PROBABILITY AND STATISTICS (UCS401)

Lecture-31

(Interval of Estimation and confidence interval with illustrations)

Sampling Distributions and Theory of Estimation (Unit –V & VI)

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Interval of Estimation and Confidence limits

The probability that we associated with an interval is called the confidence level. Higher the probability, the more is the confidence.

* If t is the sample statistics used to estimate the corresponding population parameter θ, then $(1-\alpha)$ 100% confidence limits for θ are $t \pm S.E.(t)\, t_\alpha$, where α is the level of significance.

$S.E.(t)$ is the standard error of t and t_α is the significant or critical value of t at the significant level α.

$t - S.E(t)\, t_\alpha$ and $t + S.E(t)\, t_\alpha$ are the limits of confidence interval.

By central limit theorem, if t is the sample statistics, then $Z = \frac{t - E(t)}{S.E(t)}$ is standard normal variate with mean zero and variance $= 1$ as n tends to ∞.

Confidence interval of the mean :

(i) Consider a population. Let μ be the population mean and $\bar{x}$ be the sample mean of sampling distribution. Assume that the sample is large. When the standard deviation σ of the normal population is known, the $S.E.(\bar{x})$ is $\frac{\sigma}{\sqrt{n}}$ (n is large sample size).

The confidence interval

$$[\bar{x} - S.E(\bar{x}) Z_\alpha, \ \bar{x} + S.E.(\bar{x}) Z_\alpha]$$

or $$\left[\bar{x} - \frac{\sigma}{\sqrt{n}} Z_\alpha \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}} Z_\alpha\right]$$

* Confidence interval for μ, when σ^2 is unknown :

Let $X_1, X_2, X_3, \dots X_n$ are random sample from $N(\mu, \sigma^2)$ then we know that

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

and

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

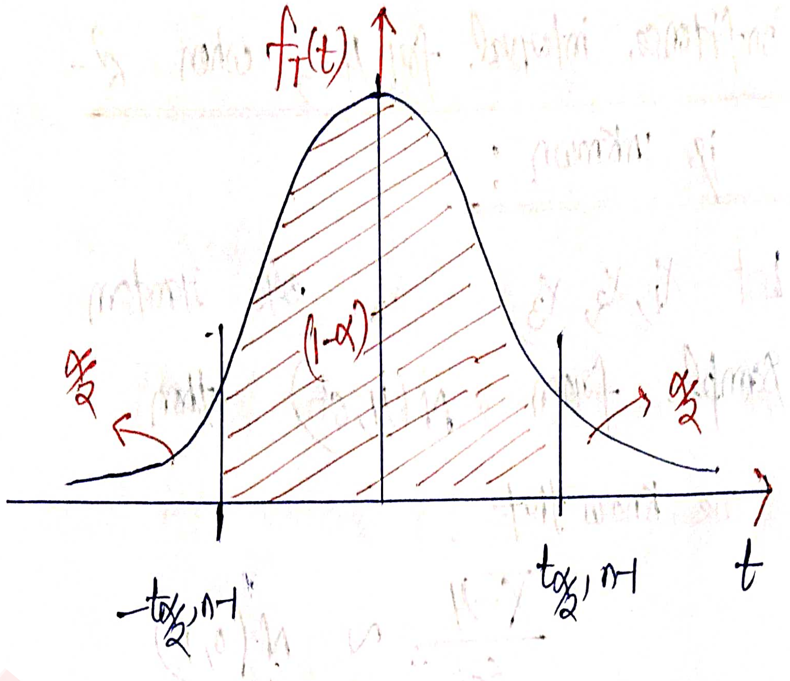
$$\Rightarrow \quad T = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{(n-1)}$$

$$\Rightarrow \quad 1-\alpha = P\left(-t_{\alpha/2, n-1} < T < t_{\alpha/2, n-1}\right)$$

$$\Rightarrow \quad 1-\alpha = P\left(-t_{\alpha/2, n-1} < \frac{\bar{X}-\mu}{s/\sqrt{n}} < t_{\alpha/2, n-1}\right)$$

$$= P\left(-t_{\alpha/2, n-1}\frac{s}{\sqrt{n}} - \bar{X} < -\mu < \frac{s}{\sqrt{n}} t_{\alpha/2, n-1} - \bar{X}\right)$$

$$1-\alpha = P\left(\bar{X} - \frac{s}{\sqrt{n}} t_{\alpha/2, n-1} < \mu < \bar{X} + \frac{s}{\sqrt{n}} t_{\alpha/2, n-1}\right)$$



* $\bar{x} = 53.92$

$s = 10.07$

$n = 24$

find a 99% Confidence interval for μ.

