PROBABILITY AND STATISTICS (UCS401)

Lecture-17

Two-dim. c.r.v.'s (Joint, Marginal and conditional distribution)

Two-dim. r.v.'s and Joint Distributions (Unit -V)

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

Dr. Rajanish Rai
Assistant Professor
School of Mathematics
Thapar Institute of Engineering and Technology, Patiala

Two-dimensional random variables - Continuous variables (Joint, marginal and conditional distribution)

Rajnish Rai

A two-dimensional random variable (X,Y) is said to be continuous, if there exists a non-negative function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for every pair $(x,y) \in \mathbb{R}^2$, we have

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dy\, dx.$$

where F is the distribution function of (X,Y).

The function f is called joint density function (J.d.f.) if

$$\boxed{\begin{gathered} f(x,y) \geq 0 \\ \int_x \int_y f(x,y)\, dy\, dx = 1 \end{gathered}}$$

Marginal density functions :

For two-dimensional r.v.'s X and Y the marginal density function of X, denoted by $f_X(x)$, is given as

$$\boxed{f_X(x) = \int_y f(x,y)\,dy} \quad \left(\begin{array}{c} x \text{ fixed,} \\ y \text{ varies} \end{array}\right)$$

* The marginal density function of Y, denoted by $f_Y(y)$, is given as

$$\boxed{f_Y(y) = \int_x f(x,y)\,dx} \quad \left(\begin{array}{c} y \text{ fixed,} \\ x \text{ varies} \end{array}\right)$$

Conditional density function :

The conditional density function for X given Y, denoted by $f(X/Y)$ or $f_{X/Y}$ is defined as

$$\boxed{f(X/Y) = \frac{f(x,y)}{f_Y(y)}}$$

where $f_Y(y)$ is marginal density function of Y.

Similarly, the conditional density function for Y given X, denoted by $f(y/x)$ or $f_{Y/X}$ is defined as:

$$\boxed{f(y/x) = \frac{f(x,y)}{f_X(x)}}$$

Where $f_X(x)$ is the marginal density function of X.

Independent random variables:

Two random variable X and Y are said to be independent if

$$f(x,y) = f_X(x) \cdot f_Y(y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Question: The joint density function of random variables X and Y is given by

$$f(x,y) = \begin{cases} kxy & ; 0<x<1, \ 0<y<1 \\ 0 & ; \text{elsewhere} \end{cases}$$

Find (i) Value of k

(ii) Marginal density function of X and Y

(iii) check weather X and Y are independent or not?

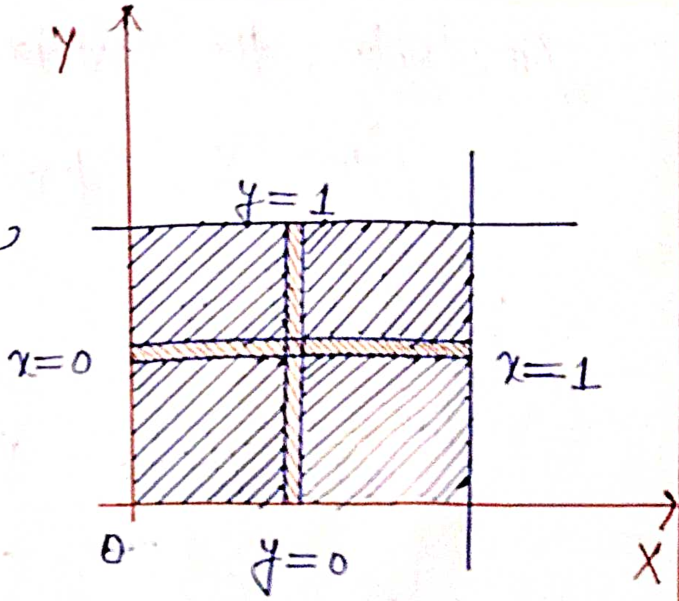
(iv) $P(X+Y \leq 1)$.

