TRANSPORTATION PROBLEM

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Q. There is a dairy firm that has 2 milk plants, daily milk production: 6 & 9 million litres respectively. There are 3 distribution centres which have the milk requirement of 7, 5, 3 million litres, respectively. Cost of transportation of 1 million litres of milk from each plant to each distribution centre is given in hundreds of rupees below. Formulate an LP model to minimize the transportation cost.

Plant/Distribution	D1	D2	D3	Supply
P1	2	3	11	6
P2	1	9	6	9
Demand →	7	5	3	

Formulation: Let x_{ij} be the units of commodity supplied from ith plant to jth destination.

$$1 \leq i \leq 2$$
$$1 \leq j \leq 3$$

Minimize $= 2x_{11} + 3x_{12} + 11x_{13} + x_{21} + 9x_{22} + 6x_{23}$

St:

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 6 \\ x_{21} + x_{22} + x_{23} &\leq 9 \end{aligned}\right\} \text{Source Constraints}$$

$$\left.\begin{aligned} x_{11} + x_{21} &\geq 7 \\ x_{12} + x_{22} &\geq 5 \\ x_{13} + x_{23} &\geq 3 \end{aligned}\right\} \text{Demand Constraints.}$$

$$x_{ij} \geq 0 \ \forall \ i, j$$

General Transportation Problem :-

Minimize $z = \sum_j \sum_i x_{ij} c_j$; c_{ij} is per unit cost of ith plant to jth destination

st :

$$\left.\begin{aligned} x_{11} + x_{12} + \cdots + x_{1n} &\le a_1 \\ x_{21} + x_{22} + \cdots + x_{2n} &\le a_2 \\ &\vdots \\ x_{m1} + \cdots + x_{mn} &\le a_m \end{aligned}\right\} \text{Source Constraint}$$

x_{ij} : units of commodity transportaten.

$$\left.\begin{aligned} x_{11} + x_{21} + \cdots + x_{m1} &\ge b_1 \\ &\vdots \\ x_{1n} + \cdots + x_{mn} &\ge b_n \quad ; \ x_{ij} \ge 0 \end{aligned}\right\} \text{Demand Constraint}$$

<u>or</u>

$$\boxed{\begin{aligned} &\text{Minimize } z = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} c_{ij} \\ &\text{st : } \sum_{j=1}^{n} x_{ij} \le a_i \leftarrow \text{Source} \\ &\qquad \sum_{i=1}^{m} x_{ij} \le b_j \ ; \ x_{ij} \ge 0 \leftarrow \text{Demand} \end{aligned}}$$

STANDARD TRASPORTATION PROBLEM

$$\text{Min } Z = \sum_j \sum_i x_{ij} c_{ij}$$

such that:

$$\left.\begin{aligned} \sum_{i=1}^{m} x_{ij} &= a_i \\ \sum_{j=1}^{n} x_{ij} &= b_j \end{aligned}\right\} \quad x_{ij} \geq 0$$

Similarly we can also have a maximization problem.

$$\text{Max } Z = \sum \sum x_{ij} c_{ij}$$

st : $\sum x_{ij} = a_i$

$\sum x_{ij} = b_j$; $x_{ij} \geq 0$

But in most cases we will get a minimization problem as we would obviously want to minimize costs on any given day for transportation.

REMARKS TO PROVE :-

1. Necessary and sufficient condition to have feasible solution is $\boxed{\text{total supply} = \text{total demand}}$

$\sum a_i = \sum b_j$ | T.P. $\rightarrow$ Balanced TP

$\sum a_i \neq \sum b_j$ | T.P $\rightarrow$ Unbalance TP

when we have an unbalanced TP then we will have to convert it into balanced TP to solve our question.

② There is always an existence of Optimal Solⁿ to transportation problem.

↳ usually when we solve LPPs then we know that LPP may have no feasible solⁿ however in the case of Transportation Problem feasible solution ALWAYS exists.

