PROBABILITY AND STATISTICS
(UCS401)
Lecture-14
(Concept of Hypergeometric Distribution with illustrations)
Random Variables and their Special Distributions (Unit –III & IV)

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Hypergeometric distribution

- **Sampling with and without replacement :-**

- An urn contains 1000 balls : 700 green & 300 reds. A sample of 7 balls is drawn. What is the probability that it has 3 green & 4 red balls ?

(i) Sampling with replacement :-

- Pick one ball & record it colour.
- Put it back in the urn & shake it up.
- Again, pick one ball and record colour.
- Repeat this n times.

From, here $P(\text{green}) = 0.7$ in each draw

$P(\text{red}) = (0.3)$ in each draw

$$P(3G \,\&\, 4R) = {}^7C_3 (0.7)^3 (0.3)^4$$

$$= 0.09724$$

Ans

Binomial distribution

$n = 7$ (finite)

Probability p of success is constant for each trail.

(ii) Sampling without replacement :- 1000 : 700G & 300R

- Pick on ball, record it colour & set it aside.
- Pick one new ball from the remaining 999 balls, record it colour & set it aside.

- Pick one new ball from the remaining 998 balls, record it colour & set it aside.
- Repeat n times

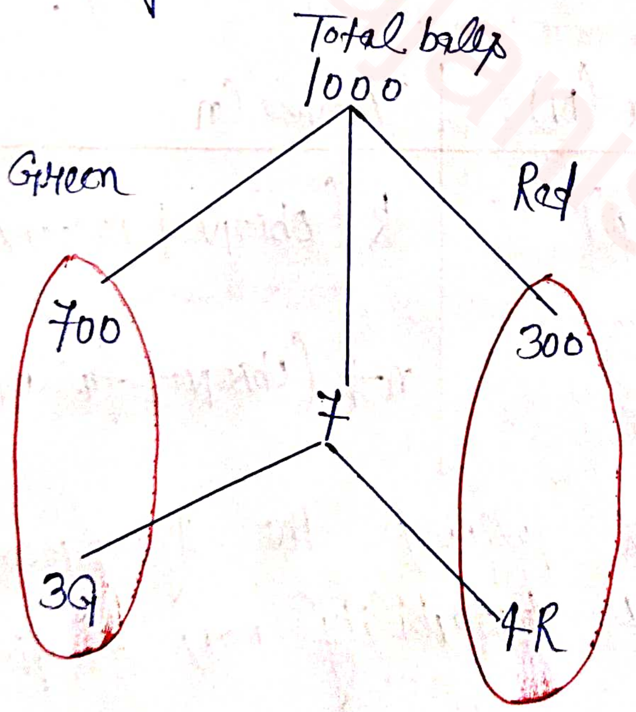
In this case

$P(\text{green}) \neq 0.7$ in each draw.

1G	2nd G	3rd G
700 G	699 G	698 G
300 R	300 R	300 G
1000	999	998

$P(1G) = \frac{700}{1000}$ $\quad P(2nd\,G) = \frac{699}{999}$ $\quad P(3rd\,G) = \frac{698}{998}$

Here each draw has different probability to be green. ⊗ Binomial



$\Rightarrow$ without replacement

$\Downarrow$

Hypergeometric distribution

$$P(3G \,\&\, 4R) = \frac{{}^{700}C_3 \times {}^{300}C_4}{{}^{1000}C_7}$$

Ans

Hypergeometric distribution -:

The following conditions characterize the hypergeometric distribution :

(i) The result of each draw (the elements of the population being sampled) can be classified into one of two mutually exclusive categories (e.g. pass / fail, success / failure, Employed / unemployed).

