PROBABILITY AND STATISTICS (UCS401)

Lecture-7

(Total Probability theorem with illustrations)

Introduction to Probability (Unit -II)

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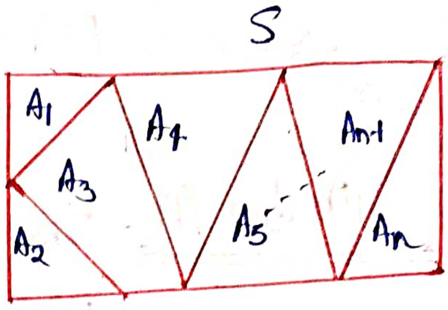
[5] Total probability theorem and Examples

Situation ① Two-thirds of the students in a Class are boys and rest are girls. It is known that the probability of a girl getting a first class is 0.25 and that of boy is 0.28. Find the probability that a student chosen at random will get first class.

Situation ② In a class of 75 students, 15 students were considered to be intelligent, 45 as medium and rest below average. The probability that a very intelligent student fails in a viva-voce examination is 0.005; the medium student fails has a probability of 0.05; and corresponding probability for a below average student is 0.15. What is the probability of the student passed the viva-voce examination;

Partition :-



Let $A = \{A_1, A_2, A_3, \dots, A_n\}$ be finite collection of events.

Then, A is a partition of sample S if following three conditions holds:

(i) $P(A_i) \geq 0$ for each i

(ii) Events A_i are pairwise disjoints, i.e., $A_i \cap A_j = \phi$ for $i \neq j$

(iii) Union of the events A_i equal to sample space S, i.e.,

$$\bigcup_{i=1}^{n} A_i = S$$

Example :- Consider an experiment of throwing a fair die. The sample space S is

$$S = \{1, 2, 3, 4, 5, 6\}$$

There are many partition for sample space S

(i) $A_1 = \{1, 2\}$ & $A_2 = \{3, 4, 5, 6\}$

Here, $A_1 \cap A_2 = \phi$

& $A_1 \cup A_2 = S$

Clearly, it forms a partition of S.

(ii) Define events $A_1 = \{1,3,5\}$ and $A_2 = \{2,4,6\}$.
Clearly, it forms a partition of S.

As, $\left.\begin{array}{l} A_1 \cap A_2 = \phi \\ \& \quad A_1 \cup A_2 = S \end{array}\right]$

(iii) Define events $A_1 = \{1,3\}$ & $A_2 = \{2,4,6\}$

$A_1 \cap A_2 = \phi$ but $A_1 \cup A_2 \neq S$

$\Rightarrow$ Not forms a partition of S.

(iv) Define events $A_1 = \{1,2\}$, $A_2 = \{3,4\}$ and $A_3 = \{5,6\}$. Clearly, it forms a partition of S.

As, here,

$$\left.\begin{array}{l} A_1 \cap A_2 = \phi \\ A_2 \cap A_3 = \phi \\ A_3 \cap A_1 = \phi \end{array}\right] \& \; A_1 \cup A_2 \cup A_3 = S.$$

(v) Define events $A_i = \{i\}$. Clearly it forms a partition of S.

$A_1 = \{1\}$ $A_2 = \{2\}$ $A_3 = \{3\}$ - - - $A_6 = \{6\}$

Here $A_i \cap A_j = \phi$ for $i \neq j$

$$\bigcup_{i=1}^{6} A_i = S$$

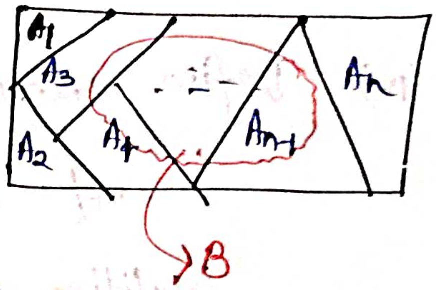
#

Total probability theorem -:

Let $A = \{A_1, A_2, A_3, \ldots, A_n\}$ be a partition of the sample space S. If B is any event, then

$$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + \cdots + P(A_n)\,P(B/A_n)$$

$$= \sum_{i=1}^{n} P(A_i)\,P(B/A_i).$$



Proof -: As $A = \{A_1, A_2, \ldots, A_n\}$ be a partition of the sample space S. Thus, events A_i are pairwise disjoint and $\bigcup_{i=1}^{n} A_i = S$.

