BIRTH AND DEATH PROCESS

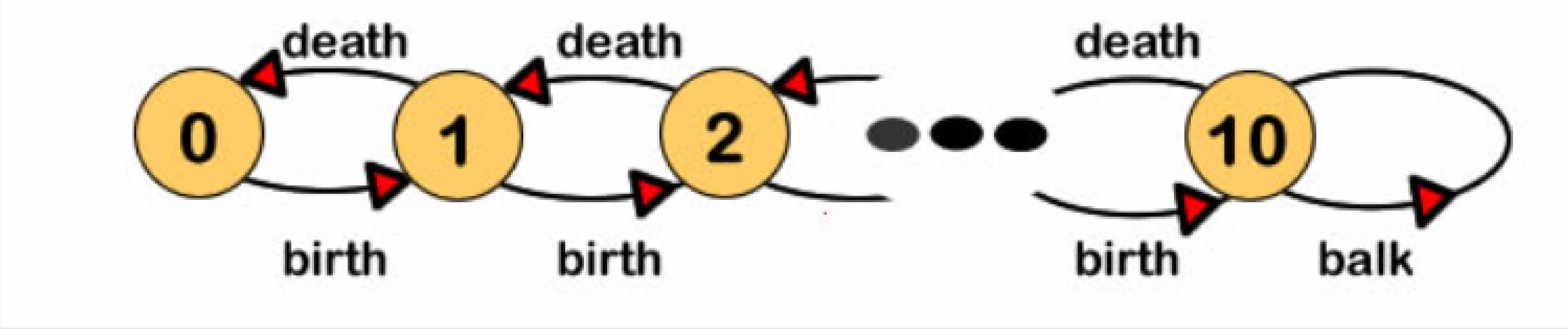
Population models

MTH371 SPA

IIIT-D

MARKOV CHAIN

INTRODUCTION TO BIRTH-DEATH PROCESS



The Birth-death process is a special case of CTMC(Continous Time Markov Chain), where the population is the number of entities that comprise a system. Each birth increases the population by 1, and each death decreases the population by 1, where each birth is a forward step from state i to i+1 and a death is a backward step from state i to i-1 . Each state in the Markov chain represents the current total population.

APPLICATION OF BIRTH-DEATH PROCESS

Scenairos where birth and death process is used to model the data:

1. **Demography:** how population of a particular community evolves over time.
2. **Queuing theory:** in a counter, look at the number of people in the queue. A person is served and leaves the queue is called a “death”, and a person joins the queue is called a “birth”.
3. **Mathematical biology:** to model the evolution of bacteria.

MODELLING

We know that

For a Continuous Markov Chain, the transition probability function for $t > 0$ can be described as

$$P_{ij}(t) = P(X(t+u) = j | X(u) = i)$$

And is independent of u>=0.

- The birth-death process is a special case of the continuous-time Markov process, where the states (for example) represent a current size of a population, and the transitions are limited to birth and death.
- The birth-and-death process is characterized by the birth rate $\{\lambda i\} i=0,\ldots,\infty$ and death rate $\{\mu i\} i=0,\ldots,\infty$, which vary according to state i of the system. We can define a Pure Birth Process as a birth-death process with $\mu i = 0$ for all i. Similarly, a Pure Death Process corresponds to a birth-death process with $\lambda i = 0$ for all i.

MODELLING

The general description of the Birth-and-Death process can be as follows: after the process enters state i, it holds in the given state for some random length of time, exponentially distributed with parameter ($\lambda_i + \mu_i$). When leaving i, the process enters either i + 1 with probability:

$$\lambda_i/(\lambda_i + \mu_i)$$

or i − 1 with probability

$$\mu_i/(\lambda_i + \mu_i)$$

- a pure birth process with the birth rate λ_i , we would name it a Poisson process with parameter $\lambda_i h$ so that λ_i is the expected number of birth events that occur per unit time. In this case, the probability of a birth over a short interval h is $\lambda_i h + o(h)$.
- Similarly, if in state $X(t) = i$ a death rate is μ_i , then the probability that an individual dies in a very small time interval of length h is $\mu_i h + o(h)$.
- Since birth and death processes are independent and have a Poisson distribution with parameters $\lambda_i h$ and $\mu_i h$, their sum is a Poisson distribution with parameter $h(\lambda_i + \mu_i)$.

