PROBABILITY AND STATISTICS (UCS401)

Lecture-28

(Chi-Square, t and F- Sampling distributions with illustrations)

Sampling Distributions and Theory of Estimation (Unit –V & VI)

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

Dr. Rajanish Rai
Assistant Professor
School of Mathematics
Thapar Institute of Engineering and Technology, Patiala

Rajesh Rai

Sampling distribution (χ^2, t & F distribution)

Content Covered :

- χ^2 – distribution
- t – distribution
- F – distribution.

Note that :

* Let $x_1, x_2, x_3, \dots x_n$ be a random sample from a population with mean μ and variance σ^2, M.g.f. $M_X(t)$. Then

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{as } n \to \infty .$$

* If population is itself Normal then

- $$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$ irrespective of large or small n.

$$\Rightarrow \bar{X}-\mu \sim N\left(0, \sigma^2/n\right)$$

$$\Rightarrow \bar{X} \sim N\left(\mu, \sigma^2/n\right).$$

* Population $A \sim N(\mu_1, \sigma_1^2)$ Population $B \sim N(\mu_2, \sigma_2^2)$

$$\Rightarrow \overline{X_1} \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \qquad \overline{X_2} \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

$$\overline{X_1} - \overline{X_2} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

Variance always gets added.

$$\frac{(\overline{X_1} - \overline{X_2}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

Note that: "If populations are not Normal then this is valid for large n."

i.e., $n > 30$.

Chi-square distribution :

A continuous random variable X is $\chi^2_{(n)}$ if its p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{1}{2^{n/2}\,\Gamma\!\left(\frac{n}{2}\right)} e^{-x/2} x^{n/2-1} & x>0,\ n>0 \\ 0 & o/w \end{cases}$$

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} e^{-x/2} x^{n/2-1}\,dx$$

$$= \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-x/2} x^{n/2-1}\,dx$$

$$\because \int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$$

$$= \frac{1}{\cancel{2^{n/2}}\,\cancel{\Gamma\left(\frac{n}{2}\right)}} \frac{\cancel{\Gamma\left(\frac{n}{2}\right)}}{\cancel{\left(\frac{1}{2}\right)^{n/2}}} = 1$$

So X is basically Gamma$\left(\frac{n}{2}, \frac{1}{2}\right)$

$X \sim G(\alpha, \beta)$

$$f(x) = \frac{\beta^\alpha e^{-\beta x} x^{\alpha-1}}{\Gamma(\alpha)} \quad x>0.$$

$E(X) = \alpha/\beta \qquad Var(X) = \frac{\alpha}{\beta^2} \qquad M_X(t) = \left(1 - t/\beta\right)^{-\alpha};\ |t|<\beta.$

$$\Rightarrow E(X) = \frac{(n/2)}{1/2} = n$$

$$\Rightarrow Var(X) = \frac{(n/2)}{(1/4)} = 2n$$

$$\Rightarrow M_X(t) = (1-2t)^{-n/2} \quad |t| < 1/2.$$

i.e., If $X \sim \chi^2_{(n)}$ then

$$E(X) = n, \quad Var(X) = 2n$$

$$\& \quad M_X(t) = (1-2t)^{-n/2}; \quad |t| < 1/2.$$

Results: If $Y \sim N(0,1)$ then $X = Y^2 \sim \chi^2_{(1)}$	If $X_i \sim \chi^2_{(1)}$, all X_i are independent $\Rightarrow \sum_{i=1}^{n} X_i \sim \chi^2_{(n)}$.

$\Downarrow$

- $P(X \leq x) = F_X(x)$

$$F_X(x) = P(Y^2 \leq x)$$

$$= P(|Y| \leq \sqrt{x})$$

$$= P(-\sqrt{x} \leq Y \leq \sqrt{x})$$

$\because \; Y \sim N(0,1)$

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= P\left(-\sqrt{x} \le Y \le \sqrt{x}\right)$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} f(y)\, dy$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$$\Rightarrow \quad F_X(x) = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

we know that

$$\frac{d}{dx}\left(F_X(x)\right) = f(x)$$

we know by Leibnitz integral rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\, dt\right) = f\left(x, b(x)\right)\frac{d}{dx}\left(b(x)\right) - f\left(x, a(x)\right)\frac{d}{dx}\left(a(x)\right) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt.$$

