PROBABILITY AND STATISTICS
(UCS401)

Lecture-8

(Bayes' Theorem with illustrations)

Introduction to Probability (Unit -II)

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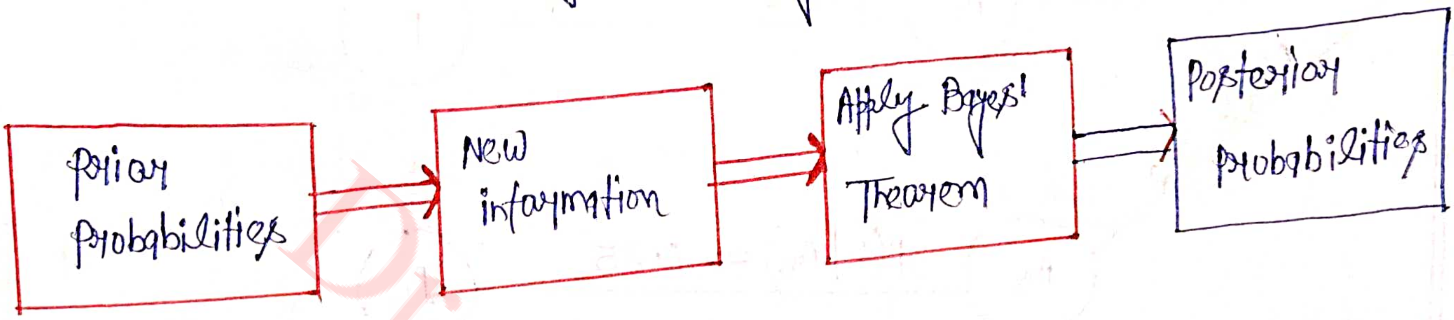
[6] Bayes' Theorem with illustrations

Bayes' Theorem was given by a British Mathematician, Thomas Bayes' in 1763.

In it, we update the prior probabilities based on the given information by calculating the revising probabilities.



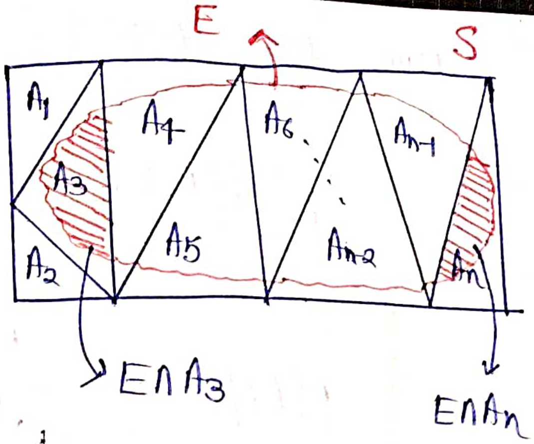
Bayes' Theorem —: If $A_1, A_2, A_3, \ldots, A_n$ are mutually disjoint events with $P(A_i) \neq 0$, then for any arbitrary event E which is subset of $\bigcup_{i=1}^{n} A_i$ such that $P(E) > 0$, we have

$$P(A_i/E) = \frac{P(A_i)\,P(E/A_i)}{\sum_{i=1}^{n} P(A_i)\,P(E/A_i)}$$

Proof —: Given that $A_1, A_2, A_3, \ldots, A_n$ are mutually disjoint

i.e.,
$$A_1 \cap A_2 \cap A_3 \cdots \cap A_n = \phi$$
$$A_i \cap A_j = \phi \quad \text{for } i \neq j$$