MODELLING

Now analyzing the changes in the process in a short time interval h. that is,

$$\begin{aligned}
P_{i,i+1}(h) &= P(X(t+h) - X(t) = 1 | X(t) = i) \\
&= \frac{(\lambda_i h)^1 e^{-\lambda_i h}}{1!} \frac{(\mu_i h)^0 e^{-\mu_i h}}{0!} + o(h) \\
&= (\lambda_i h) e^{-\lambda_i h} e^{-\mu_i h} + o(h) \\
&= (\lambda_i h) e^{-h(\lambda_i + \mu_i)} \\
&= (\lambda_i h) \sum_{n=0}^{\infty} \frac{(-h(\lambda_i + \mu_i))^n}{n!} \\
&= (\lambda_i h)(1 - h(\lambda_i + \mu_i) + \frac{1}{2} h^2 (\lambda_i + \mu_i)^2 - \ldots) + o(h) \\
&= \lambda_i h + o(h)
\end{aligned}$$

MODELLING

Similarly, we can derive it for the death process and also for the following:

(1) $P(X(t+h) - X(t) = 1 | X(t) = i) = \lambda_i h + o(h)$

(2) $P(X(t+h) - X(t) = -1 | X(t) = i) = \mu_i h + o(h)$

(3) $P(|X(t+h) - X(t)| > 1 | X(t) = i) = o(h)$

(4) $\mu_0 = 0, \lambda_0 > 0; \quad \mu_i, \lambda_i > 0, i = 1, 2, 3, \ldots$

(5) $P(X(t+h) - X(t) = 0 | X(t) = i) = 1 - (\mu_i + \lambda_i)h + o(h)$

the above equations can be used to construct the transition probability matrix.

POPULATION DYNAMICS

MTH371 SPA

IIIT-D

BIRTH AND DEATH PROCESS

We created a simple model of population dynamics as a birth and death process. We have assumed that the birth and death rates depend only on the population size.

We followed the basic process:

1. Calculate the mean rate at which events (birth and death) occur.
2. Calculate the wait time until the next event (exponential distribution).
3. Determine which event actually occurs (birth/death).
4. Adjust population and time parameters based on the outcomes of the event.
5. Repeat until the max time is reached.

