PROBABILITY AND STATISTICS (UCS401)

Lecture-15

(Chebyshev’s and Markov’s inequality with illustrations)

Random Variables and their Special Distributions (Unit –III & IV)

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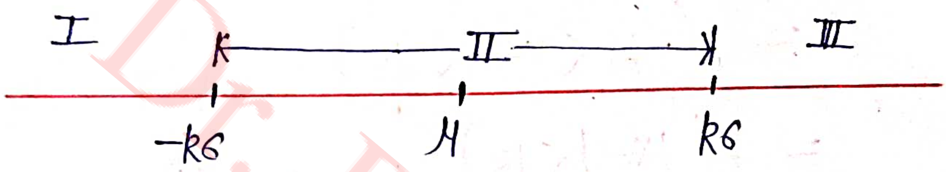
Chebyshev's inequality

Chebyshev's inequality:- Let X be a random variable with mean μ and finite variance σ^2, then for any real number $k>0$,

$$P\left(|X-\mu| < k\sigma\right) \geq 1-\frac{1}{k^2}$$



and

$$P\left(|X-\mu| \geq k\sigma\right) \leq \frac{1}{k^2}.$$

The rule is often called Chebyshev's theorem, about the range of standard deviations around the mean, in statistics.

The equality has great utility because it can be applied to any probability distribution in which the mean and variance are defined.

Question:- A random variable X has a mean 8 and varience 9 and an unknown probability distribution.

Find $P(-4 < X < 20)$ and $P(|X-8| \geq 6)$.

Solution:- Given that

$$\text{Mean } \mu = 8$$

$$\text{varience } \sigma^2 = 9$$

Thus, S.D. $\sigma = 3$.

$$P(-4 < X < 20) = P(-4-8 < X-8 < 20-8)$$

$$= P(-12 < X-8 < 12)$$

$$= P(|X-8| < 12) \quad \text{—} (1)$$

By chebyshev's inequality, we have

$$P(|X-\mu| < k\sigma) \geq 1-\frac{1}{k^2}$$

i.e., $$P(|X-8| < k\sigma) \geq 1-\frac{1}{k^2} \quad \text{—} (2)$$

From (1) & (2)

$$k\sigma = 12$$

$$k \times (3) = 12$$

$$\Rightarrow \boxed{k = 4}$$

$\therefore$

$$P(-4 < X < 20) = P(|X-8| < 12) \geq 1-\frac{1}{k^2}$$

$$\geq 1-\frac{1}{4^2} = \frac{15}{16}$$

$$\Rightarrow P(-4 < X < 20) \geq \frac{15}{16}$$

Ans

(ii) The required probability

$$= P(|X-8| \geq 6)$$

By Chebyshev's inequality, we have

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

where

$$k\sigma = 6$$

$$k(3) = 6$$

$$\Rightarrow \boxed{k=2}$$

$$\Rightarrow P(|X-8| \geq 6) \leq \frac{1}{k^2} = \frac{1}{4}$$

$$\Rightarrow P(|X-8| \geq 6) \leq 1/4$$ Ans

Question:- A random variable X has a mean 12, variance 9 and an unknown probability distribution. Find $P(6 < X < 18)$ and $P(3 < X < 21)$.

