SEARCH TECHNIQUES IN NLPP

Search Techniques NLPP

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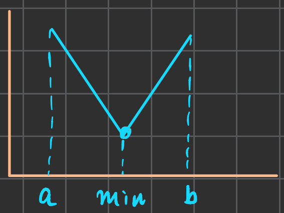
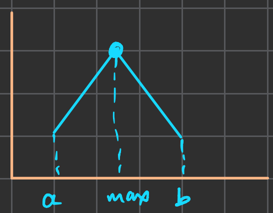
NLPP: if at least your objective f^n or constraint is NOT LINEAR is an NLPP

In search techniques we we try to find the APPROXIMATE sol^n and not the OPTIMAL SOLUTION.

☆ We are going to consider NLPP without constraints.

UNIMODAL F^n: That has one peak in the given interval $[a,b]$ ie, it has maximum or minimum value.

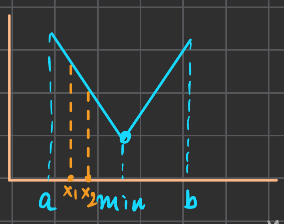


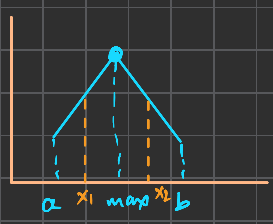


← examples of unimodal f^ns.

RESULTS:

(i) if $x_1 < x_2$ & $f(x_1) > f(x_2)$ then the interval where the minimum lies.





intervals :-

$x_1 < b$ & $f(x_1) < f(b)$ → $[x_1, b]$ & $[x_2, b]$ ← $x_2 < b$ & $f(x_2) < f(b)$

common = $[x_1, b]$

This is where minimum lies

intervals :-

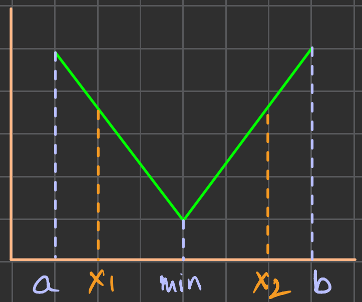
$[a, x_2]$, $[x_1, x_2]$ & $[x_2, b]$

$a < x_2$ & $f(a) < f(x_2)$

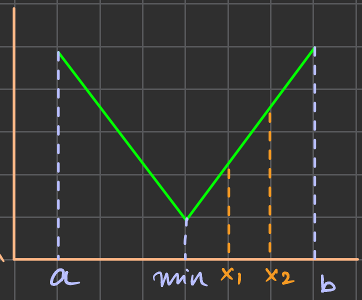
$x_1 < x_2$ & $f(x_1) < f(x_2)$

$x_2 < b$ & $f(x_2) < f(b)$

(ii) $x_1 < x_2$ implies $f(x_1) < f(x_2)$



Intervals : $[x_1, x_2]$, $[x_1, b]$, $[a, x_2]$
Common : $[a, x_2]$



Intervals : $[a, x_1]$ & $[a, x_2]$
Common : $[a, x_2]$

(iii) If $x_1 < x_2$ & $f(x_1) > f(x_2)$ then common $[a, x_2]$
If $x_1 < x_2$ & $f(x_1) < f(x_2)$ then common $[x_1, b]$

FIBONACCI SEARCH TECHNIQUE

Fibonacci No's:

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
1	1	2	3	5	8	13	21	34

$f_0 = 1$; $f_1 = 1$; $F_n = F_{n-2} + F_{n-1}$; $n \geq 2$

$\boxed{x_1 = a + \frac{F_{n-2}}{F_n} L_0}$; $\boxed{x_2 = a + \frac{F_{n-1}}{F_n} L_0}$ $\boxed{\alpha = \frac{1}{F_n}}$

L_0 ← length of interval

Measure of effectiveness :

$\boxed{\alpha = \frac{L_n}{L_0}}$

L_0 = length of interval
L_n = length of uncertainity after 'n' iterations

Q Minimize $f(x) = x(x-2)$; $0 \leq x \leq 1.5$ within the interval of uncertainity $0.25 L_0$