In particular

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\frac{d}{dx}(b(x)) - f(a(x))\frac{d}{dx}(a(x)).$$

Therefore,

$$\frac{d}{dx}F_X(x) = f(x) = \frac{d}{dx}\left(\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x/2}\frac{d}{dx}(\sqrt{x}) - \frac{1}{\sqrt{2\pi}} e^{-x/2}\frac{d}{dx}(-\sqrt{x})$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x/2}\left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{\sqrt{2\pi}} e^{-x/2}\left(\frac{1}{2\sqrt{x}}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sqrt{x}} e^{-x/2}$$

$$f(x) = \frac{1}{2^{1/2}\,\Gamma(1/2)} e^{-x/2} x^{1/2-1}; \quad 0 < x < \infty$$

$$X \sim \chi^2_{(1)} \qquad \because \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

$\because$ Square of a standard Normal $\sim \chi^2_{(1)}$

$$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

and $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2_{(1)}$.

Results: If $X_i \sim N(0,1)$ be independent random variables

then $Y_n = \sum_{i=1}^{n} X_i^2 \sim \chi^2_{(n)}$.

Result: If $\bar{X}$ and s^2 are the mean and variance of a random sample of size n from a normal population with mean μ and standard deviation σ, then $\bar{X}$ and s^2 are independent and $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Assuming (i)st part to be true

use $s^2 = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$

$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - \mu)^2$$

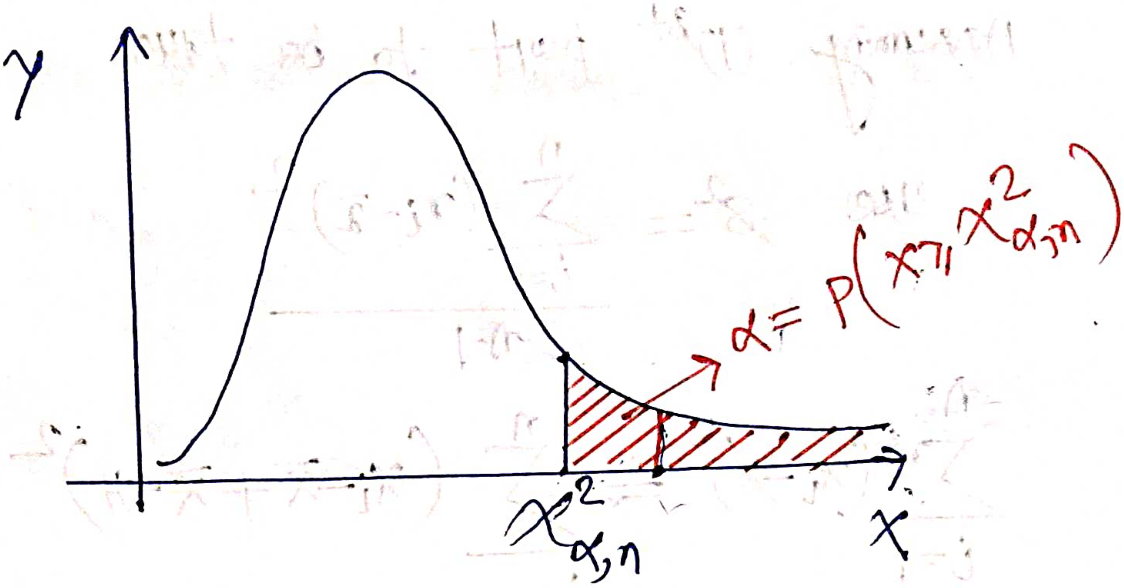
$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + \sum_{i=1}^{n} (\bar{x} - \mu)^2 + 2\sum_{i=1}^{n} \underbrace{(x_i - \bar{x})}_{0}(\bar{x} - \mu)$$

$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

$$\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} + \frac{n(\bar{x} - \mu)^2}{\sigma^2}$$

$$\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 = \frac{(n-1)s^2}{\sigma^2} + \left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2$$

$$\chi^2_{(n)} = [??] + \chi^{(2)}_{(1)} \Rightarrow \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}.$$



Generally if $n \leq 30$, we use χ^2 – probability distribution.