③ The number of Basic variables is :-

$$\boxed{\underbrace{m}_{\text{no. of sources}} + \underbrace{n}_{\text{no. of destinations}} - 1}$$

PROOFS

① Necessary and sufficient condition to have feasible solution is total supply = total demand

↑ EXISTENCE OF FEASIBLE SOLUTION

To prove :-

$$\boxed{\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j}$$

TRANSPORTATION PROBLEM

Every transportation problem can be represented as '$m \times n$' matrix where 'm' is the no of sources & 'n' is the no of destinations.

- Total no. of variables = $m \times n$
- Total no. of constr^n = $m+n$

	D_1	D_2	D_3	......	D_n	Availability
S_1	c_{11} x_{11}	c_{12} x_{12}	c_{13} x_{13}	.....	c_{1n} x_{1n}	a_1
S_2	c_{21} x_{21}	c_{22} x_{22}	c_{23} x_{23}	.....	c_{2n} x_{2n}	a_2
$\vdots$	$\vdots$					$\vdots$
S_m	c_{m1} x_{m1}	c_{m2} x_{m2}	c_{m3} x_{m3}	.....	c_{mn} x_{mn}	a_m
Req^n or Demand	b_1	b_2	b_3	.....	b_n	

$x_{ij} \rightarrow$ quantity of product supplied from source 'i' to destination 'j'

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \overbrace{c_{ij}}^{\text{cost}}\ \overbrace{x_{ij}}^{\text{quantity}}$$

such that:

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \dots m$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, 2, \dots n$$

$$x_{ij} \geq 0$$

$$i \in [1, n]; \ j \in [1, n]$$

TRANSPORTATION PROB

- BTP: Balanced, $\Sigma a_i = \Sigma b_j$
- UBTP: Unbalanced, $\Sigma a_i \neq \Sigma b_j$

Always convert unbalanced to blanced for solving.

Remark:- In a BTP, having m sources & n destinations only $m+n-1$ only will be linearly indipendent (ie, basic variables)

$$\boxed{n(BV) = m+n-1}$$

Proof By Example :-

Let us consider 2×3 TP:

	D_1	D_2	D_3	Availability
S_1	c_{11}	c_{12}	c_{13}	a_1
S_2	c_{21}	c_{22}	c_{23}	a_2
Demand	b_1	b_2	b_3	

$$\text{Min } Z = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} x_{ij}$$

s/t:

$x_{11} + x_{12} + x_{13} = a_1$ — ①

$x_{21} + x_{22} + x_{23} = a_2$ — ②

$x_{11} + x_{21} = b_1$ — ③

$x_{12} + x_{22} = b_2$ — ④

$x_{13} + x_{23} = b_3$ — ⑤

Doing :-

$(① + ②) - (③ + ④)$

$\Rightarrow (\cancel{x_{11}} + \cancel{x_{12}} + x_{13} + \cancel{x_{21}} + \cancel{x_{22}} + x_{23}) - (\cancel{x_{11}} + \cancel{x_{21}} + \cancel{x_{12}} + \cancel{x_{22}}) = (a_1 + a_2) - (b_1 + b_2)$

$\Rightarrow \; x_{13} + x_{23} = (a_1 + a_2) - (b_1 + b_2)$

But we know that $\sum a_i = \sum b_j \Rightarrow a_1 + a_2 = b_1 + b_2 + b_3$

Substituting this :-

$\Rightarrow x_{13} + x_{23} = \cancel{b_1} + \cancel{b_2} + b_3 - \cancel{b_1} - \cancel{b_2} \Rightarrow \boxed{x_{13} + x_{23} = b_3}$ this is constraint ⑤

This shows that there are we have ① dependent variable

- ☆ Total no. of variables $= m \cdot n$
- ☆ Total no. of constraints $= m + n$
- ☆ Total no. of linearly independent variables $= m + n - 1$
- ☆ Total no. of Basic variables $= m + n - 1$
- ☆ Max no. of Basic Feasible Sol^n $= / \leq \; {}^{mn}C_{m+n-1}$

DUAL OF TRANSPORTATION PROBLEM :-

$$\text{Max } z = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

st :-

$u_i + v_j \leq c_{ij}$

u_i & v_j are unrestricted

Steps to Solve Transportation Problem

Maximization $\xrightarrow{\text{convert}}$ Minimization

UBTP $\xrightarrow{\text{convert}}$ BTP

↓

find IBFS (initial Basic feasible solⁿ)

↓

Optimal

→ YES → END

→ NO → Do more iterations → Optimal

Steps :-

① Check if question is maxi (Max) or mini (Min) problem.
  ↳ if it is a maximization problem then convert into minimization problem.

