Vectors in $\mathbb{Z}_2^d$

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Number Theory

To write a non-negative integer in **base two** is to write it as a sum of distinct non-negative powers of 2.

Example

Write 13 in base 2.

$$\begin{aligned} 13 &= 8 + 4 + 1 \\ &= 2^3 + 2^2 + 2^0 \\ &= \boxed{1101_2.} \end{aligned}$$

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Two numbers a and b are **congruent** or **equivalent** modulo k if they have the same remainder when divided by k. This is denoted by $a \equiv b \pmod{k}$. In this presentation, we'll focus on $k = 2$.

XOR

Exclusive-Or (XOR)

For two non-negative integers a and b, the operation a XOR b can be performed as follows:

- Convert both a and b to base 2.
- Add each pair of bits $\pmod 2$.
- Convert the resulting base 2 number back to base 10.

If we denote XOR by $\oplus$, then

$$0 \oplus 0 = 1 \oplus 1 = 0,$$

$$1 \oplus 0 = 0 \oplus 1 = 1.$$

(The OR operation is defined similarly, except 1 OR $1 = 1$.)

XOR

Example

Calculate $5 \oplus 7$. (Express the answer in base 10.)

This is equivalent to

$$\begin{array}{r} (101)_2 \\ \oplus\ (111)_2 \\ \hline (010)_2 \end{array},$$

which is equal to $\boxed{2}$ in base 10.

Another XOR example

Example

Calculate $13 \oplus 25$.

This is equivalent to

$$\begin{array}{r} (01101)_2 \\ \oplus\ (11001)_2 \\ \hline (10100)_2 \end{array},$$

which is equal to $\boxed{20}$ in base 10.

Scalars

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- The integers ($\mathbb{Z}$) do not form a field since the inverses of any nonzero integer aside from -1 and 1 are not integers. For example, the inverse of 2 is $\frac{1}{2}$, which is not an integer.
- However, the integers modulo any prime p do form a field ($\mathbb{Z}_p$) since we can compute modular inverses. For example, the inverse of 2 modulo 7 is 4 since $2 \cdot 4 \equiv 1 \pmod 7$.

Vector Spaces

Vector Space

A **vector space** is a set V on which two operations $+$ and $\cdot$ are defined, called vector addition and scalar multiplication. The elements of V are called **vectors**.

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A **vector space** is a set V on which two operations $+$ and $\cdot$ are defined, called vector addition and scalar multiplication. The elements of V are called **vectors**.

- V must be closed under both of these operations, meaning that $x, y \in V \implies x + y \in V$ and $x \in V \implies c \cdot x \in V$ for any scalar c.
- Addition should be both associative and commutative.
- Scalar multiplication should be associative and distributive over addition.

Vector Space Examples

Wikipedia

Addition and scalar multiplication are well defined for all of these examples.

- Coordinate spaces
  - Tuples of numbers $(x_1, x_2, \ldots, x_n)$ and scalars c such that $x_i, c \in \mathbb{R}$
  - Component-wise addition and multiplication:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$$

  - Also works for any other field (ex. $\mathbb{Z}_2$ instead of $\mathbb{R}$)
- Matrices over some field
- Polynomial vector spaces
  - Operations with vector space of polynomials with real coefficients and degree less than or equal to n: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ (though the degree doesn't have to be restricted)
- Function spaces
  - Given functions $f(x)$ and $g(x)$ from any set to some field, we can define $(f+g)(x)$ and $(cf)(x)$.

$\mathbb{Z}_2^d$

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- $V = \{0, 1, \ldots, 2^d - 1\}$. Each number stands for an array of d bits.

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- $V = \{0, 1, \ldots, 2^d - 1\}$. Each number stands for an array of d bits.
- If $x, y \in V$, then $x + y$ is defined to be $x \oplus y$. Note that $x + y \in V$ is satisfied.
- For any $c \in \mathbb{Z}_2$, let $cx = \overbrace{x + x + \ldots x}^{c \text{ occurences of } x}$. So $cx = 0$ if c is even and $cx = x$ if c is odd.

$\mathbb{Z}_2^d$ is indeed closed under both addition and scalar multiplication.

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From now on, we'll do all calculations $\pmod 2$ and substitute $+$ in place of $\oplus$.

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The vector space V **spanned** by a collection of vectors $v_1, v_2, \ldots, v_n$ consists of all vectors x that can be written as a **linear combination** of $v_1, v_2, \ldots, v_n$. In other words, $x = \sum_{i=1}^{n} c_i v_i$ for some choice of scalars $c_i \in \{0, 1\}$.

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Example

What are $\text{span}(\{3, 5\}), \text{span}(\{3, 5, 6\})$, and $\text{span}(\{\})$?