$$P(X \geq \chi^2_{\alpha,n}) = \alpha = \text{Area to the right of point } \chi^2_{\alpha,n}.$$

Question –: Semiconductors are manufactured by a Company. The manufacturing process is said to be under control if $\sigma^2 \leq 0.36$, where X is the thickness of the semi conductor. To check the manufacturing process. A random sample of size 20 are checked periodically. The process is regarded to be out of control if

$$P\left(s^2 \text{ will take on a value } \geq \text{ sample value}\right) \leq 0.01.$$

Solution-: Given

$$P(s^2 \geq x) \leq 0.01$$

$$P\left(\frac{(n-1)s^2}{\sigma^2} \geq \frac{(n-1)x}{\sigma^2}\right) \leq 0.01$$

Here $n = 20$ $\qquad \sigma^2 \leq 0.36$

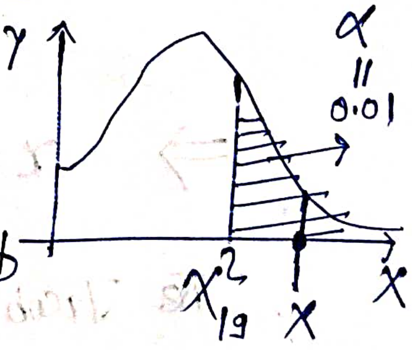
$$\frac{1}{\sigma^2} \geq \frac{1}{0.36}$$

$$\Rightarrow \frac{(n-1)x}{\sigma^2} \geq \frac{(n-1)x}{0.36}$$

$$\Rightarrow P\left(\chi^2_{(n-1)} \geq \frac{(n-1)x}{\sigma^2}\right) \leq 0.01$$

$$\Rightarrow P\left(\chi^2_{19} \geq \frac{19x}{0.36}\right) \leq 0.01$$

From χ^2-table



$$\chi^2_{0.01,19} = 36.191$$

$$\Downarrow$$

$$P(x \geq \chi^2_{19}) \leq 0.01$$

$$\Rightarrow \frac{19x}{0.36} \geq 36.191$$

$\Rightarrow \quad x \geq 0.6857$

$\Rightarrow$ If the sample variance of a specific lot ≥ 0.6857

$\Rightarrow$ Process is OUT of Control.

t-distribution (Student's t-distribution)

Let $X \sim N(0,1)$, $Y \sim \chi^2_{(n)}$ independent, then $\dfrac{X}{\sqrt{Y/n}} \sim T_{(n)}$

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{2^{n/2}\Gamma\frac{n}{2}} e^{-y/2} y^{n/2 - 1}$$

$$-\infty < x < \infty$$
$$0 < y < \infty$$

Let $T = \dfrac{X}{\sqrt{\frac{Y}{n}}}$ and $U = Y$

$\Rightarrow \quad x = \sqrt{\frac{u}{n}}\, t$ and $y = u$

The Jacobian $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial t} \end{vmatrix}$ or $\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix}$

$$J = \begin{vmatrix} \sqrt{u/n} & \frac{t}{2\sqrt{nu}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

$$f_{T,U}(t,u) = f\big(x(t,u),\, y(t,u)\big)\, |J|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{n}\frac{t^2}{2}} \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-u/2}\, u^{n/2-1} \sqrt{\frac{u}{n}}$$

$$= \frac{1}{2^{\frac{n+1}{2}} \sqrt{\pi n}\, \Gamma\frac{n}{2}} e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)} u^{\frac{n+1}{2}-1}$$

$$f_{T,U}(t,u) = \frac{1}{2^{n+1/2} \sqrt{\pi n}\, \Gamma\frac{n}{2}} e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)} u^{\frac{n+1}{2}-1}$$

$$-\infty < t < \infty$$
$$0 < u < \infty.$$

$$f_T(t) = \int_u f_{T,U}(t,u)\, du$$

$$f_T(t) = \int_0^\infty \frac{1}{2^{n+1/2}\sqrt{\pi n}\, \Gamma\frac{n}{2}} e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)} u^{\frac{n+1}{2}-1}\, du$$

$$f_T(t) = \frac{1}{2^{\frac{n+1}{2}}\sqrt{\pi n}\,\Gamma\frac{n}{2}} \int_0^\infty e^{-\frac{1}{2}\left(1+\frac{t^2}{n}\right)u}\, u^{\frac{n+1}{2}-1}\, du$$

