PROBABILITY AND STATISTICS (UCS401)

Lecture-16

(Joint probability distribution, Marginal and conditional distribution)

Two-dim. r.v.'s and Joint Distributions (Unit -V)

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Two dimensional random variable – Joint probability distributions, Marginal and Conditional dists.

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Two-dimensional random variables :

Let X and Y be random variables on a sample space S with respective image sets

$$X = \{x_1, x_2, x_3, \cdots, x_n\}$$

and $$Y = \{y_1, y_2, y_3, \cdots y_m\}$$

Make a product set

$$X \times Y = \{x_1, x_2, x_3, \cdots, x_n\} \times \{y_1, y_2, y_3, \cdots, y_m\}$$

and denote it as an ordered pair (x_i, y_j)

$i = 1, 2, 3, \cdots, n$ & $j = 1, 2, 3, \cdots, m$.

to the $P(X = x_i, Y = y_j)$

We write it as $p(x_i, y_j)$.

The function p on $X \times Y$ defined by

$$p_{ij} = p(x_i, y_j) = P(X = x_i, Y = y_j)$$

is called joint probability function of X and Y

and is represented in the tabular form as :

$X \backslash Y$	y_1	y_2	y_3	$\cdots$	y_m
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1m}
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2m}
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
x_n	p_{n1}	p_{n2}	p_{n3}	$\cdots$	p_{nm}

In particular

$$p_{11} = p(x_1, y_1) = P(X = x_1,\ Y = y_1)$$

* If (X, Y) is a two-dimensional discrete random variable, then the joint density function of X, Y also called joint probability mass function of X, Y denoted by p_{XY} is defined as :

$$p(x, y) = \begin{cases} P(X = x, Y = y) & \text{for all value of } (x, y) \text{ of } (X, Y) \\ 0 & \text{otherwise.} \end{cases}$$

(a) For discrete:

(i) $p(x,y) \geq 0$

(ii) $\sum_{x} \sum_{y} p(x,y) = 1$

(b) For Continuous:

(i) $f(x,y) \geq 0$

(ii) $\int_{x} \int_{y} f(x,y)\, dy\, dx = 1$

Marginal density function:

For two discrete r.v's X and Y the marginal function of X, denoted by $p_X(x)$ or f_X or $\underset{y \text{ varies}}{P(X=x)}$, is given as

(a) discrete:

$$p_X(x) = \sum_{y} P(X=x, Y=y) \quad (x \text{ fixed}, y \text{ varies})$$

(b) Continuous:

$$f_X(x) = \int_{y} f(x,y)\, dy \, ;$$

* The marginal of Y, denoted by $p_Y(y)$ or p_y or $\underset{x \text{ varies}}{P(Y=y)}$, is given as

(a) <u>Discrete :</u>

$$p_Y(y) = \sum_x P(X=x, Y=y) \quad (y \text{ fixed}, x \text{ varies})$$

(b) <u>Continuous :</u>

$$f_Y(y) = \int_x f(x,y)\, dx \quad (y \text{ fixed}, x \text{ varies})$$

Moreover, the joint probability function of X and Y is usually represented in the form of the following table :

$$\sum_{i=1}^{n} \sum_{j=1}^{m} P(X=x_i, Y=y_j) = 1$$

$$\sum_{j=1}^{m} f_Y(y_j) = \sum_{i=1}^{n} f_X(x_i) = 1$$

$X \backslash Y$	y_1	y_2	$\cdots$	y_j	$\cdots$	$\cdots$	y_m	$f_X(x)$
x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$	$\cdots$	p_{1m}	$f_X(x_1)$
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$	$\cdots$	p_{2m}	$f_X(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$	$\cdots$	p_{im}	$f_X(x_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	$\vdots$	p_{nj}	$\vdots$	$\vdots$	p_{nm}	$f_X(x_n)$
$f_Y(y)$	$f_Y(y_1)$	$f_Y(y_2)$	$\cdots$	$f_Y(y_j)$	$\cdots$	$\cdots$	$f_Y(y_m)$	1

From the following table we conclude that

$$p_{ij} = p(x_i, y_j) = P(X = x_i, Y = y_j)$$

$$= P(X = x_i \cap Y = y_j)$$

is the joint discrete density function of X and Y

(ii) $$f_X(x_i) = P(X = x_i) = \sum_{j=1}^{m} P(X = x_i, Y = y_j)$$

→ Marginal discrete density function of X

(iii) $$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} P(X = x_i, Y = y_j)$$

→ Marginal discrete density function of Y.