Ans : $\alpha = \frac{L_n}{L_0} = \frac{0.25 \cancel{L_0}}{\cancel{L_0}} = \boxed{0.25}$

$\frac{1}{F_n} \leq 0.25 \Rightarrow \frac{1}{F_n} \leq \frac{1}{4} \Rightarrow \boxed{F_n \geq 4}$

From the fibonacci series we know $F_4 = 5$

So; $F_n \geq 4 \Rightarrow F_4 \geq 4 \Rightarrow n = 4$ ← no' of iterations

$F_4 \to 5$

First Iteration :

$n = 4$

interval $= [0, 1.5]$ ($a = 0$, $b = 1.5$)

$L_0 = 1.5 - 0 = 1.5$

$x_1 = a + \frac{F_{4-2}}{F_4} L_0 = 0 + \frac{F_2}{F_4}(1.5) = 0 + \frac{2}{5} \times 1.5$
$= 0.6$

$$x_2 = a + \frac{F_{4-1}}{F_4} L_0 = 0 + \frac{F_3}{F_4}(1.5) = 0.9$$

$$f(x_1) = f(0.6) = -0.84 \qquad f(x_2) = -0.99$$

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ so, interval: $[x_1, b] = [0.6, 1.5]$

2nd Iteration :-

$n = 3$; $[a,b] = [0.6, 1.5]$; $L_0 = 1.5 - 0.6$

$$x_1 = a + \frac{F_{3-2}}{F_3} L_0 = 0.9 \qquad x_2 = a + \frac{F_{3-1}}{F_3} L_0 = 1.2$$

$$f(x_1) = -0.99 \qquad f(x_2) = -0.96$$

$$\boxed{x_1 < x_2}$$

$$\boxed{f(x_1) < f(x_2)}$$

∴ New interval: $[a, x_2] = [0.6, 1.2]$

3rd Iteration :-

$n = 2$; $[a,b] = [0.6, 1.2]$; $L_0 = 1.2 - 0.6 = 0.6$

$$x_1 = a + \frac{F_{2-2}}{F_2} L_0 = 0.9$$

$$x_2 = a + \frac{F_{2-1}}{F_2} L_0 = 0.9$$

$$\boxed{x_1 = x_2}$$

We can consider $x_2 = 0.91$ (just greater than x_1) so that we can find new interval.

Thus, $x_1 < x_2$

$f(x_1) = -0.99 \qquad f(x_2') = -0.981$

$f(x_1) < f(x_2)$; $\Rightarrow$ interval $= [a, x_1] = [0.6, 0.9]$

$x_2 = 0.9$ (we took $x_2 = 0.91$ just to check interval)

We will not perform any more iterations as Fib. series is only valid till $n \geq 2$; so our :-

Optimal Solⁿ $= \frac{0.6 + 0.9}{2} = 0.75$

Ans

Approx. Optimal Value $= f(0.75)$

$= 0.75(0.75 - 2) = -0.9375$

STEEPEST DESCENT SEARCH TECHNIQUE

Updating:-

$x = (x_1, x_2, x_3 \cdots)$ gradient

$$\boxed{\begin{aligned} X_{i+1} &= x_i + \lambda_i s_i \\ &= x_i - \lambda_i \nabla f(x_i) \; ; \; s_i = -\nabla f(x_i) \end{aligned}}$$

Termination process :-

1) $|f(x_{i+1}) - f(x_i)| \leq \epsilon_1$

2) $|x_{i+1} - x_i| \leq \epsilon_2$

3) $\left| \dfrac{f(x_{i+1}) - f(x_i)}{f(x_i)} \right| \leq \epsilon_3$

4) $\left| \dfrac{\partial f}{\partial x_i} \right| \leq \epsilon_4$

$\boxed{\epsilon_i = \text{tolerance.}}$

← We can choose any of the termination criteria. It will be according to what is available to us in the question.