We know that

$$\int_0^\infty e^{-ax}\, x^{n-1}\, dx = \frac{\Gamma n}{a^n}$$

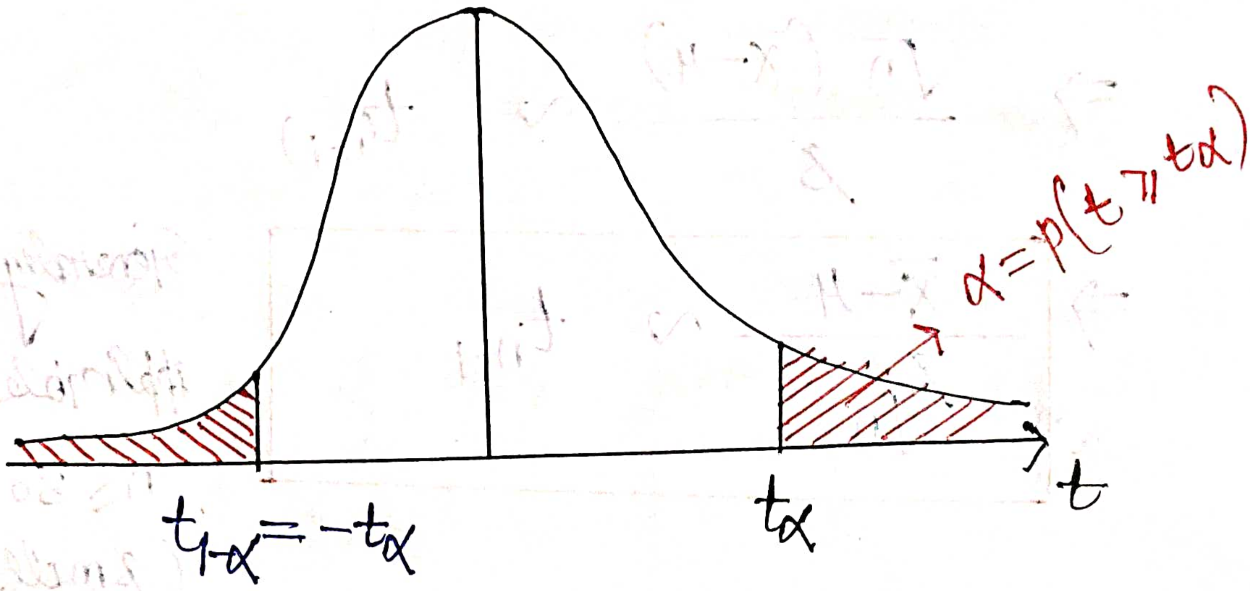
$$f_T(t) = \frac{1}{2^{n+1/2}\sqrt{\pi n}\,\Gamma\frac{n}{2}} \frac{\Gamma\frac{n+1}{2}}{\left[\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right]^{\frac{n+1}{2}}}$$

$$f_T(t) = \frac{\cancel{2^{n+1/2}}}{\cancel{2^{n+1/2}}\sqrt{\pi n}\,\Gamma\frac{n}{2}} \frac{\Gamma\frac{n+1}{2}}{\left(1+\frac{t^2}{n}\right)^{n+1/2}}$$

$$f_T(t) = \frac{\Gamma\frac{n+1}{2}}{\sqrt{\pi n}\,\Gamma\frac{n}{2}} \left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} ; -\infty < t < \infty$$

This is known as t- distribution with n- degree of freedom.

$E(t) = 0 \quad \because f_T(t)$ is symmetric about 0.



* Assuming normal population with finite mean and variance $\bar{X}$ and s^2 are independent.