Conditional density function :

* The conditional density function for X given Y, denoted by $p(X/Y)$ or $p_{X/Y}$ is defined as

$$p(X/Y) = \frac{P(X=x \cap Y=y)}{P(Y=y)}$$

$$\boxed{p(X/Y) = \frac{P(X=x,\ Y=y)}{p_Y(y)}}$$

$$\Rightarrow \quad p(X/Y) = \frac{p(x,y)}{p_Y(y)}$$

$p_Y(y) \longrightarrow$ Marginal density function of y.

Similarly, the conditional density function for Y given X, denoted by $p(Y/X)$ or $p_{Y/X}$ is defined as

$$p(Y/X) = \frac{P(Y=y \cap X=x)}{P(X=x)}$$

$$\boxed{p(Y/X) = \frac{P(X=x,\ Y=y)}{p_X(x)}}$$

$$\Rightarrow \quad p(Y/X) = \frac{p(x,y)}{p_X(x)}$$

$p_X(x) \longrightarrow$ Marginal density function of x.

Independent random variables :

Two random variable X and Y are said to be independent if

$$p(x,y) = p_X(x) \cdot p_Y(y)$$

i.e., $$P(X=x, Y=y) = p(x,y) = p_X(x) \cdot p_Y(y).$$

where, $p_X(x) \longrightarrow$ Marginal function of X.

$p_Y(y) \longrightarrow$ Marginal function of Y.

Question :- Consider two random variables X and Y such that

$$P(X=x, Y=y) = \begin{cases} 1/3 & x=-1,\ y=0 \\ 1/3 & x=0,\ y=1 \\ 1/3 & x=1,\ y=0 \\ 0 & o/w \end{cases}$$

Find (i) Marginal distribution functions of X and Y.

(ii) Conditional distribution function of X given $Y=1$.

Solution :

X \ Y	0	1	$p_X(x)$	Marginal function of X
-1	1/3	0	1/3	$p_X(-1)$
0	0	1/3	1/3	$p_X(0)$
1	0	1/3	1/3	$p_X(1)$
$p_Y(y)$	1/3	2/3	1	
	$p_Y(0)$	$p_Y(1)$		

Marginal function of Y.

$$p_X(-1) = \sum_{y=\{0,1\}} P(X=-1, Y=y) = \frac{1}{3},$$

$$p_X(0) = \sum_{y\in\{0,1\}} P(X=0, Y=y) = \frac{1}{3},$$

$$p_X(1) = \sum_{y\in\{0,1\}} P(X=1, Y=y) = \frac{1}{3}$$

i.e., Marginal distribution function of X

X	-1	0	1
$p_X(x)$	1/3	1/3	1/3

$$p_Y(0) = \sum_{x \in \{-1,0,1\}} P(X=x, Y=0) = 1/3$$

$$p_Y(1) = \sum_{x \in \{-1,0,1\}} P(X=x, Y=1) = 2/3$$

i.e., Marginal distribution function of Y

Y	0	1
$p_Y(y)$	$1/3$	$2/3$

(ii) Conditional distribution of X given $Y=1$

$$P(X \mid Y=1) = \frac{P(X=x, Y=1)}{p_Y(1)} \; ; \; x = -1, 0, 1$$

For

$x=-1$ $\quad P(X=-1 \mid Y=1) = \dfrac{P(X=-1, Y=1)}{p_Y(1)} = \dfrac{0}{2/3} = 0$