Solution:- Given that

Mean $\mu = 12$

Variance $\sigma^2 = 9$

Thus S.D. $\boxed{\sigma = 3}$

$$P(6 < X < 18) = P(6-12 < X-12 < 18-12)$$

$$= P(-6 < X-12 < 6)$$

$$= P(|X-12| < 6) \quad \text{——} (1)$$

By Chebyshev's inequality, we have

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \quad \text{——} (2)$$

From ① and ②, we have

$$k\sigma = 6$$

$$k(3) = 6$$

$$\Rightarrow \boxed{k=2}$$

$$\therefore P(6<X<18) = P(|X-12|<6) \geq 1-\frac{1}{k^2}$$

$$\geq 1-\frac{1}{4} = \frac{3}{4}$$

$$\therefore P(6<X<18) \geq \frac{3}{4}$$ Ans

(ii) $$P(3<X<21) = P(3-12 < X-12 < 21-12)$$

$$= P(-9 < X-12 < 9)$$

$$= P(|X-12| < 9) \quad \text{—①}$$

By chebyshev's inequality

$$P(|X-\mu| < k\sigma) \geq 1-\frac{1}{k^2}$$

i.e., $$P(|X-12| < k\sigma) \geq 1-\frac{1}{k^2} \quad \text{—②}$$

Thus we have $$k\sigma = 9$$

$$\Rightarrow k(3) = 9 \quad \Rightarrow \boxed{k=3}$$

$$= P(|X-12|<9) \geq 1-\frac{1}{k^2} = 1-\frac{1}{9} = \frac{8}{9}$$

$$\therefore P(3<X<21) \geq \frac{8}{9}$$ Ans

Question:- suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50. If the variance of a weekly production is known and equal to 25, then what can be said about the productivity that will be between 40 and 60?

Solution:- Given that

Mean $\mu = 50$

Variance $\sigma^2 = 25$

Thus, S.D. $\sigma = 5$.

Thus, required probability

$$= P(40 < X < 60)$$

$$= P(40-50 < X-50 < 60-50)$$

$$= P(-10 < X-50 < 10)$$

$$= P(|X-50| < 10) \quad \text{--- (1)}$$

By Chebyshev's inequality, we have

$$P(|X-\mu| < k\sigma) \geq 1-\frac{1}{k^2}$$

i.e., $$P(|X-50| < k\sigma) \geq 1-\frac{1}{k^2} \quad \text{--- (2)}$$

(1) & (2) $\Rightarrow$ $k\sigma = 10 \Rightarrow k(5) = 10 \Rightarrow \boxed{k=2}$

$$= P(|X-50| < 10) \geq 1-\frac{1}{k^2} = 1-\frac{1}{4} = \frac{3}{4}$$

$$= P(|X-50| < 10) \geq \frac{3}{4}$$ Ans

↳ lowest bound

Question:- A random variable X has the probability distribution.

x	0	1	2	4
$P(x)$	$1/2$	$1/4$	$1/8$	$1/8$

(a) Find an upper bound for $p(|X-1| \geq 2)$ by chebyshev's inequality.

(b) Find $p(|X-1| > 2)$ by direct computation.

Solution:- Since mean and varience are not given, so we firstly calculate it.

$$E(x) = \text{Mean} = \sum_{x=0}^{4} x\,P(x) = 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{8}\right) + 4\left(\tfrac{1}{8}\right)$$

$$= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{2} = 1$$

$$\boxed{\text{Mean} = E(X) = 1}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{x=0}^{4} x^2 P(x) = 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{4}\right) + 4\left(\tfrac{1}{8}\right) + 16\left(\tfrac{1}{8}\right)$$

$$= \tfrac{1}{4} + \tfrac{1}{2} + 2 = \tfrac{11}{4}$$

$$Var(X) = E(X^2) - (E(X))^2 = \tfrac{11}{4} - 1 = \tfrac{7}{4}$$

$$\sigma^2 = \frac{7}{4}$$

Thus $\boxed{S.D.(\sigma) = \frac{\sqrt{7}}{2}}$

Thus, required probability

$$= P(|X-1| > 2)$$

By Chebyshev's inequality, we have

$$P(|X-\mu| > k\sigma) \leq \frac{1}{k^2}$$

i.e., for $\mu = 1$, $P(|X-1| > k\sigma) \leq \frac{1}{k^2}$

where, $k\sigma = 2$

$$k\left(\frac{\sqrt{7}}{2}\right) = 2$$

$$\Rightarrow \boxed{k = \frac{4}{\sqrt{7}}}$$

$$P(|X-1| > 2) \leq \frac{1}{k^2} = \frac{7}{16}$$

$$\therefore \; P(|X-1| > 2) \leq \frac{7}{16}$$ Ans

(b) To find $P(|X-1| > 2)$ by direct computation.

The only X which satisfy $P(|X-1| > 2)$ is $X = 4$

$|0-1| = 1 \ngtr 2$
$|1-1| = 0 \ngtr 2$
$|2-1| = 1 \ngtr 2$
$|4-1| = 3 > 2$

Thus, required probability is

$$P(|X-1| > 2) = P(X = 4) = \frac{1}{8} \text{ (exact value)}$$ Ans

which is less than $\frac{7}{16}$, an upper bound computed by Chebyshev's inequality.

