PROBABILITY AND STATISTICS (UCS401)

Lecture-5

(Conditional Probability with illustrations)

Introduction to Probability (Unit -II)

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③ Conditional probability with illustration

⊛ We know that how to calculate the probability of various events on the assumption that No information was available about the random experiment other than sample space.

If, however, it is known that an event B has occured already, the probability of finding the event A under the condition of even B, we need concept of Conditional probability.

Examples-: (i) What is the probability of being colorblind given that the person is female.

(ii) Two digits are selected from 1 to 9. If sum is even, find the probability that both digits are odd.

(iii) There are 12 balls in a bag, 8 red and 4 green. Three balls are drawn successively without replacement, find the probability that they are alternatively of the same order.

8R & 4G ⇒ you pick = 3

$\boxed{\begin{matrix} 3R \\ 3G \\ RRG \end{matrix}}$ ✗ . ⇓ Conditional probability.

RGR ✓

GRG ✓

Conditional probability :-

The probability of an event A, given that the event B has already occurred, is called the Conditional probability of A.

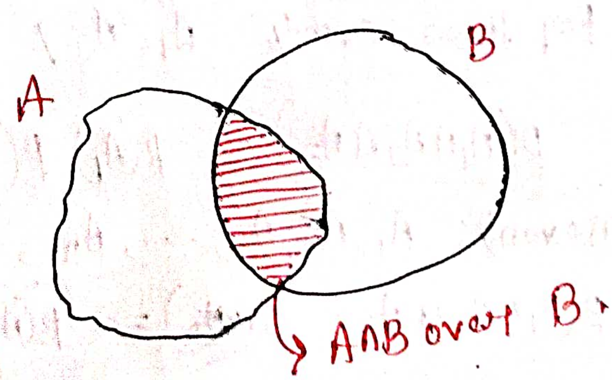
It is denoted by $P(A/B)$
↳ given that

Note that :- $P(A/B)$ read as probability of A given that B has occured.

"The vertical bar is read "given" and the events appearing to the right of the bar are those that you know have already occured."

"The Conditional probability of event A, given that B has occured, is defined as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad ; \quad P(B) \neq 0$$



Multiplication rule of probability :-

The probability of simultaneous happening of two events A and B is defined as;

$$P(A \cap B) = P(A)\, P(B/A) \;;\quad P(A) \neq 0$$

Where, $P(B/A)$ is the conditional probability of happening of event B under the condition that A has happened.

or

$$P(A \cap B) = P(B) \cap P(A/B) \;;\quad P(B) \neq 0.$$

Where, $P(A/B)$ is the conditional probability of happening of event A under the condition that B has happened.

Generalization of multiplication rule of probability :-

The multiplication theorem of probability can be extended to more than two events.

For three events A_1, A_2 & A_3, we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2);$$

For n events, $A_1, A_2, A_3, \ldots, A_n$, we have

$$P(A_1 \cap A_2 \cap A_3 \cdots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \cdots\cdots P(A_n/A_1 \cap A_2 \cdots \cap A_{n-1});$$

Question ① Suppose that in general population, there are 51% men and 49% women, and that proportions of the colorblind men and women are shown in the probability table:

	Men(B)	Women(W)
Colorblind	0.04	0.002
Not Colorblind	0.47	0.488

(i) If a person is drawn at random from this population and is found to be men, then what is probability that man is colorblind.

(ii) What is the probability of being colorblind, given that person is female.

Solution:-

	Men(B)	Women(W)	Total
Colorblind (A)	0.04	0.002	0.042
Not colorblind (CB)	0.47	0.488	0.958
Total	0.51	0.49	1.00

⇓ Must be one.

(i) If a person is drawn at random from this population and is found to be man (B), what is probability that man is colorblind (A), i.e., $P(A/B)$

Thus, required probability $= P(A/B)$

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.04}{0.51}$$ Ans

(ii) What is the probability of being colorblind (A), given that the person is female (W), i.e., $P(A/W)$

Thus, required probability $= P(A/W)$

$$= \frac{P(A \cap W)}{P(W)}$$

$$= \frac{0.002}{0.49}$$

$$= 0.004$$ Ans

Question (2) Consider an experiment of tossing two fair dice. Considering the two events A and B defined by

A: the sum of the numbers on the dice is 6 or 8.

B; at least one die shows 4.

