Partial Differential equations

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$z\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2}$$

$$\frac{\partial z}{\partial y} = \frac{y}{b^2}$$

$$2z = \frac{x\partial z}{\partial x} + \frac{y\partial z}{\partial y}$$

$$2z = px + qy$$

2) $z = (x+y)\phi(x^2-y^2)$.

$$\frac{\partial z}{\partial x} = y\phi'(x^2-y^2)(2x) + \phi(x^2-y^2)$$

$$\frac{\partial z}{\partial y} = x\phi'(x^2-y^2)(-2y) + \phi(x^2-y^2)$$

$$\cancel{\partial z} \quad p+q = \phi(x^2-y^2)$$

$$z = (x+y)(p+q)$$

$$\frac{\partial z}{\partial x} = y\phi(x^2-y^2) + (x+y)\phi'(x^2-y^2)2x$$

$$\frac{\partial z}{\partial y} = x\phi(x^2-y^2) + (x+y)\phi'(x^2-y^2)-2y$$

$$p = x$$

$$\frac{\partial z}{\partial x} = x$$

$$z = \frac{x^2}{2} + \phi(y).$$

$$yp + xq = (y+x)\phi(x^2-y^2)$$

$$\frac{yp+xq}{\cancel{y+x}}\cancel{(x+y)} = z$$

$$z = yp + xq$$

1) $\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x-y) = 0$. $\longrightarrow$ Direct Integration.

$$\frac{\partial z}{\partial y} + 3x^2y^3 - \frac{1}{2}\sin(2x-y) = 0 + \phi(x)$$

$$z + x^3y^3 - \cos(2x-y) + \int\phi(x) + g(y) = 0$$

$$z + x^3y^3 - \cos(2x-y) + F(x) + g(y) = 0$$

* $Pp + Qq = R \rightarrow$ Lagrange's eq'n.

where P, Q, R are functions of x, y, z.

Ex:-

1) $\underbrace{\frac{y^2z}{x}}_{P} p + \underbrace{xz}_{Q} q = \underbrace{y^2}_{R}$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$

$\frac{dx}{dy} = \frac{y^2}{x^2}$ $\qquad$ $\frac{dx}{dz} = \frac{z}{x}$

$x^3 - y^3 = c$ $\qquad$ $x^2 - z^2 = c_1$

$f(x^2 - z^2) = x^3 - y^3$ (or)

$f(x^3 - y^3) = x^2 - z^2$

2) $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$.

$\frac{x\partial x}{x(mz-ny)} = \frac{y\partial y}{y(nx-lz)} = \frac{z\partial z}{z(ly-mx)}$ } using multipliers x, y, & z we have.

each fraction $= \frac{x\partial x + y\partial y + z\partial z}{0} \Rightarrow x\partial x + y\partial y + z\partial z = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$

each fraction $= \frac{l\partial x + m\partial y + n\partial z}{l(mz-ny) + m(nx-lz) + n(ly-mx)}$

$= lx + my + nz = c_1$

$f(lx + my + nz) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$

$\cancel{\frac{\partial x}{\partial y} = \frac{mz - ny}{nx - lz}}$.

$\cancel{n\frac{x^2}{2} - lzx - mzy - n\frac{y^2}{2} = c}$

$\cancel{\frac{\partial x}{\partial}}$

3) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

using multipliers x, y, & z we have.

each fraction $= \frac{2x dx + 2y dy + 2z dz}{2x^3 - 2xy^2 - 2xz^2 + 2xy^2 + 2xz^2}$

each fraction $= \frac{x dx + y dy + z dz}{x^3 + xy^2 + xz^2}$

each fract $= \frac{2x dx + 2y dy + 2z dz}{\cancel{x}(x^2 + y^2 + z^2)} = \frac{dy}{\cancel{x}y}$

$\ln(x^2 + y^2 + z^2) = \ln y$

$\ln$

$\frac{dy}{y\,dz} = \frac{1}{z}$ $\qquad$ $\ln\left(\frac{y}{z}\right) = c$.

