Sigma algebra

A collection of subsets of S is called a sigma algebra, denoted by $\mathcal{B}$, if it satisfies the following properties

(i) $\phi \in \mathcal{B}$

(ii) if $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$

(iii) if $A_1, A_2, \ldots \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

(closed under countable unions)

Eg.

$\mathcal{B} = \{\phi, S\}$ — trivial sigma algebra

if S is finite then $\mathcal{B} = \{$all subsets of S, including S itself$\}$

Probability function

Given a sample space S and an associated sigma algebra $\mathcal{B}$, a probability function P with domain $\mathcal{B}$, that satisfies

is a Prob. funct

(i) $P(A) \geq 0$ for all $A \in \mathcal{B}$

(ii) $P(S) = 1$

(iii) if $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint

Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Kolmogorov's Axioms

Axioms of Probability

Example. Consider a simple experiment of tossing a fair coin.

So $S = \{H, T\}$

$$P(\{H\}) = P(\{T\}) \quad \text{--- (1)} \quad \{\because \text{given fair coin}$$

Since $S = \{H\} \cup \{T\}$

$$P(S) = P(\{H\} \cup \{T\})$$

$$1 = P(\{H\}) + P(\{T\}) \quad \text{--- (2)}$$

Solving (1) & (2) gives $P(\{H\}) = P(\{T\}) = \frac{1}{2}$

if not given fair coin, any function P that satisfies (2) can be considered as a probability function

Thm. if P is a probability function and A is any set in $\mathcal{B}$ Then

(i) $P(\phi) = 0$ (ii) $P(A) \leq 1$ (iii) $P(A^c) = 1 - P(A)$

Proof Since A & A^c forms a partition of the sample space

$S = A \cup A^c \Rightarrow P(S) = P(A \cup A^c)$

$1 = P(A) + P(A^c) \Rightarrow P(A^c) = 1 - P(A)$ $\{\because$ of (iii) axioms

Since $A \in B \Rightarrow A^c \in B$

$P(A^c) \geq 0 \Rightarrow 1 - P(A) \geq 0 \Rightarrow P(A) \leq 1$ $\{\because$ of (i) axiom

Also $S = S \cup \phi \Rightarrow P(S) = P(S \cup \phi)$

$1 = P(S) + P(\phi) \Rightarrow P(\phi) = 0$

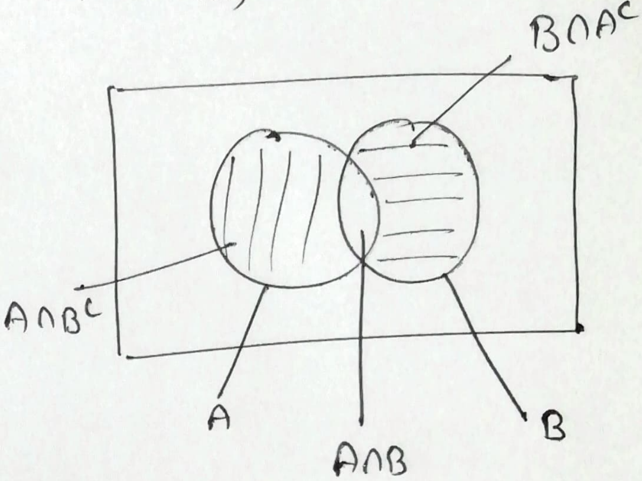
Th^m: if P is a probability function, and A and B are any sets in $\mathcal{B}$ Then

(i) $P(B \cap A^c) = P(B) - P(A \cap B)$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(iii) if $A \subset B$ Then $P(A) \leq P(B)$

Proof.



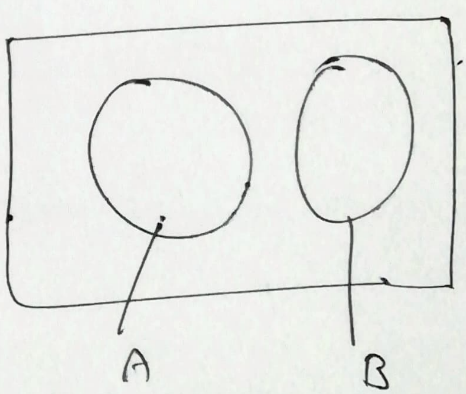
(i) $B = \{B \cap A\} \cup \{B \cap A^c\}$