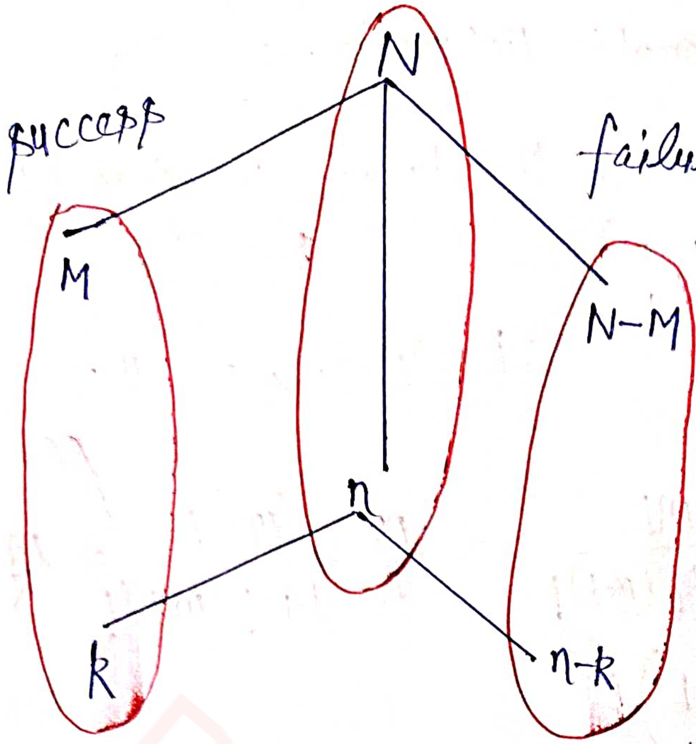
(ii) The probability of success changes on each draw, and each draw decreases the population (sampling without replacement from a finite population).

Notation :

Population (N)	Sample (n)
M (success)	k (observed success)
N−M (failure)	n−k (observed failure).

* A random variable X follows the Hypergeometric distribution if its probability mass function (p.m.f.) is given by

$$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} ; \quad k = 0, 1, 2, 3, \ldots, n$$



$$\Rightarrow P(X=k) = \frac{{}^{M}C_{k} \times {}^{N-M}C_{n-k}}{{}^{N}C_{n}}$$

Where,
- N is the population size.
- M is the number of successes states in the population
- n is the number of draws.
- k is the number of observed successes,
- $\binom{a}{b} = {}^{a}C_{b}$ is a Binomial coefficient.

$\because$ $P(X=k)$ is p.m.f. then

$$\sum_{k=0}^{n} \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} = 1$$

by property of p.m.f.

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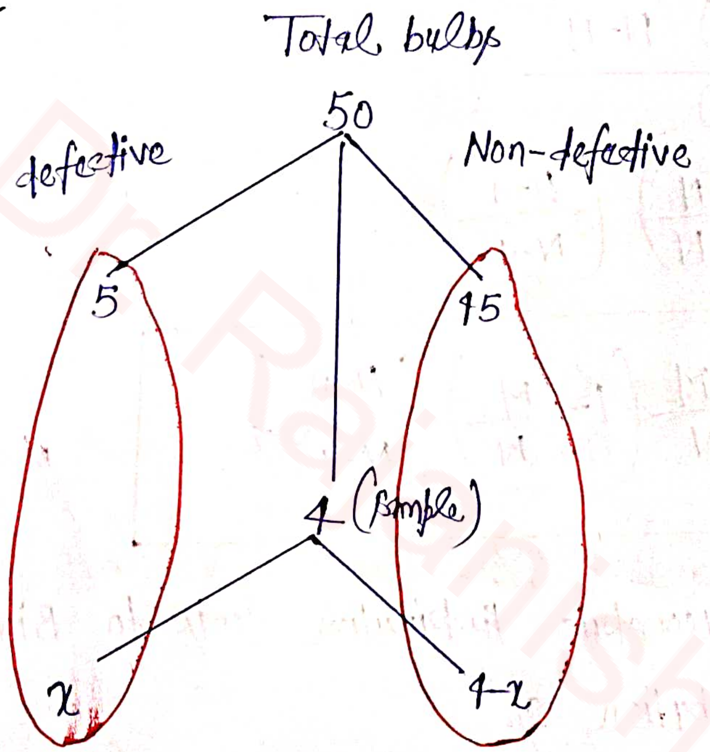
$$\text{Mean} = E(X) = \frac{nM}{N}$$