For $x=0$ $\quad P(X=0 \mid Y=1) = \dfrac{P(X=0, Y=1)}{p_Y(1)} = \dfrac{1/3}{2/3} = 1/2$

For $x=1$ $\quad P(X=1 \mid Y=1) = \dfrac{P(X=1, Y=1)}{p_Y(1)} = \dfrac{1/3}{2/3} = 1/2$

$X \mid Y=1$	-1	0	1
$P(X \mid Y=1)$	0	$1/2$	$1/2$

Question: The joint distribution of X and Y is given by

$$p(x,y) = \frac{x+y}{21} \; ; \quad x = 1,2,3 \quad y = 1,2.$$

Find the marginal distribution of X and Y.

Also, find the conditional distribution of Y given $X=3$.

Solution: We have given

$$p(x,y) = \frac{x+y}{21} \qquad x = 1,2,3 \quad y = 1,2$$

Marginal of X

$X \backslash Y$	1	2	$p_X(x)$	
1	2/21	3/21	5/21	$p_X(1)$
2	3/21	4/21	7/21	$p_X(2)$
3	4/21	5/21	9/21	$p_X(3)$
$p_Y(y)$	9/21	12/21	1	
	$p_Y(1)$	$p_Y(2)$		

Marginal of Y

$$p_X(1) = \sum_{y\in\{1,2\}} P(X=1, Y=y) = 5/21$$

$$p_X(2) = \sum_{y\in\{1,2\}} P(X=2, Y=y) = \frac{7}{21}$$

$$p_X(3) = \sum_{y \in \{1,2\}} P(X=3, Y=y) = \frac{9}{21}$$

i.e., Marginal distribution function of X

X	1	2	3
$p_X(x)$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$

$$p_Y(1) = \sum_{x \in \{1,2,3\}} P(X=x, Y=1) = \frac{9}{21}$$

$$p_Y(2) = \sum_{x \in \{1,2,3\}} P(X=x, Y=2) = \frac{12}{21}$$

i.e., Marginal distribution function of Y

Y	1	2
$p_Y(y)$	$\frac{9}{21}$	$\frac{12}{21}$

(ii) Conditional probability of Y given X = 3

$$P(Y \mid X=3) = \frac{P(Y=y \cap X=3)}{P(X=3)}$$

$$P(Y \mid X=3) = \frac{P(Y=y, X=3)}{p_X(3)} \quad y = 1, 2.$$

For $y=1$:

$$P(Y=1|X=3) = \frac{P(Y=1, X=3)}{p_X(3)}$$

$$P(Y=1|X=3) = \frac{4/21}{9/21} = 4/9$$

For $y=2$

$$P(Y=2|X=3) = \frac{P(Y=2, X=3)}{p_X(3)}$$

$$= \frac{5/21}{9/21} = 5/9$$

$$P(Y=2|X=3) = 5/9$$

$Y/X=3$	1	2
$P(Y/X=3)$	$4/9$	$5/9$

Ans

Question: The joint density function of X and Y is given by

$$p(x,y) = k(2x+y) \; ; \; x = 0,1,2 \; ; \; y = 0,1,2$$

Find (i) the value of k,

(ii) Marginal distribution of X and Y,

(iii) Are X and Y independent variables?

(iv) Find the conditional distribution of Y for X = x.

Solution: Given that

$$p(x,y) = k(2x+y) \quad x = 0,1,2 \quad y = 0,1,2.$$

X \ Y	0	1	2
0	0	1k	2k
1	2k	3k	4k
2	4k	5k	6k

For joint density function, we have

$$\sum_x \sum_y p(x,y) = 1$$

$\Rightarrow \quad 0 + 1k + 2k + 2k + 3k + 4k + 4k + 5k + 6k = 1$

$\Rightarrow \quad 27k = 1$

$\Rightarrow \quad \boxed{k = \frac{1}{27}}$

Marginal of X

X \ Y	0	1	2	$p_X(x)$	
0	0	1/27	2/27	3/27	$p_X(0)$
1	2/27	3/27	4/27	9/27	$p_X(1)$
2	4/27	5/27	6/27	15/27	$p_X(2)$
$p_Y(y)$	6/27	9/27	12/27	1	
	$p_Y(0)$	$p_Y(1)$	$p_Y(2)$		

