PROBABILITY AND STATISTICS (UCS401)

Lecture-21

(Exponential distribution with illustrations)

Random Variables and their Special Distributions(Unit –III & IV)

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Exponential distribution [Continuous distribution]

A Continuous random variable X which has the following probability density function (p.d.f.)

$$f(x) = \theta e^{-\theta x}, \quad \theta > 0 \qquad 0 < x < \infty$$

is called exponential variate and its distribution is called exponential distribution.

Or

A random variable X is said to follow exponential distribution with parameter θ if its probability density function (p.d.f.) is given by:

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & o/w \end{cases}$$

Question:- Show that $f(x) = \theta e^{-\theta x}$, is the p.d.f for exponential distribution.

Soln:- For exponential distribution

$$f(x) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & o/w \end{cases}$$

we know by definite $f(x)$ is p.d.f if

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \theta e^{-\theta x}dx$$

$$= 0 - \frac{1}{\theta}\left(e^{-\theta x}\right)_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right] = 1$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)dx = 1}$$

$\Rightarrow$ $f(x)$ is p.d.f for exponential distribution

Mean :- The mean of the exponential distribution is given by

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{0} x \times 0\,dx + \int_{0}^{\infty} x\theta e^{-\theta x} dx$$

$$= \theta \int_{0}^{\infty} x e^{-\theta x} dx$$

we know that by Gamma function

$$\Gamma n = \int_{0}^{\infty} x^{n-1} e^{-x} dx$$

$$\frac{\Gamma n}{a^n} = \int_{0}^{\infty} e^{-ax} x^{n-1} dx$$

$$\Gamma n+1 = n\Gamma n$$

$$\Gamma n+1 = n!$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

$$= \theta \int_0^\infty x^{2-1} e^{-\theta x} dx$$

$$= \cancel{\theta} \frac{\Gamma 2}{\theta^{\cancel{2}}} = \frac{1}{\theta}$$

$$\Rightarrow \boxed{\text{Mean} = E(X) = 1/\theta}$$

(ii) Varience –:

The varience is defined as :

$$Var(X) = E(X^2) - (E(X))^2$$

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \int_{-\infty}^{0} 0 \cdot x^2 dx + \int_0^\infty x^2 \theta e^{-\theta x} dx$$

$$= \theta \int_0^\infty x^2 e^{-\theta x} dx$$

$$= \theta \int_0^\infty x^{3-1} e^{-\theta x} dx$$

$$= \theta \frac{\Gamma 3}{\theta^3}$$

$$E(X^2) = \frac{2}{\theta^2}$$

Now $Var(X) = \frac{2}{\theta^2} - 1/\theta^2 = 1/\theta^2$

$$\boxed{Var(X) = \frac{1}{\theta^2}}$$

Note that :-

For exponential distribution

$$\text{Varience} = \frac{\text{mean}}{\theta} = \frac{1}{\theta^2}$$

If $\theta = 1$, $E(X) = V(X)$

If $\theta > 1$, $E(X) > V(X)$

If $\theta < 1$ $E(X) < V(X)$

③ Moment Generating function (MGF)

The moment generating function $M_X(t)$ is defined as :

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} P[X=x] & ; \text{discrete} \\ \int_x e^{tx} f(x) dx & ; \text{Continuous} \end{cases}$$

Now for exponential distribution (Continuous distribution)

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{tx}\, 0\, dx + \int_{0}^{\infty} e^{tx}\, \theta e^{-\theta x}\, dx$$

$$= \theta \int_{0}^{\infty} e^{-(\theta - t)x}\, dx$$

$$= -\frac{\theta}{\theta - t}\left[e^{-(\theta - t)x}\right]_0^{\infty} \qquad |t| < \theta$$

$$= \frac{-\theta}{\theta - t}\left[e^{-\infty} - e^{0}\right] = \frac{\theta}{\theta - t}$$

$$\boxed{M_X(t) = \left(1 - \frac{t}{\theta}\right)^{-1}} \qquad |t| < \theta$$

$$M_X(t) = 1 + \frac{t}{\theta} + \frac{t^2}{\theta^2} + \frac{t^3}{\theta^3} + \cdots + \frac{t^r}{\theta^r} + \cdots$$

(iv) Characteristic function $(\phi_X(t))$:-

The characteristic function is given by :

$$\phi_X(t) = E(e^{itX}) = \begin{cases} \sum_x e^{itx} P[X=x] & ; \text{discrete} \\ \int_x e^{itx} f(x)\,dx & ; \text{Continuous} \end{cases}$$