Find $P(A/B)$

Solution :- A sample space of tossing two fair dice consists $6^2 = 36$ outcomes.

$$S = \{ (11)\ (12)\ (13)\ (14)\ (15)\ (16)$$
$$(21)\ (22)\ (23)\ (24)\ (25)\ (26)$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$(61)\ (62)\ (63)\ (64)\ (65)\ (66) \}$$

$$A = \{ (15), (24), (26)\ (33)\ (35)\ (42)\ (44)$$
$$(51)\ (53)\ (62) \}$$

$$B = \{ (14)\ (24)\ (34)\ (41)\ (42)\ (43)\ (44)\ (45)$$
$$(46)\ (54)\ (64) \}$$

Thus $A \cap B = \{ (24)\ (42)\ (44) \}$

$$P(A \cap B) = \frac{3}{36}$$

$$P(B) = \frac{11}{36}$$

Thus, required probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{3/36}{11/36}$$

$$\boxed{P(A/B) = \frac{3}{11}}$$

Ans

Question (3) Consider an experiment of tossing four sided dice. Consider the two events A and B defined by

$$A = \{(x,y) \mid \max(x,y) = m\}$$

$$B = \{(x,y) \mid \min(x,y) = 2\}$$

where $m = 1, 2, 3, 4$. Find $P(A/B)$.

Solution:- For two four sided dice

$$S = \{(11)\ (12)\ (13)\ (14)$$
$$(21)\ (22)\ (23)\ (24)$$
$$(31)\ (32)\ (33)\ (34)$$
$$(41)\ (42)\ (43)\ (44)\}$$

Exhaustive Cases $= 4^2 = 16$.

Given $B = \{(xy) \mid \min(x,y) = 2\}$

$B = \{(22)\ (23)\ (24)\ (32)\ (42)\}$

$P(B) = \frac{5}{16}$

Given $A = \{(xy) \mid \max(xy) = m\}$

For $m = 1$ $\quad A = \{(11)\}$ $\quad A \cap B = \phi$

For $m = 2$ $\quad A = \{(12)\ (21)\ (22)\}$ $\quad A \cap B = \{(22)\}$

For $m = 3$ $\quad A = \{(13)\ (31)\ (23)\ (32)\ (33)\}$

$A \cap B = \{(32)(23)\}$

$m = 4$ $\qquad A \cap B = \{(2\,1)\ (1\,2)\}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A/B) = \begin{cases} 0 & \text{if } m = 1 \\ \frac{1}{5} & \text{if } m = 2 \\ 2/5 & \text{if } m = 3 \\ 2/5 & \text{if } m = 4 \end{cases}$$

Ans

Question-[9] A box contains 4 bad and 6 good tubes. Two are drawn out from the box, at a time. Out of them is tested and found to be good. What is the probability that other one is also good.?

Solution-: Let

A : first tube is good.

B ; second tube is good.

Find $(B/A) = ?$

Given that

4 bad tubes

6 Good tubes

Total tubes = 10

One pick $\Rightarrow {}^{10}C_1$

two pick $\Rightarrow {}^{10}C_2$.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

A; first tube is good

$$P(A) = \frac{{}^6C_1}{{}^{10}C_1} = \frac{6}{10} = \frac{3}{5}$$

and $\underbrace{P(A \cap B)}_{\text{both good}} = \frac{{}^6C_2}{{}^{10}C_2} = \frac{\frac{6 \times 5}{2 \times 1}}{\frac{10 \times 9}{2 \times 1}} = \frac{1}{3}$

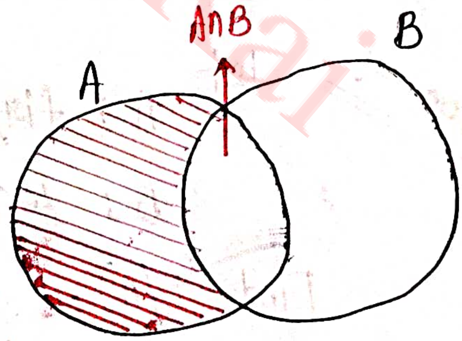
$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/3}{3/5} = \frac{5}{9}$$

Hence, the required probability

$$\boxed{P(B/A) = \frac{5}{9}}$$ Ans

Question-[5] If $P(A) = 0.5$, $P(B) = 0.3$, $P(A \cap B) = 0.15$.

Find $P(A/\overline{B}) = ?$



$$P(A/\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})}$$

$\because A \cap \overline{B} = A - (A \cap B)$

$A \cap \overline{B} = A - (A \cap B)$

$$P(A/\overline{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{0.5 - 0.15}{1 - 0.3} = \frac{0.35}{0.70}$$

$$\boxed{P(A/\overline{B}) = 0.5}$$ Ans

Question [6] A box of light bulbs contains 30 bulbs, of which 5 bulbs are defective.

If 3 of the bulbs are selected at random and taken out from the box in succession without replacement, find the probability that all three bulbs are defective.

Solution-: Define the events;

A; the first bulb selected is defective.

B; the second bulb selected is defective.

C; the third bulb selected is defective.

