PROBABILITY AND STATISTICS
(UCS401)

Lecture-11

(Geometric Distribution with Examples)

Random Variables and their Special Distributions(Unit –III & IV)

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

Dr. Rajanish Rai
Assistant Professor
School of Mathematics
Thapar Institute of Engineering and Technology, Patiala

Geometric distribution & Examples

(*) You have to perform the experiment → until
⇓
Geometric distribution.

(*) What is the probability of getting 1st head in 7th attempt during tossing a coin repeatedly?

1, 2, 3, 4, 5, 6 — Tail
7 → 1st Head

(*) A person shot the target in an independent manner. What is the probability that a target would be hit on his 10th attempt.

1, 2, 3, 4, 5, 6, 7, 8, 9 — until the target.
10th → hit the target

(*) Let X be the number of births in a hospital until the first girl is born. Find the distribution function of X.

" Geometric distribution is the probability distribution of the number X of independent Bernoli trails performed until a success occurs, where Bernolli trails have a constant probability of success p.

* Binomial $\left\langle \begin{matrix} n \longrightarrow x \text{ success occurs.} \\ p \end{matrix} \right.$

* Geometric $(n) \longrightarrow x$ until 1st success occurs.

When Geometric distribution is used :-

Geometric distribution is applicable to find the probability, where we perform an experiment until a success occurs.

Examples :-

(i) Tossing a coin repeatedly until the first head appears.

(ii) Shot the target until it hits.

(iii) Give the test until he will pass it.

(iv) Throwing a die repeatedly until first times a six appears.

q 1
q 2
q 3
q 4
q 5 $(x-1)^{th}$ step $\xrightarrow{\text{failure}}$ q^{x-1}
6
⋮
q ⋮

$x = 1, 2, 3, 4, \ldots \infty$

$x^{th} \xrightarrow{\text{success}} p$
⋮
∞

The required probability $= pq^{x-1}$

Geometric distribution -:

The probability mass function (p.m.f.) of the geometric distribution is

$$P(x) = P(X=x) = \begin{cases} pq^{x-1} & ; \quad x = 1, 2, 3, \dots, \infty \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

such that $p + q = 1$, $p \geq 0$.

$p \rightarrow$ probability of success until xth success occurs

$q \rightarrow$ probability of failure upto $(x-1)$th step.

Note that -: The number of unknown parameter in Geometric distribution is only p.

Question -: Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is hit in any shot is 0.7. What is the

(i) Probability that the target would be hit on 10th attempt.

(ii) Probability it takes him less than 4 shoots.

(iii) Probability that it takes him an even number of shoots.

solution:- Let X be the no. of attempt to hit the target

$$p \to P(\text{hitting}) = 0.7$$

$$q = 0.3$$

Thus, required probability

$$= P(X=10)$$

1
2
3
4
5
6
7
8
9

Not hit

$\Leftarrow$ Geometric distribution

$10^{th} \to$ hit

For geometric distribution, the p.m.f. is given by

$$P(x) = P(X=x) = p\,q^{x-1} \qquad x = 1, 2, 3, \ldots \infty$$

$$P(X=10) = (0.7)(0.3)^9$$

$$\boxed{P(X=10) = 1.3777 \times 10^{-5}}$$

Ans

(ii) probability it takes him less than 4 shoots.

The required probability

$$= P(X<4)$$

For geometric distribution, the p.m.f. is given by

$$P(X=x) = p\,q^{x-1} = (0.7)(0.3)^{x-1}$$

$$P(X<4) = P(X=1) + P(X=2) + P(X=3)$$

$$= (0.7)(0.3)^0 + (0.7)(0.3)^1 + (0.7)(0.3)^2$$

$$P(X<4) = 0.9730$$

Ans

(iii) Probability that it takes him an even number of shots:

Thus, required probability

$$= P(X=\text{even})$$

$$= P(X=2) + P(X=4) + P(X=6) + P(X=8) + \dots$$

$$= pq + pq^3 + pq^5 + pq^7 + \dots$$

$$= pq(1+q^2+q^4+q^6+\dots)$$

↳ infinite geometric series.

$$= \frac{pq}{1-q^2}$$

$$= \frac{(0.3)(0.7)}{1-(0.3)^2} = 0.2308 \quad \text{Ans}$$

Question :- Suppose that the probability for an applicant for a driver's licence to pass the road test on any given attempt is $\frac{2}{3}$. What is the probability that the applicant will pass the road test on the third attempt ?