Marginal of Y

$$p_X(0) = \sum_{y \in \{0,1,2\}} P(X=0, Y=y) = \frac{3}{27}$$

$$p_X(1) = \sum_{y \in \{0,1,2\}} P(X=1, Y=y) = \frac{9}{27}$$

$$p_X(2) = \sum_{y \in \{0,1,2\}} P(X=2, Y=y) = \frac{15}{27}$$

X	0	1	2
$p_X(x)$	3/27	9/27	15/27

$\Rightarrow$ Marginal dist of X.

(ii) Marginal distribution of Y

$$p_Y(0) = \sum_{x \in \{0,1,2\}} P(X=x, Y=0) = \frac{6}{27}$$

$$p_Y(1) = \sum_{x \in \{0,1,2\}} P(X=x, Y=1) = \frac{9}{27}$$

$$p_Y(2) = \sum_{x \in \{0,1,2\}} P(X=x, Y=2) = \frac{12}{27}$$

y	0	1	2
$p_Y(y)$	$6/27$	$9/27$	$12/27$

(iii) Are X and Y are independent variables?

we know that two random variables X and Y are said to be independent if

$$p(x,y) = p_X(x)\, p_Y(y) \quad \text{for all pair } (x,y)$$

Since $p(0,0) \neq p_X(0) \cdot p_Y(0)$

$\because p(0,0) = P(X=0, Y=0) = 0$

$p_X(0) = 3/27$,

$p_Y(0) = 6/27$

Thus, X and Y are not independent variables.

Note that

$$p(0,1) = p_X(0) \cdot p_Y(1)$$

(iv) The conditional distribution for Y given $X=x$

$$P(Y|X=x) = \frac{P(Y=y \cap X=x)}{P(X=x)}$$

$$P(Y|X=x) = \frac{P(Y=y, X=x)}{p_X(x)}$$

Case-I when $x=0$

$X \backslash Y$	0	1	2	$p_X(x)$
0	0	$1/27$	$2/27$	$3/27$
1	$2/27$	$3/27$	$4/27$	$9/27$
2	$4/27$	$5/27$	$6/27$	$15/27$
$p_Y(y)$	$6/27$	$9/27$	$12/27$	1

$$P(Y|X=0) = \frac{P(Y=y, X=0)}{p_X(0)}$$

For $y=0$

$$P(Y=0|X=0) = \frac{P(Y=0, X=0)}{p_X(0)} = \frac{0}{3/27} = 0$$

$y=1$ $$P(Y=1|X=0) = \frac{P(Y=1, X=0)}{p_X(0)} = \frac{1/27}{3/27} = 1/3$$

$y=2$ $$P(Y=2|X=0) = \frac{P(Y=2, X=0)}{p_X(0)} = \frac{2/27}{3/27} = 2/3$$

$Y\|X=0$	0	1	2
$P(Y\|X=0)$	0	$1/3$	$2/3$

Case-II When $x = 1$

$x \backslash y$	0	1	2	$p_X(x)$
0	0	$1/27$	$2/27$	$3/27$
1	$2/27$	$3/27$	$4/27$	$9/27$
2	$4/27$	$5/27$	$6/27$	$15/27$
$p_Y(y)$	$6/27$	$9/27$	$12/27$	(1)