Solution:

Since $f(x,y)$ is J.d.f.,

so

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx = 1$$



$$\Rightarrow \int_0^1\int_0^1 kxy\,dx\,dy = 1$$

$$\Rightarrow \int_0^1 ky\left(\frac{x^2}{2}\right)_0^1 dy = 1$$

$$\Rightarrow \frac{k}{2}\left(\frac{y^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \boxed{k=4}$$ Ans

* Marginal density function of X

$$f_X(x) = \int_y f(x,y)\,dy$$

$$= \int_0^1 4xy\,dy = 4x\left(\frac{y^2}{2}\right)_0^1$$

$$\boxed{f_X(x) = 2x \;;\; 0<x<1}$$ Ans

and Marginal density function of Y

$$f_Y(y) = \int_x f(x,y)\,dx = \int_0^1 4xy\,dx$$

$$= 4y\left(\frac{x^2}{2}\right)_0^1$$

$$\boxed{f_Y(y) = 2y \;;\quad 0<y<1}$$

(iii) Check for the independence of X and Y

We have

$$f(x,y) = \begin{cases} 4xy & ; 0<x<1,\ 0<y<1 \\ 0 & o/w \end{cases}$$

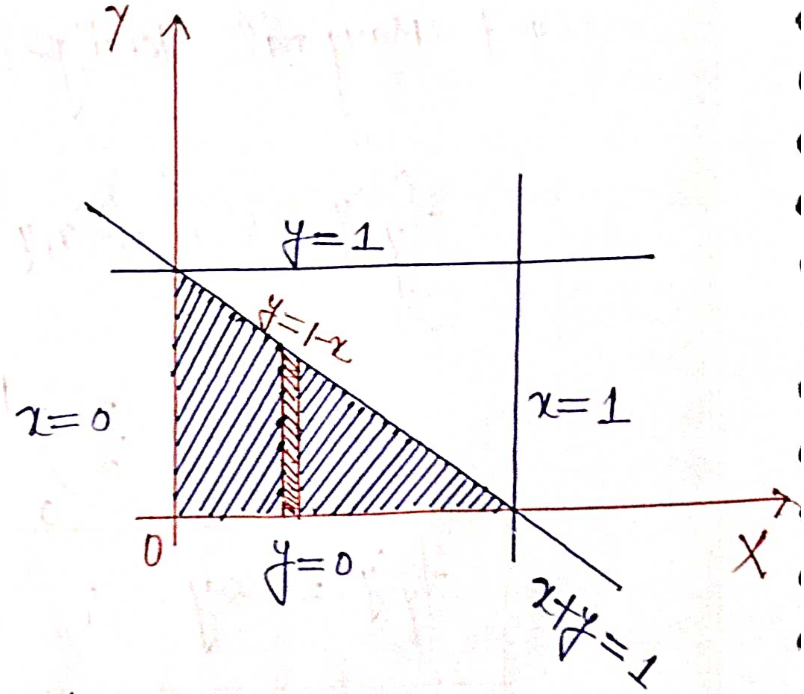
$$f_X(x) = 2x \qquad 0<x<1$$

$$f_Y(y) = 2y \qquad 0<y<1$$

Since $f(x,y) = f_X(x)\cdot f_Y(y)$

Thus, variables X and Y are independent.

(iv) $P(X+Y \le 1)$



$$= \int_x \int_y f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_{y=0}^{y=1-x} 4xy\, dy\, dx$$

$$= \int_0^1 (4x)\left(\frac{y^2}{2}\right)_0^{1-x} dx = \int_0^1 2x(1-x)^2\, dx$$