Answer: Note that $3 \oplus 5 = 6$.

$$\text{span}(\{3, 5\}) = \text{span}(\{3, 5, 6\}) = \{0, 3, 5, 6\}.$$

$$\text{span}(\{\}) = \{0\}.$$

Span

Useful properties:

Firstly, $v_{m+1} \in \text{span}(\{v_1, \ldots, v_m\})$ implies

$$\text{span}(\{v_1, \ldots, v_m, v_{m+1}\}) = \text{span}(\{v_1, \ldots, v_m\}).$$

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$$\text{span}(\{v_1, \ldots, v_m, v_{m+1}\}) = \text{span}(\{v_1, \ldots, v_m\}).$$

Secondly,

$$\text{span}(\{v_1, v_2, \ldots, v_m\}) = \text{span}(\{v_1, v_1 + v_2, \ldots, v_m\}).$$

Basis

Basis

A collection of vectors $v_1, \ldots, v_n$ is a **basis** for a vector space V if $\text{span}(\{v_1, \ldots, v_n\}) = V$ and no nontrivial linear combination of $v_1, \ldots, v_n$ sums to 0 (nontrivial means that you ignore $\sum_{i=1}^{n} 0 \cdot v_i$, which is obviously 0). n is called the **dimension** of V (and is the same regardless of the basis chosen, see here). This is denoted by $\dim(V)$.

Note that 0 should never be in the basis.

Example

If $V = \{0, 3, 5, 6\}$ then $\{3, 5\}$ is a basis for V, but $\{3, 5, 6\}$ is not because $3 + 5 + 6 = 3 \oplus 5 \oplus 6 = 0$. Of course, $\{3, 6\}$ and $\{5, 6\}$ are also bases for V.

Basis

Example

In terms of n, how many distinct elements are contained within $\text{span}(\{v_1, \ldots, v_n\})$, if $v_1, \ldots, v_n$ form a basis?

Recall that $\text{span}(\{3, 5\}) = \{0, 3, 5, 6\}$.

Basis

Example

In terms of n, how many distinct elements are contained within $\text{span}(\{v_1, \ldots, v_n\})$, if $v_1, \ldots, v_n$ form a basis?

Recall that $\text{span}(\{3, 5\}) = \{0, 3, 5, 6\}$.

Answer: If any x could be written as a linear combination of $\{v_1, v_2, \ldots, v_n\}$ in more than one of the 2^n possible ways, this would contradict the definition of basis. It follows that $|V| = \boxed{2^n.}$

Matrices

Consider any $n \times m$ matrix (n rows, m columns)

$$M = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & \cdots & | \end{bmatrix}.$$

The v_i are each **column vectors** of dimension n. This represents a **linear transformation** from a vector space of dimension m to a vector space of dimension n.

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The v_i are each **column vectors** of dimension n. This represents a **linear transformation** from a vector space of dimension m to a vector space of dimension n.

So for any column vector x of dimension m,

$$M \cdot x = \sum_{i=1}^{m} v_i x_i$$

is a column vector of dimension n that is a linear combination of the columns of M.

Dimension

Dimension

The **column dimension** of M is equal to

$$\mathrm{cdim}(M) = \dim(\mathrm{span}(\{v_1, v_2, \ldots, v_m\})).$$

Recall that this is the number of linearly independent columns. The **row dimension** can be defined similarly.

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Since it can be shown that $\mathrm{cdim}(M) = \mathrm{rdim}(M)$, we can use $\dim(M)$ to refer to both $\mathrm{cdim}(M)$ and $\mathrm{rdim}(M)$. This is also known as the **rank** of the matrix M, denoted by $\mathrm{rank}(M)$.

Link to Explanations (Optional)

Nullity

Null Space

The **null space** of a matrix M is the set that consists of all column vectors v such that

$$M \cdot v = 0.$$

Note that this set is actually a vector space since

$$M \cdot a = M \cdot b = 0 \implies M \cdot (a+b) = 0$$

$$M \cdot (ca) = c(M \cdot a) = 0$$

The **nullity** of matrix M is the dimension of this set.

Rank-Nullity Theorem

The Rank-Nullity Theorem states that for an $n \times m$ matrix M,

$$\text{rank}(M) + \text{null}(M) = m.$$

For example,

$$\text{rank}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 2, \text{null}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

$$\text{rank}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = 1, \text{null}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = 2$$

We won't prove this (it would require the introduction of additional notation), but you should at least intuitively understand why this is true in the context of linear transformations.

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Programming Note

For all of the following problems, we'll assume that integer values are in the range $[0, 2^{60})$ unless otherwise stated (so that they fit into a `long long`).

