$P(X=x) = F_X(x) - F_X(x-1)$

$$\boxed{CDF = F_X(x) = P(X \leq x) = \sum^{x} P(X=x)}$$

Random Variable X

Discrete

Probability Mass function (PMF)

$P(X=x)$ — (i) $P(X=x) \geq 0$ (ii) $\sum_x P(X=x) = 1$

Continuous

$f(x)$ (PDF) — (i) $f(x) \geq 0$ (ii) $\int_x f(x)\,dx = 1$

Probability Density function

$\frac{d}{dx} F_X(x) = f(x)$

$$\boxed{CDF = F_X(x) = \int_{-\infty}^{x} f(x)\,dx}$$

$E(X)$

Expectation

$E(X) = \sum_x x\,P(X=x)$

PMF

PDF

$E(X) = \int_x x\,f(x)\,dx$

$M_X(t)$

Moments

$M_X(t) = E(e^{tX})$

$= \sum_x e^{tx} P(X=x)$

Provided $E(X)$ & $M_X(t)$ exists

$M_X(t) = E(e^{tX})$

$= \int_x e^{tx} f(x)\,dx$

$Var(X) = E(X^2) - [E(X)]^2$

OR

$E(X - E(X))^2$

$\frac{d^n}{dt^n} M_X(t) \Big|_{t=0} = E(X^n)$

Statistical Distribution

Discrete

- Uniform
- Bernoulli
- Binomial
- Negative Binomial
- Geometric
- Poisson
- Hyper-geometric

Continuous

- Uniform
- Gamma
- Exponential
- Weibull
- Normal
- Log-normal

1-parameter
2-parameter

Discrete Uniform

A random variable X has a discrete uniform $(1, N)$ distribution if its PMF is given by

$$P(X = x \mid N) = \frac{1}{N}, \quad x = 1, 2, \ldots N$$

where N is a specified +ve integer.

This distribution puts equal mass on each of the outcomes $1, 2, \ldots N$.

Mean.

$$E(x) = \sum_{x=1}^{N} x\, P(X = x \mid N)$$

$$= \frac{1}{N} \sum_{x=1}^{N} x$$

$$= \frac{1}{N} \frac{N(N+1)}{2}$$

$$= \frac{N+1}{2}$$

Variance $\quad V(x) = E(x^2) - [E(x)]^2$

Now $E(x^2) = \sum_{x=1}^{N} x^2\, P(X = x \mid N) = \frac{1}{N} \sum_{x=1}^{N} x^2$

$$= \frac{1}{N} \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6}$$

and so $\quad V(x) = \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 = \frac{(N-1)(N+1)}{12}$

Moment Generating function (MGF)

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^{N} e^{tx} P(X=x|N)$$

$$= \frac{1}{N} \sum_{x=1}^{N} e^{tx}$$

Note: This distribution can be generalized so that the sample space is any range of integers $N_0, N_0+1, \ldots N_1$ with PMF given by

$$P(X=x|N_0, N_1) = \frac{1}{N_1-(N_0-1)}$$

Bernoulli trial

A Bernoulli trial is an experiment with two, and only two, possible outcomes.

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

A random variable X has a Bernoulli(p) distribution if its PMF is given by

$$P(X=x|p) = p^x (1-p)^{1-x} \quad ; \quad x = 0, 1$$

The value $X=1$ is often termed as success, p is prob. of success
$X=0$ failure $(1-p)$ failure

$$E(x) = \sum_{x=0}^{1} x \, P(x=x|p) = \sum_{x=0}^{1} x \, p^{x}(1-p)^{1-x}$$

$$= 0 + 1 . \, p \, (1-p)^{0}$$

$$= p$$

$$E(x^2) = \sum_{x=0}^{1} x^2 \, P(x=x|p) = \sum_{x=0}^{1} x^2 \, p^{x}(1-p)^{1-x}$$

$$= 0 + 1^2 \, p \, (1-p)^{0}$$

$$= p$$

$$V(x) = E(x^2) - [E(x)]^2 = p - p^2 = p(1-p)$$

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} \, P(x=x|p)$$

$$= \sum_{x=0}^{1} e^{tx} \, p^{x} \, (1-p)^{1-x}$$

$$= (pe^t)^0 \, (1-p) + (pe^t)^1 \, (1-p)^0$$

$$= (1-p) + pe^t$$

Bernoulli $\longrightarrow$ Binomial

Y = total number of successess in n independent & identical Bernoulli trials

Then $Y \sim$ Binomial (n, p) distribution.