② Check if Unbalanced (UBTP) or Balanced (BTP) Transportation problem.
  ↳ if unbalanced then convert to balanced

③ for our Minimization Balanced Transportation Problem find the IBFS → initial Basic feasible Solⁿ.

④ Check for optimality
  ↳ if optimality not satisfied then continue iterations.

Ways to Solve IBFS

- North-West Corner Rule (N-WC)
- Least Cost Entry Method (LCEM)
- Vogel's Approximation (VAM)

$\boxed{\text{NWC} > \text{LCEM} > \text{VAM}}$ ← solutions given by each method

NORTH WEST CORNER RULE (NWCR)

Q. find IBFS of given transportation problem for minimizing cost by NWCR.

North West corner

	D_1	D_2	D_3	D_4	Availability
S_1	* 14 7 x_{11}	56 x_{12}	48 x_{13}	27 x_{14}	~~13~~ 6 (13-7)
S_2	82	35	21	81 x_{24}	19
S_3	99	31	71	63 x_{34}	16
Demand	7	14	21	6	48 / 48

① We see that the question is minimize so no conversion required.

② Checking BTP

$$\Sigma a_i = \Sigma b_j \ ?$$

$$13+19+16 = 48 \qquad 7+14+21+6 = 48$$

equal

So this is a Balanced Transportation Problem.

③ Take min of a_i or b_j for a given entry

for $(1,1) = \min(7, 13) = 7$

= Replace 13 with $13-7 = 6$

	D_1	D_2	D_3	D_4	Availability
S_1	7	6 ~~56~~	~~48~~ x_{13}	~~27~~ x_{14}	~~13~~ 6 (13-7)
S_2	82	~~35~~ 8	~~21~~ 11	81 x_{24}	~~19~~ 11 (19-8) = 11
S_3	99	31	~~71~~ 10	$\boxed{63}$	~~16~~ 16-10 = 6
Demand	~~7~~	~~14~~	~~21~~	6	48 / 48

14-6 = 8

21-11 = 10 ✓

same

↳ last entry always has same value in both sides

$m = 3$

$n = 4$

$m+n-1 = 6$ ← from the highlighted values we can see we have 6 BVs as :—

$X_{11} = 7$	$X_{22} = 8$	$X_{33} = 10$
$X_{12} = 6$	$X_{23} = 11$	$X_{34} = 6$

← $m+n-1$ Basic Variables

Cost for IBFS :—

$$14 \times 7 + 6 \times 56 + 8 \times 35 + 21 \times 11 + 71 \times 10 + 63 \times 6$$

$$= \underline{2033}$$

☆ DEGENERATE BASIC FEASIBLE SOL$^{\underline{n}}$

↳ when any of the basic variables comes '$\underline{\underline{0}}$'

Q. find IBFS by NWCR to maximize returns.

	D_1	D_2	D_3	Availability
S_1	4	3	2	10
S_2	5	6	1	8
S_3	6	4	3	5
S_4	3	5	4	6
Demand	9	12	8	

① This is a maximization problem.
↳ we need to convert to MINIMIZATION
There are 2 ways to do that :-
(i) Multiply every cost with −1
(ii) Choose the maximum cost and subtract all costs from that to create a new table.

We will use method (ii)

	D_1	D_2	D_3	Availability
S_1	6−4= 2	6−3= 3	6−2= 4	10
S_2	6−5= 1	6−6 0	6−1 5	8
S_3	6−6= 0	6−4= 2	6−3= 3	5
S_4	6−3= 3	6−5= 1	6−4= 2	6
Demand	9	12	8	

new table

Now our maximization problem is converted to minimization problem so we now need to check BTP.