$$= \int_0^1 2x(1+x^2-2x)\, dx = \int_0^1 2\left(x+x^3-2x^2\right) dx$$

$$= 2\left[\frac{x^2}{2}+\frac{x^4}{4}-\frac{2}{3}x^3\right]_0^1 = \frac{1}{6}$$

$$\therefore \quad \boxed{P(X+Y \le 1) = \frac{1}{6}}$$

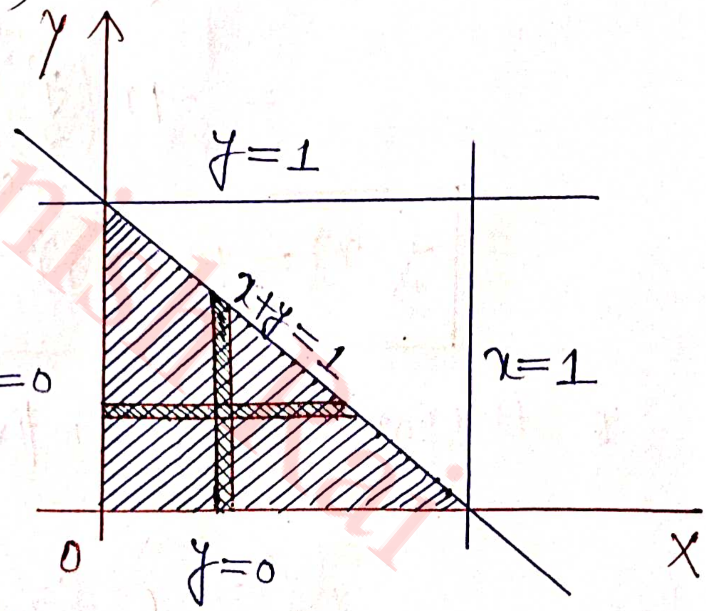
Question: The joint density function of random variables X and Y is given by

$$f(x,y)=\begin{cases} kxy & ; \ 0<x,y<1, \ x+y\leq 1 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

Find (i) value of k (ii) Marginal density function of X and Y

(iii) check whether X and Y are independent or not ?

(iv) $P\left(X+Y\leq \frac{1}{2}\right)$.



Solution:

Since $f(x,y)$ is J.d.f., so

$$\int_0^1\int_0^{1-x} f(x,y)\, dy\, dx = 1$$

$$\Rightarrow \int_0^1\int_0^{1-x} kxy\, dy\, dx = 1$$

$$\Rightarrow \int_0^1 kx\left(\frac{y^2}{2}\right)_0^{1-x} dx = 1$$

$$\Rightarrow \quad \frac{k}{2}\int_0^1 x(1-x)^2\,dx = 1$$

$$\Rightarrow \quad \frac{k}{2}\left(\frac{x^2}{2}+\frac{x^4}{4}-\frac{2}{3}x^3\right)_0^1 = 1$$

$$\Rightarrow \quad \frac{k}{2}\left(\frac{1}{12}\right) = 1$$

$$\Rightarrow \quad \boxed{k = 24}$$ Ans

* Marginal density function of X

$$f_X(x) = \int_y f(x,y)\,dy = \int_0^{1-x} 24xy\,dy$$

$$= 24x\left(\frac{y^2}{2}\right)_0^{1-x}$$

$$\boxed{f_X(x) = 12x(1-x)^2 \qquad 0<x<1}$$

* Marginal density function of Y

$$f_Y(y) = \int_x f(x,y)\,dx$$

$$= \int_0^{1-y} 24xy\,dx = 24y\left(\frac{x^2}{2}\right)_0^{1-y}$$

$$\boxed{f_Y(y) = 12y(1-y)^2 \; ; \quad 0<y<1}$$

Check for the independence of X and Y

Since
$$f(x,y)=\begin{cases}24xy\ ; & 0<x,y<1\ ,\ x+y\le 1\\ 0\ ; & \text{elsewhere}\end{cases}$$

