PROBABILITY AND STATISTICS (UCS401)

Lecture-19

Transformation of random variable-Discrete and Continuous r.v.'s Random Variables and their Special Distributions(Unit –III & IV)

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Transformation of discrete random variables

Let X be a discrete random variable with p.m.f. $p(x)$.
If the transformation $y = g(x)$ is defined for all values with in the range of X.

Type 1: when $Y = g(x)$ is one-to-one	Type 2: when relation $Y = g(x)$ is not one to one.

Case–I when relation $Y = g(x)$ is one to one.

Question–: Let X be a random variable with probability distribution

$$f(x) = \begin{cases} 1/3 & ; \quad x = 1, 2, 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the probability distribution of the random variable $Y = 2X - 1$.

Solution:- The density function of X is

X	1	2	3
$P(x)$	$1/3$	$1/3$	$1/3$

When $x=1$; $y=1$
When $x=2$; $y=3$ one to one
When $x=3$; $y=5$

Thus, the function $Y=2X-1$ is one to one.

Hence the p.d.f of Y obtained as

Y	1	3	5
$P(y)$	$1/3$	$1/3$	$1/3$

Question:- Let X be the number of heads when two coins are tossed simultaneously. Find the probability distribution of $Y=(X+1)^3$.

Solution:- $S=\{HH\ HT\ TH\ TT\}$

X : Number of heads

X : 0, 1, 2.

$P(X=0) = 1/4, \quad P(X=1) = 1/2, \quad P(X=2) = 1/4$

X	0	1	2
$P(X=x)$	$1/4$	$1/2$	$1/4$

$$\left.\begin{array}{lll} \text{when } x=0; & & y=1 \\ \text{when } x=1; & & y=8 \\ \text{when } x=2; & & y=27 \end{array}\right\} \text{one to one}$$

Thus, the function $y=(x+1)^3$ is one to one.

y	1	8	27
$P(Y=y)$	$1/4$	$1/2$	$1/4$

Question:- Let X be binomial random variable with density function

$$P(x) = {}^3C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{3-x} \quad ; x = 0, 1, 2, 3$$

Find probability distribution of the random variable $Y = X^2$.

Solution:- We have given

$$P(x) = {}^3C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{3-x} \quad ; x = 0, 1, 2, 3$$

The p.d.f. of X is

X	0	1	2	3
$P(x)$	$\frac{27}{125}$	$\frac{54}{125}$	$36/125$	$8/125$

When $x = 0$; $y = 0$

When $x = 1$; $y = 1$

When $x = 2$; $y = 4$

When $x = 3$; $y = 9$

One to one

Thus, the function $y = x^2$ is one to one

Y	0	1	4	9
$P(y)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

Question: Let X be a geometric random variable with the probability distribution

$$f(x) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}; \quad x = 1, 2, 3, 4, \ldots$$

Find the probability distribution of the variable $Y = X^2$.

Solution: Since $Y = X^2$ is one to one and hence the probability distribution of Y is

$$f(y) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{\sqrt{y}-1}; \quad y = 1, 4, 9, 16, \ldots$$

Case-II When the relation $Y = g(X)$ is NOT one to one.

Question: Let X be the number of heads in two tosses of a fair coin. Find the probability distribution of $Y = (X-1)^2$.

Solution: $S = \{HH, HT, TH, TT\}$

X: Number of heads

The probability density function of X is

X	0	1	2
$P(X)$	$1/4$	$1/2$	$1/4$

When $x=0$; $y=1$

When $x=1$; $y=0$ — Not one to one

When $x=2$; $y=1$

Thus, the function $y=(x-1)^2$ is NOT one to one

because $y(0)=y(2)$

$y=0 \rightarrow x=1$

$y=1 \rightarrow x=0$, or $x=1$

Y	0	1
$P(Y)$	$1/2$	$1/4+1/4=1/2$

Y	0	1
$P(Y)$	$1/2$	$1/2$

is the required p.d.f. of Y.