Q Minimize $\overbrace{f(x) = x_1^2 - x_1 x_2 + x_2^2}^{\text{NLPP}}$ so that error does not exceed by 0.05 (ϵ); ie; $\underbrace{|f(x_{i+1}) - f(x_i)| \leq 0.05}_{\text{termination criteria}}$. The initial approximation is $\underbrace{X_1 = (1, 1/2)}$

Soln : $x_{i+1} = x_i + \lambda_i \nabla f(x_i)$

$f(x) = x_1^2 - x_1 x_2 + x_2^2$

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) = (2x_1 - x_2, -x_1 + 2x_2)$$

①

$$\begin{aligned} X_2 &= X_1 - \lambda_1 \nabla f(X_1) \\ &= (1, 1/2) - \lambda_1 (2(1) - 1/2, -1 + 2(1/2)) \\ &= (1, 1/2) - \lambda_1 (3/2, 0) \end{aligned}$$

$$= \left(1 - \frac{\lambda_1 3}{2}, \frac{1}{2}\right) = \left(\frac{2 - 3\lambda_1}{2}, 1/2\right) \quad \text{—①}$$

$$f(X_2) = \left(\frac{2 - 3\lambda_1}{2}\right)^2 - \left(\frac{2 - 3\lambda_1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = \frac{4 - 6\lambda_1 + 9\lambda_1^2}{4} - \frac{2 - 3\lambda_1}{4} + \frac{1}{4}$$

$$= \frac{4 - 6\lambda_1 + 9\lambda_1^2 - 2 + 3\lambda_1 + 1}{4} = \frac{3 - 3\lambda_1 + 9\lambda_1^2}{4}$$

★ $\frac{\partial f}{\partial \alpha_1} = 0 \rightarrow$ to find stationary point.

$$\Rightarrow \frac{-3+18\alpha_1}{4} = 0 \Rightarrow \boxed{\alpha_1 = \frac{1}{2}}$$

Substituting this value in ①

$$X_2 = \left(\frac{2-3(1/2)}{2}, \frac{1}{2}\right) = \left(\frac{1}{4}, \frac{1}{2}\right)$$

$$f(X_2) = \left(\frac{1}{4}\right)^2 - \frac{1}{4}\times\frac{1}{2} + \left(\frac{1}{2}\right)^2 = \frac{1}{16} - \frac{1}{8} + \frac{1}{4}$$

$$= \frac{1-2+4}{16} = \frac{3}{16} = 0.1875$$

$$f(X_1) = (1)^2 - 1\times\frac{1}{2} + \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} + \frac{1}{4} = \frac{4-2+1}{4}$$

$$= 0.75$$

$$|f(X_2) - f(X_2)| = 0.5625 \nleq 0.05 \rightarrow$$ so we can not terminate here.

② $X_3 = X_2 - \alpha_2 \nabla f(X_2)$

$$\nabla f(X_2) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (2x_1 - x_2, -x_1 + 2x_2)$$

$$= \left(\frac{1}{4}, \frac{1}{2}\right) - \alpha_2\left(2*\frac{1}{4} - \frac{1}{2}, \frac{-1}{4} + 2*\frac{1}{2}\right)$$

$$= \left(\frac{1}{4}, \frac{1}{2}\right) - \alpha_2\left(0, \frac{3}{4}\right)$$

$$= \left(\frac{1}{4}, \frac{2-3\alpha_2}{4}\right) = \left(\frac{1}{4}, \frac{1}{8}\right)$$

$$f(X_3) = \frac{1}{16} - \left(\frac{2-3\alpha_2}{16}\right) + \left(\frac{2-3\alpha_2}{4}\right)^2 \Rightarrow \frac{\partial f}{\partial \alpha_2} = 0 \Rightarrow \boxed{\alpha_2 = \frac{1}{2}}$$

$$f(X_3) = \left(\frac{1}{4}\right)^2 + \frac{1}{4}\times\frac{1}{8} + \left(\frac{1}{8}\right)^2 = \frac{3}{64}$$

$$|f(X_3) - f(X_2)| = 9/64 \leq 0.05$$ ✓

we have obtained optimal solⁿ at $\underline{\underline{X_4}}$

NLPP

- Without Contraints
  - **Hessian Matrix**
- With Constraints
  - Sign of all constraints are of '=' type
    - **Lagrange Multiplier**
  - Sign of at least one constraint is not of '=' type ($\geq$, $\leq$)
    - **KKT Method**

NLPP without Constraints
HESSIAN MATRIX

Stationary Points : $f(x) = f(x_1, x_2, x_3 \cdots x_n)$

$$\boxed{\frac{\partial f}{\partial x_i} = 0 \; ; \quad i = 1, 2 \cdots n}$$

→ solve these to find the $x_1, x_2 \cdots x_n$.