PARAMETERS

1. Initial population
2. Birth rate (λ)
3. Death rate (μ)
4. Max time (total time)

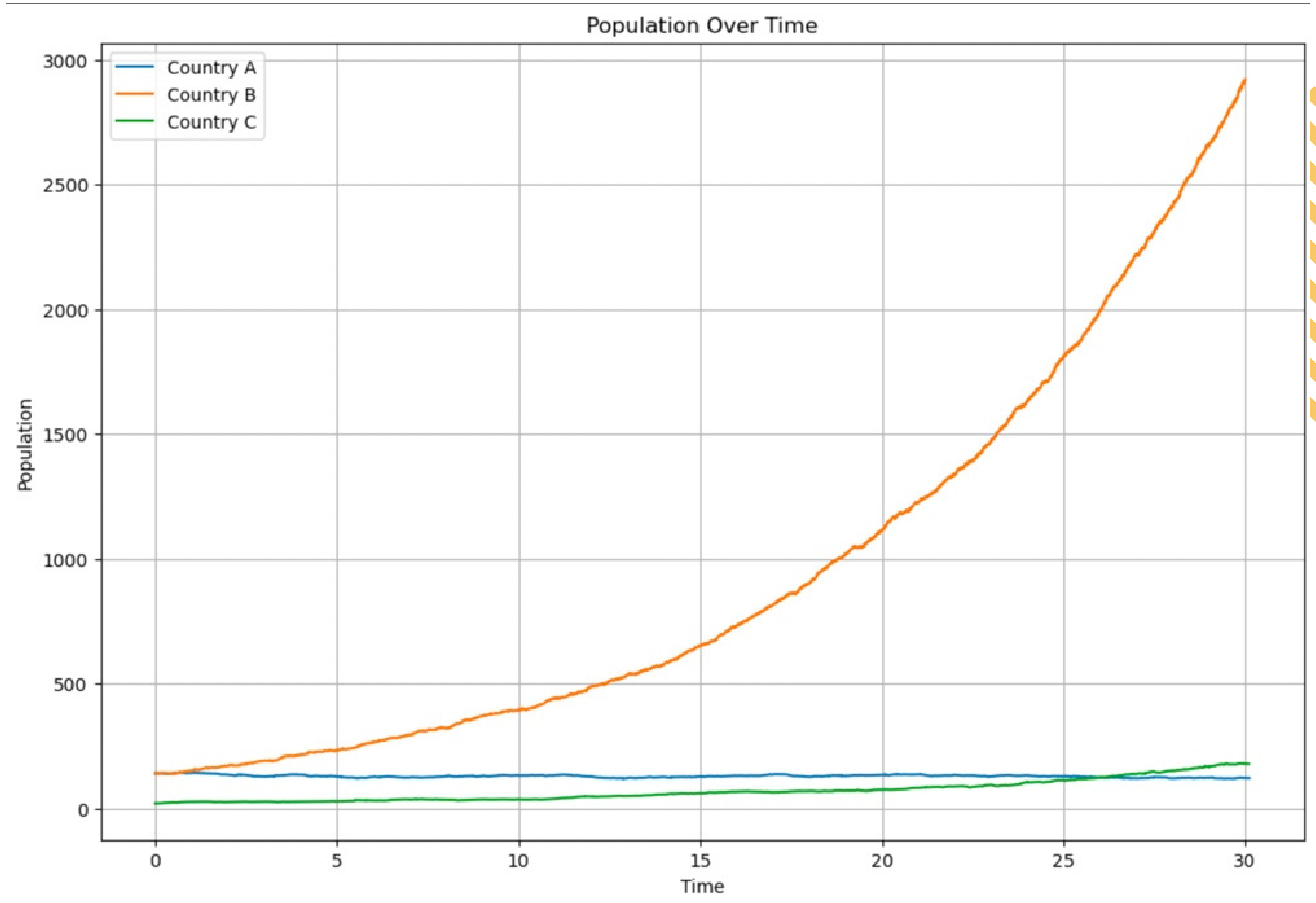
Initial population	Birth rate (λ) (% per year)	Death rate (μ) (% per year)	Max time (total time in years)
A: 142	7.52	7.18	30
B: 140	17.23	7.34	30
C: 21	13.31	6.67	30

INFERENCE

The graph represents the population dynamics for three countries over the span of 30 years.