$$p(y|x=1) = \frac{p(y=y, x=1)}{p_X(1)}$$

When $y=0$

$$p(y=0|x=1) = \frac{p(y=0, x=1)}{p_X(1)} = \frac{2/27}{9/27} = 2/9$$

For $y=1$

$$p(y=1|x=1) = \frac{p(y=1, x=1)}{p_X(1)} = \frac{3/27}{9/27} = 1/3$$

For $y=2$

$$p(y=2|x=1) = \frac{p(y=2, x=1)}{p_X(1)} = \frac{4/27}{9/27} = 4/9$$

$y\|x=1$	0	1	2
$p(y\|x=1)$	$2/9$	$1/3$	$4/9$

Case-III For $x=2 \Rightarrow p(y=y|x=2) = \dfrac{p(y=y, x=2)}{p_X(2)}$

||ly

$y\|x=2$	0	1	2
$p(y\|x=2)$	$4/15$	$5/15$	$6/15$

Ans

Question : The joint density function of X and Y is given by

$$p(x,y) = \frac{x+y}{21} \quad ; \quad x = 1,2,3 \quad y = 1,2.$$

Find the mean of X, Y, XY, X+Y.

Solution :- We know that

- Mean of X

$$E(X) = \sum_{x} x\, p_X(x)$$

where $p_X(x)$ is a marginal density of X.

- Mean of Y

$$E(Y) = \sum_{y} y\, p_Y(y)$$

where, $p_Y(y)$ is a marginal density of Y.

- Mean of XY and X+Y

$$E(XY) = \sum_{x} \sum_{y} xy\, p(x,y)$$

$$E(X+Y) = \sum_{x} \sum_{y} (x+y)\, p(x,y).$$

where $p(x,y) = P(X=x, Y=y)$ is a joint density function of X and Y.

Given that

$$p(x,y) = \frac{x+y}{21} \qquad \left. \begin{array}{l} x = 1,2,3 \\ y = 1,2 \end{array} \right]$$

$X \backslash Y$	1	2	$p_X(x)$	
1	$2/21$	$3/21$	$5/21$	$p_X(1)$
2	$3/21$	$4/21$	$7/21$	$p_X(2)$
3	$4/21$	$5/21$	$9/21$	$p_X(3)$
$p_Y(y)$	$9/21$	$12/21$	1	
	$p_Y(1)$	$p_Y(2)$		

$\therefore$ Marginal distribution function of X

X	1	2	3
$p_X(x)$	$5/21$	$7/21$	$9/21$
	$p_X(1)$	$p_X(2)$	$p_X(3)$

Thus, Mean of X is

$$E(X) = \sum_x x\, p_X(x) = 1 \times p_X(1) + 2 \times p_X(2) + 3 \cdot p_X(3)$$

$$= 1 \times \frac{5}{21} + 2 \times \frac{7}{21} + 3 \times \frac{9}{21}$$

$$\boxed{E(X) = \frac{46}{21}}$$

$\therefore$ Marginal distribution function of Y

Y	1	2
$p_Y(y)$	$\frac{9}{21}$	$\frac{12}{21}$
	$p_Y(1)$	$p_Y(2)$

Thus, Mean of Y is

$$E(Y) = \sum_{y} y\,p_Y(y) = 1 \times p_Y(1) + 2 \times p_Y(2)$$

$$= 1 \times \frac{9}{21} + 2 \times \frac{12}{21}$$

$$= \frac{\cancel{33}^{11}}{\cancel{21}_{7}} = \frac{11}{7}$$

$$\boxed{E(Y) = \frac{11}{7}}$$

$X \backslash Y$	1	2	$f_X(x)$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{5}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{7}{21}$
3	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{9}{21}$
$p_Y(y)$	$\frac{9}{21}$	$\frac{12}{21}$	1

$$E(XY) = \sum_{x}\sum_{y} xy\,p(x,y)$$

$$= (1)(1)\,P(X=1, Y=1)$$
$$+ (1)(2)\,P(X=1, Y=2) + \cdots$$
$$+ (3)(2)\,P(X=3, Y=2).$$

$$= (1)(1)\left(\frac{2}{21}\right) + (1)(2)\frac{3}{21} + \cdots + (3)(2)\frac{5}{21} = \frac{24}{7}$$

and $$E(X+Y) = \sum_{x}\sum_{y} (x+y)\,p(x,y) = (1+1)\,P(X=1, Y=1)$$

$$+ (1+2)\,P(X=1, Y=2) + \cdots + (3+2)\,P(X=3, Y=2)$$

$$= 2 \times \frac{2}{21} + 3 \times \frac{3}{21} + \cdots + 5 \times \frac{5}{21}$$

$$\boxed{E(X+Y) = \frac{79}{21}}$$

Question: For a bivariate probability distribution of (X, Y) given below;

