PROBABILITY AND STATISTICS (UCS401)

Lecture-12

(Negative Binomial with illustrations)

Random Variables and their Special Distributions (Unit –III & IV)

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

Dr. Rajanish Rai
Assistant Professor
School of Mathematics
Thapar Institute of Engineering and Technology, Patiala

(E) Negative-Binomial / Pascal distribution with examples.

(*) Negative-Binomial distribution is applicable when we need to performed an experiment until a total r successes are obtained.

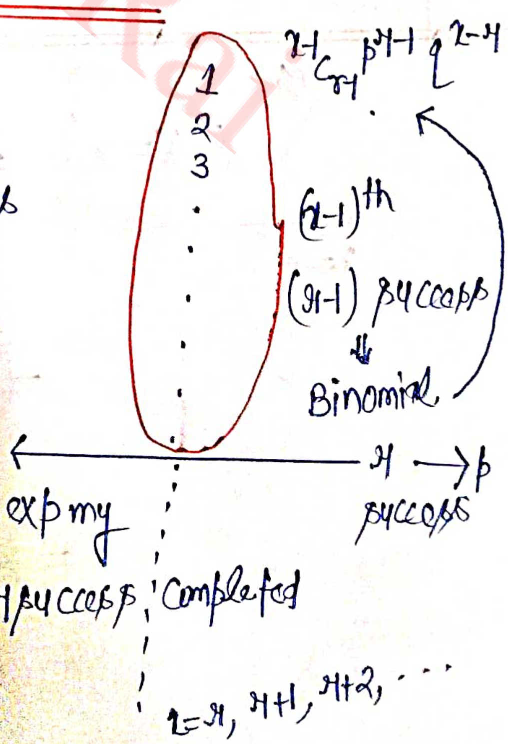
Remember-: (i) If $r=1$, means we perform an experiment till we obtain first success (which is the case of geometric distribution.

(ii) Negative-Binomial distribution is a generalization of the geometric distribution.

(iii) This distribution is also known as Pascal distribution.

Distribution functions for negative Binomial-:

$P(X=x) = P$($(r-1)$ success in the first $(x-1)$ trails AND a success in the xth trail)



$$= \left({}^{x-1}C_{r-1}\, p^{r-1} q^{(x-1)-(r-1)} \right) p$$

$$= \left({}^{x-1}C_{r-1}\, p^{r-1} q^{x-r} \right) p$$

$$= \binom{x-1}{r-1} p^{r} q^{x-r}.$$

Thus, for Negative Binomial distribution

$$\boxed{P(X=x) = \binom{x-1}{r-1} p^r q^{x-r} \quad ; \quad x = r, r+1, r+2, \dots}$$

Thus, for negative Binomial distribution, the p.m.f. is given by -:

$$P(X=x) = \begin{cases} \binom{x-1}{r-1} p^r q^{x-r} & ; \quad x = r, r+1, r+2, \dots \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

* Clearly if $r=1$, then

$$P(X=x) = p q^{x-1} \rightarrow \text{Geometric distribution.}$$

Why we call negative Binomial :

$$\sum_{x=r}^{\infty} P(X=x) = \sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r q^{x-r}$$

$$= p^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} q^{x-r}$$

$$= p^r (1-q)^{-r} \rightarrow \text{Negative Binomial}$$

$$\because (1-x)^n = \sum_{r=1}^{\infty} {}^nC_r x^r$$

$$= p^r p^{-r} = 1$$

Thus, the terms of the negative Binomial probability laws for $x = r, r+1, r+2, \dots$ are successive terms of the negative Binomial expansion.

This explain the reason why the random variable X with given density is called a negative random variable.

Mean and variance of negative-Binomial distribution-:

The p.m.f. is

$$P(X=x) = \binom{x-1}{r-1} p^r q^{x-r} \qquad x = r, r+1, r+2, r+3, \dots$$

having $\boxed{\text{Mean} = \frac{r}{p}}$ and $\boxed{\text{variance} = \frac{rq}{p^2}}$

Question-: If the probability is 0.40 that a child exposed to a certain disease will contain it, what is the probability that the tenth child exposed to the disease will be the third to catch.

Solution-: Let X be the number of child exposed to a certain disease.

$p = P(\text{child exposed to a disease}) = 0.40$

$\boxed{q = 0.60}$ Given that $r = 3$

The required probability

$= P(X=10)$

1
2
3
4
5
6
7
8
9

2 child
Binomial

$\boxed{10}$ 3rd child → p

$$= \left({}^9C_2\, p^2 q^7 \right) p$$

$$= {}^9C_2\, p^3 q^7$$

$$= \frac{9 \times 8}{2 \times 1} (0.4)^3 (0.6)^7$$

Thus, required probability

$$\boxed{P(X=10) = 0.0645}$$ Ans

Question-: Let X be the number of births in a family until the second daughter is born. If probability of having a male child is $\frac{1}{2}$. Find the probability that the sixth child in the family is the second daughter.