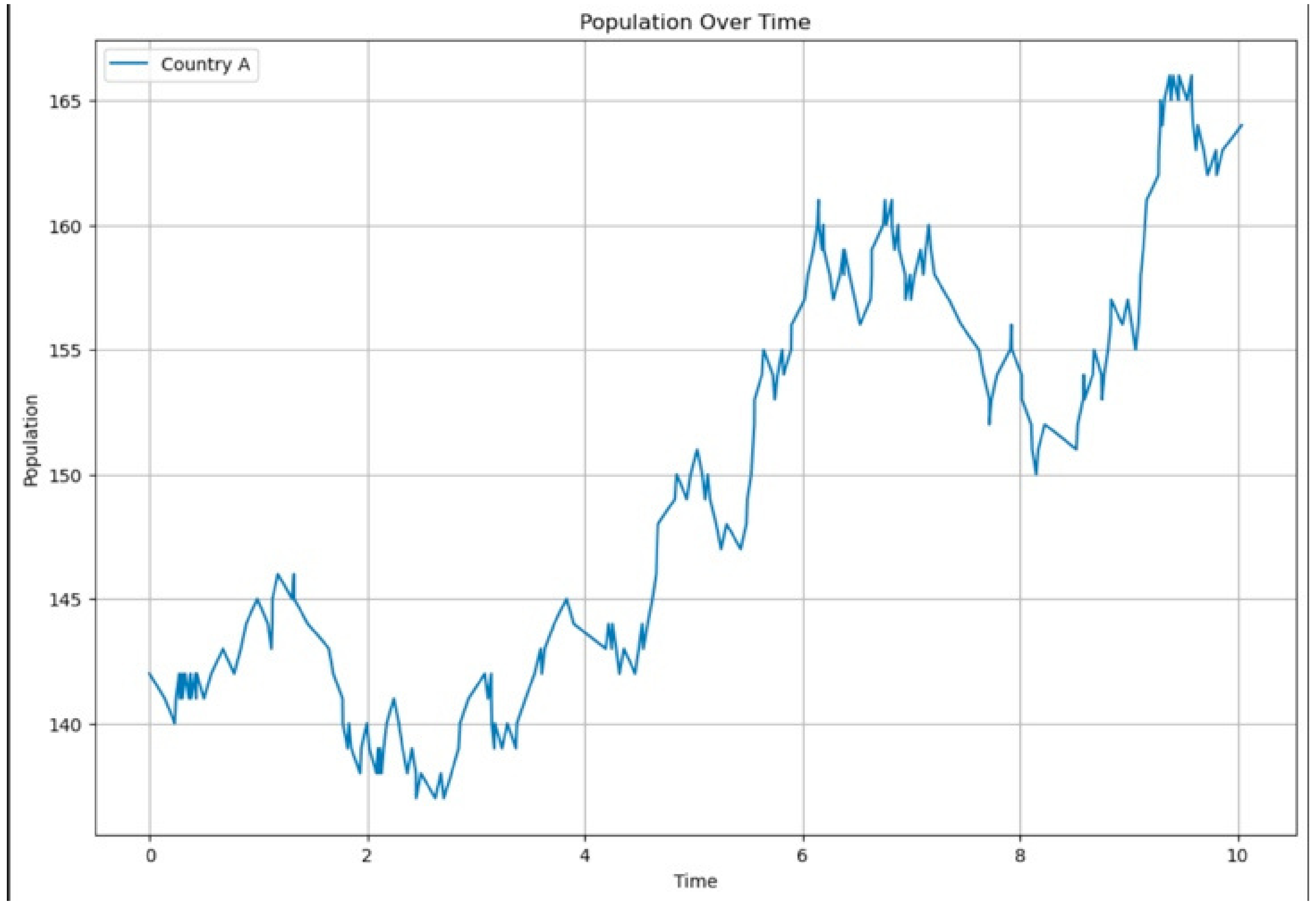
For country A as the values of birth and death rates are similar we can observe there is not any major fluctuation in its population size.