$\cancel{\ln\left(\frac{y}{z}\right) = c}$.

$\therefore f\left(\ln\left(\frac{y}{z}\right)\right) = \ln\left(\frac{x^2 + y^2 + z^2}{y}\right)$.

$= f\left(\frac{y}{z}\right) = \frac{x^2 + y^2 + z^2}{y}$

4) $$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

using multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

each fraction $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$$c = -\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

→ using multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

each fraction $= \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz$.

$c_1 = \ln(xyz)$.

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \ln(xyz).$$

$u = \phi(v)$
$\phi(u, v) = 0$ } → complete solution.

* $f(x, y, z, p, q) = 0$

Here z depends upon x & y.

So $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$$\boxed{dz = px + qy}$$

Charpit's eq'n:

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-(pf_p + qf_q)} = \frac{dp}{pf_z + f_x} = \frac{dq}{qf_z + f_y}$$

1) $(p^2+q^2)y = qz$ → i)

$(p^2+q^2)y - qz = 0$

$f = (p^2+q^2)y - qz$.

$f_p = 2py$

$f_q = 2qy - z$

$f_x = 0 \quad f_y = p^2+q^2 \quad f_z = -q$.

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dp}{-pq} = \frac{dq}{-q^2 + p^2 + q^2}$$

use multipliers $p, q, -1$

each fraction $= \dfrac{p\,dx + q\,dy - dz}{-2p^2y + zq - 2q^2y + 2p^2y + 2q^2y - qz}$

$px + qy - z = c_1$

use multipliers p, q.

each fraction $= \dfrac{p\,dp + q\,dq}{0}$

$p^2 + q^2 = c_2$

~~$f(p^2+q^2) = px + qy - z$.~~

$cy = qz$ from i)

$\boxed{q = \frac{cy}{z}}$

$\frac{x}{2y} - c_3 = \frac{pz}{cy}$.

$\left(\frac{x}{2y} - c_3\right)\frac{cy}{z} = \boxed{p = \frac{xc}{2z} \ldots}$

$\frac{x}{2y} - c_4 = \frac{x}{2y} - c_3$

$$\frac{dx}{+2py} = \frac{dp}{-pq} \qquad \frac{dx}{-2py} = \frac{dq}{p^2}$$

$$\frac{x}{2y} - \frac{p}{q} = c_3 \qquad +\frac{x}{2y} + \frac{q}{p} = c_4$$

$$\frac{x}{2y} - c_3 = \frac{p}{q}$$

$$\left(\frac{x}{2y} - c_3\right)\left(\frac{1}{\frac{x}{2y} - c_4}\right) = 1$$

$= \frac{x}{2y}$

$cy = qz$

$q = \frac{cy}{z}$

$p^2 + \frac{c^2y^2}{q^2} = c^2$

$p^2 = c^2\sqrt{y^2 - q^2}$

$p = c\sqrt{\frac{q^2z^2}{c^2y^2} - q^2}$

$p = q\sqrt{z^2 - c^2y^2}$

We have $dz = pdx + qdy$.

$\cancel{z\,dz = \frac{c}{z}\sqrt{z^2-c^2y^2} + \frac{cy}{z}dy}$

$\cancel{\frac{z^2}{2} = c\sqrt{z^2-c^2y^2}\,x + \frac{c^2y^2}{2}}$

$dz - qdy = pdx$.

$zdz - c^2ydy = c\sqrt{z^2-c^2y^2}\cdot dx$

$\Rightarrow \frac{zdz - c^2ydy}{2z\sqrt{z^2-c^2y^2}} = dx$

$\Rightarrow \frac{1}{2}\frac{dt}{t} = cdx$

$\sqrt{z^2-c^2y^2} = cx + a$.

$(\sqrt{z^2-c^2y^2} - cx) = a$.

$z^2 - c^2y^2 = (cx+a)^2$.