$\because$ A_i's are disjoint, i.e., $A_i \cap A_j = \phi$ for $i \neq j$

$\therefore$ $(B \cap A_1) \cap B \cap A_2$

$= B \cap (A_1 \cap A_2) = B \cap \phi$

$= \phi$

$\Rightarrow$ $B \cap A_i$ are also pairwise disjoint $\forall\ i = 1, 2, 3, \ldots n$.

As $B \subseteq S$

$\Rightarrow$ $B = B \cap S$

$\Rightarrow$ $B = B \cap \left(\bigcup_{i=1}^{n} A_i\right)$

$\Rightarrow$ $B = \bigcup_{i=1}^{n} (B \cap A_i)$

$$\Rightarrow P(B) = \sum_{i=1}^{n} P(B \cap A_i) \quad \text{---} (1)$$

By Conditional probability, we have

$$P(B/A_i) = \frac{P(B \cap A_i)}{P(A_i)} \quad \forall i$$

$$\Rightarrow P(B \cap A_i) = P(A_i)\, P(B/A_i)$$

$$(1) \Rightarrow \boxed{P(B) = \sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$ Proved

Question (1) Two-thirds of the students in a class are boys and rest are girls. It is known that the probability of a girl getting a first class is 0.25 and that of a boy is 0.28. Find the probability that a student chosen at random will get first class.

Solution:- Define the events

A_1 : student is boy (Cause)

A_2 : student is girl. (Cause)

B ; student will get first class (Effect)

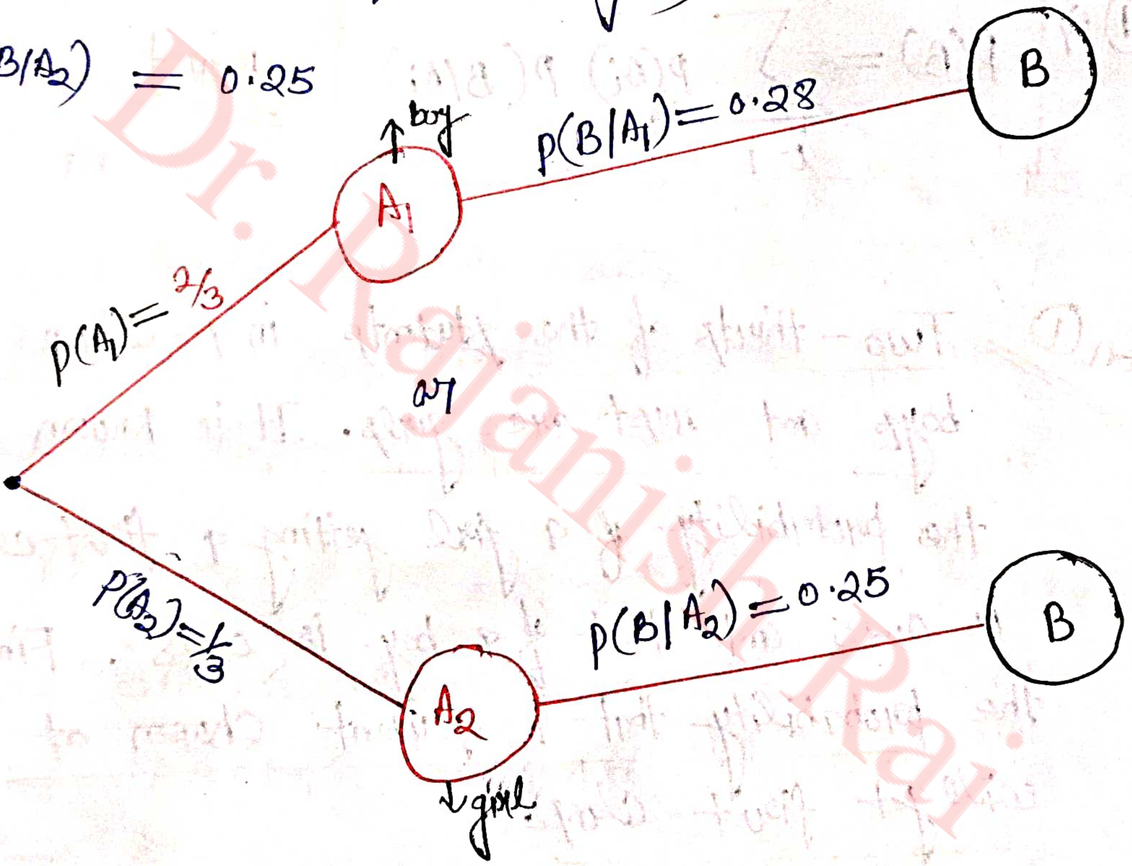
$P(A_1) = \frac{2}{3}$ & $P(A_2) = \frac{1}{3}$