$$A_1 \cup A_2 \cup A_3 \cdots \cup A_n = S$$



Given that

$$E \subset \bigcup_{i=1}^{n} A_i ,$$ we have

$$E = E \cap \left(\bigcup_{i=1}^{n} A_i \right)$$

$$E = \bigcup_{i=1}^{n} (E \cap A_i) \quad \text{—(1)}$$

$\because$ $E \cap A_i \subset A_i$ are mutually disjoint events,

$$P(E) = P\left(\bigcup_{i=1}^{n} (E \cap A_i) \right)$$

$$= \sum_{i=1}^{n} P(E \cap A_i)$$

$$= \sum_{i=1}^{n} P(A_i)\, P(E/A_i)$$

$$P(E) = \sum_{i=1}^{n} P(A_i)\, P(E/A_i) \quad \text{—(2)}$$

$\because$ $A \subset B$

$A = A \cap B$

$(E \cap A_1) \cap (E \cap A_2)$

$= E \cap (A_1 \cap A_2)$

$= E \cap \phi = \phi$

$E \cap A_i = \phi \quad \forall i$

$\Rightarrow$ $E \cap A_i$ is mutually disjoint $\forall i$

$\because$ $P(E/A_i) = \dfrac{P(E \cap A_i)}{P(A_i)}$

$P(E \cap A_i) = P(A_i)\, P(E/A_i) \quad \forall i$

or

$P(A \cap B) = P(A)\, P(B/A)$

Now, by definition of Conditional probability, we have

$$P(A_i/E) = \frac{P(A_i \cap E)}{P(E)}$$

$$\boxed{P(A_i/E) = \frac{P(A_i)\,P(E/A_i)}{\sum_{i=1}^{n} P(A_i)\,P(E/A_i)}}$$

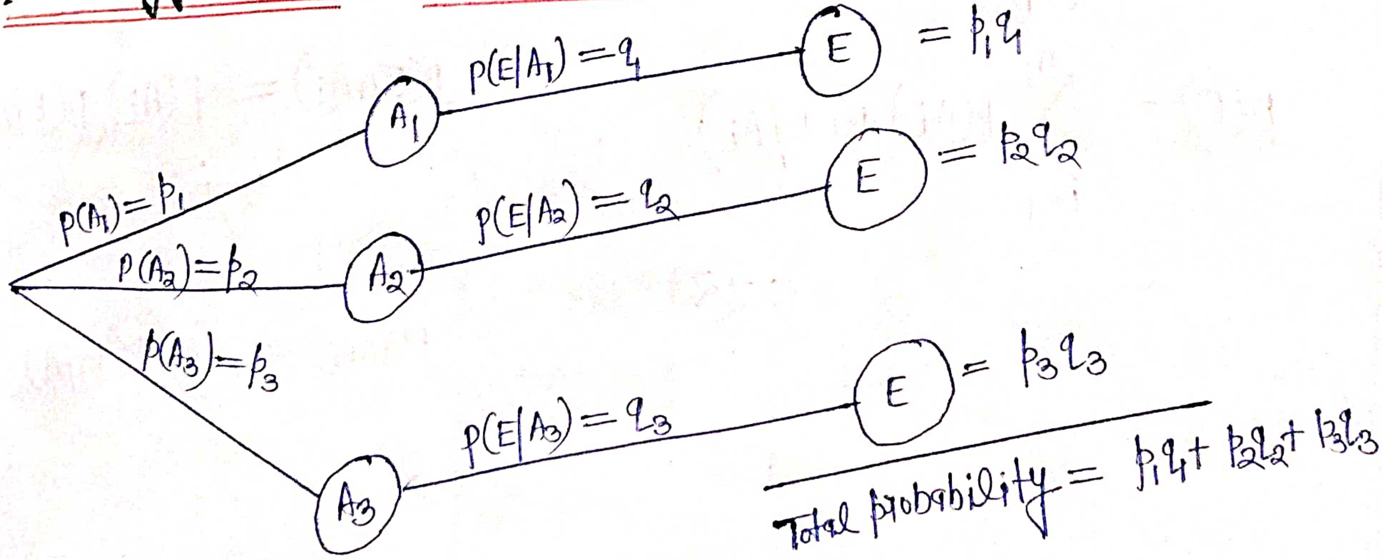
Proved

Remarks –: (i) The probabilities $P(A_1), P(A_2), P(A_3), \ldots, P(A_n)$ are termed as "a prior probabilities" because they exists before we gain any information.

(ii) The probabilities $P(E/A_i)$ are called "likelihoods" because they indicates how likely the event E occure given each and every a prior probabilities.

(iii) The probability $P(A_i/E)$ are called "posterior probability" because they are determined after the result of experiment are known.

Strategy used for solving problems by Bayes' Theorem –:



$$P(A_1/E) = \frac{P(A_1)\,P(E/A_1)}{\sum_{i=1}^{3} P(A_i)P(E/A_i)} = \frac{p_1 q_1}{p_1q_1 + p_2q_2 + p_3q_3}$$

and so-on.

||ly $P(A_2/E) = \frac{p_2 q_2}{p_1q_1 + p_2q_2 + p_3q_3}$.

