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School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

End-Term Examination, December 2022

B.E. V Semester | UCS410 : Probability & Statistics
Time Limit: 03 Hours | Maximum Marks: 40
Instructor(s) (Dr.) : Jatinderdeep Kaur, Jolly Puri, Kavita, Mamta Gulati, Paramjeet Singh, Rajanish Kumar

Instructions: You are expected to answer all questions. Non-programmable calculators are permitted. The standard normal table is given at the end of the paper.

1. (a) The time one has to wait for a bus at a downtown bus stop is observed to be a random variable X with the following probability density function:

$$f(x) = \begin{cases} \frac{1}{9}(x+1); & 0 \le x < 1, \\ \frac{4}{9}\left(x - \frac{1}{2}\right); & 1 \le x < 3/2, \\ \frac{4}{9}\left(\frac{5}{2} - x\right); & 3/2 \le x < 2, \\ \frac{1}{9}(4-x); & 2 \le x < 3 \\ \frac{1}{9}; & 3 \le x < 6 \\ 0, & \text{otherwise.} \end{cases}$$

Let the events A and B be defined as: A = One waits between 0 to 1 minutes inclusive; B = One waits more than 2 minutes inclusive. Find the value of $P(\overline{A} \cap \overline{B})$, where $\overline{A}$ and $\overline{B}$ are the complements of events A and B respectively. [3 marks]

(b) The refusal rate for telephone polls is known to be approximately 20%. A newspaper report indicates that 50 people were interviewed before the first refusal.
(i) Comment on the validity of the report. Use probability in your argument.
(ii) What is the expected number of people interviewed before a refusal? [3 marks]

(c) Compute $P(\mu - 2\sigma < X < \mu + 2\sigma)$, where X has the density function

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

and compare with the result given in Chebyshev's theorem. [4 marks]

2. (a) The time required to assemble a piece of machinery is a random variable having approximately a normal distribution with $\mu = 13.2$ minutes and $\sigma = 3.0$ minutes. What are the probabilities that assembly of a piece of machinery of this kind will take
(i) at least 11.1 minutes.
(ii) anywhere from 10.35 to 16.05 minutes? [3 marks]

(b) Let X be a continuous uniformly distributed random variable with mean 1 and variance $\frac{4}{3}$. Find the (i) probability distribution function and (ii) $P(X < 0)$. [3 marks]

(c) A random variable X has the Poisson distribution $p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}$, for $x = 0, 1, 2 \ldots$
(i) Show that the moment-generating function of X is $M_x(t) = e^{\mu(e^t - 1)}$. (ii) Using $M_x(t)$, find mean and variance of the Poisson distribution. (iii) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$ when $\mu = 4$. [4 marks]

3. ~~(a)~~ Let X_1 and X_2 be independent random variables each having the probability distribution

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Show that the random variables Y_1 and Y_2 are independent when $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$. [4 marks]

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(b) A population has a density function given by

$$f(x) = \begin{cases} (k+1)x^k; & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For n independent observations $x_1, x_2, \ldots, x_n$ made from this population, find the maximum likelihood estimate of k.

[3 marks]

(c) Show that the sample statistics defined by $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is a biased estimator for the population variance σ^2.

[3 marks]

4. (a) A random sample of 64 bags of seeds weighed, on average, 5.23 grams. Test the hypothesis that $\mu = 5.5$ grams against the alternative hypothesis, $\mu < 5.5$ grams at the 0.05 level of significance. Assume that the standard deviation of the population is 0.24 grams. [3 marks]

(b) A random sample of size 100 is taken from the population whose mean is 60 and variance is 400. Using central limit theorem, with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4? [3 marks]

(c) The contents of seven similar containers of sulphuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2 and 9.6 litres. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution. Given that $t_{0.025} = 2.447$, $t_{0.05} = 1.943$ with 6 degree of freedom and $t_{0.025} = 2.365$, $t_{0.05} = 1.895$ with 7 degree of freedom. [4 marks]

Table of the Standard Normal Cumulative Distribution Function

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817