$$Var(X) = \frac{nM}{N}\left(1-\frac{M}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Question :- A crater contains 50 light bulbs of which 5 are defective and 45 are not. A Quality control inspector randomly samples 4 bulbs without replacement. Let X be the number of defective bulbs selected. Find the probability mass function of discrete random variable X.

Solution :-



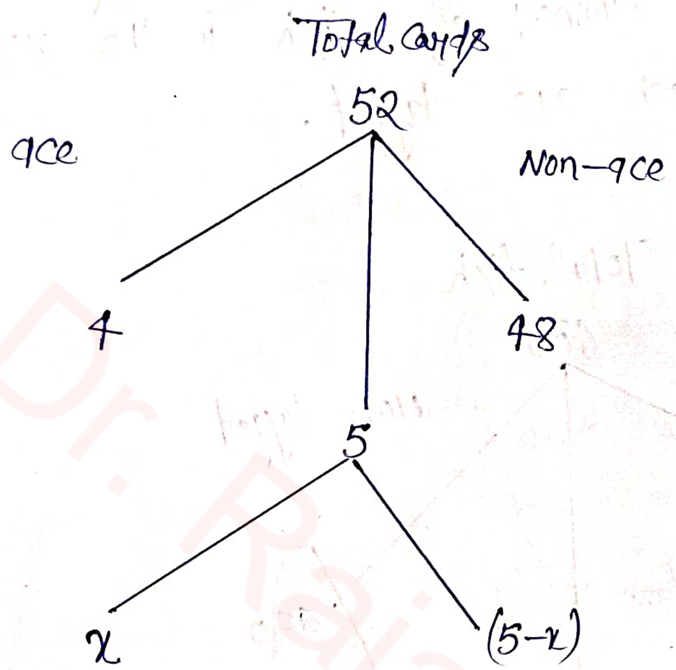
Thus, required probability mass function (p.m.f) is given by

$$P(X=x) = \frac{{}^{5}C_x \times {}^{45}C_{4-x}}{{}^{50}C_4}$$

$$x = 0, 1, 2, 3, 4$$

Question: Let the random variable X denote the number of aces in a five-card hand dealt from a standard 52-cards deck. Find a formula for the probability mass function of X.

Solution:



Thus, required probability mass function (p.m.f.) is given by

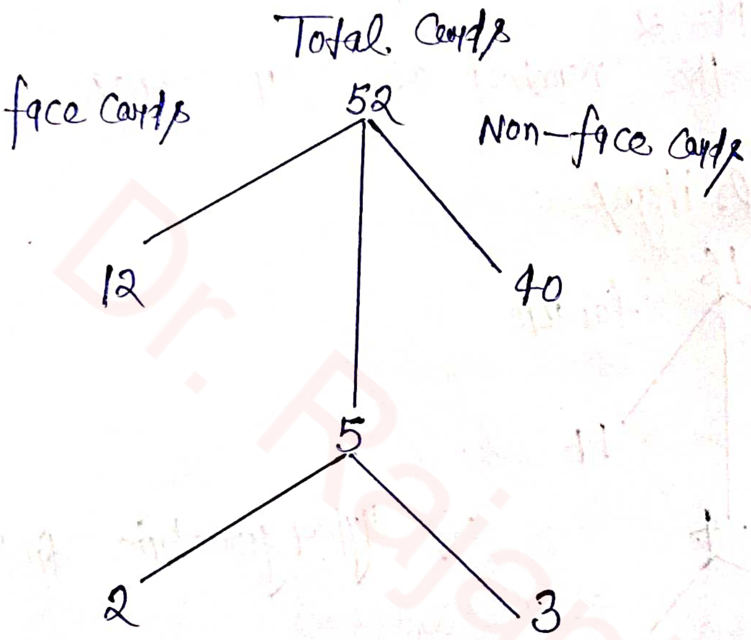
$$p(X=x) = \frac{{}^{4}C_x \times {}^{48}C_{5-x}}{{}^{52}C_5} \quad ; \quad x = 0, 1, 2, 3, 4$$