Solution :- X be no. of attempt to pass road test. $\equiv$ Not clear

Given that

और $\boxed{-}$ clear the test

$$P(\text{pass road test}) = \frac{2}{3}$$

⇓

$$p = \frac{2}{3} \;\&\; q = \frac{1}{3}.$$

Geometric distribution

∴ For Geometric distribution, the p.m.f. is given by

$$P(X=x) = pq^{x-1} \qquad x = 1, 2, 3, \dots \infty$$

$$P(X=x) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{x-1} \quad \text{---(1)}$$

Thus, required probability

$$P(X=3) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{2}{27} \quad \text{Ans}$$

Practice sheets

Question ①: Suppose that a trainee soldier shoots a target according to geometric distribution. If probability that a target is shot in any shot is 0.8, find the probability that it takes an odd number of shots.

Solution -: Given that

$$p = P(\text{shot the target}) = 0.8$$

$$q = 0.2$$

$\therefore$ For geometric distribution, the p.m.f. is given by

$$P(x) = P(X=x) = pq^{x-1}$$

Thus, required probability

$$= P(X = \text{odd})$$

$$= P(X=1) + P(X=3) + P(X=5) + P(X=7) + \cdots$$

$$= pq^0 + pq^2 + pq^4 + pq^6 + \cdots$$

$$= p(1 + q^2 + q^4 + q^6 + \cdots)$$

$$= \frac{p}{1-q^2}$$

↳ infinite geometric series

$$P(X = \text{odd}) = \frac{(0.8)}{1-(0.2)^2}$$

Question :- Let X be the number of births in a hospital until the first girl is born. Determine the probability and distribution function of X. Assume ~~that~~ the probability that the baby born is a girl is $\frac{1}{2}$.

Solution :- Let X be the ~~probability~~ number of birth until first girl born at xth attempt. $p \to \frac{1}{2}$

$\therefore$ For geometric distribution, the p.m.f. is given by – $q \to \frac{1}{2}$

$$p(x) = P(X=x) = pq^{x-1}$$

$$P(X=x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{x}$$

$$\boxed{P(X=x) = \left(\frac{1}{2}\right)^{x}}$$

Ans

Mean and variance of the geometric distribution :-

The probability mass function (p.m.f.) of the geometric distribution is

$$p(x) = P(X=x) = \begin{cases} pq^{x-1} & ; x = 1, 2, 3, \ldots \infty \\ 0 & ; \text{otherwise.} \end{cases}$$

such that $p+q=1$, $p \geq 0$

- The number of unknown parameter in Geometric distribution is only p.

	p.m.f.	Mean	Variance
For Binomial distribution	$p(x) = {}^nC_x p^x q^{n-x}$	np	npq
For Poisson distribution	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ

Result: If X is a geometric random variable with parameter p, then

$$\text{Mean}(X) = \frac{1}{p} ; \quad \text{Variance}(X) = \frac{q}{p^2}.$$

proof: If a random variable X follows a geometric distribution, then its p.m.f. is given by

$$p(x) = P(X=x) = \begin{cases} pq^{x-1} & ; x = 1,2,3,\ldots\infty \\ 0 & o/w \end{cases}$$

$$\text{Mean}(X) = E(X) = \sum_{x=1}^{\infty} x\, p(x)$$

$$= \sum_{x=1}^{\infty} x\, pq^{x-1}$$

$$= 1pq^0 + 2pq^1 + 3pq^2 + 4pq^3 + \ldots$$

$$= p\left(1 + 2q + 3q^2 + 4q^3 + \ldots\right)$$

$$= p(1-q)^{-2}$$

↳ Binomial expansion

$$= \frac{p}{p^2} = \frac{1}{p}$$

$p+q=1$

$$\boxed{\text{Mean}(X) = E(X) = \frac{1}{p}}$$

Ans

(ii) $$\text{Variance}(X) = E(X^2) - (E(X))^2$$

$$= E(X^2 - X + X) - (E(X))^2$$

$$= E(X(X-1) + X) - (E(X))^2$$

$$= E(X(X-1)) + E(X) - (E(X))^2$$

$$= \sum_{x=1}^{\infty} x(x-1)\, p q^{x-1} + \frac{1}{p} - \frac{1}{p^2}$$

$$= [0 + 2pq^1 + 6pq^2 + 12pq^3 + 20pq^4 + \dots] + \frac{1}{p} - \frac{1}{p^2}$$

$$= 2pq[1 + 3q + 6q^2 + 10q^3 + \dots] + \frac{1}{p} - \frac{1}{p^2}$$

$$= 2pq(1-q)^{-3} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2pq}{(1-q)^3} + \frac{(p-1)}{p^2}$$

$$= \frac{2pq}{p^3} - \frac{q}{p^2} = \frac{2q}{p^2} - \frac{q}{p^2}$$

$$\boxed{\text{Var}(X) = \frac{q}{p^2}}$$

Thus, for geometric distribution,

$$\text{Mean} = \frac{1}{p}$$

$$\text{Variance} = \frac{q}{p^2}$$

Question :- Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is hit in any shot is 0.7, what is the

(i) Average number of shots needed to hit the target ?