$X_i \sim$ Bernoulli(p)

$Y = \sum_{i=1}^{n} X_i$

$Y \sim$ Binomial (n, p)

with PMF given by

$$P(Y = y \mid n, p) = {}^{n}C_{y}\, p^{y} (1-p)^{n-y} \; ; \; y = 0, 1, 2 \ldots n$$

Proof

if n independent & identical Bernoulli trials are performed

Then define

$$A_i = \{ X = 1 \text{ on the } i\text{th trial} \}, \; i = 1, 2, \ldots n$$

Then Y = total number of successes in n trials

The event $\{Y = y\}$ will occur only if, out of the events $A_1, A_2, \ldots A_n$, exactly y of them occur, and necessarily $(n-y)$ of them do not occur.

Here $Y = 0, 1, 2, \ldots n$ That is zero number, one, two .. or n number of successes

Let us consider one particular outcome of the n Bernoulli trials might be

$$A_1 \cap A_2^c \cap A_3 \text{ --- } \cap A_{n-1}^c \cap A_n$$

This has probability of occurence

$$P(A_1 \cap A_2^c \cap A_3 \text{ --- } \cap A_{n-1}^c \cap A_n)$$

$\{\because$ Independent

$$= P(A_1)\, P(A_2^c)\, P(A_3) \text{ --- } P(A_{n-1}^c)\, P(A_n)$$

$$= p\,(1-p)\,p \text{ --- } (1-p)\,p$$

$$= p^y (1-p)^{n-y}$$

However, ~~there are~~ total possible no. of outcomes are $\binom{n}{y}$

Therefore

$$P(Y=y \mid n, p) = {}^nC_y\, p^y (1-p)^{n-y}, \quad y = 0, 1, \ldots n$$

Note: if $Y \sim \text{Binomial}(n, p)$

and $n = 1$

Then $Y \sim \text{Bernoulli}(p)$

Remember:

$$\sum_{y=0}^{n} {}^{n}C_y \, p^y (1-p)^{n-y} = [p + (1-p)]^n$$

OR.

$$\sum_{i=0}^{n} {}^{n}C_i \, x^i y^{n-i} = (x+y)^n$$

$x = p$

$y = 1-p$

<u>Mean.</u>

$$E(Y) = \sum_{y=0}^{n} y \, {}^{n}C_y \, p^y (1-p)^{n-y}$$

$$= \sum_{y=1}^{n} y \frac{n!}{y!\,(n-y)!} p^y (1-p)^{n-y}$$

$$= \sum_{y=1}^{n} y \frac{n\,(n-1)!}{y\,(y-1)!\,(n-y)!} p \cdot p^{y-1} (1-p)^{n-y}$$

$$= np \sum_{y=1}^{n} {}^{n-1}C_{y-1} \, p^{y-1} (1-p)^{n-y}$$

$$= np$$

See Put ~~n~~ $z = y-1$ Then $z=0$ when $y=1$, $z = n-1$ when $y = n$

$$\therefore E(Y) = np \sum_{z=0}^{n-1} {}^{n-1}C_z \, p^z (1-p)^{n-1-z} = np[p + (1-p)]^{n-1} = np$$

$$E(y^2) = E(y(y-1) + y) \leftarrow \text{(*)}$$

Remember This step
$\because y! = y(y-1)\cdots 1.$

Now

$$E(y(y-1)) = \sum_{y=0}^{n} y(y-1)\, {}^{n}C_{y}\, p^{y} (1-p)^{n-y}$$

$$= \sum_{y=2}^{n} y(y-1) \frac{n(n-1)}{y(y-1)}\, {}^{n-2}C_{y-2}\, p^{2} p^{y-2} (1-p)^{n-y}$$

$$= n(n-1)p^2 \sum_{y=2}^{n} {}^{n-2}C_{y-2}\, p^{y-2} (1-p)^{n-y}$$

$$= n(n-1)p^2$$

Therefore.