	D_1	D_2	D_3	Availability
S_1	~~4~~ 9	~~3~~ 1	4	10 = 10 − 9 = 1
S_2	1	~~6~~ 8	5	8
S_3	0	~~4~~ 3	~~3~~ 2	~~5~~ 5 − 3 = 2
S_4	3	1	~~4~~ 6	6
Demand	9	12	8	

12 − 1 = 11, 11 − 8 = 3; 8 − 2 = 6

same (END)

$\Sigma a_i = 29$

$\Sigma b_i = 29$

∵ $\Sigma a_i = \Sigma b_i$ our problem is a $\boxed{\text{BTP}}$ so we can proceed.

we apply NWCR on the problem.

From the table :-

Basic variables:

$$\boxed{\begin{array}{ll} x_{11} = 9 & x_{12} = 1 \\ x_{22} = 8 & x_{32} = 3 \\ x_{33} = 2 & x_{43} = 6 \end{array}}$$

No. of BV = 6
$(m+n-1 = 4+3-1 = 6)$

Maximizing IBFS = Σ(multiply the BV with the actual cost)

$= 9\times4 + 1\times3 + 8\times6 + 3\times4 + 2\times3 + 6\times4$

$= \boxed{129}$

LEAST COST ENTRY METHOD (LCEM)

Q. Find IBFS of given transportation problem for minimizing cost by NWCR.

	D_1	D_2	D_3	D_4	Availability
S_1	14	56	48	27	13
S_2	82	35	21	81	19
S_3	99	31	71	63	16
Demand	7	14	21	6	$\Sigma a_i = 48$, $\Sigma b_j = 48$

1. It is a minimizing problem.
2. It is BTP $\Sigma a_i = \Sigma b_j$
3. LCEM :

	D_1	D_2	D_3	D_4	Availability
S_1	~~14~~ 7	56	~~48~~ 0	~~27~~ 6	13 13−7 = 6 6−6 = 0 ✓
S_2	82	35	~~21~~ 19	81	19 ✓
S_3	99	~~31~~ 14	~~71~~ 2	63	16 16−14 = 2
Demand	7	14	21	6	$\Sigma a_i = 48$, $\Sigma b_j = 48$

21 − 19 = 2   2 − 0 = 2

Here there is no direction rule just choose the minimum cost from the table & do the same prev. steps as in NWCR

Basic Variables:

$$\boxed{\begin{array}{ll} x_{11} = 7 & x_{13} = 0 \\ x_{14} = 6 & x_{23} = 19 \\ x_{32} = 14 & x_{33} = 2 \end{array}}$$

→ since one of our BVs is '0' so we have degenerate IBFS

Minimum Cost: $7 \times 14 + 0 \times 48 + 6 \times 27 + 19 \times 21 + 14 \times 31 + 71 \times 2$

$= \boxed{1235}$ → this is lesser than the value we got with NWC

Q. find IBFS by LCER to maximize returns.

	D_1	D_2	D_3	Availability
S_1	4	3	2	10
S_2	5	6	1	8
S_3	6	4	3	5
S_4	3	5	4	6
Demand	9	12	8	

① This is a maximization problem.
↳ we need to convert to MINIMIZATION
There are 2 ways to do that :-
(i) Multiply every cost with -1
(ii) Choose the maximum cost and subtract all costs from that to create a new table.

We will use method (ii)

	D_1	D_2	D_3	Availability
S_1	6-4= 2	6-3= 3	6-2= 4	10
S_2	6-5= 1	6-6 0	6-1 5	8
S_3	6-6= 0	6-4= 2	6-3= 3	5
S_4	6-3= 3	6-5= 1	6-4= 2	6
Demand	9	12	8	

new table

Now our maximization problem is converted to minimization problem so we now need to check BTP.