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

and $$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

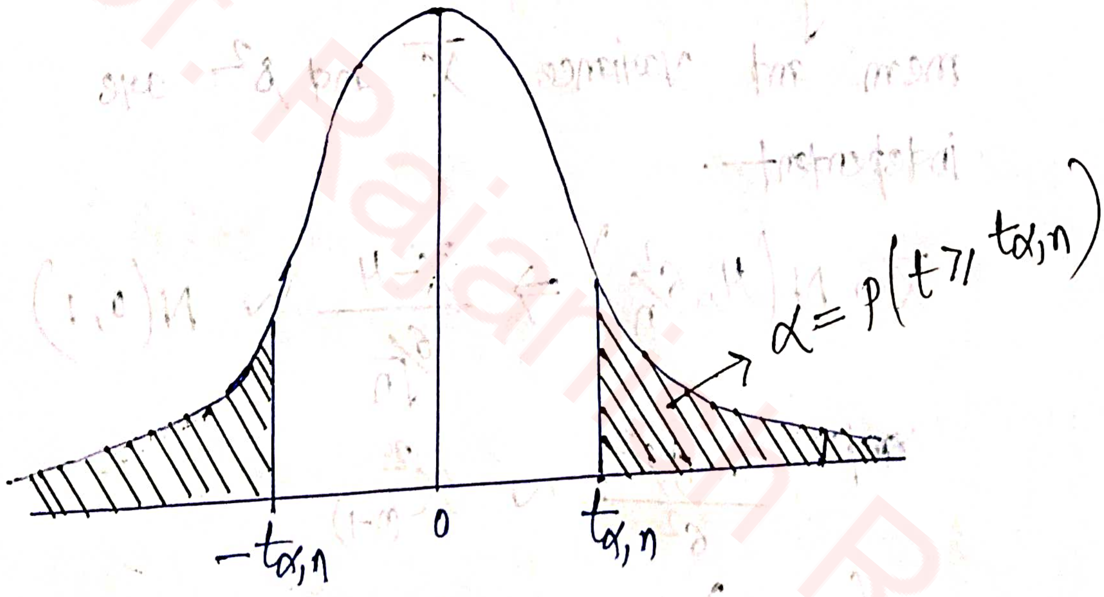
$$\frac{\left(\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\dfrac{\cancel{(n-1)}s^2}{\cancel{\sigma^2}\cancel{(n-1)}}}} \sim t_{(n-1)}$$

$$\Rightarrow \frac{\sqrt{n}\,(\bar{X}-\mu)}{\sqrt{s^2}} \sim t_{(n-1)}$$

$$\Rightarrow \frac{\sqrt{n}\,(\bar{x}-\mu)}{s} \sim t_{(n-1)}$$

$$\Rightarrow \boxed{\frac{\bar{x}-\mu}{s/\sqrt{n}} \sim t_{n-1}}$$

Generally applicable for $n \leq 30$ (small sample).



$$\alpha = P(t \geq t_{\alpha,n})$$

$$\alpha = P(t \leq -t_{\alpha,n})$$

$$\because \alpha = P(t \geq t_{\alpha,n})$$

$$\alpha = 1 - P(t \leq t_{\alpha,n})$$

$$P(t \leq t_{\alpha,n}) = 1-\alpha .$$

Question: One hour test of 16 students the marks of the student averaged 16.4. with S.D. 2.1
Test the claim that the average marks of this students exceeds 12.

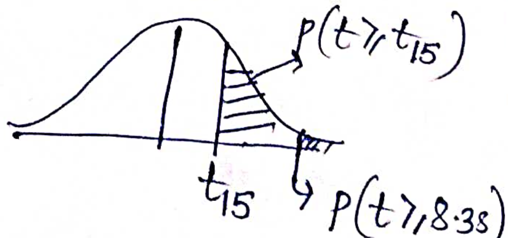
Solution: Given

$$n = 16$$
$$\bar{x} = 16.4$$
$$s = 2.1$$
$$\mu = 12$$

According to question

$$P(\mu \geq 12)$$
$$\Rightarrow P(-\mu \leq -12)$$
$$\Rightarrow P\left(\frac{\bar{x}-\mu}{s/\sqrt{n}} \leq \frac{16.4-12}{2.1/\sqrt{16}}\right)$$
$$\Rightarrow P(t \leq 8.38)$$
$$= 1 - P(t \geq 8.38)$$
$$= 1 - \text{Negligible}$$
$$= 1$$