$P(B) = P(\{B \cap A\} \cup \{B \cap A^c\})$

$= P(B \cap A) + P(B \cap A^c)$

$\because \{B \cap A\}$ and $\{B \cap A^c\}$ are disjoint

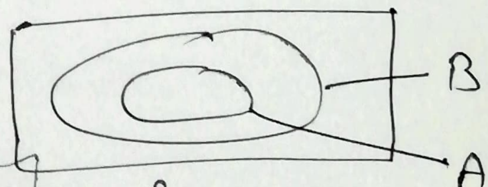
$$\boxed{\therefore\ P(B \cap A^c) = P(B) - P(A \cap B)} \quad \text{—(1)}$$



Also (ii) $A \cup B = A \cup \{B \cap A^c\}$

$P(A \cup B) = P(A \cup \{B \cap A^c\})$

$= P(A) + P(B \cap A^c)$



$$\boxed{= P(A) + P(B) - P(A \cap B)}$$ {using (1)

(iii) if $A \subset B$ Then $A \cap B = A$

$\therefore$ using (1) $P(B \cap A^c) = P(B) - P(A \cap B)$

$= P(B) - P(A)$

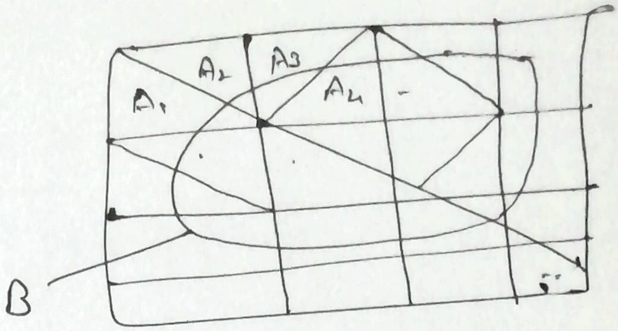
We know $P(B \cap A^c) \geq 0$

$P(B) - P(A) \geq 0$

$P(B) \geq P(A) \Rightarrow \boxed{P(A) \leq P(B)}$

Thm: if P is a probability function, Then
$$P(B) = \sum_{i=1}^{\infty} P(B \cap A_i) \text{ for any partition } A_1, A_2, \ldots$$

Proof



$$B = B \cap S$$

$$B = B \cap \left(\bigcup_{i=1}^{\infty} A_i \right)$$

Since A_i is any partition

$\therefore \; S = \bigcup_{i=1}^{\infty} A_i$ and $A_i \cap A_j = \phi$ for all $i \neq j$

Therefore
$$P(B) = P\left(B \cap \left(\bigcup A_i\right)\right)$$
$$= P\left(\bigcup_{i=1}^{\infty} (B \cap A_i)\right)$$
$$P(B) = \sum P(B \cap A_i)$$
$\because$ of (iii) axiom

Since A_i's are disjoint, The sets $B \cap A_i$ are also disjoint

Conditional Probability if A and B are events in S and $P(B) > 0$ Then The Conditional Probability of A given B, written $P(A|B)$, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B)\, P(A|B)$$
$$P(A \cap B) = P(A)\, P(B|A)$$

Baye's Rule

Let $A_1, A_2, —$ be a partition of The sample space and let B be any set. Then, for each $i = 1, 2, —$

$$P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}$$

Proof

Take L.H.S

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(A_i)\, P(B \mid A_i)}{\sum_{j=1}^{n} P(B \cap A_j)}$$

$$= \frac{P(A_i)\, P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)\, P(B \mid A_j)}$$

law of total Probability

$= R.H.S$

Example A factory uses Three machines X, Y, Z to produce certain items. Suppose

(i) Machine X produces 50 percent of The items of which 3 percent are defective

(ii) Machine Y produces 30 percent of The items of which 4 percent are defective

(iii) Machine Z produces 20 percent of The items of which 5 percent are defective.

Then find

1. The probability p That a randomly selected item is defective
2. Suppose a defective item is found among The output find The probability it come from

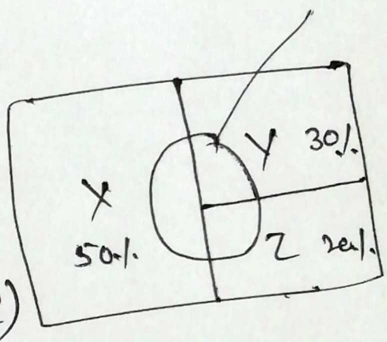
   2.1 machine X

   2.2 machine Y

   2.3 machine Z

Sol^n

(i) Let D : denote The event That an item is defective



Partition total — 100%

$$D = \{D \cap X\} \cup \{D \cap Y\} \cup \{D \cap Z\}$$

$$P(D) = P(D \cap X) + P(D \cap Y) + P(D \cap Z)$$

$$= P(X)P(D|X) + P(Y)P(D|Y) + P(Z)P(D|Z)$$

$$= 0.50 \times 0.03 + 0.30 \times 0.04 + 0.20 \times 0.05$$

$$= 0.037 = 3.7\%$$

(2.1) $$P(X|D) = \frac{P(X \cap D)}{P(D)} = \frac{P(X)P(D|X)}{P(D)} = \frac{0.50 \times 0.03}{0.037}$$

$$= 40.5\%$$

(2.2) $$P(Y|D) = \frac{P(Y \cap D)}{P(D)} = \frac{P(Y)P(D|Y)}{P(D)} = \frac{0.30 \times 0.04}{0.037}$$