	D_1	D_2	D_3	Availability
S_1	~~2~~ 4	3	~~4~~ 6	10, 10−4 = 6
S_2	1	~~0~~ −8−	5	8
S_3	~~0~~ 5	2	3	5
S_4	2	~~2~~ 4	~~2~~ 2	6, 6−4 = 2 ✓
Demand	9, 9−5 = 4	12, 12−8 = 4 ✓	8, 8−2 = 6	

In this case we can see that we have minimum value at two places at [2,2] & [1,3]

So to $\boxed{\text{Break the tie}}$ we select that cell which has maximum allocation.

Eg: in this case if we choose [2,2] we will replace 0 → 8

If we choose [1,3] we will do 0 → 5

Since [2,2] has maximum allocation we will choose that as the minimum cell. A similar case comes for [1,1] & [4,3] as both are equal to '2'

Basic Variables:-

$X_{11} = 4$	$X_{13} = 6$	$X_{22} = 8$
$X_{31} = 5$	$X_{42} = 4$	$X_{43} = 2$

Maximized Profit $= 4\times4 + 6\times2 + 8\times6 + 5\times6 + 4\times5 + 2\times4$

$= \boxed{134}$ Ans

this answer is larger than NWC method. as NWC performs worse than LCEM

VOGEL'S APPROXIMATION (VAM)

→ gives more minimum value than LCEM or NWCR

Q. five warehouses; 4 Plants; find IBFS by VAM to minimize cost

	P_1	P_2	P_3	P_4	Availability
W_1	3	11	6	0	80
W_2	3	11	7	3	110
W_3	8	13	7	5	150
W_4	10	8	12	1	100
W_5	7	10	10	6	150
Demand	150	200	175	100	$\Sigma a_i = 590$, $\Sigma b_j = 625$

① Problem is minimization so no conversion required

② $\Sigma a_i \neq \Sigma b_j \Rightarrow$ Unbalanced Transportation Problem

So we have to convert this into BTP.

We add a dummy row/col to the lesser one. In this case we add a dummy row.

	P_1	P_2	P_3	P_4	Availability	Penalty
W_1	3	11	6	0	80	3 (3-0)
W_2	3	11	7	3	110	0 (3-3)
W_3	8	13	7	5	150	2 (7-5)
W_4	10	8	12	1	100	7 (8-1)
W_5	7	10	10	6	150	1 (7-6)
Dummy	0	0	0	0	35	0 (0-0)
Demand	150	200	175	100	$\Sigma a_i = 625$, $\Sigma b_j = 625$	
Penalty	3 (3-0)	8 (8-0)	6 (6-0)	0 (0-0)		

Now our problem is balanced (BTP), so we can proceed with VAM.

(i) Calculate Penalties in each row & column

↳ find the minimum 2 quantities in a given row/column; subtract them & get the penalty for that row/column.

(ii) Choose the highest penalty to enter in the row/column.

(iii) Then apply LCEM starting from the least entry in the entering row/column.

(iv) Calculate the fresh penalties.

(v) Loop from steps (ii) → (v) till you get optimal

★ To break a tie use the method used in LCEM

↳ Choose the one that causes maximum assignment of cost

↳ if there is a tie even in the assignment then you can enter from anywhere → row or colum

	P_1	P_2	P_3	P_4	Availability	Penalty	After removing row ⑤			
~~W_1~~	~~3~~ 40	11 15	~~6~~ 25	~~0~~ 0	~~80~~ 80-0= ~~80~~ ~~40~~ 15	3 (3-0)	3	3 (enter)	3	3 (5) (11)
~~W_2~~	~~3~~ 110	11	7	3	110	0 (3-3)	0	0	4	may
~~W_3~~	8	13	~~7~~ 150	5	150	2 (7-5)	2	2	1	1 (6)
~~W_4~~	10	8	12	1 100	100	7 (8-1)	7 high			
W_5	7	150 ~~10~~	10	6	150	1 (7-6)	1	1	3	3 (0) ~~(10)~~ 10
~~Dummy~~	0	0 35	0	0	35	0 (0-0)				
Demand	~~150~~ 40	200 ~~185~~ ~~150~~ ~~35~~	175 25	~~100~~ 0	$\Sigma a_i = 625$, $\Sigma b_j = 625$					
Penalty	3 (3-0)	8 (8-0)	6 (6-0)	0 (0-0)						
(Updated)	(0)	(2)	(1)	(1)						
	(0)	highest (enter) (1)	(1)	3						
	(0)	(1)	(1)							
	(4) max	(1)	(1)							
		(1)	(1)							
		(1)	4							
		~~(1)~~ (1)								

same (last) end

since # BV = 0 thus
we have degenerate IBFS.