$\Rightarrow \quad (1-\alpha)\,100\% = 99\%$

$\Rightarrow \quad \alpha = 0.01$

$\Rightarrow \quad \alpha/2 = 0.005$

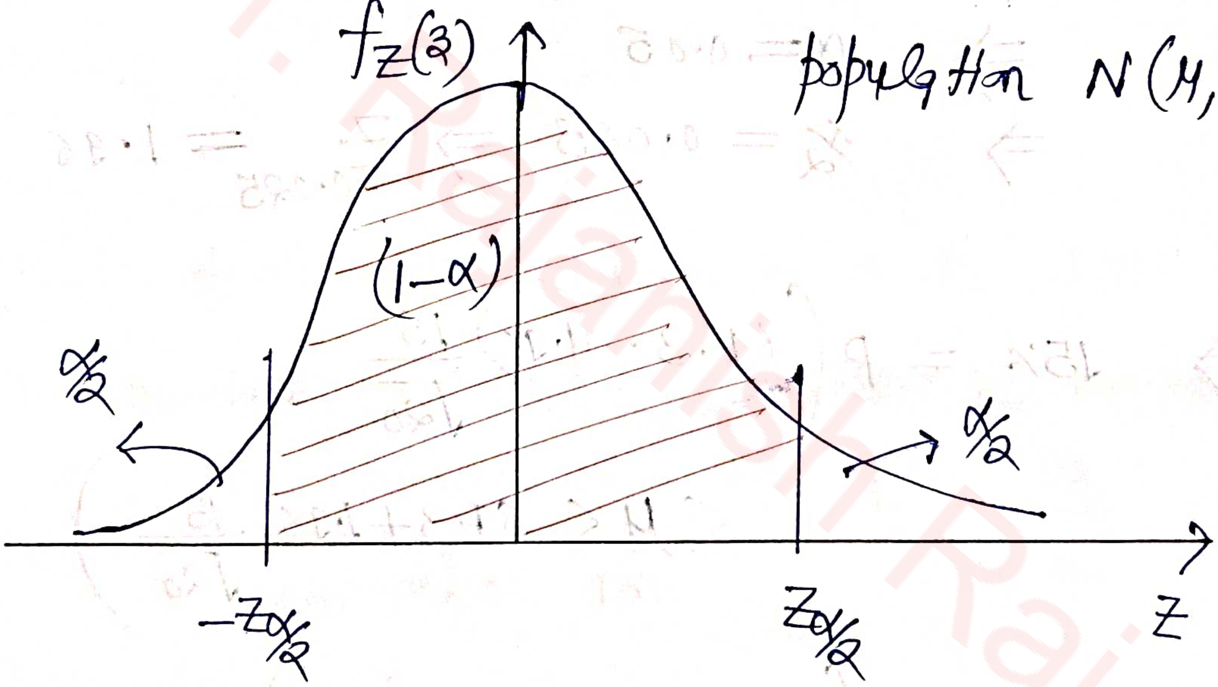
$$1 - 0.01 = P\left(53.92 - t_{0.005, 23}\frac{10.07}{\sqrt{24}} < \mu < 53.92 + t_{0.005}\frac{10.07}{\sqrt{24}}\right)$$

$$= P\left(52.92 - 2.807 \times \frac{10.07}{\sqrt{24}} < \mu < 53.92 + 2.807 \times \frac{10.07}{\sqrt{24}}\right)$$

$$\Rightarrow \quad 0.99 = P\left(48.15 < \mu < 59.69\right)$$

Ans

Case-II Confidence interval for μ if σ^2 is known from a Normal population $N(\mu, \sigma^2)$.



$$1-\alpha = P\left(-Z_{\alpha/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right)$$

$$1-\alpha = P\left(-Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X}-\mu < \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right)$$

$$1-\alpha = P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} Z_{\alpha/2} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right).$$

* Given

$n = 20$

$\sigma^2 = 225$

$\bar{x} = 64.3$

Construct a 95% Confidence interval for μ.