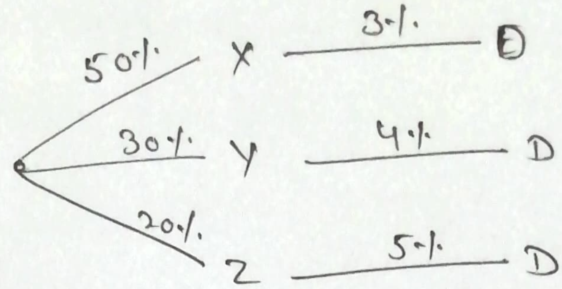
$$= 32.5\%$$

(2.3) $$P(Z|D) = \frac{P(Z \cap D)}{P(D)} = \frac{P(Z)P(D|Z)}{P(D)} = \frac{0.20 \times 0.05}{0.037}$$

$$= 27\%$$

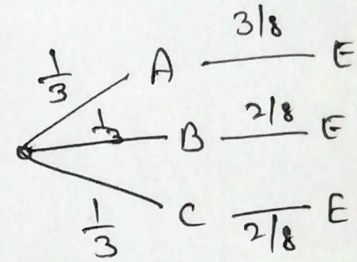
Alternatively

using tree diagram



$$P(D) = 0.50 \times 0.03$$
$$+ 0.30 \times 0.04$$
$$+ 0.20 \times 0.05$$
$$= 0.037$$
$$= 3.7\%$$

$$P(X \mid D) = \frac{0.\cancel{6}5 \times 0.03}{0.037} = 40.5\%$$



Eg. Suppose the following three boxes are given:

Box A contains 3 Red and 5 white Balls

Box B contains 2 Red and 1 white ball

Box C contains 2 Red and 3 white balls

A Box is selected at random, and a ball is drawn from the box. If the ball is red, find the probability that it came from box A.

Solⁿ. $P(A) = P(B) = P(C) = \frac{1}{3}$ Selecting a box A, B, C

E : selecting a red ball

$$P(E \mid A) = \frac{{}^3C_1}{{}^8C_1} = \frac{3}{8}$$

$$P(E \mid B) = \frac{{}^2C_1}{{}^8C_1} = \frac{2}{8}$$

$$P(E \mid C) = \frac{{}^2C_1}{{}^8C_1} = \frac{2}{8}$$

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)}$$
$$= \frac{P(A)\,P(E \mid A)}{P(E)}$$
$$= \frac{\frac{1}{3} \times \frac{3}{8}}{173/360}$$
$$\simeq 0.26$$

$$P(E) = P(A)\,P(E \mid A) + P(B)\,P(E \mid B) + P(C)\,P(E \mid C)$$
$$= \frac{1}{3} \times \frac{3}{8} + \frac{1}{3} \times \frac{2}{8} + \frac{1}{3} \times \frac{2}{8} = 173/360 \simeq 0.48$$

Statistically Independent

Two events, A and B, are statistically Independent if $P(A \cap B) = P(A)\,P(B)$

Th^m. if A and B are independent events, Then the following pairs are also independent:

(i) A and B^c

(ii) A^c and B

(iii) A^c and B^c

Proof Given A and B are Independent events

$\therefore \quad P(A \cap B) = P(A)\,P(B)$

To show (i) A and B^c are independent

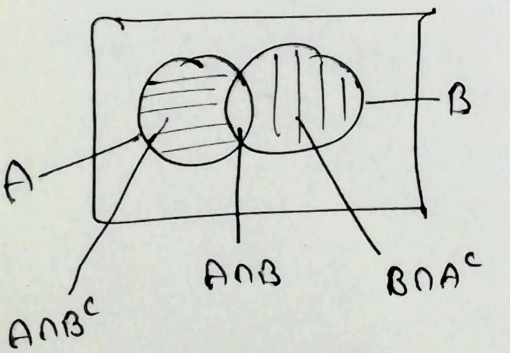
we have to show $P(A \cap B^c) = P(A)\,P(B^c)$

Now, we know $P(A \cap B^c) = P(A) - P(A \cap B)$

$= P(A) - P(A)P(B)$

$= P(A)\,[1 - P(B)]$

$= P(A)\,P(B^c)$



(ii) $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(A^c)P(B)$

(iii) $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= P(A^c) - P(B) + P(A)P(B) = P(A^c) - P(B)[1 - P(A)]$

$= P(A^c)[1 - P(B)] = P(A^c)\,P(B^c)$

* We might think that we could say A, B and C are independent if $P(A \cap B \cap C) = P(A)P(B)P(C)$

However, this is not the correct condition.