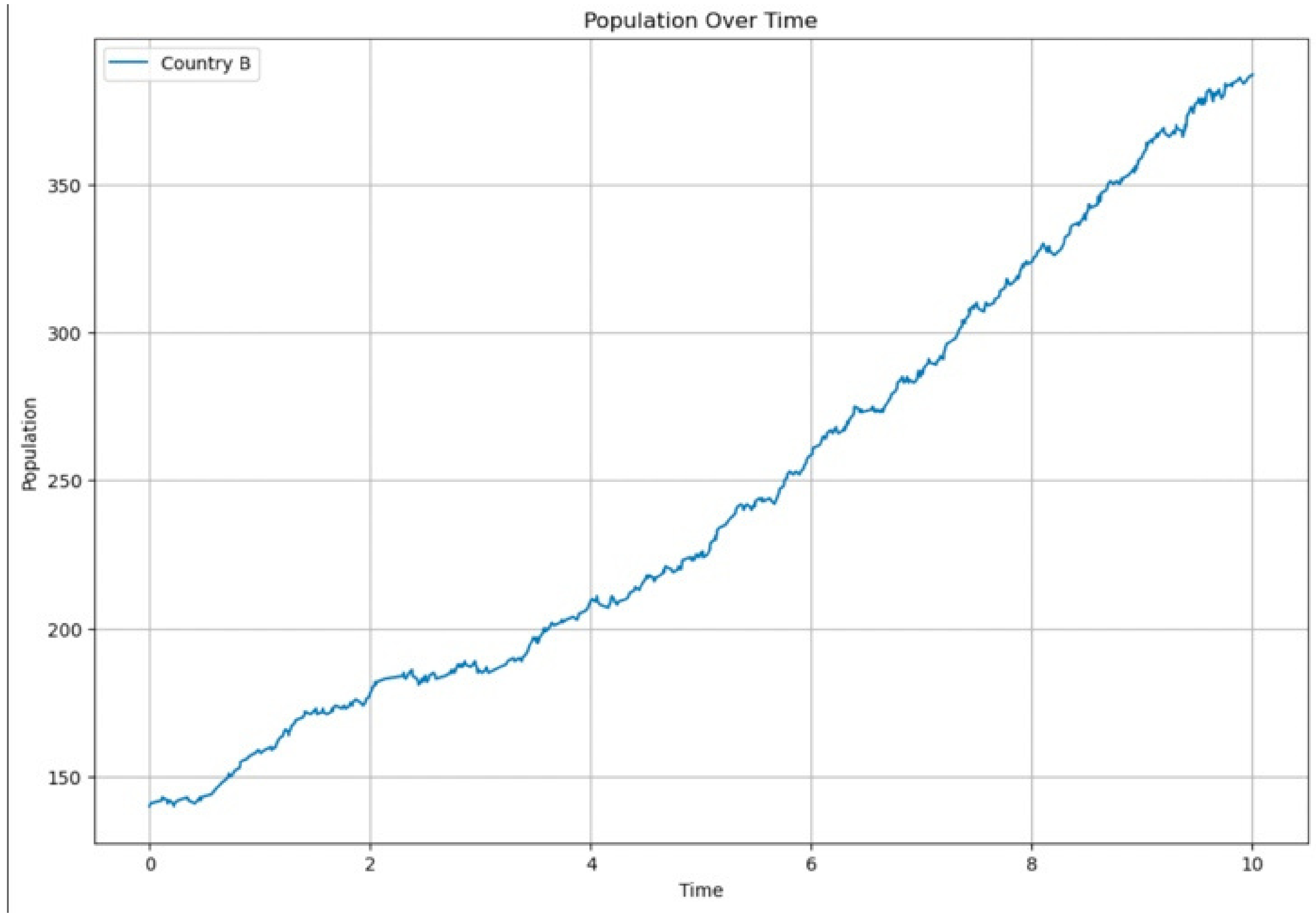
As the initial population of country B is large and the birth rate is much higher than the death rate its population is increasing much faster than the other countries.



PLOTS

The following graphs represent the population dynamics for the countries A and B respectively over a span of 10 years.

