Q

$f(x) = \frac{x^2}{18}$, $-3 < x < 3$ is a p.d.f of X

find $P(X^2 < 9) = P(|X| < 3)$ — (i)

$= P(X < 3 \text{ AND } -X < 3)$ — (ii)

$= P(-3 < X < 3)$ — (iii)

$= \int_{-3}^{3} f(x)\,dx$

$= 1$

Q. verify that $P(X=x) = \left(\frac{1}{2}\right)^x$ $x = 1, 2, 3, 4, \ldots$

is a p.m.f.

(i) $P(X=x) = \left(\frac{1}{2}\right)^x$ = Jumps at points of discontinuities

$= F(x) - F(x^-) > 0$

(ii) If it is a p.m.f then the sum of all jumps should be 1

$$\sum_{x=1}^{\infty} P(X=x) = \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$= \frac{1/2}{1 - 1/2} = 1$$

By (i) and (ii) it is a valid p.m.f

Definition:

Let X be cts R.V with c.d.f $F_X(\cdot)$ with support S being an interval.
Let F be STRICTLY increasing on S.

Then for any $p \in (0,1)$, the QUANTILE of order p of X is a value $\xi_p \in S$ such that

$$\boxed{\xi_p = F^{-1}(p) \quad \text{or} \quad F(\xi_p) = p \quad \text{or} \quad P(X \le \xi_p) = p}$$

Quantile of order p is also called $(100p)$th percentile of X

Special cases

$\xi_{1/4}$ is called .25 quantile or Lower Quartile $= q_1$

$\xi_{1/2}$ is called .50 quantile or Median. $= q_2$

$\xi_{3/4}$ is called .75 quantile or Upper Quartile $= q_3$

$Q_3 - Q_1$ = inter quartile range.

Q1 Let X have p.d.f

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \\ 0 & e.w. \end{cases}$$

find (i) CDF of X
(ii) ·25 quantile
(iii) Median
(iv) Upper Quartile
(v) Inter Quartile range

(i) $P(X \leq x) = F_X(x) = \begin{cases} 0 & , x < 0 \\ \int_0^x t/2 \, dt & , 0 \leq x < 2 \\ \int_0^2 t/2 \, dt + \int_2^\infty 0 \, dt & x \geq 2 \end{cases}$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

ii ·25 quantile of X

$\Rightarrow F(\xi_{1/4}) = 1/4$ —— (i)

$\Rightarrow P(X \leq \xi_{1/4}) = 1/4$ —— (ii)

$\Rightarrow \int_0^{\xi_{1/4}} x/2 \, dx = 1/4$ —— (iii)

$\Rightarrow x^2/4 \Big|_0^{\xi_{1/4}} = 1/4$ —— (iv)

$\Rightarrow \frac{1}{4}(\xi_{1/4}^2 - 0) = \frac{1}{4}$. $\Rightarrow \xi_{1/4}^2 = 1$

$\Rightarrow \boxed{\xi_{1/4} = 1}$

iii Median $\Rightarrow$ ·50 quantile

$\Rightarrow F(\xi_{1/2}) = 1/2$

$\Rightarrow P(X \leq \xi_{1/2}) = 1/2$

$\Rightarrow \int_0^{\xi_{1/2}} x/2 \, dx = 1/2 \Rightarrow \boxed{\xi_{1/2} = \sqrt{2}}$

iv Upper Quartile = 0·75 quantile

$\Rightarrow F(\xi_{3/4}) = 0.75 = P(X \leq \xi_{3/4})$

$\Rightarrow \int_0^{\xi_{3/4}} x/2 \, dx = 0.75$

$\Rightarrow \frac{x^2}{\cancel{4}} \Big|_0^{\xi_{3/4}} = 3/\cancel{4}$

$\Rightarrow \xi_{3/4}^2 = 3 \Rightarrow \boxed{\xi_{3/4} = \sqrt{3}}$

v Interquartile Range

$\Rightarrow$ find $\underset{\downarrow \, \xi_{3/4}}{Q_3} - \underset{\downarrow \, \xi_{1/4}}{Q_1} = \boxed{(\sqrt{3} - 1)}$

Ex Let X be discrete **R.V** with pmf.

p.m.f. $P(X=x) = {}^2C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{2-x}$ $\quad x = 0,1,2$

$\Rightarrow P(X=0) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ — (i)

$\Rightarrow P(X=1) = {}^2C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{2-1} = \frac{1}{2}$ — (ii)

$\Rightarrow P(X=2) = {}^2C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{2-2} = 1/4$ — (iii)

(i) + (ii) + (iii) = 1 and $P(X=x) > 0$ $\Rightarrow$ valid p.m.f.

