Continuous

Uniform Distribution

A random variable X is said to follow uniform distribution, Uniform (a, b), if its PDF is given by

$$f(x \mid a, b) = \begin{cases} \frac{1}{b-a} & , \text{ if } x \in [a, b] \\ 0 & , \text{ otherwise} \end{cases}$$

it's easy to check $\int_a^b f(x \mid a, b) = 1$

$$E(X) = \int_a^b x \, f(x \mid a, b) \, dx = \frac{1}{b-a} \int_a^b x \, dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b+a)}{2}$$

$$E(X^2) = \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \frac{b^3 - a^3}{3} = \frac{b^2 + ab + a^2}{3}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

$$M_x(t) = E(e^{tx}) = \frac{1}{b-a}\int_a^b e^{tx}\,dx = \frac{1}{b-a}\,\frac{e^{tx}}{t}\Big|_a^b$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}$$

$$F_X(x) = \int_a^x f(x|a,b)\,dx$$

$$= \frac{1}{b-a}(x-a)$$

Gamma Distn

A random variable T following Gamma(α) distribution has PDF

$$f(t|\alpha) = \frac{1}{\Gamma\alpha}\, t^{\alpha-1} e^{-t}, \quad t > 0,\ \alpha > 0$$

where $\Gamma\alpha = \int_0^\infty t^{\alpha-1} e^{-t}\,dt$

$$\Gamma\alpha+1 = \alpha\Gamma\alpha,\ \alpha > 0$$

$$\Gamma n = (n-1)!$$

for +ve integer n

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$E(T) = \int_0^\infty t\, f(t|\alpha)\,dt$$

$$= \frac{1}{\Gamma\alpha}\int_0^\infty t\cdot t^{\alpha-1} e^{-t}\,dt$$

$$= \frac{1}{\Gamma\alpha}\int_0^\infty t^{\alpha} e^{-t}\,dt$$

$$= \frac{1}{\Gamma\alpha}\,\Gamma(\alpha+1)$$

$$= \frac{\alpha\,\Gamma\alpha}{\Gamma\alpha}$$

$$= \alpha$$

$$E(T^2) = \frac{1}{\Gamma\alpha} \int_0^\infty t^{\alpha+1} e^{-t} dt = \frac{\Gamma(\alpha+2)}{\Gamma\alpha}$$

$$= \frac{\alpha(\alpha+1)\Gamma\alpha}{\Gamma\alpha}$$

$$= \alpha(\alpha+1)$$

Therefore $V(T) = E(T^2) - [E(T)]^2$

$$= \alpha(\alpha+1) - \alpha^2$$

$$= \alpha$$

Transformation of a random variable

X — random variable

$Y = g(X)$ — also a random variable

$\mathcal{X} = \{x : f_X(x) > 0\}$

$\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$

Depending on the nature of $g(x)$ & sometimes possible to obtain a tractable expression in terms of probability distribution of Y

Th^m. let X have CDF, $F_X(x)$, let $Y = g(X)$, and $\mathcal{X}$ and $\mathcal{Y}$ be defined earlier.

a. if g is an increasing function on $\mathcal{X}$.

Then $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$

b. if g is an decreasing function on $\mathcal{X}$

Then $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

Further

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & ; y \in \mathcal{Y} \\ 0 & ; \text{otherwise} \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

g is monotone increasing
1-1 & onto

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

Q. If $T \sim \text{Gamma}(\alpha)$ distribution, and we take a transformation $X = \beta T$, $\beta > 0$

find the distribution of X

Sol:

We are given $X = \beta T \quad = g(T)$

$$T = \frac{X}{\beta} \quad = g^{-1}(x)$$

Since $g(T) = \beta T$

$$\frac{d}{dT} g(T) = \beta > 0$$

$\therefore$ g is an increasing function

Also $X > 0$ as $\beta > 0$, and $T > 0$

Therefore, according to previous Th^m.

$$f_X(x \mid \alpha, \beta) = f_T(g^{-1}(x)) \frac{d}{dT} g^{-1}(x)$$

$$= \frac{1}{\Gamma\alpha} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)} \cdot \frac{1}{\beta}$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta}, \quad \alpha > 0,\ \beta > 0,\ x > 0$$

which is PDF of two-parameter Gamma(α, β) distribution

Now

$$f(x \mid \alpha, \beta) = \frac{1}{\beta^{\alpha} \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} \quad , \quad x > 0, \ \alpha > 0, \ \beta > 0$$

Here remember

$$\int_0^{\infty} x^{\alpha-1} e^{-x/\beta} \, dx = \beta^{\alpha} \Gamma\alpha$$

Now $E(X) = E(\beta T) = \beta E(T) = \alpha\beta$

$V(X) = V(\beta T) = \beta^2 V(T) = \alpha\beta^2$

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if $X \sim \text{Gamma}(\alpha, \beta)$ distribution

and we take $\alpha = 1$

Then $X \sim$ exponential(β) distribution

with PDF

$$f(x \mid \beta) = \frac{1}{\beta} e^{-x/\beta} \ ; \ \beta > 0, \ x > 0$$

$E(X) = \beta$

$V(X) = \beta^2$

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if $X \sim$ exponential(β)

$Y = X^{\frac{1}{r}}$, $r > 0$

Then $Y \sim$ Weibull(r, β)

Thm: Let X have pdf $f_X(x)$, let $Y = g(X)$, and define the sample space $\mathcal{X}$ as defined earlier.