$$V(y) = E(y^2) - [E(y)]^2$$

{from (*)

$$= E(y(y-1)) + E(y) - [E(y)]^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

MGF $M_Y(t) = E(e^{tY}) = \sum_{y=0}^{n} e^{ty}\, {}^{n}C_y\, p^y (1-p)^{n-y}$

$$= \sum_{y=0}^{n} {}^{n}C_y\, (pe^t)^y (1-p)^{n-y}$$

$$= \left[pe^t + (1-p)\right]^n$$

Eg. Consider an experiment of tossing a unbiased coin 10 times. Find probability of getting no head.

Now tossing a coin can be seen as a Bernoulli trial with two outcomes Head {say success} and Tail {say failure} with respective probabilities p and $(1-p)$

Therefore $Y = \sum X_i$ = total no. of successes in n Bernoulli trials

$\sim$ Binomial (n, p)

if we consider unbiased coin then $p = \frac{1}{2}$

So $Y \sim$ Binomial $(10, \frac{1}{2})$

$P(\text{getting no head}) = P(Y=0) = {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10-0} = \left(\frac{1}{2}\right)^{10}$

Negative Binomial Distribution

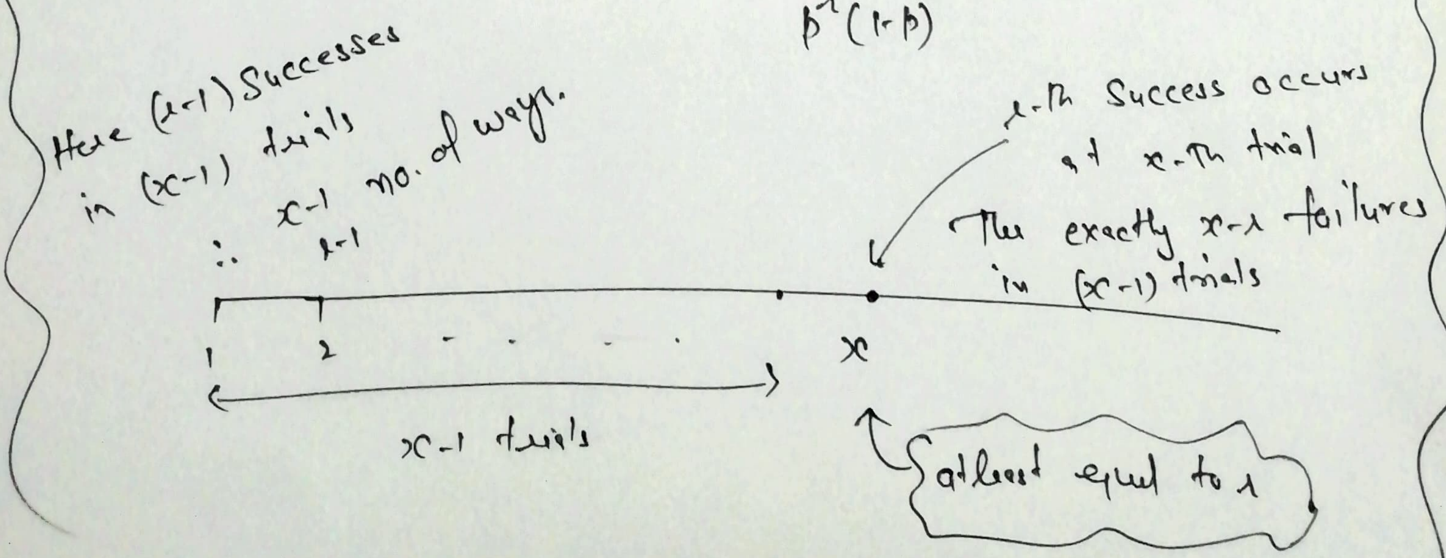
* The Binomial distribution counts the number of successes in a fixed number of Bernoulli trials. Suppose that, instead, we count the number of Bernoulli trials required to get a fixed number of successes. This later formulation leads to the negative binomial distribution