$P_Y(0)=1/2$

$P_Y(1)=1/4+1/4=1/2$

Function of random variable (Transformation method)

(Continuous random variable)

Let X be a continuous random variable with p.d.f. $f(x)$. If the transformation $y = u(x)$ is

- Continuously differentiable and
- either non-decreasing or non-increasing

for all values within the range of X for which $f(x) \neq 0$, then the p.d.f. of $Y = u(X)$ is given by

$$\boxed{g(y) = f(x(y)) \left| \frac{dx}{dy} \right|}$$

Remark: When the equation $y = u(x)$ has a finite number of solution, say $x_1, x_2, x_3, \ldots, x_n$, we can find the p.d.f of W by

$$g(y) = f(x_1)\left|\frac{dx_1}{dy}\right| + f(x_2)\left|\frac{dx_2}{dy}\right| + \cdots + f(x_n)\left|\frac{dx_n}{dy}\right|$$

Question: If X is uniform random variable over $(0,1)$. Find the distribution function of $Y = e^x$.

Solution: If $X \sim U(a,b)$ then p.d.f. of uniform distribution over (a,b) is

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \quad a<x<b \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Here, $X \sim U(0,1)$. Then the p.d.f of X is

$$f(x) = \begin{cases} 1 & ; \quad 0<x<1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Since $y = e^x$ is continuously differentiable and non-decreasing for $0<x<1$ so we apply transformation method.

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right| .$$

For $y = e^x \Rightarrow x = \log y$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{1}{y}}$$

When

$$0 < x < 1 \Rightarrow 0 < \log y < 1$$

$$\Rightarrow e^0 < e^{\log y} < e^1$$

$$\Rightarrow 1 < y < e$$

Thus, for these values of y, we have

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right|$$

$$= 1 \times \frac{1}{y} \quad ; \quad 1 < y < e$$

$$g(y) = 1/y \quad ; \quad 1 < y < e$$

Hence, the p.d.f. of Y is

$$g(y) = \begin{cases} 1/y & ; \quad 1 < y < e \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Question: If p.d.f. of random variable X is given by

$$f(x) = \begin{cases} 2x & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the p.d.f. of $Y = 8X^3$.

Solution: Given

$$f(x) = \begin{cases} 2x & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Since $y = 8x^3$ is continuously differentiable and non-decreasing for $0 < x < 1$, so we apply transformation method.

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right|$$

For $y = 8x^3 \Rightarrow x = \frac{1}{2} y^{1/3}$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{6} y^{-2/3}$$

When $0 < x < 1 \Rightarrow 0 < \frac{1}{2} y^{1/3} < 1$

$$\Rightarrow 0 < y^{1/3} < 2$$

$$\Rightarrow 0 < y < 8$$

Thus, for these value of y, we have

$$g(y) = f(x(y))\ \frac{dx}{dy}$$

$$= (2x)\ \frac{1}{6}\, y^{-2/3}$$

$$= 2\left(\frac{y^{1/3}}{2}\right)\frac{1}{6}\, y^{-2/3}$$

$$\boxed{g(y) = \frac{1}{6}\, y^{-1/3}\ ; \quad 0 < y < 8}$$

Thus, the p.d.f of Y is

$$g(y) = \begin{cases} \frac{1}{6}\, y^{-1/3} & ; \quad 0 < y < 8 \\ 0 & ; \quad \text{else where} \end{cases}$$

Question : If X is a continuous random variable with p.d.f.

$$f(x) = \begin{cases} x/12 & ; \quad 1 < x < 5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the p.d.f. of $Y = 2X - 3$.

Solution : Given

$$f(x) = \begin{cases} \frac{x}{12} & ; \quad 1 < x < 5 \\ 0 & ; \quad o/w \end{cases}$$

Since $Y = 2x - 3$ is continuously differentiable and non-decreasing for $1 < x < 5$, so we apply the transformation method.

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right|$$

For $y = 2x - 3 \Rightarrow x = \frac{y+3}{2}$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2}$$

Also $1 < x < 5 \Rightarrow 1 < \frac{y+3}{2} < 5$

$$\Rightarrow 2 < y+3 < 10$$

$$\Rightarrow \boxed{-1 < y < 7}$$

Thus, for these values of y, we have

$$g(y) = f(x(y))\left|\frac{dx}{dy}\right|$$

$$= \frac{1}{12}\left(\frac{y+3}{2}\right)\left|\frac{1}{2}\right| \quad ; \quad -1<y<7$$

$$g(y) = \frac{1}{48}(y+3) \quad ; \quad -1<y<7$$

Hence, p.d.f of Y is

$$g(y) = \begin{cases} \frac{1}{48}(y+3) & ; \quad -1<y<7 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Question-: If X is uniformly distributed over $(0,1)$, then find the p.d.f. of $Y = -2\log X$.