$\downarrow$

max ace $= 4$

Question :- Rahul likes to play cards. He draws 5 cards from a pack of 52 cards. What is the probability of that from the 5 cards drawn, Rahul draws only 2 face cards?

Solution :- Let X denotes the number of face cards



Thus, required probability

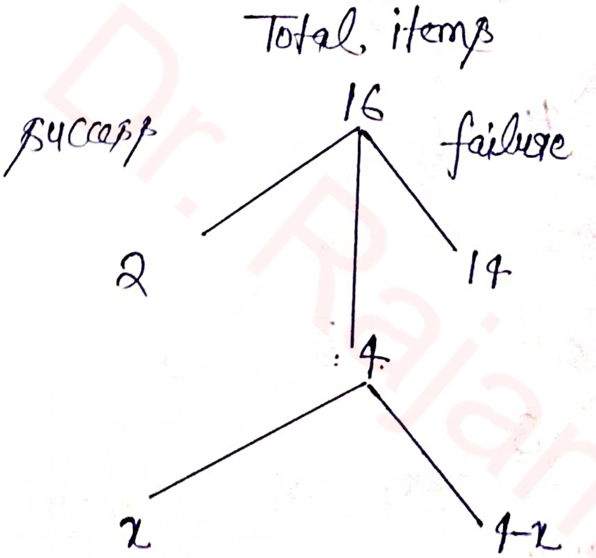
$$P(X=2) = \frac{{}^{12}C_2 \times {}^{40}C_3}{{}^{52}C_5}$$ Ans

Question :- Find the expectation of a Hypergeometric distribution such that the probability that a 4-trail hypergeometric experiment results in exactly 2 success, when population consists of 16 items.

Solution :- Given that

$$N = 16 \qquad n = 4$$
$$M = 2$$

let X denotes the number of successes



$\Rightarrow$ Hypergeometric distribution

Thus, the p.m.f. is given by

$$P(X=x) = \frac{{}^{2}C_{x} \times {}^{14}C_{4-x}}{{}^{16}C_{4}} \qquad x = 0, 1, 2, 3, 4$$

For hypergeometric distribution, the Expected value

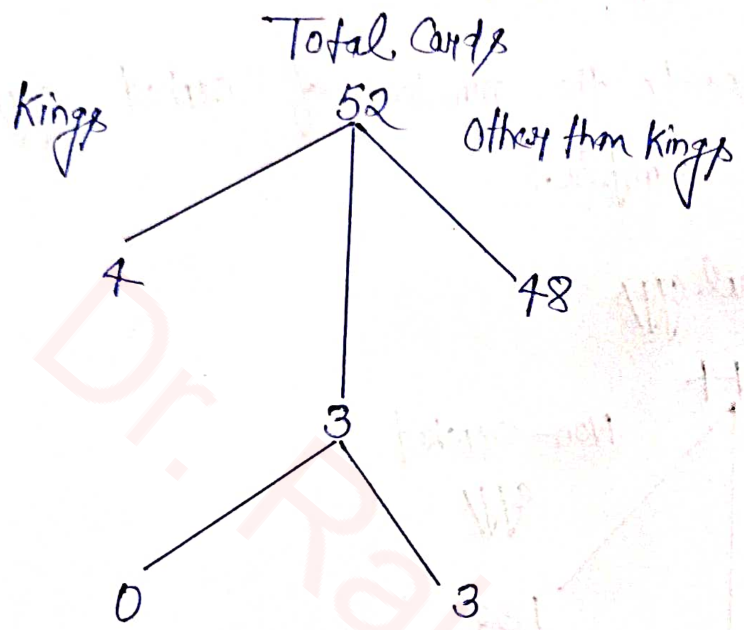
$$E(X) = \frac{nM}{N} = \frac{4 \times 2}{16} = \frac{1}{2}$$

$$\boxed{\text{Mean} = E(X) = \frac{1}{2}}$$

Ans

Question :- Consider Harish draws 3 cards from a pack of 52 cards. What is the probability of getting no kings.