In a sequence of independent Bernoulli(p) trials, let the random variable X denote the trial at which the r-th success occurs, where r is a fixed integer.

Then

$$P(X = x \mid r, p) = \binom{x-1}{r-1} p^{r} (1-p)^{x-r} \; ; \quad x = r, r+1, \ldots$$

and we say that X has a negative binomial (r, p) distribution



$$E(X) = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$\boxed{\sum_{x=r} \binom{x-1}{r-1} p^r (1-p)^{x-r} = 1}$$

$$= \sum_{x=r}^{\infty} \frac{r}{p} \binom{x}{r} p^{r+1} (1-p)^{x-r}$$

$$= \frac{r}{p} \sum_{x=r+1}^{\infty} \binom{x}{r} p^{r+1} (1-p)^{x-r}$$

$$= \frac{r}{p}$$

Now $E(X^2) = E\left(X(X+1) - X \right)$ Remember this step

$$\therefore \quad E\left(X(X+1)\right) = \sum x(x+1) \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$= \sum \frac{r(r+1)}{p^2} \binom{x}{r} p^{r+2} (1-p)^{x-r}$$

$$= \frac{r(r+1)}{p^2}$$

Therefore $V(X) = E(X(X+1)) - E(X) - [E(X)]^2$

$$= \frac{r(r+1)}{p^2} - \frac{r}{p} - \frac{r^2}{p^2}$$

$$= \frac{r}{p}\left(\frac{1}{p} - 1\right) = \frac{r(1-p)}{p^2}$$

Alternative form of negative binomial

Y = number of failures before the r-th success

This formulation is statistically equivalent ~~to~~ in terms of X

X = trial at which the r-th success occurs.

Since $Y = X - r$

So $Y = 0$ when $X = r$

and therefore alternate form of the negative binomial distribution has PMF, given by

$$P(Y = y \mid r, p) = {}^{r+y-1}C_y \, p^r (1-p)^y, \quad y = 0, 1, \ldots$$

Note.

$$E(Y) = E(X - r) = E(X) - r$$

$$= \frac{r}{p} - r = \frac{r(1-p)}{p}$$

Also $V(Y) = V(X - r) = V(X) = \dfrac{r(1-p)}{p^2}$

Geometric Distribution

X : The trial at which the first success occurs

we are waiting for a success

This is a special case of negative binomial distribution, if we set $r=1$

Therefore, PMF is given by

$$P(X=x \mid p) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots$$

Mean. $E(X) = \dfrac{1}{p}$

$V(X) = \dfrac{1-p}{p^2}$

See P-53, $X \sim NB(r, p)$

Put $r=1$, Then

$E(X) = \dfrac{r}{p}$

$V(X) = \dfrac{r(1-p)}{p^2}$

* The geometric distribution has an interesting property, known as the "memoryless" property.

For Integer $s > t$, it is the case that

$$P(X > s \mid X > t) = P(X > s-t)$$

That is, the geometric distribution "forgets" what has occured. The probability of getting an additional $s-t$ failures, having already observed t failures, is the same as prob. of observing $(s-t)$ failures at the start of the sequence.