Nature of Stationary points :-

① A point $(x_1, x_2, \cdots x_n)$ will be pt. of maxima, if the Hessian matrix corresponding to $(x_1, x_2 \cdots x_n)$ is -ve semi definite or -ve definite.

② A point $(x_1, x_2 \cdots x_n)$ will be pt of minima, if the Hessian matrix corresponding to $(x_1, x_2 \cdots x_n)$ is +ve semidefinite or +ve define.

③ A point $(x_1, x_2 \cdots x_n)$ will be a saddle point, if the Hessian matrix corresponding to $(x_1, x_2, \cdots x_n)$ is INDEFINITE.

Understanding +ve/-ve definite :-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$D_1 = |a_{11}| \qquad D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

for a given Hessian Matrix when :-

(i) $D_1, D_2 \cdots D_m > 0 \rightarrow$ +ve definite

(ii) $D_1 > 0$ & $D_2, D_3 \cdots D_n \geq 0 \rightarrow$ +ve semidefinite

(iii) $D_1 < 0$, $D_2 > 0$, $D_3 < 0 \cdots$ alternate signs :-
$D_i(\text{odd}) < 0$ & $D_i(\text{even}) < 0 \rightarrow$ -ve DEFINITE

(iv) $D_1 < 0$, $\Delta_2 \geq 0$, $D_3 \leq 0 \cdots$ alternate
↳ D_1 should be less than 0
D_i, $i \neq 1 \Rightarrow$ may have alternate $\geq / \leq 0$
↳ -ve semidefinite.

HESSIAN MATRIX :-

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{n \times n}$$

Q $f(x) = 2 + 2x_1 + 3x_2 - x_1^2 - x_2^2 \rightarrow$ identify station. pt & nature

(i) find stationary point :-

$$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 2 - 2x_1 = 0 \Rightarrow -2x_1 = -2 \Rightarrow x_1 = 1$$

$$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 3 - 2x_2 = 0 \Rightarrow -2x_2 = -3 \Rightarrow x_2 = \frac{3}{2}$$

Stationary point $= (1, 3/2)$

(ii) finding the nature $\rightarrow$ Construct Hessian matrix

$$\underset{(x_1, x_2)}{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\underbrace{\odot\ D_1 = -2 < 0 \qquad \odot\ D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0}$$

Thus H is <u>-ve definite</u> (alternate D_i (<0)(>0))

Thus $(1, 3/2)$ is a point of <u>maxima</u>.

Q find stationary point & identify the nature of the given f^n :-

$$f(x) = x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + x_1x_3 + 16$$

① finding a stationary point.

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2x_2 + x_3 = 0 \qquad \frac{\partial f}{\partial x_2} = 2x_2 - 2x_1 = 0 \Rightarrow \boxed{x_2 - x_1 = 0} \text{ (ii)}$$

$$\Rightarrow \boxed{2x_1 - 2x_2 + x_3 = 0} \text{ (i)} \qquad \frac{\partial f}{\partial x_3} = 6x_3 + x_1 = 0 \Rightarrow \boxed{6x_3 + x_1 = 0} \text{ (iii)}$$

Solving ①, ② & ③:

Stationary point $= (0, 0, 0)$

(ii) finding nature → constructing Hessian matrix :-

$$H = \begin{bmatrix} \frac{\partial f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1 \partial x_3} \\ \frac{\partial f}{\partial x_2 \partial x_2} & \frac{\partial f}{\partial x_2^2} & \frac{\partial f}{\partial x_2 \partial x_3} \\ \frac{\partial f}{\partial x_3 \partial x_1} & \frac{\partial f}{\partial x_3 \partial x_2} & \frac{\partial f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Leftarrow H$$

$D_1 = \boxed{2 > 0}$ $\quad D_2 = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 4 - 4 = 0$

$D_3 = \begin{vmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \boxed{-2 < 0}$

H is indefinite

↓

Thus the obtained Stationary point is a saddle point.