For exponential distribution

$$\phi_X(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$$= \int_{-\infty}^{0} e^{itx}\, 0\, dx + \int_{0}^{\infty} e^{itx}\, \theta e^{-\theta x}\,dx$$

$$= \theta \int_{0}^{\infty} e^{-(\theta - it)x}\,dx$$

$$= \frac{-\theta}{\theta - it}\left[e^{-(\theta - it)x}\right]_0^{\infty}$$

$$= \frac{-\theta}{\theta - it}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{\theta}{\theta - it}$$

$$\Rightarrow \boxed{\phi_X(t) = \left(1 - \frac{it}{\theta}\right)^{-1}} \qquad |t| < \theta$$

Mean and Varience by Moment generating function -:

We know that

$$E(X^r) = \begin{cases} \left.\dfrac{d^r}{dt^r}\left(M_X(t)\right)\right|_{t=0} & \text{if } M_X(t) \text{ given} \\ \left.(-i)^r \dfrac{d^r}{dt^r}\left(\phi_X(t)\right)\right|_{t=0} & \text{if } \phi_X(t) \text{ given} \end{cases}$$

For exponential distribution

$$M_X(t) = \left(1 - t/\theta\right)^{-1}$$

$$\text{Mean} = E(X) = \left.\frac{d}{dt}\left(M_X(t)\right)\right|_{t=0}$$

$$= \frac{d}{dt}\left[\left(1 - t/\theta\right)^{-1}\right]_{t=0}$$

$$= \left.(-1)\left(1 - t/\theta\right)^{-2}\left(-1/\theta\right)\right|_{t=0}$$

$$\boxed{\text{Mean} = E(X) = \frac{1}{\theta}}$$

We know that $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \left.\frac{d^2}{dt^2}\left(M_X(t)\right)\right|_{t=0}$$

$$= \frac{d}{dt}\left[(-1)\left(1-\frac{t}{\theta}\right)^{-2}\left(-\frac{1}{\theta}\right)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\left(\frac{1}{\theta}\right)\left(1-\frac{t}{\theta}\right)^{-2}\right]_{t=0}$$

$$= \left[\left(\frac{-2}{\theta}\right)\left(1-\frac{t}{\theta}\right)^{-3}\left(-\frac{1}{\theta}\right)\right]_{t=0}$$

$$E(X^2) = \frac{2}{\theta^2}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2}$$

$$\Rightarrow \boxed{\text{Var}(X) = \frac{1}{\theta^2}}$$

Question :- The length of a telephone conversation is an exponential variate with mean 3 min. Find the probability of that call

1. end less than 3 min
2. texts between 3 to 5 min.

Solution :- For exponential distribution

$$\text{Mean} = E(X) = \frac{1}{\theta} = 3 \Rightarrow \boxed{\theta = \tfrac{1}{3}}$$

The probability density function for exponential distribution

$$f(x) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & o/w \end{cases}$$

(1) End less than 3

$$P(X<3) = \int_{-\infty}^{3} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{3} \frac{1}{3} e^{-x/3}\,dx$$

$$= \frac{1}{3}(-3)\left(e^{-x/3}\right)_0^3 = -\left(e^{-1}-1\right)$$

$$\boxed{P(X<3) = 1 - e^{-1}}$$

(2) Texts between 3 to 5

$$P(3<X<5) = \int_{3}^{5} f(x)\,dx = \int_{3}^{5} \frac{1}{3} e^{-x/3}\,dx$$

$$= \frac{1}{3}(-3)\left[e^{-x/3}\right]_3^5 = -\left(e^{-5/3} - e^{-1}\right)$$

$$\boxed{P(3<X<5) = e^{-1} - e^{-5/3}}$$

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Memoryless property-:

If X is a exponential distribution, then

$$P(X > m / X > n) = P(X > m-n)$$

for any $m, n > 0$.

Question-: A fast-food chain finds that the average time customers have to wait for service is 45 second. If the waiting time can be treated as an exponential random variable, what is the probability that a customer will have to wait more than 5 minutes given that already waited for 2 minutes.