Solution-: We know that if $X \sim U(a,b)$, so its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \quad a<x<b \\ 0 & ; \quad o/w \end{cases}$$

i.e., if $X \sim U(0,1)$, so its p.d.f. is given by

$$f(x)=\begin{cases}1 & ; \quad 0<x<1 \\ 0 & ; \quad o/w\end{cases}$$

Since $y=-2\log(x)$ is continuously differentia[ble]

and non-increasing for $0<x<1$,

So we apply transformation method

$$g(y)=f(x(y))\left|\frac{dx}{dy}\right|$$

For $y=-2\log x \Rightarrow x=e^{-y/2}$

$$\Rightarrow \frac{dx}{dy}=-\frac{1}{2}e^{-y/2}$$

When $0<x<1 \Rightarrow 0<e^{-y/2}<1$

$$\Rightarrow \log(0)<-y/2<\log(1)$$

$$\Rightarrow -\infty<-y/2<0$$

$$\Rightarrow 0<y<\infty$$

Thus, for these values of y, we have

$$g(y)=f(x(y))\left|\frac{dx}{dy}\right|=1\times\left|-\frac{1}{2}e^{-y/2}\right|$$

$$g(y)=\frac{1}{2}e^{-y/2} \quad ; \quad 0<y<\infty.$$

Hence, p.d.f. of Y is

$$g(y) = \begin{cases} \frac{1}{2} e^{-y/2} & ; \quad 0 < y < \infty \\ 0 & ; \quad o/w \end{cases}$$

i.e., Y follows the exponential distribution

$$Y \sim \exp(1/2).$$

Question: If the random variable X is uniformly distributed over $(-1, 1)$, find the density function of $Y = \sin\left(\frac{\pi x}{2}\right)$.

Solution: We know that if

$X \sim U(a, b)$, so its p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \quad a < x < b \\ 0 & ; \quad o/w \end{cases}$$

If $X \sim U(-1, 1)$, so its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{2} & ; \quad -1 < x < 1 \\ 0 & ; \quad o/w \end{cases}$$

Since $y = \sin\left(\frac{\pi x}{2}\right)$ is continuously differentiable and non-decreasing for $-1 < x < 1$, so we apply transformation method

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right|$$

For $y = \sin\left(\frac{\pi x}{2}\right) \Rightarrow x = \frac{2}{\pi} \sin^{-1}(y)$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}$$

Also, when

$$-1 < x < 1 \Rightarrow -1 < \frac{2}{\pi} \sin^{-1}(y) < 1$$

$$\Rightarrow -\frac{\pi}{2} < \sin^{-1}(y) < \frac{\pi}{2}$$

$$\Rightarrow -\sin\left(\frac{\pi}{2}\right) < y < \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow -1 < y < 1.$$

Thus, for these values of y, we have

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}}{\pi} \frac{1}{\sqrt{1-y^2}} \quad ; \quad -1 < y < 1$$

Hence, the p.d.f. of Y is

$$g(y)=\begin{cases}\frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} & ; -1<y<1\\ 0 & ; \text{otherwise}\end{cases}$$

Question: For a random variable X with p.d.f.

$$f(x)=\begin{cases}e^{-x} & ; x>0\\ 0 & ; \text{otherwise}\end{cases}$$

Find the density function of $Y=X\sqrt{X}$.

Solution: Given that

$$f(x)=\begin{cases}e^{-x} & ; x>0\\ 0 & ; \text{otherwise}\end{cases}$$

Clearly $y=x\sqrt{x}=x^{3/2}$ is continuously differentiable and non-decreasing for $x>0$, so we apply transformation method.

$$g(y)=f(x(y))\left|\frac{dx}{dy}\right|.$$

For $y = x^{3/2} \Rightarrow x = y^{2/3}$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{3} y^{-1/3}$$

When $x > 0 \Rightarrow y^{2/3} > 0$.

$$\Rightarrow y > 0.$$

Thus, for these values of y, we have

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right|$$

$$g(y) = e^{-y^{2/3}} \frac{2}{3} y^{-1/3} \quad ; \quad y > 0$$

$$g(y) = \frac{2}{3} y^{-1/3} e^{-y^{2/3}} \quad ; \quad y > 0.$$

Hence, the p.d.f. of Y is

$$g(y) = \begin{cases} \frac{2}{3} y^{-1/3} e^{-y^{2/3}} & ; \quad y > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$