$\underline{(0, 0, 0)}$

↑

Neither max nor min.

NLPP with EQUALITY constraints
LAGRANGE METHOD

NLPP format :

$$\text{Optimize } f(x)$$
$$\text{s/t; } g_i(x) = 0 \text{ ; } i = 1, 2 \cdots n$$

Lagrange eqⁿ formed :

$$\boxed{L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)}$$

This lagrange fⁿ will help us to convert our NLPP with constraints into a NLPP without constraints and we can then apply our previous method to solve this new lagrange fⁿ.

So we find :-

$$\frac{\partial L}{\partial x_i} = 0 \text{ ; } \frac{\partial L}{\partial \lambda_i} = 0 \text{ ; } i = 1, 2, \cdots m.$$

In the previous Hessian method we constructed the Hessian matrix. In this case we will be constructing the Bordered Hessian Matrix.

Bordered Hessian Matrix :-

$$\boxed{H^B = \begin{bmatrix} O_{m \times m} & P_{m \times n} \\ P^T_{n \times m} & Q_{n \times n} \end{bmatrix}}$$

differentiate every constraint with every variable

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 x_n} \\ \vdots & & & \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}_{n \times n}$$

Hessian for $L(x)$

How to determine nature?

(i) If D_i starts from sign $(-1)^{m+1}$ & changes alternatively when $i \geq 2m+1 \rightarrow$ POINT OF MAXIMA

(ii) If D_i starts from sign $\boxed{(-1)^m}$ & $\boxed{\text{remains same}}$ when $i \geq 2m+1 \longrightarrow$ POINT OF MINIMA.

(iii) If no such pattern is formed $\rightarrow$ SADDLE POINT

Q Optimize the function: $f(x) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

s/t: $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$.

Ans: Since all constraints are of equal to type thus we will apply Lagrange's method.

(i) Find L

$$\boxed{L(x,\lambda) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 + \lambda_1(x_1 + x_2 + x_3 - 15) + \lambda_2(2x_1 - x_2 + 2x_3 - 20)}$$

(ii) To find the stationary point.

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 8x_1 - 4x_2 + \lambda_1 + 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 4x_1 + \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 + \lambda_1 + 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 + x_3 - 15 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 2x_1 - x_2 + 2x_3 - 20 = 0$$

$$x_1 = \frac{11}{3},\ x_2 = \frac{10}{3},\ x_3 = 8,\ \lambda_1 = \frac{-40}{9},\ \lambda_2 = -\frac{52}{9}$$

(iii) Constructing the bordered Hessian Matrix.

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

$m = 2$

$i = 2m+1 = 4+1 = \underline{\underline{5}}$

(iv) Studying the nature by calculating determinants :-

$i = 2m+1 = 5$.

$$D_5 = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix} = 90 > 0 \leftarrow \text{same as } \underline{\underline{(-1)^{m=2}}}$$

$\underline{\underline{\text{true}}}$

sign: $\underline{\underline{(-1)^m}}$

we can not go further as 5x5 is max

$\hookrightarrow$ Thus the point that we have calculated;

$$\left(\frac{11}{3}, \frac{10}{3}, 8\right)$$

This is pt of minima.