XOR Closure

Example

You are given a set of N $(1 \leq N \leq 10^5)$ distinct integer values $x_1, x_2, \ldots, x_N$. Find the minimum number of values that you need to add to the set such that the following will hold true:
For every two integers A and B in the set, $A \oplus B$ is also in the set.

`csacademy.com/contest/archive/task/xor-closure/`

Extension: Compute the answer for each prefix of $x_1, \ldots, x_N$.

Thoughts?

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Extension: Compute the answer for each prefix of $x_1, \ldots, x_N$.

Thoughts?

Solution: Let $X = \{x_1, x_2, \ldots, x_N\}$. We need to compute $\dim(\text{span}(X))$. Then the answer will be $2^{\dim(\text{span}(X))} - N$.

XOR Closure

It suffices to add the integers of X one by one.

- If $x_{i+1} \in \text{span}(x_1, x_2, \ldots, x_i)$, then adding it will leave the span unchanged.
- Otherwise, the size of the basis increases by one (and the number of elements in the span of the basis is multiplied by two).

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Question 1

How can you quickly test whether $x_{i+1} \in \text{span}(x_1, \ldots, x_i)$? Obviously, going through all 2^i possible linear combinations of $x_1, \ldots, x_i$ is too slow.

XOR Closure

For a positive integer x, let $msb(x)$ be the index of the **m**ost **s**ignificant **b**it in x (one less than the length of x when written in base 2).

Example

$$msb(4) = msb(7) = 2$$

$$msb(3) = 1, msb(1) = 0$$

XOR Closure

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$$msb(3) = 1, msb(1) = 0$$

Easier Version: Suppose that we have a basis $\{v_1, v_2, \ldots, v_n\}$ of $x_1, x_2, \ldots, x_i$ such that $msb(v_1) > msb(v_2) > \cdots > msb(v_n)$. Can you easily test whether a vector a is in $\text{span}(\{v_1, \ldots, v_n\})$?

XOR Closure

Example

Is it true that $20 \in \text{span}(\{26, 15, 3, 1\})$?

$$\begin{aligned} 26 &= 11010_2 \\ 15 &= 01111_2 \\ 3 &= 00011_2 \\ 1 &= 00001_2 \\ 20 &= 10100_2 \end{aligned}$$

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Can verify that $20 = 26 \oplus 15 \oplus 1$.

XOR Closure

Solution: Let $A = a$. Iterate over all i from $1 \dots n$. If $A + v_i < A$, replace A with $A + v_i$. If $A = 0$ at the end of this process, then $a \in \text{span}(\{v_1, \dots, v_n\})$.

Note

More generally, this allows us compute the minimum value of $a + v$ over all $v \in \text{span}(\{v_1, \dots, v_n\})$.

XOR Closure

Example

Is it true that $9 \in \text{span}(\{26, 15, 3, 1\})$?

$$\begin{aligned} 26 &= 11010_2 \\ 15 &= 01111_2 \\ 3 &= 00011_2 \\ 1 &= 00001_2 \\ 9 &= 01001_2 \end{aligned}$$

XOR Closure

Example

Is it true that $9 \in \text{span}(\{26, 15, 3, 1\})$?

$$\begin{aligned} 26 &= 11010_2 \\ 15 &= 01111_2 \\ 3 &= 00011_2 \\ 1 &= 00001_2 \\ 9 &= 01001_2 \end{aligned}$$

No! But can you find the minimum value of $9 \oplus v$ over all $v \in \text{span}(\{26, 15, 3, 1\})$?

XOR Closure

Example

Is it true that $9 \in \text{span}(\{26, 15, 3, 1\})$?

The minimum possible XOR of 9 with some subset is

$$9 \oplus 15 \oplus 3 \oplus 1 = 4 = 100_2.$$

XOR Closure

Example

Is it true that $9 \in \text{span}(\{26, 15, 3, 1\})$?

If we add 4 to the basis and maintain the sorted order, then it looks like this:

$$
\begin{aligned}
26 &= 11010_2 \\
15 &= 01111_2 \\
4 &= 00100_2 \\
3 &= 00011_2 \\
1 &= 00001_2
\end{aligned}
$$

The most significant bits are still in decreasing order!

XOR Closure

Recap:

Go through all elements $a \in X$ in any order. For each a, calculate A, which is the minimum value of $a + v$ over all v in the span of the elements that precede a.

XOR Closure

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Go through all elements $a \in X$ in any order. For each a, calculate A, which is the minimum value of $a + v$ over all v in the span of the elements that precede a.

- If a is already in the span of the elements that precede it, then $A = 0$, continue.
- Otherwise, add A to the basis while maintaining the decreasing msb condition. This is how **Gaussian elimination** is used to convert a matrix into **row echelon form** (meaning that as you go down the rows, the leftmost one always moves to the right).