Basic Variables :

$x_{11} = 40$	$x_{12} = 15$	$x_{13} = 25$	$x_{14} = 0$	$x_{Dummy\,2} = 35$
$x_{21} = 110$	$x_{33} = 150$	$x_{44} = 100$	$x_{52} = 150$	

$$\text{Min Cost} = 40 \times 3 + 15 \times 11 + 25 \times 6 + 0 \times 0 + 110 \times 3 + 150 \times 7 + 100 \times 1 + 150 \times 10 + 35 \times 0$$
$$= 3415$$

Optimal Solution of T.P.

Q. Find Optimal Solution of the given TP to minimize cost.

		$v_1=1$	$v_2=2$	$v_3=1$	$v_4=1$	
		D_1	D_2	D_3	D_4	Availability
$u_1=0$	S_1	1 15	2 15	3	4	30
$u_2=4$	S_2	7	6 5	2	5 45	50
$u_3=1$	S_3	4	3 10	2 25	7	35
	Demand	15	30	25	45	

$m = 3$

$n = 4$

$m+n-1 = 3+4-1 = 6$

* Only when we have $m+n-1$ B.V. then we can find our O.S.

⓪ We need to find the values of u_i & v_j.
We know that for B.V. $z_j - c_j = 0$.
In this case :—

$$\boxed{u_i + v_j - c_{ij} = 0} \longrightarrow \text{for basic cells}$$

$$\Rightarrow \boxed{u_i + v_j = c_{ij}}$$

(i) We assume $u_1 = 0$ then we can easily find the other values using the above formula.

So for BV
$[1,1] \Rightarrow u_1 + v_1 = C_{11} \Rightarrow 0 + v_1 = 1 \Rightarrow v_1 = 1$
$[1,2] \Rightarrow u_1 + v_2 = C_{12} \Rightarrow 0 + v_2 = 2 \Rightarrow v_2 = 2$
$[2,2] \Rightarrow u_2 + v_2 = C_{22} \Rightarrow u_2 + 2 = 6 \Rightarrow u_2 = 4$
$[2,4] \Rightarrow u_2 + v_4 = C_{24} \Rightarrow 4 + v_4 = 5 \Rightarrow v_4 = 1$
$[3,2] \Rightarrow u_3 + v_2 = C_{32} \Rightarrow u_3 + 2 = 3 \Rightarrow u_3 = 1$
$[3,3] \Rightarrow u_3 + v_3 = C_{33} \Rightarrow 1 + v_3 = 2 \Rightarrow v_3 = 1$

(ii) Now unlike simplex in this case we will check the South West entries $(u_i + v_j - c_{ij})$ for non-basic cells.

$(1+0-3)$ $\theta + 1 - 4 = -3$

		$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 1$	
		D_1	D_2	D_3	D_4	Availability
$u_1 = 0$	S_1	1 (15)	2 (15)	3 | -2	4 | -3	30
$u_2 = 4$	S_2	7 | -2	6 (5)	2 | 3	5 (45)	50
$u_3 = 1$	S_3	4 | -2	3 (10)	2 (25)	7 | -5	35
	Demand	15	30	25	45	

$4+1-7 = -2$

$4+1-2 = 3$

$1+1-7 = -5$

$1+1-4 = -2$

$$\boxed{u_i + v_j - c_{ij}}$$

$\Downarrow$

		$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 1$	
		D_1	D_2	D_3	D_4	Availability
$u_1 = 0$	S_1	1 (15)	2 (15)	(-2) 3	(-3) 4	30
$u_2 = 4$	S_2	(-2) 7	6 (5)	(3) 2 θ	5 (45)	50
$u_3 = 1$	S_3	(-2) 4	3 (10)	2 (25)	(-5) 7	35
	Demand	15	30	25	45	

$m+n-1$

$\theta = 5$

(iii) In case of simplex we used to stop when we saw that all $z_j - c_j < 0$; in this case we stop when all South West Entries are ≤ 0.