Solution-: Given that

$$\text{Mean} = 45 \text{ second}.$$

For exponential distribution,

$$\text{Mean} = \frac{1}{\lambda} = 45$$

$$\boxed{\lambda = \frac{1}{45}}$$

For exponential distribution, the p.d.f. is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; o/w \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{45} e^{-x/45} & ; x \geq 0 \\ 0 & ; o/w \end{cases}$$

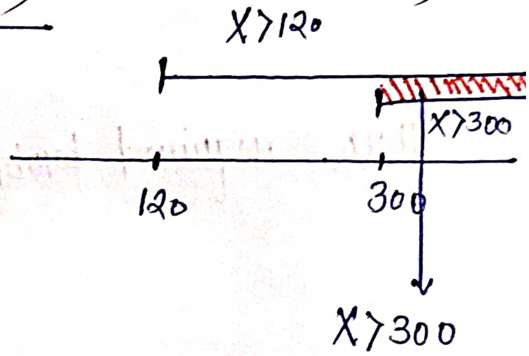
Ist method -:

By PDF -: Thus, required probability is given by

$$= P\left(X > 3\,\text{min} \,\middle|\, X > 2\,\text{min}\right)$$

$$= P\left(X > 300 \,\middle|\, X > 120\right)$$

$$= \frac{P\left((X>300) \cap (X>120)\right)}{P(X>120)} \qquad P(A|B) = \frac{P(A\cap B)}{P(B)}$$



$$= \frac{P(X>300)}{P(X>120)}$$

$$= \frac{\displaystyle\int_{300}^{\infty} \frac{1}{45} e^{-x/45}\,dx}{\displaystyle\int_{120}^{\infty} \frac{1}{45} e^{-x/45}\,dx}$$

$$= \frac{\frac{1}{45}(-45)\left(e^{-x/45}\right)_{300}^{\infty}}{\frac{1}{45}(-45)\left(e^{-x/45}\right)_{120}^{\infty}}$$

$$= \frac{e^{-\infty} - e^{\frac{-300}{45}}}{e^{-\infty} - e^{-120/45}} = \frac{e^{-20/3}}{e^{-8/3}}$$

$$= e^{-\frac{20+8}{3}} = e^{-4}$$

$$\boxed{P\left(X > 5\,\text{min} \,\middle|\, X > 2\,\text{min}\right) = e^{-4}}$$

By using c.d.f :- For exponential distribution, the c.d.f is given by

$$F(x) = P(X \leq x) = \begin{cases} 1-e^{-\lambda x} & ; \; x \geq 0 \\ 0 & ; \; o/\omega \end{cases}$$

$$F(x) = \begin{cases} 1-e^{-x/45} & ; \; x \geq 0 \\ 0 & ; \; o/\omega \end{cases}$$

Thus, required probability $= P(X>300 / X>120)$

$$= \frac{P((X>300) \cap (X>120))}{P(X>120)}$$

$$= \frac{P(X>300)}{P(X>120)}$$

$$= \frac{1-P(X \leq 300)}{1-P(X \leq 120)}$$

$$= \frac{1-F(300)}{1-F(120)}$$

$$= \frac{1-\left(1-e^{-300/45}\right)}{1-\left(1-e^{-120/45}\right)} = \frac{e^{-300/45}}{e^{-120/45}}$$

$$= \frac{e^{-20/3}}{e^{-8/3}} = e^{\frac{-20+8}{3}}$$

$$\boxed{P(X>300 / X>120) = e^{-4}}$$ Ans

2nd method using memoryless property :-

By PDF :-

$$P(X>300 \mid X>120) = P(X> 300-120)$$
$$= P(X>180)$$
$$= \int_{180}^{\infty} f(x)\,dx$$
$$= \frac{1}{45}\int_{180}^{\infty} e^{-x/45}\,dx$$
$$= \frac{1}{45}(-45)\left(e^{-x/45}\right)_{180}^{\infty}$$
$$= -\left[e^{-\infty} - e^{-\cancel{180}^{\,12}/\cancel{45}_{\,3}}\right]$$
$$= -\left[0 - e^{-4}\right]$$
$$P(X>300 \mid X>120) = e^{-4}$$ Ans

By using c.d.f. :-

$$P(X>300 \mid X>120) = P(X>180)$$
$$= 1 - P(X \leq 180)$$
$$= 1 - F(180)$$
$$= 1 - \left(1 - e^{-\frac{180}{45}}\right)$$
$$= 1 - 1 + e^{-4}$$
$$P(X>300 \mid X>120) = e^{-4}$$ Ans