$$f_X(x)=12x(1-x)^2\ ; \quad 0<x<1$$

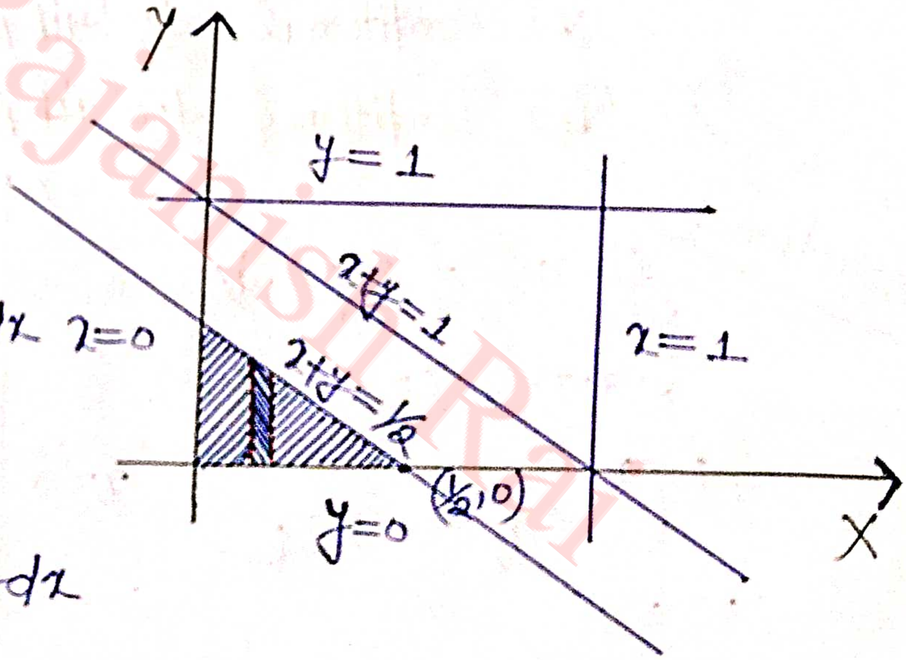
$$f_Y(y)=12y(1-y)^2\ ; \quad 0<y<1$$

Since, $f(x,y)\neq f_X(x)\cdot f_Y(y)$.

Thus, variables X and Y are NOT independent.

(iv)

$$P\left(X+Y\le \tfrac{1}{2}\right)$$



$$=\int_x\int_y f(x,y)\,dy\,dx$$

$$=\int_0^{1/2}\int_0^{1/2-x} 24xy\,dy\,dx$$

$$=\int_0^{1/2} 24x\left(\frac{y^2}{2}\right)_0^{1/2-x}dx=\int_0^{1/2}12x\left(\tfrac{1}{2}-x\right)^2dx=\frac{1}{16}$$

$$\boxed{P\left(X+Y\le \tfrac{1}{2}\right)=\frac{1}{16}}$$

Ans

Question: The joint density function of random variables X and Y is given by

$$f(x,y)=\begin{cases}\alpha & ; \; 0<x<1, \; 0<y<x \\ 0 & ; \; \text{elsewhere}\end{cases}$$

Find (i) value of α,

(ii) Marginal density function of X and Y,

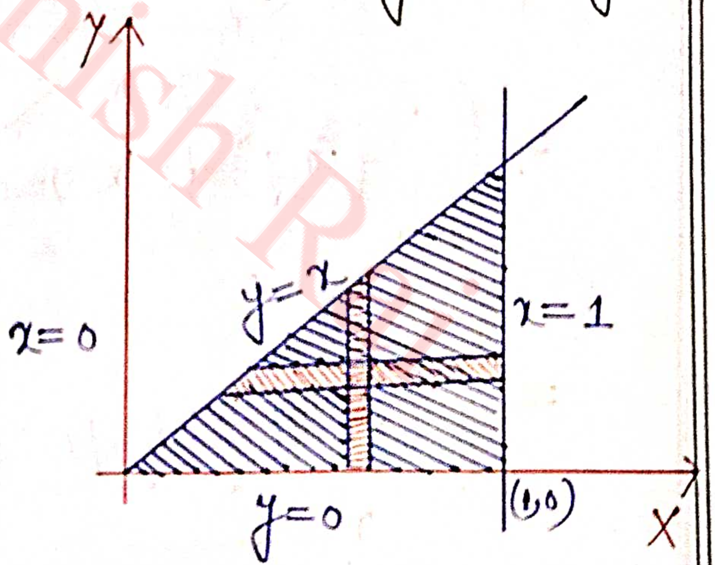
(iii) check whether X and Y are independent or not?

(iv) $P\left(X \leq \frac{1}{4}\right)$.

(v) Conditional distribution of Y given $X=x$.

(vi) Conditional distribution of X given $Y=y$.