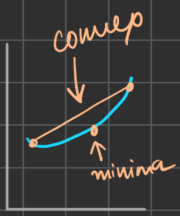
$(\because \text{sign} = (-1)^m)$

CONVEX & CONCAVE f^ns

REMARK : Since we are working with NLPP, the maxima or minima obtained may be LOCAL and NOT GLOBAL
But if your NLPP is a convex programming problem then SURELY your maxima/minima obtaind is GLOBAL

Convex function :

Let $S \in \mathbb{R}^n$ be convex set then $f : S \to \mathbb{R}^n$ is said to be convex if :-



$$\boxed{f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)} \to \text{CONVEX}$$

$x_1 = (x_{11}, x_{12} \cdots x_{1n})$ $\quad x_2 = (x_{21}, x_{22} \cdots x_{2n})$

If sign $<$ then STRICTLY CONVEX

Concave function :-

$$\boxed{f(\alpha x_1 + (1-\alpha) x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2)}$$

$>$: strictly concave

$x_1 = (x_{11}, x_{12} \cdots x_{1n})$ $\quad x_2 = (x_{21}, x_{22} \cdots x_{2n})$

Remark :- If your f^n is MINIMIZATION then your $f(x)$ should be CONVEX.
For MAXIMIZATION then your $f(x)$ should alway be CONCAVE

① if $f(x)$ is convex $\longrightarrow$ $-f(x)$ will be CONCAVE (& vice versa)

To prove the above point we take the below example :-

Q Check if $f(x) = |x|$ is convex or not.

Ans x_1, x_2 be two points
$\alpha(x_1) + (1-\alpha)x_2$ be the CLC of 2 points $(0 \leq \alpha \leq 1)$

To proove : $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

LHS:

$f(\alpha x_1 + (1-\alpha)x_2) = |\alpha x_1 + (1-\alpha)x_2|$

we know : $\boxed{|a+b| \leq |a| + |b|}$

so, $|\alpha x_1 + (1-\alpha)x_2| \leq |\alpha x_1| + |(1-\alpha)x_2|$

$\leq \alpha|x_1| + (1-\alpha)|x_2|$

$\leq \alpha f(x_1) + (1-\alpha) f(x_2) = \underline{\underline{RHS}}$

Thus proven : $\boxed{f(\alpha(x_1) + (1-\alpha)x_2)}$

Another way to check convex/concave : HESSIAN MATRIX

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

* If H is +ve definite or +ve semidefinite
$\Downarrow$
f(x) is ⊙ CONVEX (+ve semidef)
⊙ STRICTLY CONVEX (+ve definite)

* If H is -ve semidefinite : CONCAVE
or -ve definite : Strictly CONCAVE

Q $f(x) = x_1^2 + x_2^2 + x_3^2$ is convex or not.

Ans

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_3} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underbrace{D_1 = 2 > 0 \qquad D_2 = 4 > 0 \qquad D_3 = 8 > 0}$$

Since all $D_i > 0$, thus H is +ve definite.
Thus the given f^n is is <u>STRICTY CONVEX</u>

Q $f(x) = x_1^2 - 2x_2^2$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

$D_1 = 2 > 0$

$D_2 = -8 < 0$

$\because D_1 \not< 0$ & $D_2 \not> 0$
Thus H is <u>indefinite</u>

Thus, our f^n $f(x)$ is NEITHER CONVEX NOR CONCAVE.

Structure of a sample question :-

$$\boxed{\begin{aligned} &\text{Optimize } f(x) \\ &\quad s/t \quad g_i(x) \ (\leq, =, \geq)\ 0 \\ &\qquad\qquad x \geq 0 \end{aligned}}$$

convex programming probl.

① for minimize $f(x)$ should be CONVEX and the set formed by the constraints is a convex set

② for Maximize $f(x)$ SHOULD be CONCAVE and the set formed should be concave set

Q Check whether the given problem is CONVEX PROGRAMMING PROBLEM or not.

$$\text{Min } Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$$

s/t :-

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 15 \\ 2x_1 - x_2 + 2x_3 &= 20 \end{aligned}\right\} \rightarrow \text{these are eq}^{\underline{n}} \text{ of hyperplane}$$

$$x_1, x_2, x_3 \geq 0$$

Ans we know that the eq$^{\underline{n}}$ of hyperplane is a convex set. Since both the eq$^{\underline{n}}$s in the constraints are equations of hyperplanes thus the constraints are convex.

for our Minimization objective f$^{\underline{n}}$ our $f(x)$ should be convex for the problem to be CPP.