$\boxed{z^2 = c^2y^2 + (cx+a)^2}$

28/11/22 nth order linear eqn with constant co-efficient.

$$\frac{\partial^n z}{\partial x^n} + \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + \frac{\partial^n z}{\partial y^n} = F(x,y)$$

$[D^n + k_1D^{n-1}D' + \cdots + k_nD'^n] = F(x,y)$

$\Rightarrow f(D,D')z = F(x,y)$

complete soln of ① is $CF + PI$

$f(D,D')z = 0$.

To find C.F :- $f(D,D')z = 0$.

Eg :- $(D^2 + k_1DD' + k_2D'^2)z = 0$

$(D - m_1D')(D - m_2D')z = 0$

If $(D - m_1D')z = 0$, then $p - m_1q = 0$.

$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{0}$

So, $z = a$, $dy + m_1dx = 0$

$y + m_1x = b$

Hence $z = \phi(y + m_1x)$

Simillarly $z = \psi(y + m_2x)$

Hence, C.S of z is $\phi(y+m_1x) + \psi(y+m_2x)$.

Ex :-

1) $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

put $D = m$, $D' = 1$

$2m^2 + 5m + 2 = 0$

$2m^2 + 4m + m + 2$

$m = -2, -\frac{1}{2}$

$CS = \phi(y - 2x) + \phi(y - \frac{1}{2}x)$

P.I =

Case II :- If roots in the auxillary eqn are equal.

$$C.S = \phi(y+m_1x) + x\cancel{\phi(y+m_1x)}$$
$$\psi(y+m_1x)$$

2) $4r+12s+9t=0$

$$4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$$

put $D=m$; $D'=1$

$$4m^2+12m+1=0$$
$$4m^2+6m+6m+1=0$$
$$m = -\frac{3}{2}, -\frac{3}{2}$$
$$C.S = \phi(y-\tfrac{3}{2}x) + x\cancel{\phi(y-\tfrac{3}{2}x)}$$
$$\psi(y-\tfrac{3}{2}x)$$

Rules for P.I

If $F(x,y)$ i.e., is e^{ax+by}, then

$$De^{ax+by} = \frac{\partial}{\partial x}(e^{ax+by}) = a e^{ax+by}$$
$$D^2 e^{ax+by} = a^2 e^{ax+by}$$
$$DD' e^{ax+by} = ab e^{ax+by}$$
$$D'^2 e^{ax+by} = b^2 e^{ax+by}$$

So, $f(D,D') e^{ax+by} = f(a,b) e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$$

Case-II $F(x,y) = \sin(mx+ny)$

$$D^2 \sin(mx+ny) = -m^2 \sin(mx+ny)$$
$$D'^2 \sin(mx+ny) = -n^2 \sin(mx+ny)$$
$$DD' \sin(mx+ny) = -mn \sin(mx+ny)$$
$$f(D^2, DD', D'^2)\sin(mx+ny) = f(-m^2, -mn, -n^2)\sin(mx+ny)$$
$$P.I = \frac{\sin(mx+ny)}{f(D^2, DD', D'^2)} = \frac{\sin(mx+ny)}{f(-m^2, -mn, -n^2)}$$

Case-III $F(x,y) = x^m y^n$

$$P.I = \frac{1}{f(D,D')}[x^m y^n]$$
$$= f(D,D').[x^m y^n]$$

1) Solve $(D^2-DD') = \cos x \cos 2y$.

C.F = put $D=m$, $D'=1$

$$\Rightarrow m^2-m=0$$
$$m(m-1)=0$$
$$m=0,1$$
$$C.F = \phi(y)+\psi(y+x)$$
$$P.I = \frac{\cos x \cos 2y}{D^2-DD'}$$
$$= \frac{\cos(x+2y)}{2(D^2-DD')} + \frac{\cos(x-2y)}{2(D^2-DD')}$$
$$= \frac{\cos(x+2y)}{2(-1+2)} + \frac{\cos(x-2y)}{2(-1-2)}$$
$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

$C.S = C.F + P.I$

$$= \phi(y)+\psi(y+x)+\frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

2) solve $(D^3-2D^2D')z = 2e^{2x}+3x^2y$.