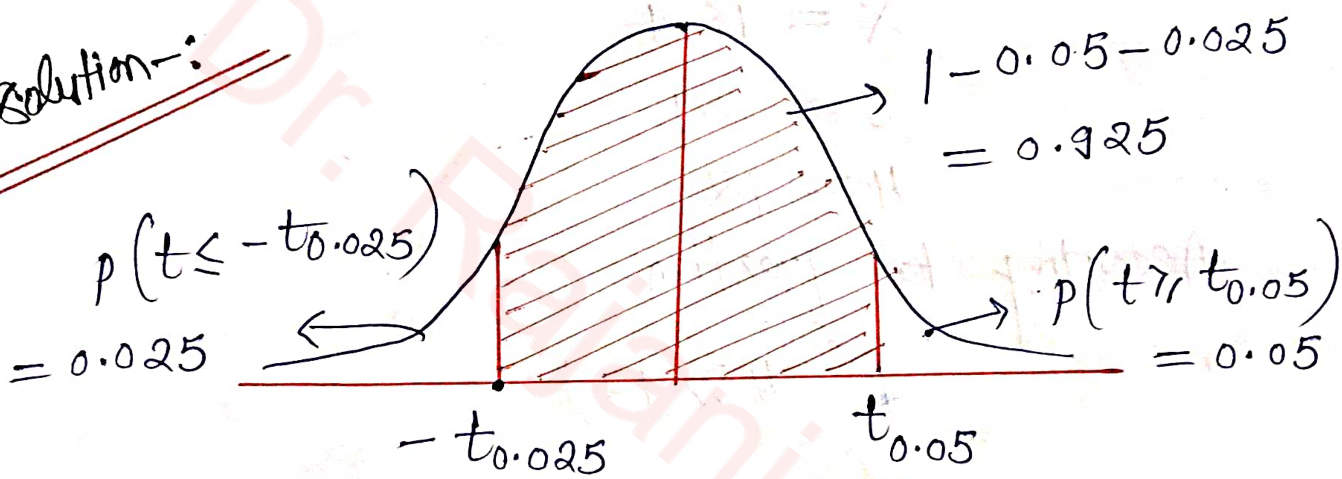
(See table $t \sim t_{15}$)



⇒ It is safe to assume that the average marks of the student are indeed greater than 12.

Question : Find $p(-t_{0.025} < T < t_{0.05})$

Solution :-



Since $t_{0.05}$ leave an area of 0.05 to the right, and $-t_{0.025}$ leaves an area of 0.025 to the left, we find a total area of

$$1 - 0.05 - 0.025 = 0.925$$

between $-t_{0.025}$ and $t_{0.05}$. Hence

$$p(-t_{0.025} < T < t_{0.05}) = 0.925$$

#

$$\because \quad \sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

$$\& \quad s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{(n-1)}$$

$$s^2 = \frac{n\sigma^2}{n-1}$$

$$\Rightarrow \boxed{s^2 = \frac{n\sigma^2}{n-1}} \quad (n \leq 30) \text{ (for small sample)}$$

when $n \to \infty$

$$s^2 = \lim_{n \to \infty} \left(\frac{n}{n-1}\right) \sigma^2 \quad (n > 30)$$

$$\Rightarrow \boxed{s^2 = \sigma^2} \quad \text{(for large sample)}.$$

---

66 use when POPUTION VARIANCE is NOT KNOWN so σ^2 is replaced or approximated by s^2 or σ by s.

F-distribution :

Let $U \sim \chi^2_{(m)}$, $V \sim \chi^2_{(n)}$

Both independent from one another.

Then $F = \dfrac{U/m}{V/n}$ is a R.V. with $F_{m,n}$.

We know that

$$\frac{(n_1-1)s_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}$$

$$\frac{(n_2-1)s_2^2}{\sigma_2^2} \sim \chi^2_{n_2-1}$$

$\Rightarrow$ Assuming these quantities are from independent random samples then

$$\left(\frac{s_1^2}{\sigma_1^2}\right) \Big/ \left(\frac{s_2^2}{\sigma_2^2}\right) \sim F_{n_1-1,\, n_2-1}$$

$$\Rightarrow \boxed{\frac{s_1^2\,\sigma_2^2}{s_2^2\,\sigma_1^2} \sim F_{n_1-1,\, n_2-1}}$$

$$\boxed{F_{n_1-1,\, n_2-1} = \frac{1}{F_{n_2-1,\, n_1-1}}}$$

ex $\dfrac{1}{F_{3,2}} = F_{2,3}$.