To check the objective f$^{\underline{n}}$:-

$$H = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \left.\begin{aligned} D_1 &= 8 > 0 \\ D_2 &= 16 > 0 \\ D_3 &= 32 > 0 \end{aligned}\right\} D_i > 0$$

↳ +ve definite

↓ thus

the $f(x)$ is convex

Since the constraints along with the minimization objective f$^{\underline{n}}$ is convex thus this is a convex programming problem. (PROOVED)

NLPP with constraints ($\geq, \leq$)

KKT METHOD

↳ Karush Kuhn Tucker Conditions.

Question structure :-

$$\boxed{\begin{aligned} &\min f(x) \\ &\quad s/t: \ g_i(x) \leq b_i \\ &\qquad\qquad x \geq 0 \end{aligned}}$$

Conditions to apply KKT

① The given problem should be a convex programming problem.

② Conditions for Minimization & $g_i(x)$ is '$\leq$'

i = no. of constraints
j = no. of variables

(i) $\frac{\partial L}{\partial x_j} = 0$ where $\boxed{L(x,\lambda) = f(x) + \lambda_i(g_i(x))}$

(ii) $\lambda_i(g_i(x) - b_i) = 0$

(iii) $g_i(x) \leq b_i$

(iv) $\lambda_i \geq 0$

Q Solve the following NLPP :-

$$\text{Max } f(x) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 + 49$$

s/t :

$$\left.\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}\right\} \text{convex set}$$

↳ $f(x)$ is concave ($\because$ max)

⇓ convert to minimization.

$$\text{Min } f(x) = -4x_1 - 6x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2 - 49$$

s/t :

$$x_1 + x_2 - 2 \leq 0$$

$$x_1, x_2 \geq 0$$

$$L(x, \lambda) = -4x_1 - 6x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2 - 49 + \lambda_1(x_1 + x_2 - 2)$$

KKT conditions:

1. $\frac{\partial L}{\partial x_1} = 0$; $\frac{\partial L}{\partial x_2} = 0$ $\Rightarrow$ $\frac{\partial L}{\partial x_1} = 0 \Rightarrow -4 + 4x_1 + 2x_2 + \lambda_1 = 0$ — (I)

   $\frac{\partial L}{\partial x_2} = 0 \Rightarrow -6 + 4x_2 + 2x_1 + \lambda_2 = 0$ — (II)

2. $\lambda_1(x_1 + x_2 - 2) = 0 \Rightarrow \lambda_1(x_1 + x_2 - 2) = 0$ — (III)

3. $x_1 + x_2 \leq 2$ — (IV)

4. $\lambda_1 \geq 0$ — (V)

from (III) :-

$$\lambda_1(x_1 + x_2 - 2) = 0$$

$\Rightarrow \lambda_1 = 0$ or $x_1 + x_2 - 2 = 0$

from (I) & (II)

$$\left.\begin{aligned} -4 + 4x_1 + 2x_2 &= 0 \\ -6 + 4x_2 + 2x_1 &= 0 \end{aligned}\right\} \begin{aligned} x_1 &= 1/3 \\ x_2 &= 4/3 \end{aligned}$$

$x_1 + x_2 = 2 \Rightarrow x_1 = 2 - x_2$

Substituting in (I) :-

(I) $-4 + 4(2 - x_2) + 2x_2 + \lambda_1 = 0$

To verify this point we put this value in (IV) :-

$x_1 + x_2 = \frac{5}{3} \leq 2$ (✓) true

in (V) : $\lambda_1 \geq 0$ $(\because \lambda_1 = 0)$ (✓) true

$\Rightarrow -4+8-4x_2+2x_2+\lambda_1 = 0$

$\Rightarrow 4 - 2x_2 + \lambda_1 = 0$ —(a)

Substituting in (II)

$-2+2x_2+\lambda_1 = 0$ —(b)

Adding (I) & (II)

$2+2\lambda_1 = 0 \Rightarrow \boxed{\lambda_1 = -1}$ (✗)

Not allowed $\because \lambda \geq 0$ (always)

This is the legitimate point

$$\left(\frac{1}{3}, \frac{4}{3}\right)$$

This is the optimal Sol^n

Optimal value (max) = $\frac{483}{9}$ (ANS)

> NOTE: If we get 2 points then it is important that we put both the points in the objective f^n and whichever point gave a more optimal sol^n we would choose that point.