Question ① A student knew only 60% of the questions in a test each with 5 answers. He is simply guessed while answering the test. What is the probability that he knew the answer to a given question given that he answered it correctly.

Solution:- Define the events

A_1 : student knew the answer (Cause)

A_2 : student guessed the answer (Cause)

E : Student answered it correctly (Effect)

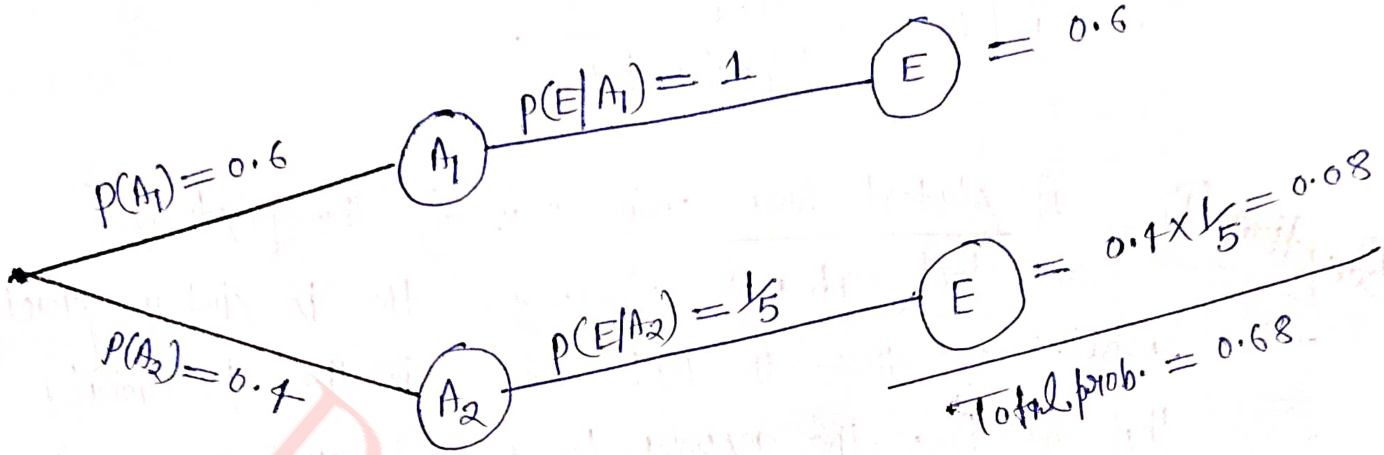
$P(A_1) = P(\text{students knew the answer}) = 0.6$

$P(A_2) = P(\text{students guessed the answer}) = 0.4$

$P(E/A_1)$ = probability of correct answer given that he knew the answer $= 1$

$P(E/A_2)$ = Probability of correct answer given that he guessed the answer.

Qus — (a), (b), (c), (d), (e) $\Rightarrow P(\text{Correct ans. given that he guessed the ans.}) = \frac{1}{5}$



Thus, required probability that knew the answer to a question given that he answered it correctly

$$P(A_1|E) = \frac{P(A_1)\,P(E/A_1)}{\sum_{i=1}^{3} P(A_i)\,P(E/A_i)}$$

$$P(A_1/E) = \frac{0.6}{0.68}$$

$$\boxed{P(A_1|E) = 0.88235}$$

Moreover, $P(A_2/E) = \frac{0.08}{0.68}$

$$P(A_2/E) = \frac{8}{68}$$ Ans.

Question ② The probability of X, Y, Z becoming the manager is $\frac{4}{9}$, $\frac{2}{9}$, $\frac{1}{3}$, respectively. The probability that the bonus scheme will be introduced if X, Y and Z becomes manager are $\frac{3}{10}$, $\frac{1}{2}$ and $\frac{4}{5}$, respectively. If the Bonus is introduced, what is the probability that the manager appointed was Y?