Thus, required probability that all three are defective

$$= P(A \cap B \cap C)$$

Now by multiplication rule

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$$

		Ist	IInd
defective bulbs =	5	4	3
undefective bulbs =	25	25	25
Total bulbs =	30	29	28

$$P(A \cap B \cap C) = \frac{5}{30} \times \frac{4}{29} \times \frac{3}{28} = \frac{1}{406}$$

Thus, required probability

$$\boxed{P(E) = \frac{1}{406}}$$

Question [7] The probability that a management trainee will remains with a company is 0.60. The probability that an employee earn more than Rs 10,000 per month is 0.50. The probability that an employee is a management trainee who remained with the company or who earn more than Rs 10,000 per month is 0.70. What is the probability that an employee earns more than Rs 10,000 per month, given that he is a management trainee who stayed with the company.

Solution :-

Let,

A ; management trainee will remains with the company

B ; an employee earn more than Rs 10,000 per month

Given $P(A) = 0.60$, $P(B) = 0.50$

$P(A \cup B) = 0.70$

Find $P(B/A) = \dfrac{P(B \cap A)}{P(A)}$ —— (1)

$\because$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.70 = 0.6 + 0.5 - P(A \cap B)$

$\boxed{P(A \cap B) = 0.4}$

Thus, required probability

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$\boxed{P(B/A) = 2/3}$$ Ans

Question (8) A box of 100 gaskets contains 10 gaskets with type A defects, 5 gaskets with type B defects and 2 gaskets with both types of defects.

Find the probability that

(i) A gasket to be drawn has a type B defect under the condition that it has a type A defect.

(ii) A gasket to be drawn has no type B defect under the condition that it has no type A defect.

Solution –: Let E_1; the gasket has type A defect.

E_2; the gasket has type B defect.

$$P(E_1) = \frac{10}{100} = 0.1$$

$$P(E_2) = \frac{5}{100} = 0.05$$

$$P(E_1 \cap E_2) = \frac{2}{100} = 0.02.$$

(i)

(i) $P(E_2/E_1) = \dfrac{P(E_2 \cap E_1)}{P(E_1)}$

$= \dfrac{0.02}{0.1}$

$\boxed{P(E_2/E_1) = 0.2}$ Ans

(ii) $P(\overline{E_2}/\overline{E_1}) = \dfrac{P(\overline{E_2} \cap \overline{E_1})}{P(\overline{E_1})}$

$= \dfrac{P(\overline{E_1 \cup E_2})}{P(\overline{E_1})}$

$P(\overline{E_2}/\overline{E_1}) = \dfrac{1 - P(E_1 \cup E_2)}{1 - P(E_1)}$. —(1)

$\because P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.10 + 0.05 - 0.02$

$\boxed{P(E_1 \cup E_2) = 0.13}$

Thus, the required probability

$P(\overline{E_2}/\overline{E_1}) = \dfrac{1 - 0.13}{1 - 0.1} = \dfrac{0.87}{0.90} = 0.97 \,(\approx)$

$\boxed{P(\overline{E_2}/\overline{E_1}) = 0.97 \approx}$ Ans

Question-[9] Two digits are selected at random from the digits 1 through 9. If sum is even, find the probability that both the digits are odd.

Solution:-

Let

A ; both the digits are odd.

B ; sum is even.

$$P(A/B) = ?$$

$\because$ two digits are selected at random out of 9 digits 1 through 9.

$\therefore$ Exhaustive Cases $= {}^{9}C_2 = 36$

Even digits are 2, 4, 6 & 8

Odd digits are 1 3 5 7 9

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{—— (1)}$$

$P(B) = P(\text{sum is even})$

$= P(\text{both the digits are even}) + P(\text{both the digits are odd})$

$$= \frac{{}^{4}C_2}{36} + \frac{{}^{5}C_2}{36} = \frac{6}{36} + \frac{5 \times 2}{36}$$

$$\boxed{P(B) = 2/3}$$

Thus,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(\text{both the digits are odd} \cap \text{sum is even})}{P(\text{sum is even})}$$

$$= \frac{P(\text{both the digits are odd})}{P(\text{sum is even})}$$

$$= \frac{{}^5C_2/36}{2/3}$$

$$= \frac{{}^5C_2}{36} \times \frac{3}{2} = \frac{5\cancel{10}}{\cancel{36}12} \times \frac{\cancel{3}}{\cancel{2}} = \frac{5}{12}$$

Thus, required probability

$$\boxed{P(A/B) = \frac{5}{12}}$$ Ans

Qus-10 There are 12 balls in a bag, 8 red and 4 green. Three balls are drawn successively without replacement. Find the probability that they alternatively of same color.

Solution-: Given that

Red balls = 8
Green balls = 4

Total balls = 12
you pick = 03

3R	RGR ✓
3G	GRG ✓
RRG	
GRR	
GGR	
RGG	

Here, two cases occurs:

(i) The balls are red, green and red
(ii) The balls are green, red and green

disjoint.