Eg. Let an experiment consist of tossing two dice. Define following events

$A = \{\text{double appears}\}$

$B = \{\text{The sum is between 7 and 10}\}$

$C = \{\text{The sum is 2 or 7 or 8}\}$

Now $P(A) = \frac{1}{6}$

(1,1) (2,2) (3,3) (4,4)
(5,5) (6,6)

$P(B) = \frac{1}{2}$

(1,6) (2,5) (3,4) — Sum 7
(2,6) (3,5) (4,4) — 8
(3,6) (4,5) — 9
(5,5) — 10

$P(C) = \frac{1}{3}$

(1,1)*
(1,6) (2,5) (3,4)
(2,6) (3,5) (4,4)*

Also $P(A \cap B \cap C)$

$= P(\text{Sum is 8, Composed of double 4's})$

$= \frac{1}{36}$

$= \frac{1}{6} \times \frac{1}{2} \times \frac{1}{3} = P(A)\,P(B)\,P(C)$ → Not a strong condition to guarantee Pairwise independence

However, $P(B \cap C) = P(\text{Sum equals 7 or 8})$

$= \frac{11}{36} \neq P(B)\,P(C)$

Similarly $P(A \cap B) \neq P(A)\,P(B)$

Definition: A Collection of events $A_1, A_2, \ldots, A_n$ are mutually indebendent if for any Subcollection $A_{i_1}, \ldots, A_{i_k}$, we have

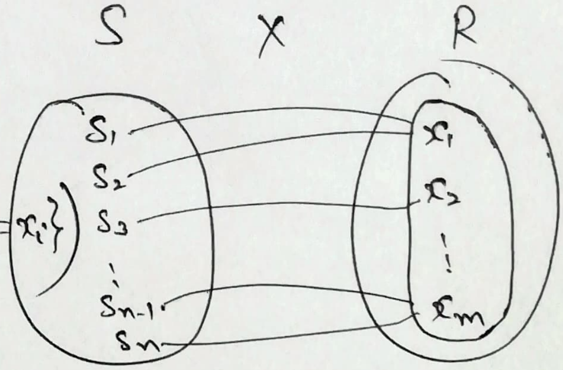
$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})$$

Random Variable

easy to deal with a Summary variable Than with The original probability structure

A random variable is a function from a sample space S into The real numbers.