Solution :- Define the events

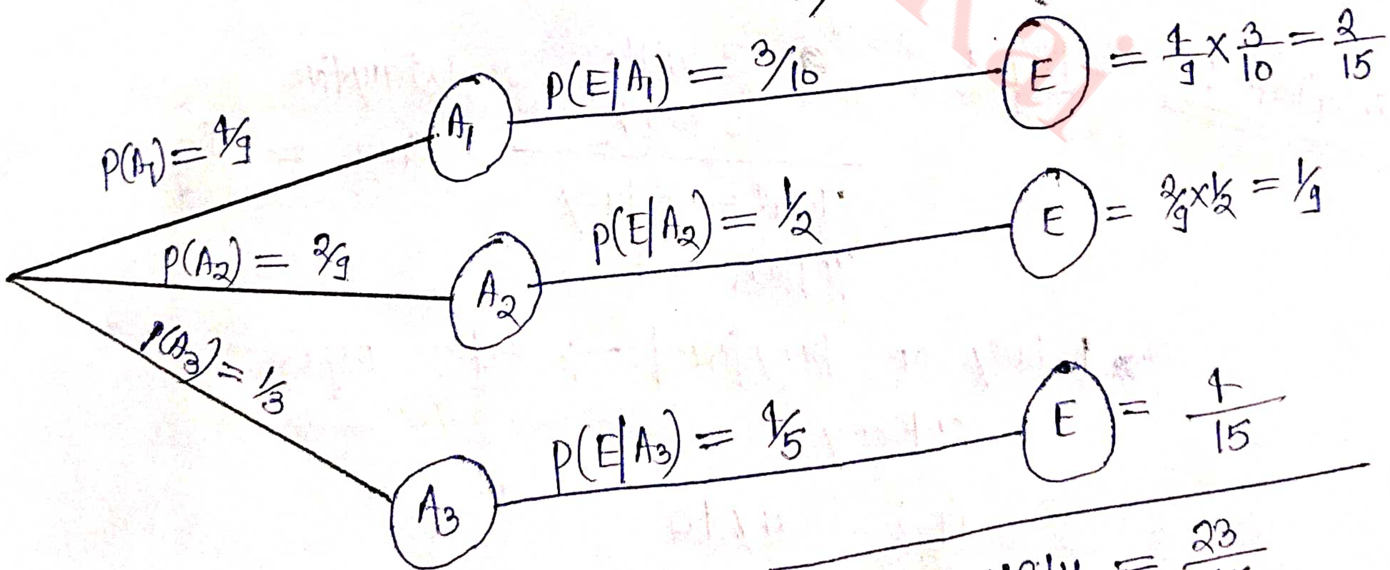
A_1 : X becomes the manager

A_2 : Y becomes the manager

A_3 : Z becomes the manager

E : Bonus is introduced

$$P\left(\frac{\text{manager is appointed was Y}}{\text{Bonus is introduced}}\right) = P(A_2/E) = ?$$



Total probability $= \frac{23}{45}$

Thus, required probability

$$P(A_2/E) = \frac{P(A_2)\,P(E/A_2)}{\sum_{i=1}^{3} P(A_i)\,P(E/A_i)}$$

$$= \frac{1/9}{23/45} = \frac{1}{9} \times \frac{45}{23} = \frac{5}{23}$$

$P(A_2/E) = P\left(\text{Manager appointed was } Y \mid \text{Bonus is introduced}\right) = \frac{5}{23}$

Ans

Question (3)

From a bag containing 3 white and 5 black balls, 4 balls are transferred into an empty bag. From this bag, a ball is drawn and is found to be white. What is probability that out of four balls transferred 3 are white and 1 is black.

Solution-:

Bag 1 — 3 white, 5 black

Total = 8 balls

You pick = 4

Exhaustive Cases = $^8C_4$

4 balls are transferred → empty bag (2)

outcomes

A_1 ; 0W & 4B

A_2 : 1W & 3B

A_3 : 2W & 2B

A_4 ; 3W & 1B

Condition → E : ball found to be white

Define the events :

A_1 : transfer of 0 white and 4 black balls

A_2 : transfer of 1 white and 3 black balls

A_3 : transfer of 2 white and 2 black balls

A_4 : transfer of 3 white and 1 black ball.