$X \backslash Y$	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

Find (i) $P(X \leq 1)$

(ii) $P(Y \leq 3)$

(iii) $P(X + Y \leq 4)$

Marginal of X

Solution:

$X \backslash Y$	1	2	3	4	5	6	$p_X(x)$
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$	$1/4 = p_X(0)$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$	$5/8 = p_X(1)$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$	$1/8 = p_X(2)$
$p_Y(y)$	$3/32$ ‖	$3/32$	$11/64$	$13/64$	$3/16$	$1/4$	1
Marginal of Y	$p_Y(1)$	$p_Y(2)$	$p_Y(3)$	$p_Y(4)$	$p_Y(5)$	$p_Y(6)$	

(i) $P(X \leq 1) = P(X=0) + P(X=1)$

x fixed y varies

$\Downarrow$ Marginal distribution of X

$= p_X(0) + p_X(1)$

$= 1/4 + 5/8 = 7/8$

$$\boxed{P(X \leq 1) = 7/8}$$

(ii) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3)$

y fixed, x varies (Marginal dist. of y)

$= p_Y(1) + p_Y(2) + p_Y(3)$

$= 3/32 + 3/32 + \frac{11}{64}$

$$\boxed{P(Y \leq 3) = \frac{23}{64}}$$

(iii) $P(X+Y \leq 4)$

The pair of (x, y) which satisfy $X+Y \leq 4$ are

(0 1) (0 2) (0 3) (0 4)
(1, 1) (1 2) (1 3)
(2 1) (2 2)

Thus, required probability

$$P(X+Y \leq 4) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=0, Y=4)$$
$$+ P(X=1, Y=1) + P(X=1, Y=2)$$
$$+ P(X=1, Y=3) + P(X=2, Y=1)$$
$$+ P(X=2, Y=2).$$

$$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$$

$$\boxed{P(X+Y \leq 4) = \frac{13}{32}}$$

Ans

Question: Three balls are drawn at random from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn. Find the joint probability distribution of (X, Y).

Solution: We have
2 white
3 red
4 black $\Rightarrow$ Exhaustive

Total balls = 9 balls   Cases $= {}^9C_3$.
you pick = 3 balls

Since three balls are drawn at random from a box.

X be a number of white balls, so $X = 0, 1, 2$

Y be a number of red balls, so $Y = 0, 1, 2, 3$.

$$p(0,0) = P(0W, 0R)$$
$$= P(3 \text{ black balls}) = \frac{{}^4C_3}{{}^9C_3} = \frac{1}{21}$$

$$p(0,1) = P(0W, 1R, 2B) = P(1R, 2B)$$
$$= \frac{{}^3C_1 \times {}^4C_2}{{}^9C_3} = \frac{3}{14}.$$

$$p(1,3) = P(1W, 3R)$$
$$= 0 \quad (\text{since only three balls are drawn})$$

$$p(2,3) = P(2W, 3R)$$
$$= 0 \quad (\text{since only 3 balls are drawn})$$

$$p(0,2) = P(0W, 2R, 1B) = P(2R, 1B)$$
$$= \frac{{}^3C_2 \times {}^4C_1}{{}^9C_3} = \frac{\cancel{3\times2\times4}}{\cancel{2\times1}} \cdot \frac{\cancel{3\times2\times1}}{\cancel{9\times8\times7}} = \frac{1}{7}$$

$$p(0,2) = P(0W, 2R, 1B) = \frac{1}{7}$$

$$p(0,3) = P(0W, 3R) = \frac{{}^3C_3}{{}^9C_3} = \frac{\cancel{3\times2}}{3\,\cancel{3}\times\cancel{8}\times7} = \frac{1}{84}$$

$$p(0,3) = \frac{1}{84}$$

$$p(1,0) = P(1W, 0R, 2B)$$

$$= \frac{{}^2C_1 \times {}^4C_2}{{}^9C_3} = \frac{\cancel{2\times1} \times \cancel{4}\times3}{\cancel{2\times1}} \cdot \frac{(3\times2)}{9\times7\times8}$$

$$p(1,0) = \frac{1}{7}$$

$$p(1,1) = P(1W, 1R, 1B) = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2}{7}$$

Illy calculate others

Hence, the required joint probability distribution of (X,Y).