Q2 Max $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$ → Checking Hessian matrix we find that this f^n is CONCAVE (max)

s/t :

$2x_1 + x_2 \leq 5$ ← hyperplane (convex)

$x_1, x_2 \geq 0$

⇓ converting to minimization f^n.

$$\left.\begin{aligned} \text{Min } z &= -10x_1 - 4x_2 + 2x_1^2 + x_2^2 \\ \text{s/t: } & 2x_1 + x_2 \leq 5 \end{aligned}\right\} \text{CPP (✓)}$$

Putting the KKT conditions :-

$$\boxed{L = -10x_1 - 4x_2 + 2x_1^2 + 2x_2^2 + \lambda_1(2x_1 + x_2 - 5)} \Leftarrow L(x, \lambda)$$

① $\frac{\partial L}{\partial x_1} = 0 \Rightarrow -10 + 4x_1 + 2\lambda_1 \Rightarrow 4x_1 + 2\lambda_1 = 10$ — (i)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow -4 + 4x_2 + \lambda_1 = 0 \Rightarrow 4x_2 + \lambda_1 = 4$ — (ii)

② $\lambda_1 (2x_1 + x_2 - 5) = 0$ — (iii)

③ $2x_1 + x_2 \leq 5$ — (iv) ④ $\lambda_1 \geq 0$ — (v)

From (iii) :-

$$\lambda_1 (2x_1 + x_2 - 5) = 0$$

$\lambda_1 = 0$

Putting this in (i) & (ii)

$4x_1 = 10 \Rightarrow x_1 = 10/4$

$4x_2 = 4 \Rightarrow x_2 = 1$

Putting $(\frac{10}{4}, 1)$ in (iv) & (v)

(iv) $\Rightarrow 2x_1 + x_2 = 5 + 1 = 6 \not\leq 0$

NOT SATISFIED

$2x_1 + x_2 - 5 = 0$

$\Rightarrow 2x_1 + x_2 = 5 \Rightarrow x_2 = 5 - 2x_1$

Substituting in (i) & (ii) :-

(i) : $4x_1 + 2\lambda_1 = 10$ — (a)

(ii) : $4(5 - 2x_1) + \lambda_1 = 4$

$\Rightarrow 20 - 8x_1 + \lambda_1 = 4$

$\Rightarrow \lambda_1 - 8x_1 = -16$ — (b)

Solving (a) & (b) simultaneously :-

$\left(\frac{11}{6}, \frac{4}{3}\right)$ ← optimal point

Optimal value : $\frac{91}{6}$

Q Sample qⁿ : Min $Z = -2x_1 - x_2$ ← convex

s/t: $x_1 - x_2 \leq 0$

$x_1^2 + x_2^2 \leq 4$ } convex set } CPP

$x_1, x_2 \geq 0$

$$\boxed{L = -2x_1 - x_2 + \lambda_1 (x_1 - x_2) + \lambda_2 (x_1^2 + x_2^2 - 4)}$$

① $\frac{\partial L}{\partial x_1} = -2 + \alpha_1 + 2\alpha_2 x_1$ $\qquad$ $\frac{\partial L}{\partial x_2} = -1 - \alpha_1 + 2\alpha_2 x_2$

② $\alpha_1 (x_1 - x_2) = 0$ $\quad$ & $\quad$ $\alpha_2 (x_1^2 + x_2^2 - 4) = 0$

③ $x_1 - x_2 \leq 0$ $\quad$ & $\quad$ $x_1^2 + x_2^2 \leq 4$

④ $\alpha_1, \alpha_2 \geq 0$

Points after solving: $\left.\begin{matrix} (0,0) \\ (0,2) \\ (\sqrt{2}, \sqrt{2}) \end{matrix}\right\}$ POINTS

↑
optimal solution

& $\boxed{\text{Optimal value} = -3\sqrt{2}}$ (min value)