$$cdf_X = P(X \le x) = F_X(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \le x < 1 \\ \frac{1}{4}+\frac{1}{2} & 1 \le x < 2 \\ \frac{1}{4}+\frac{1}{2}+\frac{1}{4} & 2 \le x \end{cases}$$

can you find $\xi_{1/2}$

i.e can you find Median

i.e $F(\xi_p) = 1/2$

So in general for discrete R.V the median is defined as a point 'm' such that

$P(X \le m) \ge 1/2 \Rightarrow F(m) \ge 1/2$

and

$P(X < m) \le 1/2 \Rightarrow F(m^-) \le 1/2$

$\Big\} F(m^-) \le \frac{1}{2} \le F(m)$

check $F(1^-)$ and $F(1)$

$= 1/4 \le \frac{1}{2} \le 1/4 + \frac{1}{2}$

$\rightarrow$ 1 is the median

Q: X be a Discrete R.V with p.m.f

$P(X=0) = \frac{1}{2}$, $P(X=1) = 1/2$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

$F(0^-) \le \frac{1}{2} \le F(0)$

$0 \le \frac{1}{2} \le \frac{1}{2}$

$\boxed{\text{0 is a median}}$

$F(1^-) \le \frac{1}{2} \le F(1)$

$\frac{1}{2} \le \frac{1}{2} \le 1$

$\boxed{\text{1 is also median}}$

$F(0.5^-) \le \frac{1}{2} \le F(0.5)$

$\frac{1}{2} \le \frac{1}{2} \le \frac{1}{2}$

$\boxed{\text{0.5 is also a median}}$

For a Random variable X (cts or discrete)

Medians ALWAYS Exist (either unique or infinite)

Q Find the $50^{th}, 25^{th}$ percentile of the distribution having p.d.f

$$f(x) = \begin{cases} \frac{|x|}{4} & -2 < x < 2 \\ 0 & e.w \end{cases}$$

verify f is a p.d.f (i) $f(x) = \begin{cases} \frac{|x|}{4} & > 0 \\ 0 \end{cases} \geq 0$ ✓

(ii) $\int_{-2}^{2} \frac{|x|}{4}\, dx = \int_{-2}^{0} -\frac{x}{4}\, dx + \int_{0}^{2} \frac{x}{4}\, dx = \frac{1}{2} + \frac{1}{2} = \boxed{1}$ ✓

50^{th} percentile $\Rightarrow$ $.50$ quantile $\Rightarrow \xi_{1/2} \Rightarrow q_2 = \text{Median} = 0$

25^{th} percentile $\Rightarrow$ $.25$ quantile $\Rightarrow \xi_{1/4} = q_1 = \text{Lower Quartile}$

$\Rightarrow F(q_1) = 1/4$

$= \mathbb{P}(X \leq q_1) = 1/4$

$= \int_{-2}^{q_1} \frac{|x|}{4}\, dx = 1/4 \Rightarrow \int_{-2}^{q_1} -\frac{x}{\cancel{4}}\, dx = \frac{1}{\cancel{4}} \Rightarrow -\frac{x^2}{2}\Big|_{-2}^{q_1} = 1$

$\Rightarrow -\frac{1}{2}(q_1^2 - 4) = 1$

$\Rightarrow q_1^2 = 2 \Rightarrow q_1 = \pm\sqrt{2}$

$\Rightarrow \boxed{q_1 = -\sqrt{2}}$

not $+\sqrt{2}$

Let X be a R.V with p.d.f

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & e.w \end{cases}$$

find (i) q_2

(ii) $\mathbb{P}(X \geq 3/4 \mid X \geq 1/4)$

(i) $q_2 = \text{Median} = F(q_2) = 1/2 \Rightarrow \mathbb{P}(X \leq q_2) = 1/2$

$\Rightarrow \int_0^{q_2} 2x\, dx = 1/2 \Rightarrow q_2^2 = \frac{1}{2} \Rightarrow q = \pm\sqrt{1/2}$

$\Rightarrow q = \sqrt{1/2}$

(ii) $\mathbb{P}(X \geq 3/4 \mid X \geq 1/4) = \dfrac{\mathbb{P}(X \geq 3/4 \cap X \geq 1/4)}{\mathbb{P}(X \geq 1/4)}$

$$= \frac{\mathbb{P}(X \geq 3/4)}{\mathbb{P}(X \geq 1/4)} = \frac{\int_{3/4}^{1} 2x\,dx}{\int_{1/4}^{1} 2x\,dx} = \frac{1 - 9/16}{1 - 1/16}$$

$$= \frac{7}{15}$$

Q

Let $f(x) = \dfrac{1}{\pi(1+x^2)}$ $\quad -\infty < x < \infty$

(i) verif f is a p.d.f of some R.V. X

(ii) find c.d.f of X

(iii) find median of X

(ii) $$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt = \frac{1}{\pi}\Big[\tan^{-1} t\Big]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left\{\tan^{-1} x + \pi/2\right\}$$

$$\mathbb{P}(X \leq x) = F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$$

(iii) $F(p) = 1/2 \Rightarrow F_X(p) = \cancel{\frac{1}{2}} + \frac{1}{\pi}\tan^{-1} x = \cancel{\frac{1}{2}}$

$$\Rightarrow \tan^{-1}(x) = 0$$

$$\Rightarrow x = 0$$