u)}$$

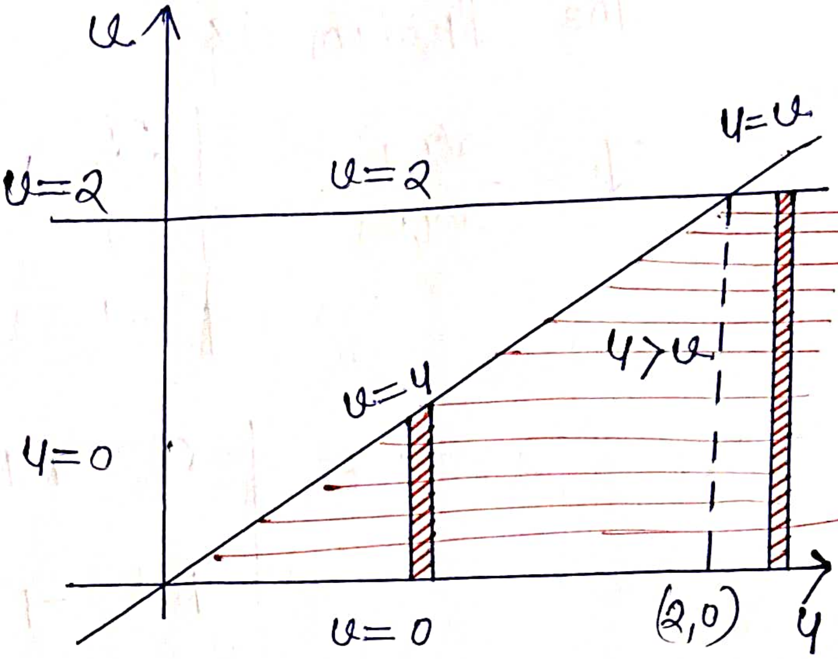
when $0<x<2 \Rightarrow 0<v<2$

when $y>0 \Rightarrow v>u$

Thus, required joint p.d.f. is

$$g(v,u) = \begin{cases} \frac{1}{2} v e^{-(v-u)} ; & 0<v<2,\ v>u \\ 0 ; & \text{otherwise.} \end{cases}$$

Thus, Marginal distribution of U



$$g(y)=\int_{v} g(y,v)\,dv$$

$$g(y)=\int_{[\,]}^{[\,]} \frac{1}{2} v e^{-(y-v)}\,dv = \frac{1}{2}e^{-y}\int_{[\,]}^{[\,]} v e^{v}\,dv$$

Case-I when

$0<v<y$

$$g(y)=\int_0^y \frac{1}{2}e^{-y}\, v e^{v}\,dv$$

$$=\frac{1}{2}e^{-y}\left(ve^{v}-e^{v}\right)_0^y$$

$$=\frac{1}{2}e^{-y}\left(ye^{y}-e^{y}+1\right)$$

$; \quad 0<y<2$

Case-II

$0<v<2$

$$g(y)=\frac{1}{2}e^{-y}\int_0^2 v e^{v}\,dv$$

$$=\frac{1}{2}e^{-y}\left(ve^{v}-e^{v}\right)_0^2$$

$$=\frac{1}{2}e^{-y}\left(2e^{2}-e^{2}+1\right)$$

$; \quad y>2$

$$=\frac{1}{2}e^{-y}\left(e^{2}+1\right) \quad 2<y<\infty.$$

Hence, required p.d.f. of U is

$$g(u) = \begin{cases} \frac{1}{2} e^{-u}(ue^{u} - e^{u} + 1) & ; \quad 0 < u < 2 \\ \frac{1}{2} e^{-u}(e^{2} + 1) & ; \quad 2 < u < \infty \end{cases}$$

Ans

Question: Suppose X_1, X_2 are independent exponential random variables whose density function is defined as

$$f(x_i) = e^{-x_i} \quad ; \quad x_i > 0.$$

Find Joint density function of $X_1 - X_2$, $X_1 + X_2$.