Solution-: Given $P(\text{male child}) = \frac{1}{2}$ $\quad r = 2$

$$\Rightarrow p \rightarrow P(\text{daughter}) = \frac{1}{2}$$

Let X be the no. of birth in a family until the 2nd daughter is born

$^5C_1\, p^1 q^4$

$\|$

1, 2, 3, 4, 5 → Binomial

one daughter

$\boxed{6}$ → 2nd daughter

→ p

Thus, required probability

$$= P(X=6)$$

$$= \left({}^5C_1\, p^1 q^4 \right) p = {}^5C_1\, p^2 q^4$$

$$= 5 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4 = \frac{5}{64}$$ Ans

Question :- In a company, 5% defective components are produced. What is the probability that at least 5 components are to be examined in order to get 3 defective?

Solution :- Let X be the number of defective product

$p = P(\text{defective}) = 0.05$

$q = 1 - 0.05 = 0.95$

& $r = 3$

$x = r, r+1, r+2 \ldots$

$x = 3, 4, 5 \ldots$

The required probability

$$= P(X \geq 5)$$

$$= 1 - P(X < 5)$$

$$= 1 - [P(X=3) + P(X=4)] \quad \text{--- (1)}$$

For negative Binomial distribution, the p.m.f. is given by

$$P(X=x) = \binom{x-1}{r-1} p^r q^{x-r} \; ; \; x = r, r+1, r+2, \ldots$$

$r = 3$

$$P(X=x) = \binom{x-1}{2} p^3 q^{x-3}$$

$$P(X=x) = {}^{x-1}C_2 (0.05)^3 (0.95)^{x-3} \quad \text{--- (2)}$$

From (1) & (2), the required probability

$$P(X \geq 5) = 1 - [{}^{2}C_2 (0.05)^3 (0.95)^0 + {}^{3}C_2 (0.05)^3 (0.95)^1]$$

$$= 1 - (0.05)^3 - 3(0.05)^3(0.95)$$

$$\boxed{P(X \geq 5) = 0.9995}$$

Ans

Question-: If the probability that a child exposed to a certain viral fever will be infected is 0.3, find the probability that the eight child exposed to the disease will be the fourth to the infected.

Solution-: Let X be the no. of child exposed to a disease

Given that

$$\left.\begin{aligned} p &\rightarrow P(\text{child exposed to viral}) = 0.3 \\ q &= 0.7 \; ; \quad r = 4 \end{aligned}\right] \text{Negative Binomial.}$$

The required probability

$$= P(X=8) = ?$$

For negative Binomial distribution, the p.m.f. is given by

$$P(X=x) = \binom{x-1}{r-1} p^r q^{x-r} \; ; \; x = r, r+1, r+2, \ldots$$

$$P(X=x) = \binom{x-1}{3} p^4 q^{x-4} \; ; \; x = 4, 5, 6, 7, \ldots$$

$$P(X=x) = {}^{x-1}C_3 (0.3)^4 (0.7)^{x-4}$$

Thus $P(X=8) = {}^{7}C_3 (0.3)^4 (0.7)^4$

or

$$P(X=8) = \left({}^{7}C_3 p^3 q^4\right) p$$

$$= {}^{7}C_3 p^4 q^4$$

$$= {}^{7}C_3 (0.3)^4 (0.7)^4$$

$$= 0.0681$$

Ans

1, 2, 3, 4, 5, 6, 7 — 3 child Binomial

$\boxed{8} \rightarrow$ 4th child $\rightarrow p$ infected

Question:- In a company, 3% of defective components are produced. What is the probability that at least 6 components are examined in order to get 3 defective.

Solution:- Let X be the number of defective components

Given that

$$\left.\begin{aligned} p &= P(\text{defective}) = 0.03 \\ q &= 0.97 \;\&\; r = 3 \end{aligned}\right] \text{Negative-Binomial}$$

Thus, required probability $= P(X \geq 6)$

$$= 1 - P(X < 6)$$

$\because$ For Negative Binomial distribution, the p.m.f. is given by

$$P(x) = P(X = x) = \binom{x-1}{r-1} p^r q^{x-r} \; ; \quad x = r, r+1, r+2, \ldots$$

$$P(X = x) = \binom{x-1}{2} p^3 q^{x-3} \quad x = 3, 4, 5, 6, \ldots$$

$$P(X = x) = {}^{x-1}C_2 (0.03)^3 (0.97)^{x-3} \quad \text{—— (1)}$$

Thus, required probability

$$P(X \geq 6) = 1 - P(X < 6)$$

$$= 1 - [P(X=3) + P(X=4) + P(X=5)]$$

$$= 1 - \left[{}^{2}C_2 (0.03)^3 (0.97)^0 + {}^{3}C_2 (0.03)^3 (0.97)^1 + {}^{4}C_2 (0.03)^3 (0.97)^2 \right]$$

Extra

Question:– If the probability of having a male child is 0.5, find the probability that in a family, the eighth child is the third boy.

Solution:– Let X be the number of male child

$p = P(\text{Male child}) = \frac{1}{2}$

$q = \frac{1}{2}$ & $r = 3$

1
2
3
4
5
6
7

Binomial

2 boys out of 7

$\boxed{8} \to$ 3rd boy $\to p$

Thus, the required probability

$$= P(X=8)$$

$$= \left({}^7C_2\, p^2 q^5\right) p$$

$$= {}^7C_2\, p^3 q^5$$

$$= {}^7C_2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5 = \frac{7\times 6}{2\times 1}\left(\frac{1}{2}\right)^8 = 0.0820$$

$$\boxed{P(X=8) = 0.0820}$$