Let $P(x > n) = P(\text{no successes in } n \text{ trials})$

$$= (1-p)^n$$

and hence take L.H.S $P(x > s \mid x > t)$

$$= \frac{P(x > s \text{ and } x > t)}{P(x > t)}$$

$$= \frac{P(x > s)}{P(x > t)}$$

$$= \frac{(1-p)^s}{(1-p)^t}$$

$$= (1-p)^{s-t}$$

$$= P(x > (s-t))$$

$$\therefore \; P(x > s \mid x > t) = P(x > s-t)$$

The geometric distribution is sometimes used to model "lifetimes" or "time until failure" of components.

For example, if the probability is .001 that a light bulb will fail on any given day, then the probability that it will last at least 30 days is

$$P(x > 30) = \sum_{x=31}^{\infty} .001\,(1-.001)^{x-1}$$

$$= (.999)^{30}$$

$$= .970$$

$(1-p)^n$

<u>Note:</u>

The memoryless property of the geometric distribution describes a very special "lack of aging" property. it indicates that the geometric distribution is not applicable to modeling lifetimes for which the probability of failure is expected to increase with time.

Poisson Distribution

A random variable $X \sim$ Poisson(λ) distribution has PMF

$$P(X = x \mid \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

Remember

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

Mean.

$$E(X) = \sum_{x=0}^{\infty} x \, P(X = x \mid \lambda)$$

$$= \sum_{x=0}^{\infty} x \, \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \lambda \, \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{\lambda}$$

Put $x - 1 = y$

$$\sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

Now $E(X^2) = E(X(X-1) + X) = E(X(X-1)) + E(X)$

Here $E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \, P(X = x \mid \lambda) = \sum_{x=2}^{\infty} \lambda^2 e^{-\lambda} \frac{\lambda^{x-2}}{(x-2)!}$

$$= \lambda^2$$

Therefore $V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$

MGF $M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X=x \mid \lambda)$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

Example Consider a telephone operator who, on the average, handles five calls every 3 minutes. What is the probability that there will be no calls in the next minute?

Sol^n:

if we let X = number of calls in a minute

Then X has a Poisson distribution with $E(X) = \lambda = 5/3$

So $P(\text{no calls in the next minute}) = P(X=0) = \frac{e^{-5/3}(5/3)^0}{0!}$

$$= e^{-5/3}$$

$$= 0.189$$

Hyper geometric distribution

Suppose we have a large urn filled with N balls that are identical in every way except that M are red and N-M are green. We reach in, blindfolded and select k balls at random (without replacement). What is the probability that exactly x of the balls are red?

total no. of samples of size k that can be drawn from the N balls is $\binom{N}{k}$

Required — x no. of balls be red, and this can be accomplished in $\binom{M}{x}$ ways, leaving $\binom{N-M}{k-x}$ ways of filling out the sample with k-x green balls.

Thus if we let X denote the number of red balls in a sample of size k, then X has a hypergeometric distribution given by

$$P(X=x \mid N, M, k) = \frac{\binom{M}{x}\binom{N-M}{k-x}}{\binom{N}{k}}, \quad x = 0, 1, \ldots$$

Provided $M \geq x$ & $N-M \geq k-x$

Combining together, we have $M-(N-k) \leq x \leq M$

$$x\binom{M}{x} = M\binom{M-1}{x-1}$$

$$\binom{N}{k} = \frac{N}{k}\binom{N-1}{k-1}$$

$$\sum_{x=0}^{k} \frac{\binom{M}{x}\binom{N-M}{k-x}}{\binom{N}{k}} = 1$$

So

$$E(x) = \sum_{x=0}^{k} x \frac{\binom{M}{x}\binom{N-M}{k-x}}{\binom{N}{k}}$$

$$= \sum_{x=1}^{k} \frac{M\binom{M-1}{x-1}\binom{N-M}{k-x}}{\frac{N}{k}\binom{N-1}{k-1}}$$

$$= \frac{kM}{N}$$

Likewise

$$V(x) = \frac{kM}{N}\left(\frac{(N-M)(N-k)}{N(N-1)}\right)$$