Solution: Since $f(x,y)$ is J.d.f., so



$$\int_0^1 \int_0^x \alpha \, dy \, dx = 1$$

$$\Rightarrow \int_0^1 \alpha \, (y)_0^x \, dx = 1$$

$$\Rightarrow \alpha \int_0^1 x \, dx = 1$$

$$\Rightarrow \alpha \left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow \boxed{\alpha = 2}$$

$$\therefore f(x,y) = \begin{cases} 2 & 0<x<1,\ 0<y<x \\ 0 & o/w \end{cases}$$

$\therefore$ Marginal density function of X

$$f_X(x) = \int_y f(x,y)\,dy = \int_0^x 2\,dy = 2x$$

$$\boxed{f_X(x) = 2x \quad ; \quad 0<x<1}$$ Ans

and Marginal density function of y

$$f_Y(y) = \int_x f(x,y)\,dx = \int_{x=y}^1 2\,dx = 2(x)_y^1$$

$$\boxed{f_Y(y) = 2(1-y) \quad ; \quad 0<y<1}$$ Ans

Check for independence of X and Y

Since, $f(x,y) = \begin{cases} 2 & ;\ 0<x<1,\ 0<y<x \\ 0 & ;\ o/w \end{cases}$

$$f_X(x) = 2x \quad ; \quad 0<x<1$$

$$f_Y(y) = 2(1-y) \quad ; \quad 0<y<1$$

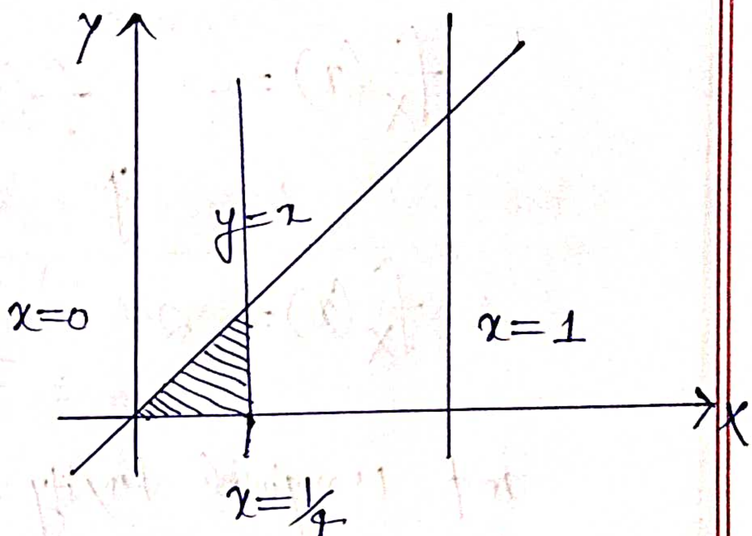
Since,

$$f(x,y) \neq f_X(x) \cdot f_Y(y)$$

Thus, variables X and Y are <u>NOT independent</u>.

(iv)

$$P\left(X \leq \tfrac{1}{4}\right) = \int_x f_X(x)\,dx$$



$$= \int_0^{1/4} 2x\,dx$$

$$= \left(x^2\right)_0^{1/4} = \frac{1}{16}$$

$$\therefore \quad \boxed{P\left(X \leq \tfrac{1}{4}\right) = \frac{1}{16}}$$

(v) Conditional distribution of Y given $X=x$

$$f(y|X=x) = \frac{f(y,x)}{f_X(x)}$$

$$= \frac{2}{2x}$$

$\because$ $f(x,y) = 2$

$f_X(x) = 2x$

$f_Y(y) = 2(1-y)$

$0 < x < 1$

$0 < y < x$

$$\boxed{f(y|X=x) = \frac{1}{x} \qquad 0 < x < 1}$$

Conditional distribution of X given $Y=y$

$$f(X|Y=y) = \frac{f(x,y)}{f_Y(y)}$$

where, $f_Y(y)$ is the marginal distribution function of Y.