Solution :- Let X represents the number of the kings



The required probability of getting no kings

$$= P(X=0)$$

$$= \frac{{}^{4}C_0 \times {}^{48}C_3}{{}^{52}C_3}$$

$$P(X=0) = 0.7826$$ Ans

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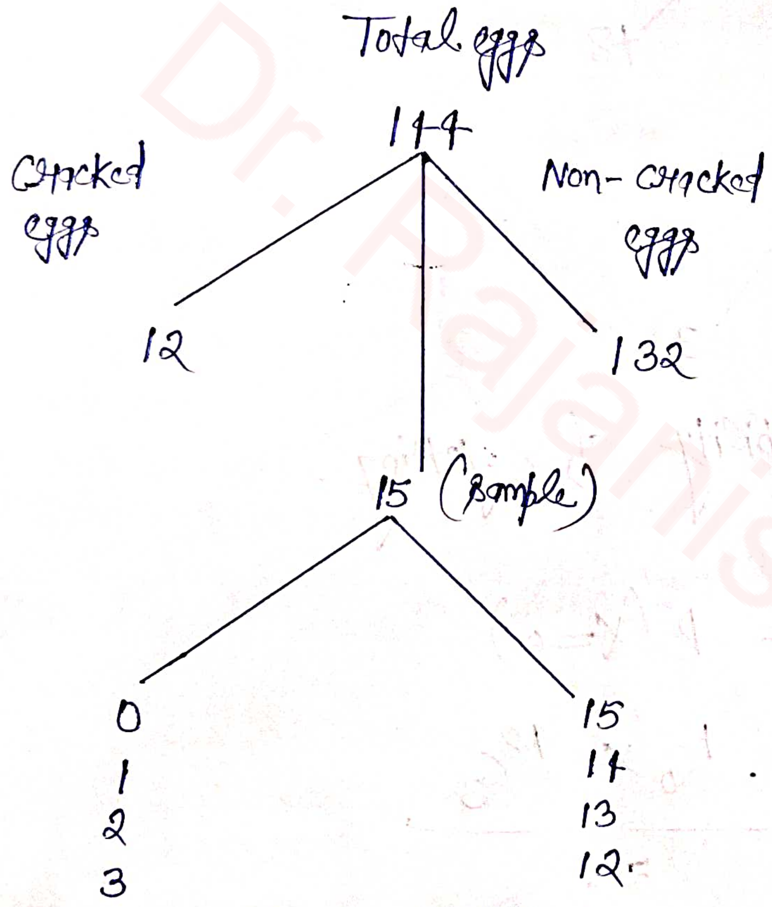
Question:- A gross of eggs contain 144 eggs. A particular gross is known to have 12 cracked eggs. An inspector randomly chooses 15 for inspection. He wants to know the probability that, among the 15 sample, at most 3 are cracked.

Solution:- Let X represents the number of cracked eggs in the sample.



Thus the required probability

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{{}^{12}C_0 \times {}^{132}C_{15}}{{}^{144}C_{15}} + \frac{{}^{12}C_1 \times {}^{132}C_{14}}{{}^{144}C_{15}} + \frac{{}^{12}C_2 \times {}^{132}C_{13}}{{}^{144}C_{15}}$$

$$+ \frac{{}^{12}C_3 \times {}^{132}C_{12}}{{}^{144}C_{15}}.$$

Ans