Solution :- We know that for geometric distribution

$$\text{Average} = \text{Mean} = \frac{1}{p}$$

Given

$$P(\text{hitting the target}) = p = 0.7$$

$$\text{Average} = \text{Mean} = \frac{1}{0.7} = \frac{10}{7} = 1.4286$$

Ans

Moment generating function (m.g.f.) :-

Question :- Find the M.G.F. for geometric distribution and find its mean and variance.

Solution :- The probability mass function (p.m.f.) of the geometric distribution is

$$P(x) = pq^{x-1} \quad ; \quad x = 1, 2, 3, 4, \dots \infty$$

Thus, M.G.F. is given as,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p q^{x-1}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$\because$ infinite sum of G.P. $\sum_{n=1}^{\infty} a r^n = \frac{a}{1-r}$, provided $r<1$

$$\therefore \quad M_X(t) = \frac{p}{q}\left(\frac{qe^t}{1-qe^t}\right) \quad \text{provided } qe^t < 1$$

Hence,

$$\boxed{M_X(t) = \frac{pe^t}{1-qe^t}, \text{ provided } qe^t<1 \text{ i.e., } t< \log_e\left(\frac{1}{q}\right).}$$

To find mean and variance -:

We know that

$$E(X^r) = \frac{d^r}{dt^r}\left(M_X(t)\right)\Big|_{t=0}$$

$$\text{Mean} = E(X) = \frac{d}{dt}\left(M_X(t)\right)\Big|_{t=0}$$

$$\text{Variance} = E(X^2) - (E(X))^2.$$

$$\text{Mean} = E(X) = \frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)\Big|_{t=0}$$

$$E(X) = \frac{(1-\cancel{qe^t})\,pe^t + pe^t\,\cancel{qe^t}}{(1-qe^t)^2}\Bigg|_{t=0}$$

$$= \frac{pe^t}{(1-qe^t)^2}\Bigg|_{t=0} = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{\text{Mean} = E(X) = \frac{1}{p}}$$

Now,

$$E(X^2) = \frac{d^2}{dt^2}\left(M_X(t)\right)\Bigg|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\left(M_X(t)\right)\right]\Bigg|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]\Bigg|_{t=0}$$

$$= \frac{(1-qe^t)^2\,pe^t - pe^t\,2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\Bigg|_{t=0}$$

$$= \frac{pe^t(1-qe^t) + 2pqe^{2t}}{(1-qe^t)^3}\Bigg|_{t=0}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}\Bigg|_{t=0}$$

$$= \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}\Bigg|_{t=0}$$

$$= \frac{pe^t(1+qe^t)}{(1-qe^t)^3}\Bigg|_{t=0}$$

$$E(x^2) = \frac{pe^0(1+qe^0)}{(1-qe^0)^3} = \frac{p(1+q)}{(1-q)^3}$$

$$E(x^2) = \frac{p(1+q)}{p^3}$$

$$\boxed{E(x^2) = \frac{1+q}{p^2}}$$

$$\because \quad Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{q+1}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} + \cancel{\frac{1}{p^2}} - \cancel{\frac{1}{p^2}}$$

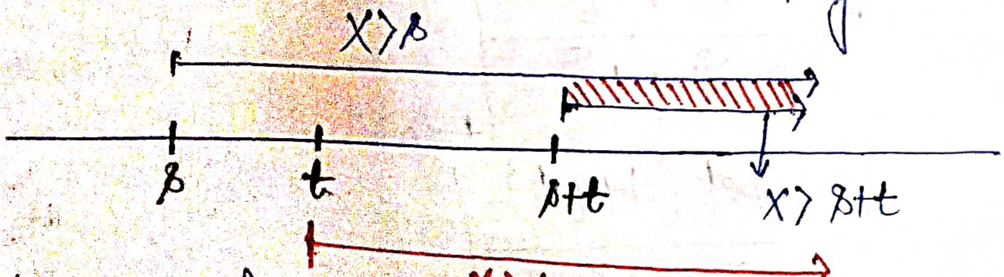
$$\boxed{Var(x) = \frac{q}{p^2}}$$

Ans

Memoryless property of the geometric distribution :-

Memoryless property
⇓
Discreate distribution

- Negative Binomial ⊗
- Poisson ⊗
- Binomial ⊗
- Geometric ✓
  ⇓
  follows memoryless property



- $P(X>n \mid X>m) = P(X>n-m)$.