$1-\alpha = 95\%$

$(1-\alpha)\,100\% = 95\%$

$\Rightarrow \alpha = 0.05$

$\Rightarrow \alpha/2 = 0.025 \Rightarrow Z_{0.025} = 1.96$

$$\Rightarrow 95\% = P\left(64.3 - (1.96)\frac{15}{\sqrt{20}} < \mu < 64.3 + 1.96\,\frac{15}{\sqrt{20}}\right)$$

$$\Rightarrow 95\% = P\left(57.7 < \mu < 70.9\right)$$

Question : [2] The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per ml. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assuming that the population standard deviation is 0.3 gram per ml.

Given $n = 36$

$\sigma = 0.3$

$\bar{x} = 2.6$

To construct 95% confidence interval for μ

$(1-\alpha) = 95\%$

$(1-\alpha)100\% = 95\%$

$\Rightarrow \alpha = 0.05$

$\Rightarrow \alpha/2 = 0.025 \Rightarrow Z_{0.025} = 1.96$

$$95\% = P\left(2.6 - 1.96\frac{0.3}{\sqrt{36}} < \mu < 2.6 + 1.96\frac{0.3}{\sqrt{36}}\right)$$

$$95\% = P\left(2.50 < \mu < 2.70\right),$$

To Construct 99% Confidence interval

$$(1-\alpha)\,100\% = 99\%$$

$$\Rightarrow \quad \alpha = 0.01$$

$$\Rightarrow \quad \alpha/2 = 0.005 \quad \Rightarrow Z_{0.005} = 2.575$$

$$99\% = P\left(2.6 - 2.575\frac{0.3}{\sqrt{36}} < \mu < 2.6 + 2.575\frac{0.3}{\sqrt{36}}\right)$$

$$99\% = P\left(2.47 < \mu < 2.73\right)$$

Ans

Theorem: If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed a specified amount e, when sample size is

$$n = \left(\frac{Z_{\alpha/2}\,\sigma}{e}\right)^2.$$

Question ③: How large a sample is required if we want to be 95% confident that our estimate of μ in Ques ② is off by less than 0.05.

Solution: The population standard deviation is

$\sigma = 0.3$ then we know that

$$n = \left(\frac{Z_{\alpha/2}\,\sigma}{e}\right)^2$$

$$Z_{\alpha/2} = Z_{0.025} = 1.96$$

$$n = \left[\frac{(1.96)(0.3)}{0.05}\right]^2$$

$$n = 138.3$$

Therefore, we can be 95% confident that a random sample of size 139 will provide an estimate $\bar{x}$ differing from μ by an amount less than 0.05.

The Case of σ unknown:

Question: The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

Solution-:

	x	$(x_i-\bar{x})$	$(x_i-\bar{x})^2$
	9.8	-0.2	0.04
	10.2	0.2	0.04
	10.4	0.4	0.16
	9.8	-0.2	0.04
	10.0	0	0
	10.2	0.2	0.04
	9.6	-0.4	0.16
sum	70		0.48

$$n = 7$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{70}{7} = 10$$

$$s^2 = \frac{\sum (x_i-\bar{x})^2}{n-1} = \frac{0.48}{6} = 0.08$$

$$s = \sqrt{0.08} = 0.283$$

and $(1-\alpha)100\% = 95\%$

$$\Rightarrow \alpha = 0.05$$

$$\Rightarrow \alpha/2 = 0.025 \qquad t_{0.025,6} = 2.447.$$

Hence, 95% Confidence interval for μ

$$95\% = P\left(\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

$$95\% = P\left(10 - 2.447\frac{0.283}{\sqrt{7}} < \mu < 10 + 2.447\frac{0.283}{\sqrt{7}}\right)$$

$$95\% = P\left(9.74 < \mu < 10.26\right)$$ Ans