$$= \frac{2}{2(1-y)} \qquad 0<y<x<1$$

$$\boxed{f(X|Y=y) = \frac{1}{1-y} \qquad 0<y<1}$$ Ans

Question-: The joint density function of random variables X and Y is given by

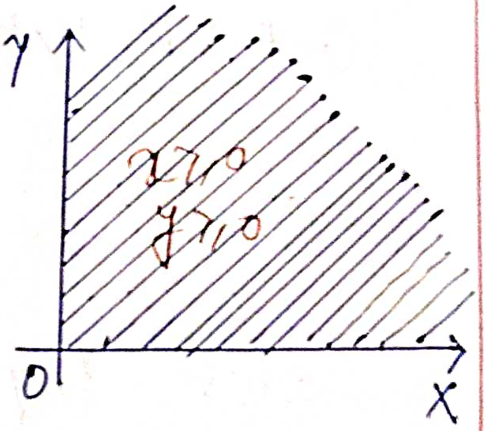
$$f(x,y) = \begin{cases} e^{-x-y} & ; \ x,y \geq 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Find $P(X>1)$, $E(X)$, $E(Y)$, $E(XY)$ and $E(X+Y)$.

Check whether X and Y are independent or not?

Solution:- Since

$$P(X>1) = \int_1^\infty f_X(x)\,dx$$



$$E(X) = \int_0^\infty x f_X(x)\,dx$$

$$E(Y) = \int_0^\infty y f_Y(y)\,dy$$

$$E(XY) = \int_0^\infty\int_0^\infty xy f(x,y)\,dy\,dx$$

$$E(X+Y) = E(X) + E(Y)$$

or

$$E(X+Y) = \int_0^\infty\int_0^\infty (x+y) f(x,y)\,dy\,dx.$$

* Marginal probability density function of X

$$f_X(x) = \int_y f(x,y)\,dy = \int_0^\infty e^{-x-y}\,dy$$

$$= e^{-x}\left(-e^{-y}\right)_0^\infty = e^{-x}\left(-\cancel{e^{-\infty}} + e^0\right)$$

$$\therefore \boxed{f_X(x) = e^{-x} \quad ; \quad x \geq 0}$$

||ly

$$f_Y(y) = \int_x f(x,y)\,dx = e^{-y} \quad y \geq 0.$$

(i) $\therefore \quad P(X>1) = \int_1^\infty f_X(x)\,dx$

$$= \int_1^\infty e^{-x}\,dx = \left(-e^{-x}\right)_1^\infty$$

$$P(X>1) = -\cancel{e^{-\infty}}^{\,0} + e^{-1}$$

$$\boxed{P(X>1) = 1/e}$$

(ii) $E(X) = \int_0^\infty x f_X(x)\,dx = \int_0^\infty x e^{-x}\,dx$

$$= \cancel{\left(-xe^{-x}\right)_0^\infty}^{\,0} + \cancel{\int_0^\infty e^{-x}\,dx}^{\,1} = \left(-e^{-x}\right)_0^\infty$$

$\therefore \quad \boxed{E(X) = 1}$

(iii) $E(Y) = \int_0^\infty y f_Y(y)\,dy = \int_0^\infty y e^{-y}\,dy$

$$= \cancel{\left(-ye^{-y}\right)_0^\infty}^{\,0} + \int_0^\infty e^{-y}\,dy = -\left(e^{-y}\right)_0^\infty$$

$$= -\left(e^{-\infty} - e^0\right)^0 = 1$$

$$\boxed{E(Y) = 1}$$

(iv) $$E(XY) = \int_0^\infty \int_0^\infty xy\, e^{-x-y}\, dy\, dx$$

$$= \int_0^\infty x e^{-x} \left(\int_0^\infty y e^{-y} dy \right) dx$$

$$= \int_0^\infty x e^{-x} (1)\, dx$$

$$\boxed{E(XY) = 1}$$

(v) $E(X+Y) = E(X) + E(Y) = 2$

or

$$E(X+Y) = \int_0^\infty \int_0^\infty (x+y) e^{-x-y}\, dy\, dx = 2$$

Check for independence of X and Y:

$f(x,y) = e^{-x-y}$

$f_X(x) = e^{-x}$

$f_Y(y) = e^{-y}$

Since $f(x,y) = f_X(x) \cdot f_Y(y)$

Thus, variables X and Y are independent. //