XOR Closure

C++ Code:

```
typedef long long ll; // represents ints in [0,2^{60})
vector<ll> basis; // list of basis elements, initially empty
for (ll a: X) { // go through elements in any order
    ll A = a;
    for (ll b: basis) A = min(A,A^b);
    if (A) { // add A to basis
        int ind = 0;
        while (ind < basis.size()
            && basis[ind] > A) ind ++;
        basis.insert(begin(basis)+ind,A);
        // preserve decreasing property
    }
}
```

XOR Closure

C++ Code (V2):

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    if (A) basis.push_back(A);
}
```

XOR Closure

C++ Code (V2):

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typedef long long ll; // represents ints in [0,2^{60})
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}
```

Doesn't maintain the *msb* condition, but still works!

Example (Again)

Example

Is it true that $9 \in \text{span}(\{26, 15, 3, 1\})$?

- Let $a = A = 9$.
- $A \oplus 26 = 19 > A$, A doesn't change.
- $A \oplus 15 = 6 < A$, set $A = 6$.
- $A \oplus 3 = 5 < A$, set $A = 5$.
- $A \oplus 1 = 4 < A$, set $A = 4$.

Then we can insert A into the basis, making it $\{26, 15, 4, 3, 1\}$.

Power Grid

Hopefully, the next few problems should be easier now that we've worked through one. If you're confused, please let me know.

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Example

A power grid consists of stations labelled $0 \ldots 2^K - 1$. You are given a list of integers $X = \{x_1, \ldots, x_M\}$ such that an edge connects stations i and j iff $i \oplus j \in X$. Compute the number of connected components in this graph.
`www.codechef.com/COOK106A/problems/XORCMPNT`

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Hint: The answer depends only on K and $\dim(X)$.

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Hint: The answer depends only on K and $\dim(X)$.

Solution: Every power station is in the same connected component as $2^{\dim(X)}$ others. The answer will be $\frac{2^K}{2^{\dim(X)}} = 2^{K-\dim(X)}$.

Square Subsets

Example

Given an array a ($|a| \leq 10^5$) find the number of different ways to select a non-empty subset of elements from it in such a way that their product is equal to a square of some integer. Two ways are considered different if sets of indexes of elements chosen by these ways are different. All elements of a are in the range $[1, 70]$.
Since the answer can be very large, you should find the answer modulo $10^9 + 7$.

codeforces.com/contest/895/problem/C

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codeforces.com/contest/895/problem/C

Solution: For each number in a, we only need to consider the primes that divide it an odd number of times. Since there are only 19 primes in $[1, 70]$, each number corresponds to a vector in $\mathbb{Z}_2^{19}$ and multiplying two numbers corresponds to an xor operation. A square corresponds to the 0 vector.

Square Subsets

The process of choosing a subset can be represented by the following linear transformation:

$$f(v) = f(v_1, v_2, \ldots, v_{|a|}) = \bigoplus_{i=1}^{|a|} (v_i \cdot a_i),$$

where $v_i = 1$ if a_i is included in the subset and $v_i = 0$ otherwise. This can be represented by a $19 \times |a|$ matrix:

$$f(v) = \begin{bmatrix} a_{1,0} & a_{2,0} & \cdots & a_{|a|,0} \\ a_{1,1} & a_{2,1} & \cdots & a_{|a|,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,18} & a_{2,18} & \cdots & a_{|a|,18} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{|a|} \end{bmatrix}$$

Our goal is to find the number of nontrivial elements in the null space of this matrix (so 2 to the power of the nullity minus one).

Square Subsets

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Let $b = \text{basis}(a)$, which we can compute in the same way as the previous problem. Then $|b|$ is the rank of f.

By the rank-nullity theorem, $|a| - |b|$ is the nullity of f, which is exactly what we want! Specifically, each of the $2^{|b|}$ elements of $\text{span}(a)$ (including 0) are attained by $2^{|a|-|b|}$ values of v.

Example

Given $a = \{3, 5\}$, $|a| - |b| = 0$, and each of $0, 3, 5, 6$ can be written as a distinct linear combination of elements in a.
Given $a = \{3, 5, 6\}$, $|a| - |b| = 1$, and each of $0, 3, 5, 6$ can be written as two different linear combinations of elements in a.

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Recap: The answer is $2^{|a|-|b|} - 1$, where one is subtracted for the empty set. Of course, slower solutions involving factors such as $70 \cdot 2^{19}$ also work due to the low constraints.

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Conclusion

These slides will be posted at:

`usaco.guide/adv/xor-basis`

Hope you learned something interesting!