Result: If X has a geometric distribution, show that for any two integers s and t,

$$P(X > s+t \mid X > s) = P(X > t).$$

This property is called memoryless property.

Proof: Since X follows the geometric distribution, the p.m.f. is given by

$$P(x) = P(X=x) = pq^{x-1}, \quad x = 1,2,3,4,\ldots\infty.$$

For any positive integer k, we have

$$P(X>k) = P(X=k+1) + P(X=k+2) + P(X=k+3) + P(X=k+4) + \ldots$$

$$= pq^{k} + pq^{k+1} + pq^{k+2} + pq^{k+3} + \ldots$$

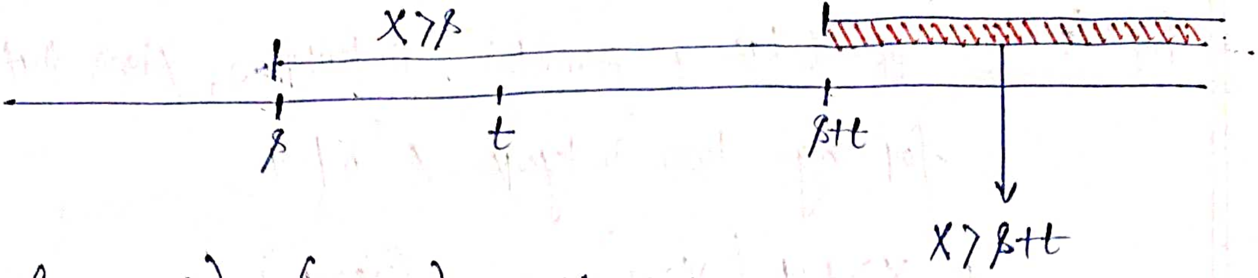
$$= pq^{k}\left(1+q+q^{2}+q^{3}+q^{4}+\ldots\right)$$

$$= \frac{pq^{k}}{1-q} = \frac{pq^{k}}{p} = q^{k}$$

$$\boxed{P(X>k) = q^{k}}$$

Hence, $$P(X > s+t \mid X > s) = \frac{P\left((X > s+t) \cap (X > s)\right)}{P(X > s)}$$

$$\because \; P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$



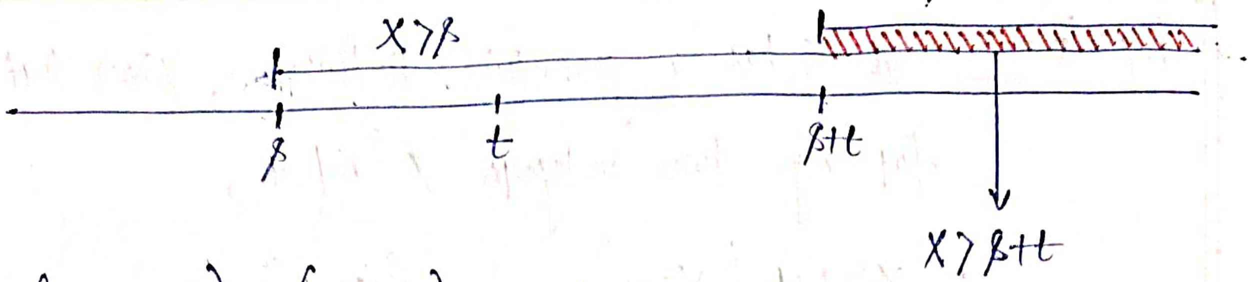
$$(X > (s+t)) \cap (X > s) = X > s+t$$

$$\therefore \quad P\left(X > (s+t) \mid X > s\right) = \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{q^{s+t}}{q^{s}} = q^{t} = P(X > t)$$

$$\therefore \quad \boxed{P(X > s+t \mid X > s) = P(X > t)}$$

Proved



$$(X>(p+t)) \cap (X>p) = X > p+t$$

$$\therefore \; P\left(X>(p+t) \,|\, X>p\right) = \frac{P(X>p+t)}{P(X>p)}$$

$$= \frac{q^{p+t}}{q^{p}} = q^{t} = P(X>t)$$

$$\therefore \; \boxed{P(X>p+t \,|\, X>p) = P(X>t)}$$

Proved