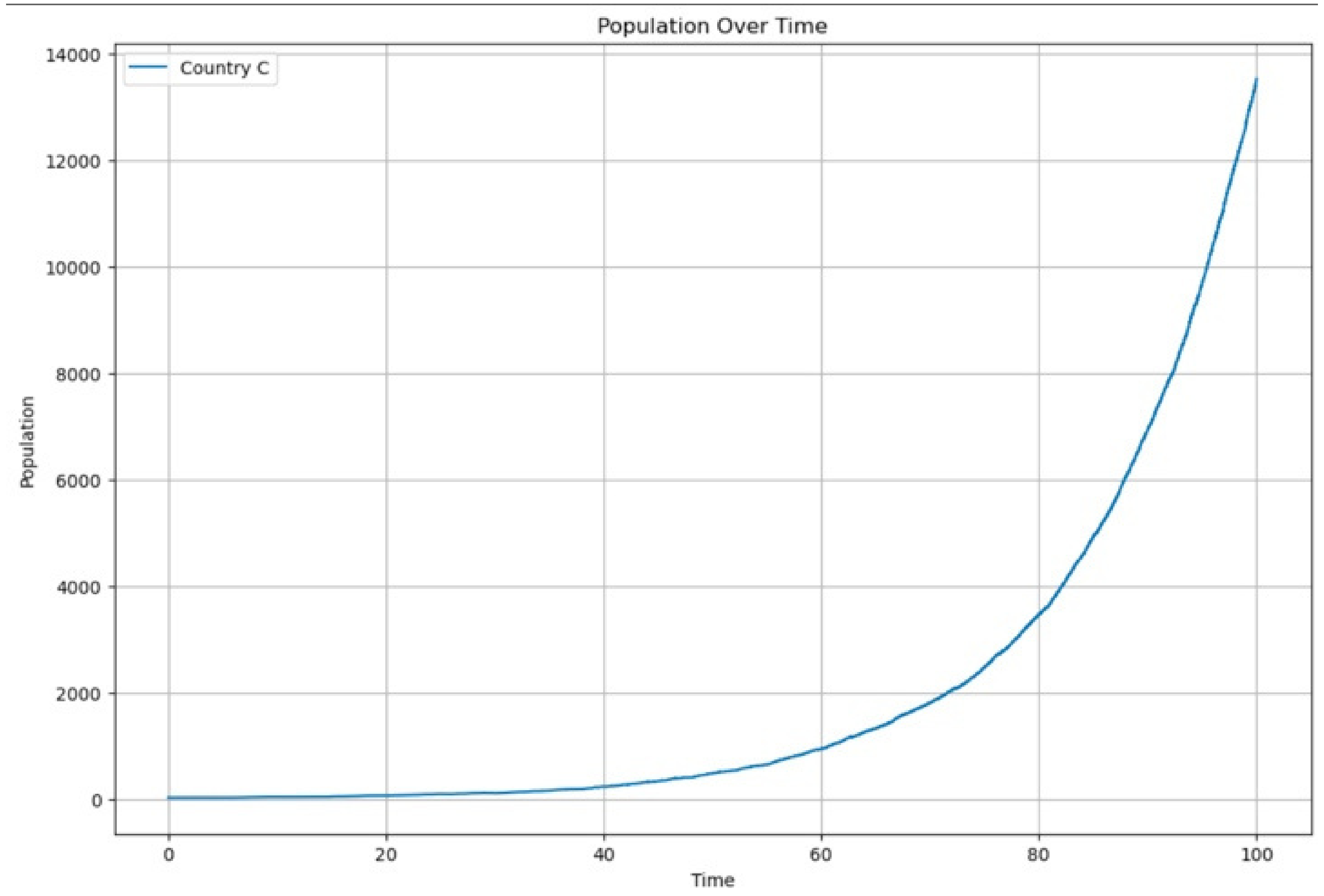




PLOTS

This graph represents the population dynamics of country C over a span of 100 years.

As the birth rate is nearly double the death rate and no restrictions are imposed we observe an exponential growth in the population size.



CONCLUSION

- We have analyzed the birth-death process and constructed it in CTMC.
- We simulated the dataset for different countries and discussed the simulation results.
- Also, to derive the required formula, we have taken several assumptions, as discussed before, to simplify the calculation.
- The datasets used and the graph discourse the random walk as in real life.
- The birth-death process helps us to analyze the scenario of different countries such as Japan or Korea, where their population is declining, and also countries like India, where the birth rate is higher than the death rate, due to which the population is increasing.

REFERENCES

- https://egyankosh.ac.in/bitstream/123456789/12592/1/Unit-6.pdf
- https://cs.nyu.edu/~mishra/COURSES/09.HPGP/scribe3
- https://www.compadre.org/picup/exercises/exercise.cfm?A=BirthDeath

Group Members:

- Neelanjan Sarkar (2021267)
- Sandeep (2021283)
- Shreshtha Kardam (2021288)
- Shruti Jha (2021289)
- Shubham Yadav (2021290)

THANK YOU