Thus, required probability

$$P(E) = P(i) + P(ii) \quad \text{——} (1)$$

$$P(i) = P(R \cap G \cap R)$$

I^st	II^nd	III^rd pick
8R	7R	7R
4G	4G	3G
12	11	10

$$= \frac{8}{12} \times \frac{4}{11} \times \frac{7}{10}$$

$$\boxed{P(i) = \frac{224}{1320}}$$

$$P(ii) = P(G \cap R \cap G)$$

I	II	III
8R	8R	7R
4G	3G	3G
12	11	10

$$= \frac{4}{12} \times \frac{8}{11} \times \frac{3}{10}$$

$$\boxed{P(ii) = \frac{96}{1320}}$$

(1) $\Rightarrow$ $$P(E) = \frac{224}{1320} + \frac{96}{1320} = \frac{320}{1320}$$

$$\boxed{P(E) = \frac{8}{33}}$$

Ans

Question-(11) A box contains 5 white and 3 black balls, and another bag contains 4 white and 5 black balls. From any of these bags, a single draw of two balls is made. Find the probability that one of them would be white and other black ball.

Solution-: Let

A ; first bag $[5W \,\&\, 3B]$ Exhaustive cases $= {}^8C_2$

$\Downarrow$

you pick $= 2$

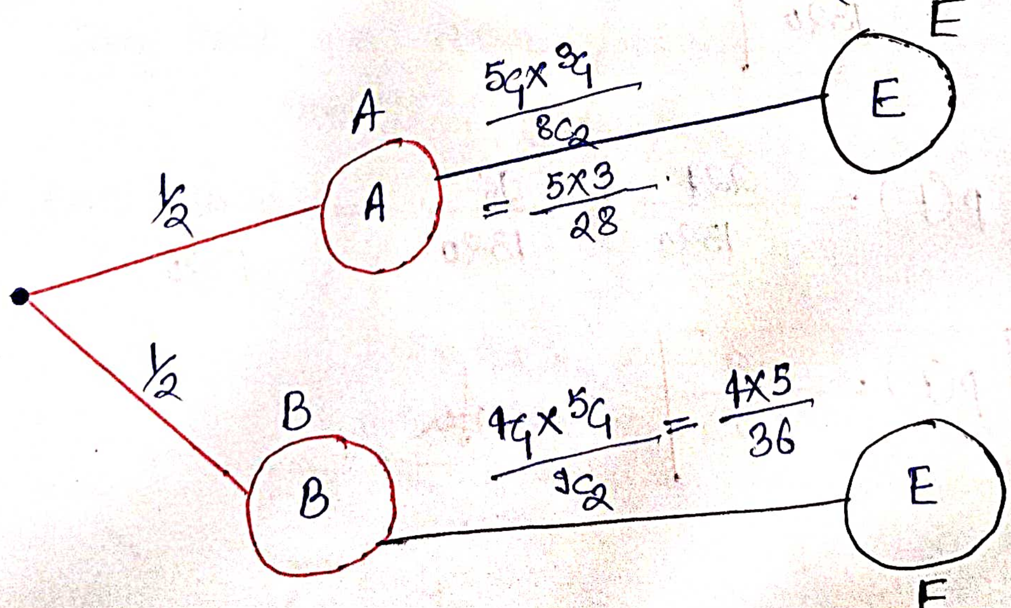
B ; second bag $[4W \,\&\, 5B]$ Exhaustive cases $= {}^9C_2$

you pick $= 2$

E ; One of them would be white and other black ball.

Since both the bag selected at random

Thus $P(A) = \frac{1}{2}$ & $P(B) = \frac{1}{2}$



Thus, required probability

$$= \frac{1}{2}\left(\frac{15}{28}\right) + \frac{1}{2}\left(\frac{20}{36}\right) \quad \underline{\text{Ans}}$$

Question-(12) A pair of die is rolled. If the sum of two digits is 9. Find the probability that one of the dice showed 3.

Solution-: Let A ; sum of two digits is 9.

B ; one of dice showed 3.

Find $P(B/A) = ?$

Die is rolled twice

$$S = \{ (11)\ (12)\ (13)\ (14)\ (15)\ (16)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$(61)\ (62)\ (63)\ (64)\ (65)\ (66) \}$$

So Exhaustive cases $= 6^2 = 36$

Now $A = \{ (36)\ (45)\ (54)\ (63) \}$

$B = \{ (13)\ (23)\ (31)\ (32)\ (33)\ (34)\ (35)\ (36)$
$(43)\ (53)\ (63) \}$

$A \cap B = \{ (36)\ (63) \}$ $\quad P(A \cap B) = 2/36$.

Thus, required probability

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{2/36}{4/36}$$

$$\boxed{P(B/A) = 1/2}$$ Ans