Solution: Since X_1, X_2 are independent random variables, so the Joint density function of X_1, X_2 is

$$f(x_1, x_2) = f(x_1) \cdot f(x_2)$$

$$= e^{-x_1} \cdot e^{-x_2}$$

$$f(x_1, x_2) = e^{-(x_1 + x_2)} \quad ; \quad x_1, x_2 > 0.$$

Take $\quad u = x_1 - x_2 \quad$ & $\quad v = x_1 + x_2$

$$\Rightarrow x_1 = \frac{u+v}{2} \quad ; \quad x_2 = \frac{v-u}{2}.$$

The Jacobian is

$$J = \frac{\partial(x_1, x_2)}{\partial(y, v)} = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/4 + 1/4 = 1/2$$

Thus, the Joint density function of U and V is

$$g(y,v) = f\left(x_1(y,v), x_2(y,v)\right) |J|$$

$$= e^{-(x_1+x_2)} \frac{1}{2}$$

$$\boxed{g(y,v) = \frac{1}{2} e^{-v}}$$

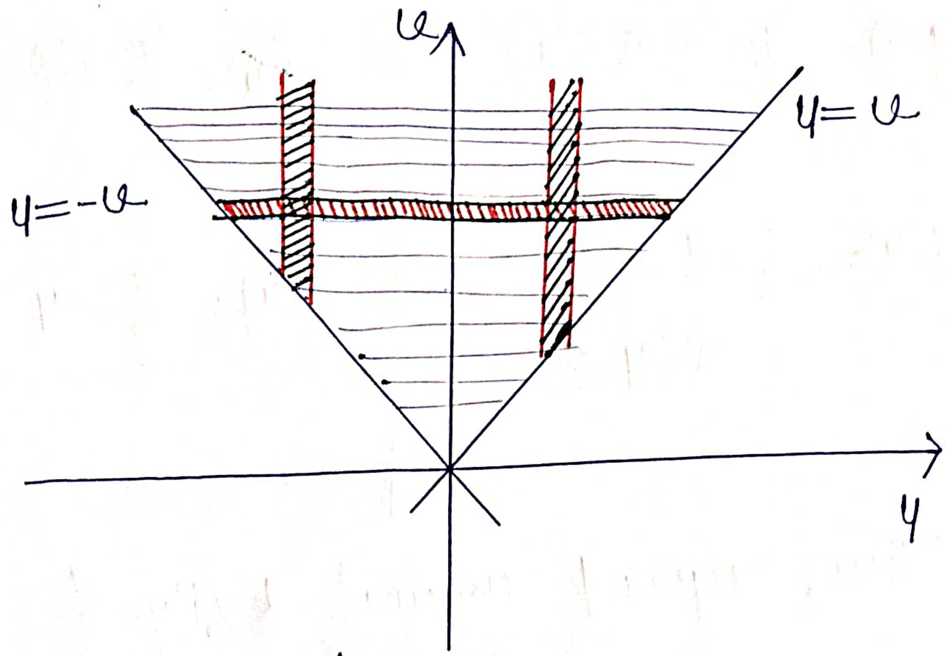
When $x_1 > 0 \Rightarrow y + v > 0 \Rightarrow y > -v$

when $x_2 > 0 \Rightarrow v - y > 0 \Rightarrow y < v$

Hence, $\boxed{-v < y < v, \quad v > 0}$

Thus, Joint density function of U and V is

$$g(y,v) = \begin{cases} \frac{1}{2} e^{-v} & ; \quad v > 0, -v < y < v \\ 0 & ; \quad \text{otherwise} \end{cases}$$



The Marginal density of V is

$$g(v) = \int_y g(y,v)\,dy = \int_{-v}^{v} \frac{1}{2}e^{-v}\,dy$$

$$\boxed{g(v) = ve^{-v} \; ; \qquad 0<v<\infty}$$

The Marginal density of U

$$g(y) = \int_v g(y,v)\,dv$$

<u>Case-I</u>

when

$-y<v<\infty$

$$g(y) = \int_{-y}^{\infty} \frac{1}{2}e^{-v}\,dv$$

$$= -\frac{1}{2}\left(e^{-v}\right)_{-y}^{\infty}$$

<u>Case-II</u>

when

$y<v<\infty$

$$g(y) = \int_{y}^{\infty} \frac{1}{2}e^{-v}\,dv$$

$$= -\frac{1}{2}\left(e^{-v}\right)_{y}^{\infty}$$

$$g(u) = -\frac{1}{2}\left(e^{-\infty} - e^{u}\right)$$

$$g(u) = \frac{1}{2}e^{u}\ ; \quad -\infty < u < 0$$

$$g(u) = -\frac{1}{2}\left(e^{-\infty} - e^{-u}\right)$$

$$g(u) = \frac{1}{2}e^{-u}\ ; \quad 0 < u < \infty$$

Thus, required Marginal density of U

$$\boxed{g(u) = \frac{1}{2}e^{-|u|} \qquad -\infty < u < \infty}$$

This probability density function is known as double exponential or Laplace p.d.f. #