In our case we see that the SW entry in (2,3) is +ve

So this is NOT OPTIMAL SOLUTION.

Now just like in simplex this is the entering variable

To do that we have to create a loop starting from the entering variable & ending there & turn can only happen when there is a Basic Variable.

		$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 1$	
		D_1	D_2	D_3	D_4	Availability
$u_1 = 0$	S_1	1 (15)	2 (15)	-2 3	-3 4	30
$u_2 = 4$	S_2	-2 7	6 (5)	3 2 (θ)	5 (45)	50
$u_3 = 1$	S_3	-2 4	3 (10)	2 (25)	-5 7	35
	Demand	15	30	25	45	

Loop

(iv) To maintain availability & demand we have to add/subtract θ from the members of the loop

		$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 1$	
		D_1	D_2	D_3	D_4	Availability
$u_1 = 0$	S_1	1 (15)	2 (15)	-2 3	-3 4	30
$u_2 = 4$	S_2	-2 7	6 (5)$-\theta$	$\boxed{3}$ 2 θ	5 (45)	50
$u_3 = 1$	S_3	-2 4	3 (10)$+\theta$	2 (25)$-\theta$	-5 7	35
	Demand	15	30	25	45	

(v) find the value of θ as

$$\min\{5, 10, 25\} = 5$$

Update the table & freshly find the values of u_i & v_j again.

(vi) STOP: when all $\boxed{v_j + u_i - c_{ij} = 0}$

Finding the new u_i & v_j.

Since this entry = 0 thus alternate opt sol^n exists. $\theta = 5$

		$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 4$	
		D_1	D_2	D_3	D_4	Availability
$u_1 = 0$	S_1	1 (15)	2 (15)	-2 | 3	0 | 4	30
$u_2 = 1$	S_2	-5 | 7	~~θ~~ -3 | 6	2 (5)	5 (45)	50
$u_3 = 1$	S_3	-2 | 4	3 (15)	2 (20)	-2 | 7	35
	Demand	15	30	25	45	

Remove this cell as it is equal to '0'

Now all our S-W non basic assignments are less than or equal to 0 so we can STOP.

⇓

This is our Optimal Solution.

- $u_1 + v_1 = 1 \Rightarrow 0 + v_1 = 1 \Rightarrow \boxed{v_1 = 1}$
- $u_1 + v_2 = 2 \Rightarrow 0 + v_2 = 2 \Rightarrow \boxed{v_2 = 2}$
- $u_3 + v_2 = 3 \Rightarrow u_3 + 2 = 3 \Rightarrow \boxed{u_3 = 1}$
- $u_3 + v_3 = 2 \Rightarrow 1 + v_3 = 2 \Rightarrow \boxed{v_3 = 1}$
- $u_2 + v_3 = 2 \Rightarrow u_2 + 1 = 2 \Rightarrow \boxed{u_2 = 1}$
- $u_2 + v_4 = 5 \Rightarrow 1 + v_4 = 5 \Rightarrow \boxed{v_4 = 4}$

Optimal Solution :-

$x_{11} = 15$, $x_{12} = 15$, $x_{23} = 5$, $x_{24} = 45$

$x_{32} = 15$, $x_{33} = 20$

Minimum $Z = 1 \times 15 + 2 \times 15 + 5 \times 2 + 45 \times 5 + 15 \times 3 + 20 \times 2$

$= \boxed{365}$

★ **Alternate Optimal Solution :** Alternate Optimal solution exists when any of the S-W entries assigned to any non-basic variable is '0'.

↳ To find this alternate optimal solution, we would make that '0' entry as 'θ' & repeat the above steps.