$Y(R)$ \ $X(W)$	0	1	2
0	$\frac{1}{21}$	$\frac{1}{7}$	$\frac{1}{21}$
1	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{1}{28}$
2	$\frac{1}{7}$	$\frac{1}{14}$	0
3	$\frac{1}{84}$	0	0

Question: If X denotes the number of kings and Y denotes the number of aces when two cards are drawn at random without replacement from a pack of 52 cards, find

(i) The joint probability distribution of (X, Y).

(ii) Marginal distribution of X and Y.

(iii) $P(X=2 \mid Y=1)$

(iv) $P(X<2 \mid 0<Y<2)$

(v) $P(1 \leq X \leq 2 \mid Y=0,2)$.

Solution:

$\left.\begin{array}{l} X: \text{Number of kings} \\ Y: \text{Number of aces} \end{array}\right]$

(i) Joint P.M.F.

Marginal dist. of Y

$Y \backslash X$	0	1	2	$p_Y(y)$
0	$\frac{^{44}C_2}{^{52}C_2} = \frac{476}{663}$	$\frac{^{44}C_1 \times {}^4C_1}{^{52}C_2} = \frac{88}{663}$	$\frac{^4C_2}{^{52}C_2} = \frac{3}{663}$	$\frac{561}{663}$ $p_Y(0)$
1	$\frac{^{44}C_1 \times {}^4C_1}{^{52}C_2} = \frac{88}{663}$	$\frac{^4C_1 \times {}^4C_1}{^{52}C_2} = \frac{8}{663}$	0	$\frac{96}{663}$ $p_Y(1)$
2	$\frac{^4C_2}{^{52}C_2} = \frac{3}{663}$	0	0	$3/663$ $p_Y(2)$
$p_X(x)$	$188/221 = p_X(0)$	$33/221 = p_X(1)$	$1/221 = p_X(2)$	1

Marginal dist. of X

(iii)

$$P(X=2 \mid Y=1) = \frac{P(X=2 \cap Y=1)}{P(Y=1)}$$

$$= \frac{P(X=2, Y=1)}{p_Y(1)}$$

$$= \frac{0}{96/663} = 0$$

$$\boxed{P(X=2 \mid Y=1) = 0}$$

(iv) $P(X<2 \mid 0<Y<2)$

$$= \frac{P(X=0,1 \cap Y=1)}{P(Y=1)}$$

$$= \frac{P(X=0, Y=1) + P(X=1, Y=1)}{p_Y(1)}$$

$$= \frac{88/663 + 8/663}{96/663} = 1$$

$$\Rightarrow \boxed{P(X<2 \mid 0<Y<2) = 1}$$

(v) $P(1 \leq X \leq 2 \mid Y = 0, 2)$

$$= \frac{P(X = 1, 2 \cap Y = 0, 2)}{P(Y = 0, 2)}$$

$$= \frac{P(X=1, Y=0) + P(X=1, Y=2) + P(X=2, Y=0) + P(X=2, Y=2)}{\underset{p_Y(0)}{\underset{\parallel}{P(Y=0)}} + \underset{p_Y(2)}{\underset{\parallel}{P[Y=2]}}}$$

$$= \frac{\frac{88}{663} + 0 + \frac{3}{663} + 0}{\frac{564}{663} + \frac{3}{663}}$$

$$= \frac{91/663}{567/663} = \frac{13}{81} \text{ Ans}$$

$$\boxed{P(1 \leq X \leq 2 \mid Y = 0, 2) = \frac{13}{81}}$$

Ans