Question: The claim that the variance of a normal population is $\sigma^2 = 25$ is to be rejected if the variance of a random sample of size 16 exceeds 54.668 or is less than 12.102. What is the probability that this claim will be rejected even though $\sigma^2 = 25$?

Solution:- Given

$$\sigma^2 = 25, \quad n = 16$$

$$P(\text{rejected}) = P(s^2 \geq 54.668) + P(s^2 \leq 12.102)$$

$$= P\left(\frac{(n-1)s^2}{\sigma^2} \geq \frac{(15)(54.668)}{25}\right) + P\left(\frac{(n-1)s^2}{\sigma^2} \leq \frac{(15)(12.102)}{25}\right)$$

$$= P\left(\chi^2_{15} \geq 32.8008\right) \quad \text{where } X \sim \chi^2_{15}$$

$$+ P\left(\chi^2_{15} \leq 7.2612\right)$$

$$= (0.005) + \left(1 - P\left(\chi^2_{15} \geq 7.2612\right)\right)$$

$$= 0.005 + (1 - 0.95) = 0.005 + 0.05$$

$$= 0.055.$$

Question: A random sample of size $n = 81$ is taken from an infinite population with the mean $\mu = 128$ and the standard deviation $\sigma = 6.3$. With what probability can we assert that the value we obtain for $\bar{X}$ will not fall between 126.6 and 129.4 if we use

(a) Chebyshev's theorem

(b) the central limit theorem?

Solution:

Given

$n = 81$ (large sample)

$\mu = 128$

$\sigma = 6.3$

The required probability

$$= 1 - P(126.6 < \bar{X} < 129.4)$$

Thus $$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-128}{6.3/\sqrt{81}}$$

$$= 1 - P\left(\frac{126.6-128}{6.3/\sqrt{81}} < \frac{\bar{X}-128}{6.3/\sqrt{81}} < \frac{129.4-128}{6.3/\sqrt{81}}\right)$$

$$= 1 - P(-2 < Z < 2)$$

$$= 1 - 2P(Z \leq 2)$$

$$= 0.0455$$ Ans

Question-: The claim that the variance of a normal population is $\sigma^2 = 4$ is to be rejected if the variance of a random sample of size 9 exceeds 7.7535. What is the probability that this claim will be rejected even though $\sigma^2 = 4$?

Solution-: Given

$$n = 9$$

$$\sigma^2 = 4$$

The required probability

$$P(\text{rejected}) = P(s^2 \geq 7.7535)$$

$$= P\left(\frac{(n-1)s^2}{\sigma^2} \geq \frac{(8)(7.7535)}{4}\right)$$

$$= P(\chi^2 \geq 15.507) \quad \chi^2 \sim \chi^2_8$$

$$= 0.05$$ Ans

Question: A random sample of size $n = 12$ from a normal population has the mean $\bar{x} = 27.8$ and the variance $s^2 = 3.24$. If we base our decision on the statistic of Theorem 13, can be say that the given information supports the claim that the mean of the population is $\mu = 28.5$?

Solution: Given that

$$n = 12$$

$$\bar{x} = 27.8$$

$$s^2 = 3.24$$

Calculate t- statistic t_{11}

$$\frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{27.8 - 28.5}{\sqrt{\frac{3.24}{12}}} = -1.3471$$

$$= -t_{0.10, 11}$$

$$\because -t_{\alpha} = t_{1-\alpha}$$

$$= t_{0.90, 11}$$

Interpretation: The claim can not be rejected.

⇓

$\therefore$ probility is 90%.

#

Question: A random sample of size $n = 25$ from a normal population has the mean $\bar{x} = 47$ and the standard deviation $s = 7$. If we base our decision on the statistics of Theorem 13, can we say that the given information supports the conjecture that the mean of population is $\mu = 42$?

Solution: Given

$$n = 25$$

$$\bar{X} = 47$$

$$s^2 = 49$$

Calculate t- statistics

$$\frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{47 - 42}{7/\sqrt{25}} = 3.57 > t_{0.005, 24}$$

Interpretation: The hypothesis is probably false.

Question: If s_1 and s_2 are the standard deviations of independent random sample of sizes $n_1 = 61$ and $n_2 = 31$ from normal population with $\sigma_1^2 = 12$ and $\sigma_2^2 = 18$, find $P\left(\frac{s_1^2}{s_2^2} > 1.16\right)$.