$P(B/A_1)$ = P(student will get first class given that student is a boy)

$P(B/A_1) = 0.28$

$P(B/A_2)$ = P(student will get first class given that student is girl)

$P(B/A_2) = 0.25$



By total probability theorem

$$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)$$

$$= \left(\frac{2}{3} \times 0.28\right) + \left(\frac{1}{3} \times 0.25\right)$$

$$= \frac{0.56}{3} + \frac{0.25}{3}$$

$$\boxed{P(B) = 0.27}$$ Ans

Question (2) An box contains 10 white and 3 black balls. Another box contains 3 white and 5 black balls. Two balls are drawn at random from the first box and placed in second box and then 1 ball is taken at random from the latter. What is probability that it is white ball.

Solution:—

Ist box	(2)	IInd box (3W & 5B)
10 white 3 black Total = 13 balls	2W 2B 1W&1B	(i) (2+3) W & 5B (ii) 3W & (5+2) B (iii) (3+1) W & (5+1) B

TF ... transfer

you pick = 2   you pick = 1

Exhaustive Cases = $^{13}C_2$

Define,

A_1 : both balls drawn from one box are white (2W)

A_2 : both balls drawn from one box are black (2B)

A_3 : balls drawn from one box are 1W and 1B (1W & 1B)

B : 1 white ball selected from 2nd box.

$P(B) = ?$

Ist box —:

10 W
3 B
Total = 13 balls
You pick = 2 balls

→ 2W, 2B, 1W&1B

$$P(A_1) = \frac{^{10}C_2}{^{13}C_2} = \frac{15}{26}$$

$$P(A_2) = \frac{^{3}C_2}{^{13}C_2} = \frac{1}{26}$$

$$P(A_3) = \frac{^{10}C_1 \times {^{3}C_1}}{^{13}C_2} = \frac{10}{26}.$$

IInd box —: (3W & 5B)

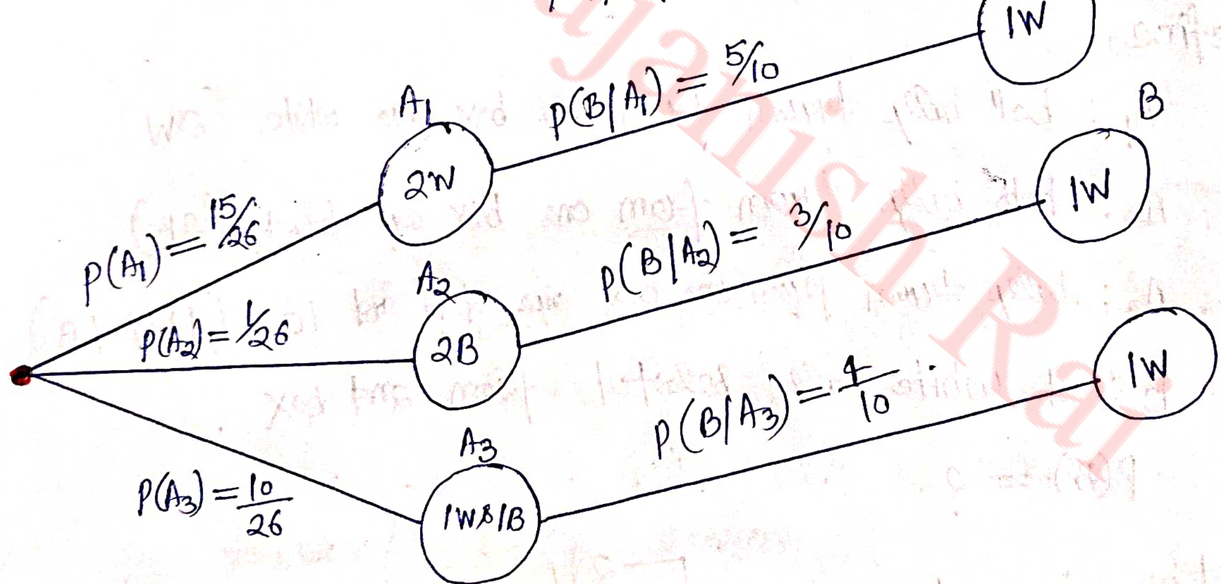
A_1 : 2W transfered $\longrightarrow$ $(3+2)$W, 5 B; 10 balls, you pick = 1W $\qquad P(B/A_1) = \dfrac{5C_1}{10C_1} = \dfrac{5}{10}$

or

A_2 : 2B transfered $\longrightarrow$ 3W, $(5+2)$B; Total = 10 balls, you pick = 1W $\qquad P(B/A_2) = \dfrac{3C_1}{10C_1} = \dfrac{3}{10}$

or

A_3 : 1B & 1W transfered $\longrightarrow$ $(3+1)$W, $(5+1)$B; Total = 10 balls, you pick = 1W $\qquad P(B/A_3) = \dfrac{4C_1}{10C_1} = \dfrac{4}{10}$