Suppose there exists a partition, $A_0, A_1, \ldots, A_k$ of $\mathcal{X}$ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further, suppose there exist functions $g_1(x), g_2(x), \ldots, g_k(x)$, defined on $A_1, A_2, \ldots, A_k$, respectively, satisfying

(i) $g(x) = g_i(x)$, for $x \in A_i$

(ii) $g_i(x)$ is monotone on A_i

(iii) The set $\mathcal{Y} = \{y : y = g_i(x) \text{ for some } x \in A_i\}$ is the same for each $i = 1, 2, \ldots, k$

(iv) $g_i^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$, for each $i = 1, 2, \ldots, k$

Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & ; y \in \mathcal{Y} \\ 0 & ; \text{Otherwise} \end{cases}$$

Normal – chi Squared relationship

Let X have a standard normal distribution, PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

Check.

$$\int_{-\infty}^{\infty} f(x) = 1 \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$$

Put $\frac{x^2}{2} = t \Rightarrow x = \sqrt{2t}$

$$x\,dx = dt$$

$$dx = \frac{1}{x} dt = \frac{1}{\sqrt{2t}} dt$$

Therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 2 \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2/2}\, dx$$

$$= 2 \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{2}} t^{-\frac{1}{2}} e^{-t}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}-1} e^{-t}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \frac{1}{\sqrt{\pi}} \sqrt{\pi}$$

$$= 1$$

Consider $Y = X^2$

The function $g(x) = x^2$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$. The set $\mathcal{Y} = (0, \infty)$

$$A_0 = \{0\}$$

$$A_1 = (-\infty, 0), \quad g_1(x) = x^2, \quad g_1^{-1}(y) = -\sqrt{y}$$

$$A_2 = (0, \infty), \quad g_2(x) = x^2, \quad g_2^{-1}(y) = \sqrt{y}$$

Therefore using previous Theorem

The PDF of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \left| -\frac{1}{2\sqrt{y}} \right|$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \left| \frac{1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{y}} e^{-y/2}, \quad 0 < y < \infty$$

which is PDF of a chi-squared random variable with 1 degree of freedom

* If $X \sim N(0,1)$ Then $Y = X^2 \sim \chi_1^2$

Normal Distribution

A random variable $X \sim N(\mu, \sigma^2)$ distribution if its PDF is given by

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

Let $Z = \frac{X-\mu}{\sigma} = g(x)$

$X = \mu + Z\sigma = g^{-1}(z)$

As $-\infty < x < \infty$

we have $-\infty < z < \infty$

Also g is monotone increasing function

$$\frac{d}{dx} g(x) = \frac{1}{\sigma} > 0$$

$\therefore$

$$f_Z(z) = f_X\left(g^{-1}(z)\right) \frac{d}{dz} g^{-1}(z)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2} z^2} \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$$ which is PDF of standard normal distn.

$$E(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z . e^{-\frac{1}{2}z^2} dz.$$

$$= 0 \quad \{\because \text{ The function is odd inside the integral}$$

$$E(z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

Put $\frac{z^2}{2} = t \Rightarrow z = \sqrt{2t}$

$z dz = dt \Rightarrow dz = \frac{1}{\sqrt{2t}} dt$

Also $t > 0$

$$\therefore E(z^2) = \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} 2t e^{-t} \frac{1}{\sqrt{2t}} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}} e^{-t} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{3}{2}-1} e^{-t} dt$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma\tfrac{1}{2} = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1$$

$$V(z) = E(z^2) - [E(z)]^2 = 1 - 0 = 1$$

Therefore $X \sim N(\mu, \sigma^2)$

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

with mean 0 and variance 1

Now $E(z) = E\left(\frac{X-\mu}{\sigma}\right)$

$$0 = \frac{1}{\sigma}\left[E(x) - \mu\right]$$

$$\boxed{E(x) = \mu}$$

Also $V(z) = V\left(\frac{X-\mu}{\sigma}\right)$

$$1 = \frac{1}{\sigma^2} V(x)$$

$$\boxed{V(x) = \sigma^2}$$

Therefore $X \sim N(\mu, \sigma^2)$

$$E(x) = \mu$$

$$V(x) = \sigma^2$$

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$$M_X(t) = E(e^{tX}) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x^2+\mu^2-2\mu x-2\sigma^2 t x\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x^2-2x(\mu+\sigma^2 t)+\mu^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x-(\mu+\sigma^2 t)\right)^2} e^{-\frac{1}{2\sigma^2}\left(\mu^2-(\mu+\sigma^2 t)^2\right)} dx$$

$$= e^{-\frac{1}{2\sigma^2}\left[\mu^2-\mu^2-\sigma^4 t^2-2\mu\sigma^2 t\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\left[-2\sigma^2\left(\mu t+\frac{\sigma^2 t^2}{2}\right)\right]}$$

$$= e^{\mu t+\frac{\sigma^2 t^2}{2}}$$

Lognormal Distribution

if X is a random variable whose logarithm is normally distributed (That is, $\log X \sim N(\mu, \sigma^2)$), Then X has a lognormal distribution

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} , \quad \begin{array}{l} x > 0 \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

*

if $X \sim LN(\mu, \sigma^2)$

$Y = \ln X \sim N(\mu, \sigma^2)$

if $Y \sim N(\mu, \sigma^2)$

$X = e^{Y} \sim LN(\mu, \sigma^2)$

Cauchy Distribution

$$f(x \mid \theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1 + \left(\frac{x - \theta}{\sigma}\right)^2} , \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \theta < \infty \\ \sigma > 0 \end{array}$$

* Mean & variance, mgf don't exist