Question:- A random variable X has mean 10 and variance 4 and an unknown probability distribution. Find the value of c such that

$$p(|X-10| \geq c) \leq 0.04 ?$$

Solution:- Given that

Mean $\mu = 10$

Varience $\sigma^2 = 4$

Thus, s.o. $\boxed{\sigma = 2}$

$\because$ $p(|X-10| \geq c) \leq 0.04$

i.e., $p(|X-10| \geq c) \leq \frac{1}{5^2}$ —— (1)

By chebyshev's inequality, we have

$$p(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

i.e., $p(|X-10| \geq k\sigma) \leq \frac{1}{k^2}$ —— (2)

Form (1) and (2), we have

$k=5$ & $k\sigma = c$ | $k>0$ real.

$\Rightarrow c = (5 \times 2) = 10$

$\Rightarrow \boxed{c = 10}$

Ans

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Markov inequality with illustrations

Chebyshev's inequality : Let X be a random variable with mean μ and finite variance σ^2, then for any positive integer $k > 0$

$$\boxed{\begin{aligned} P\left(|X-\mu| \geq k\sigma\right) &\leq \frac{1}{k^2} \\ P\left(|X-\mu| < k\sigma\right) &\geq 1 - \frac{1}{k^2} \end{aligned}}$$

or

$$\boxed{\begin{aligned} P\left(|X-\mu| \geq a\right) &\leq \frac{Var(X)}{a^2} \\ P\left(|X-\mu| < a\right) &\geq 1 - \frac{Var(X)}{a^2} \end{aligned}}$$

Markov inequality : If X is a random variable and $a > 0$ is a constant, then

$$P\left(|X| \geq a\right) \leq \frac{E\left(|X|\right)}{a}$$

- No variance required.

Note that :

(i) Markov inequality gives the bound for the probability.

(ii) chebyshev inequality gives the lower/ upper bounds for the probability

(iii) $P(X > a) \leq P(|X| > a) \leq \dfrac{E(|X|)}{a}$.

Question : Consider a random variable X that takes the value 0 with probability $24/25$ and the value 1 with probability $\frac{1}{25}$. Find a bound on the probability that X is at least 5 :

Solution :- Given that

X	0	1
$P(X=x)$	$\frac{24}{25}$	$1/25$

By Markov inequality

$$P(|X| > a) \leq \frac{E(|X|)}{a}$$

$$\therefore \; P(X > 5) \leq P(|X| \geq 5) \leq \frac{E(|X|)}{5}$$

$$E(X) = \sum_{x=0}^{1} x \, P(X=x)$$

$$E(X) = 0 \times \frac{24}{25} + 1 \times \frac{1}{25} = \frac{1}{25}$$

$$\boxed{E(x) = \frac{1}{25}}$$

Thus, required probability

$$P(X \geq 5) \leq \frac{E(|X|)}{5} = \frac{1}{5 \times 25} = \frac{1}{125}$$

$$\boxed{P(X \geq 5) \leq \frac{1}{125}}$$

Ans

Question: A Coin is weighted so that its probability of landing on heads is 20%. Suppose the coin is flipped 20 times. Find a bound for the probability it lands on heads at least 16 times.

Solution: Given that

$$\left.\begin{aligned} p &= 20\% = 1/5 \\ n &= 20 \text{ (finite)} \end{aligned}\right] \text{(Binomial distribution)}$$

we know that

$$X \sim B(n, p)$$

$$\text{Mean} = E(X) = np = 20 \times \frac{1}{5} = 4$$

$$\Rightarrow \boxed{E(X) = 4} \quad \text{--- (a)}$$

The required probability $= P(X \geq 16)$.

we know by Markov inequality

$$P(|X| \geq a) \leq \frac{E(|X|)}{a} \quad \text{--- (1)}$$

$\therefore$ The required probability

$$P(X \geq 16) \leq P(|X| \geq 16) \leq \frac{E(|X|)}{16} \quad \text{--- (2)}$$

$$\Rightarrow \quad P(X \geq 16) \leq P(|X| \geq 16) \leq \frac{4}{16}$$

$$\Rightarrow \quad \boxed{P(X \geq 16) \leq P(|X| \geq 16) \leq \frac{1}{4}}$$

<u>Ans</u>