Thus, By total probability theorem

$$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)$$

$$P(B) = \left(\frac{15}{26}\right) \times \left(\frac{5}{10}\right) + \frac{1}{26}\left(\frac{3}{10}\right) + \left(\frac{10}{26}\right)\left(\frac{4}{10}\right)$$

$$\boxed{P(B) = \frac{59}{130}}$$

Question

In a class of 75 students, 15 were considered to be intelligent, 45 as medium and rest below average. The probability that a very intelligent student fails in a viva-voce examination is 0.005; the medium student fail has a probability of 0.05; and corresponding probability for a below average student is 0.15. What is probability of the student passed the viva-voce examination.

Solution:

Define,

A_1 : The student is very intelligent (Cause).

A_2 : The student is medium. (Cause).

A_3 : The student is below average (Cause).

B : student passed in the viva-voce examination (Effect)

Given that

$$P(A_1) = \frac{15}{75} = 0.2$$

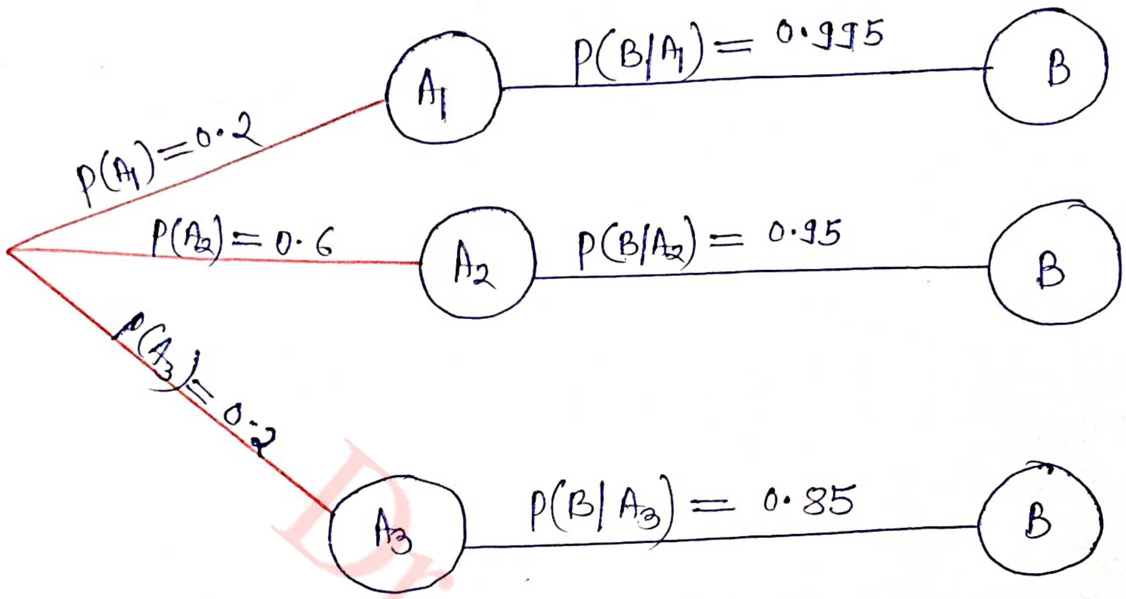
$$P(A_2) = \frac{45}{75} = 0.6$$

$$P(A_3) = \frac{15}{75} = 0.2$$

$$P(\text{fails}/A_1) = 0.005 \Rightarrow P(B/A_1) = 1-0.005 = 0.995$$

$$P(\text{fails}/A_2) = 0.05 \Rightarrow P(B/A_2) = 1-0.05 = 0.95$$

$$P(\text{fails}/A_3) = 0.15 \Rightarrow P(B/A_3) = 1 - 0.15$$
$$= 0.85$$



Thus, by total probability theorem, we get

$$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)$$

$$P(B) = (0.2)(0.995) + (0.6)(0.95) + (0.2)(0.85)$$

$$\boxed{P(B) = 0.939}$$ Ans