$$X : S \longrightarrow R$$



$$P_X(X = x_i) = P(\{s_j \in S : X(s_j) = x_i\})$$

or we can write

$$P(X = x_i) = P(\{s_j \in S : X(s_j) = x_i\})$$

$$P(X = x_1) = P(\{s_1, s_2\})$$

$$P(X = x_2) = P(\{s_3\})$$

$$P(X = x_m) = P(\{s_{n-1}, s_n\})$$

observe Carefully

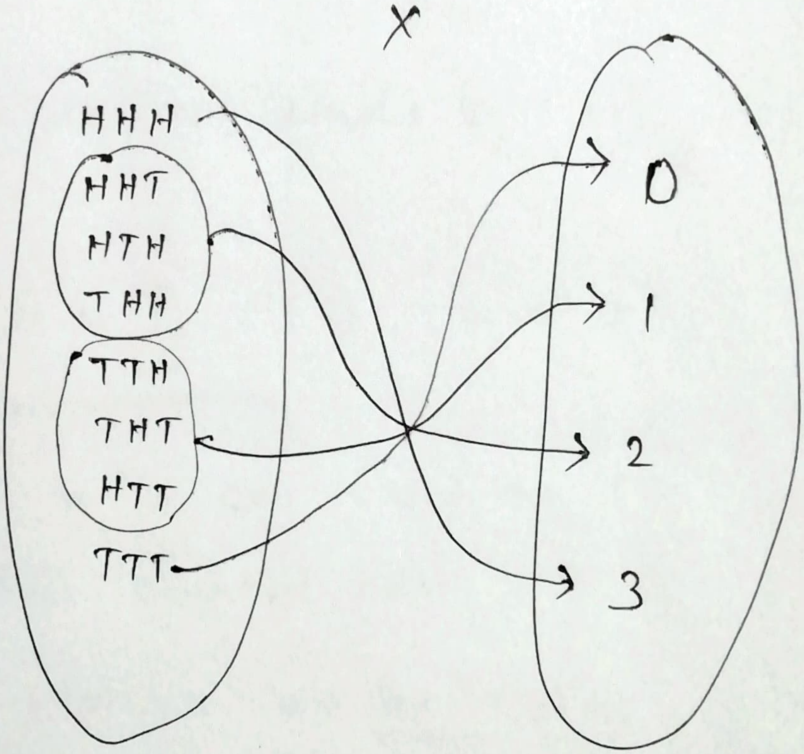
Example

Consider an experiment of tossing a fair coin three times. Define the random variable X to be the number of heads obtained in three tosses.

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

X : No. of heads obtained in three tosses

Range of r.v. X is $X = \{0, 1, 2, 3\}$



S	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
X(s)	3	2	2	2	1	1	1	0

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Random Variable

- Discrete
- Continuous
- Mixed (Discrete + Continuous)

Cumulative Distribution function (CDF)

The CDF of a r.v., X, denoted by $F_X(x)$, is defined by

$$F_X(x) = P_X(X \leq x), \text{ for all } x$$

Thm: The function $F(x)$ is a CDF if and only if the following three conditions hold:

(i) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

(ii) $F(x)$ is non-decreasing function of x

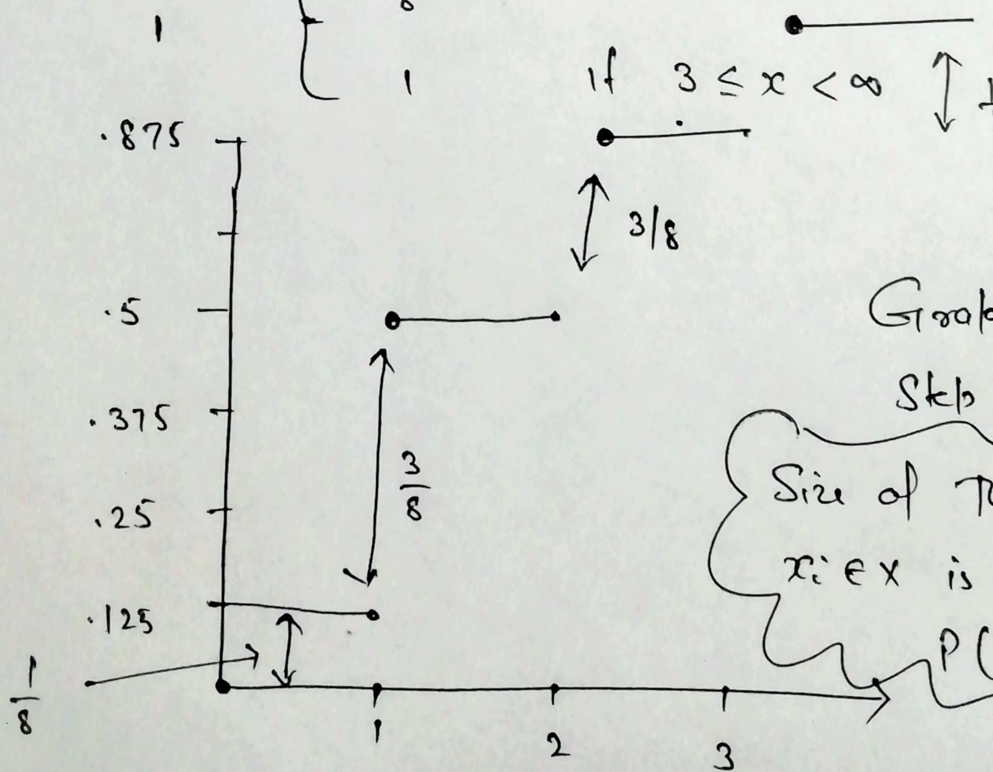
(iii) $F(x)$ is right-continuous; That is, for every number x_0, $\lim_{x \to x_0} F(x) = F(x_0)$

Eg Consider the example of tossing a fair coin three times

X : number of heads observed

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$F(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

$$F(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } 1 \le x < 2 \\ \frac{7}{8} & \text{if } 2 \le x < 3 \\ 1 & \text{if } 3 \le x < \infty \end{cases}$$



Graph is Step-function

Size of the jump at $x_i \in X$ is equal to $P(X=x_i)$