E ; Drawing a white ball from second bag

3W & 5B

A_1 : transfer of 0W & 4B

$$P(A_1) = \frac{{}^3C_0 \times {}^5C_4}{{}^8C_4} = \frac{\cancel{5\times4\times3\times2}}{\cancel{4\times3\times2}} \; \frac{\cancel{4\times3\times2}}{8\times7\times\cancel{6\times5}}$$

$$= \frac{1}{14}$$

$$\boxed{P(A_1) = \frac{1}{14}}$$

A_2 : transfer of 1W & 3B

$$P(A_2) = \frac{{}^3C_1 \times {}^5C_3}{{}^8C_4} = \frac{3\times\cancel{5\times4\times3}}{\cancel{3\times2}} \times \frac{\cancel{4\times3\times2}}{\cancel{8\times7\times6\times5}}$$

$$= 3/7$$

$$\boxed{P(A_2) = 3/7}$$

A_3 : transfer of 2W & 2B

$$P(A_3) = \frac{{}^3C_2 \times {}^5C_2}{{}^8C_4} = \frac{3}{7}$$

3W & 5B

A_4: transfer of 3W & 1B

$$P(A_4) = \frac{{}^3C_3 \times {}^5C_1}{{}^8C_4} = \frac{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}{8 \times 7 \times \cancel{6} \times \cancel{5}} = \frac{1}{14}$$

$$\boxed{P(A_4) = \frac{1}{14}}$$

A_1 : 0W & 4B $\longrightarrow$ 2nd bag
W : 0
B : 4
Total = 4
You pick = 1W

$P(E/A_1) = 0$

A_2 : 1W & 3B $\longrightarrow$ W : 1
B : 3
Total = 4
You pick = 1W

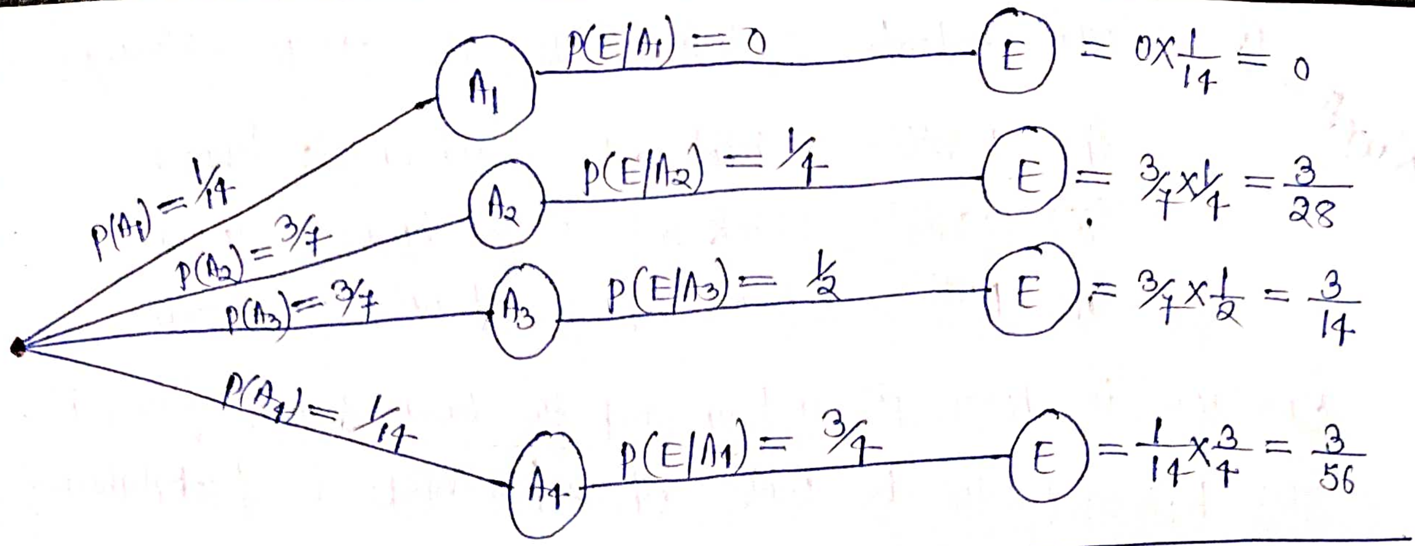
$$P(E/A_2) = \frac{{}^1C_1}{{}^4C_1} = \frac{1}{4}$$

A_3 : 2W & 2B $\longrightarrow$ W : 2
B : 2
Total = 4
You pick = 1W

$$P(E/A_3) = \frac{{}^2C_1}{{}^4C_1} = \frac{2}{4} = \frac{1}{2}$$

A_4 : 3W & 1B $\longrightarrow$ W : 3
B : 1
Total = 4
You pick = 1W

$$P(E/A_4) = \frac{{}^3C_1}{{}^4C_1} = \frac{3}{4}$$



Total probability $= \frac{21}{56}$

Thus, required probability of 3W & 1B given that 1W ball drawn from 2nd bag..