Solution: Given that

$$n_1 = 61 \; ; \; \sigma_1^2 = 12$$

$$n_2 = 31 \; ; \; \sigma_2^2 = 18.$$

To find $P\left(\frac{s_1^2}{s_2^2} > 1.16\right)$

$$= P\left(\frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} > \frac{(18)(1.16)}{(12)}\right)$$

$$= P(F > 1.74) \qquad F \sim F_{60, 30}$$

$$= 0.05$$

Question: If s_1^2 and s_2^2 are the variances of independent random samples of sizes $n_1 = 10$ and $n_2 = 15$ from normal populations with equal variances, find $P\left(\frac{s_1^2}{s_2^2} < 4.03\right)$.

Solution: Given

$n_1 = 10$; σ_1^2

$n_2 = 15$; σ_2^2

$$\sigma_1^2 = \sigma_2^2 = \sigma^2$$

$$P\left(\frac{s_1^2}{s_2^2} < 4.03\right)$$

$$P\left(\frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} < \frac{\sigma_2^2\, 4.03}{\sigma_1^2}\right)$$

$$= P(F < 4.03)$$

$$= 1 - P(F \geq 4.03) \quad F \sim F_{9, 14}$$

$$= 1 - 0.01$$

$$= 0.99$$

Table IV: Values of $t_{\alpha,\nu}$†

ν	$\alpha = .10$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$	ν
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
inf.	1.282	1.645	1.960	2.326	2.576	inf.

†Based on Richard A. Johnson and Dean W. Wichern, *Applied Multivariate Statistical Analysis*, 2nd ed., © 1988, Table 2, p. 592. By permission of Prentice Hall, Upper Saddle River, N.J.

Table V: Values of $\chi^2_{\alpha,\nu}$ †

ν	$\alpha = .995$	$\alpha = .99$	$\alpha = .975$	$\alpha = .95$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$	ν
1	.0000393	.000157	.000982	.00393	3.841	5.024	6.635	7.879	1
2	.0100	.0201	.0506	.103	5.991	7.378	9.210	10.597	2
3	.0717	.115	.216	.352	7.815	9.348	11.345	12.838	3
4	.207	.297	.484	.711	9.488	11.143	13.277	14.860	4
5	.412	.554	.831	1.145	11.070	12.832	15.086	16.750	5
6	.676	.872	1.237	1.635	12.592	14.449	16.812	18.548	6
7	.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278	7
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955	8
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589	9
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188	10
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757	11
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300	12
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819	13
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319	14
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801	15
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267	16
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718	17
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156	18
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582	19
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997	20
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401	21
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796	22
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181	23
24	9.886	10.856	12.401	13.848	36.415	39.364	42.980	45.558	24
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928	25
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290	26
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645	27
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993	28
29	13.121	14.256	16.047	17.708	42.557	45.722	49.588	52.336	29
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672	30

†Based on Table 8 of *Biometrika Tables for Statisticians*, Vol. 1, Cambridge University Press, 1954, by permission of the *Biometrika* trustees.

Table VI: Values of $f_{0.05,\nu_1,\nu_2}$[†]

ν_2 = Degrees of freedom for denominator	ν_1 = Degrees of freedom for numerator: 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07

[†]Reproduced from M. Merrington and C. M. Thompson, "Tables of percentage points of the inverted beta (F) distribution," *Biometrika*, Vol. 33 (1943), by permission of the *Biometrika* trustees.

Table VI: (*continued*) Values of $f_{0.05,\nu_1,\nu_2}$

ν_1 = Degrees of freedom for numerator; ν_2 = Degrees of freedom for denominator

ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table VI: (*continued*) Values of $f_{0.01,\nu_1,\nu_2}$

ν_2 = Degrees of freedom for denominator	ν_1 = Degrees of freedom for numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023	6,056	6,106	6,157	6,209	6,235	6,261	6,287	6,313	6,339	6,366
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.3	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	6.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87

Table VI: (*continued*) Values of $f_{0.01,\nu_1,\nu_2}$

ν_2 = Degrees of freedom for denominator	ν_1 = Degrees of freedom for numerator: 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.53	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00