$$P(A_4/E) = \frac{P(A_4)\,P(E/A_4)}{\sum_{i=1}^{4} P(A_i)\,P(E/A_i)}$$

$$= \frac{3/56}{21/56} = \frac{1}{7}$$

$$\boxed{P(A_4/E) = \frac{1}{7}}$$

Question-(4) The Contents of urn I, II, III are as follows:

(i) 1 white, 2 black, and 3 red balls; (urn I)

(ii) 2 white, 1 black and 1 red balls; (urn II)

(iii) 4 white, 5 black and 3 red balls; (urn III)

One urn is chosen at random and two balls drawn from it. They happened to be white and red. What is probability that they came from urn III?

Solution-: Define the events

A_1: selected from urn I.

A_2: selected from urn II.

A_3: selected from urn III.

E: white and red ball chosen.

$\because$ One urn chosen at random

$$P(A_1) = \frac{1}{3}; \quad P(A_2) = \frac{1}{3} \;\&\; P(A_3) = \frac{1}{3}$$

(i) 1W, 2B & 3R

Total = 6

you pick = 1W & 1R

$$P(E/A_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$$

(ii) 2W, 1B & 1R

Total = 4

you pick = 1W & 1R

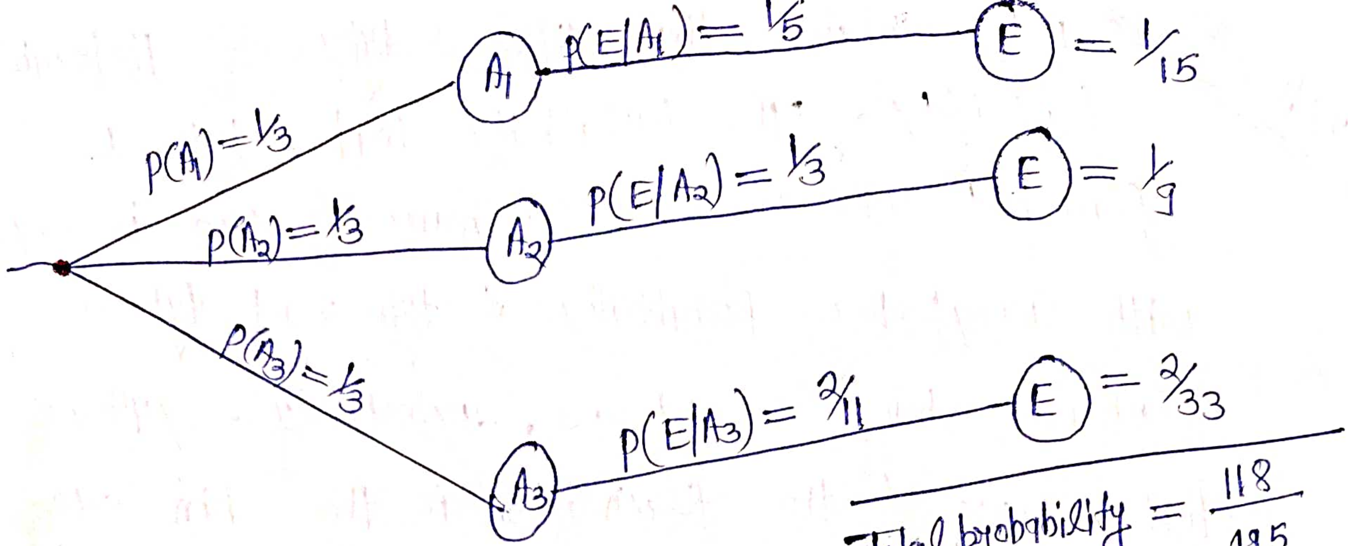
$$P(E/A_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3}$$

(iii) 4W, 5B & 3R

Total = 12

you pick = 1W & 1R

$$P(E/A_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$



Total probability $= \dfrac{118}{495}$

$P(\text{urn III selected} \mid \text{happened to be 1W \& 1R})$

$$= P(A_3|E) = \frac{P(A_3)\,P(E/A_3)}{\sum_{i=1}^{3} P(A_i)\,P(E/A_i)}$$

$$= \frac{2/33}{\dfrac{118}{495}}$$

$$\boxed{P(A_3|E) = \frac{15}{59}}$$

Ans