C.F put $D=m$ $D'=1$

$m^3-2m^2=0$

$m^2(m-2)=0$

$m=0,0,2$

$C.F = f_1(y)+f_2(y)+f_3(y+2x)$

$$P.I = \frac{2e^{2x}}{D^3-2D^2D'} + \frac{3x^2y}{D^3-2D^2D'}$$

$$= \frac{2e^{2x}}{D^2(D-2D')} + \frac{3x^2y}{D^3(1-\frac{2D'}{D})}$$

$$= \frac{1}{2(2)}e^{2x} + \frac{3}{D^3}\left(1-\frac{2D'}{D}\right)^{-1}x^2y$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\cancel{\left(1+\frac{2D'}{D}-3\left(\frac{D'}{D}\right)^2+4\left(\frac{D'}{D}\right)^3\right)x^2y}$$

$$+\frac{3}{D^3}\left(1+1\cdot\left(\frac{2D'}{D}\right)+2\cdot\left(\frac{4D'^2}{D^2}\right)+\dots\right)x^2y$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\left(x^2y+\frac{2x^2}{D}+4(0)\right)$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\left(x^2y+\frac{2x^3}{3}\right)$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^2}\left(\frac{x^3y}{3}+\frac{2x^4}{12}+c_1\right)$$

$$= \frac{e^{2x}}{4} + \frac{3}{D}\left(\frac{x^4}{12}y+\frac{x^5}{30}+c_1x+c_2\right)$$

$$= \frac{e^{2x}}{4} + 3\left(\frac{x^5}{60}y+\frac{x^6}{120}+\frac{c_1x^2}{2}+c_2x+c_3\right)$$

$$P.I = \frac{e^{2x}}{4} + 3\left(\frac{x^5}{60}y+\frac{x^6}{120}+\frac{c_1x^2}{2}+c_2x+c_3\right)$$

3) Method of separation of variables.

$(D^2-2D+D')z=0$.

for C.F put $D=m$, $D'=1$

$m^2-2m+1=0$

$m=1,1$

$C.F = \phi(y+x)+x\psi(y+x)$

P.I = put $z = X(x)Y(y) = XY$

$D^2z = X''Y$

$DD'z = X'Y'$

$D'^2z = XY''$

$X''Y - 2X'Y + XY' = 0$

$$\frac{X''-2X'}{X} = \frac{-Y'}{Y}$$

So, $$\frac{X''-2X'}{X} = \frac{-Y'}{Y} = k.$$

$\therefore$ $X''-2X'-Xk=0$ & $kY+Y'=0$.

$Y = e^{-ky}+c$

A.E $m^2-2m-k=0$.

Sol: $Y = e^{-ky}+c$

$$m = \frac{2\pm\sqrt{4+4k}}{2}$$

$m = 1\pm\sqrt{1+k}$

* wave eq^n $u_{tt} = c^2 u_{xx}$

heat eq^n $u_t = c^2 u_{xx}$

Laplace eq^n $u_{xx} + u_{yy} = 0$

Laplace

put $u = X(x) Y(y)$

$u_x = X'Y$

$u_{xx} = X''Y$

$u_{yy} = XY''$

Laplace eq^n can be written as

$X''Y + XY'' = 0$

$\frac{X''}{X} = -\frac{Y''}{Y} = k$

$X'' - Xk = 0$ & $Yk + Y'' = 0$

$\cancel{X = e^{kX}}$ $\cancel{Y = e^{-kY}}$

$\cancel{X + e^{kX} + c_1 = 0}$ $\cancel{Y + e^{-kY} + c_2 = 0}$

or Wave eq^n:-

put $u = X(x) T(t)$

$u_x = X'T$ $u_{xx} = X''T$ $u_{yy} = XT''$

wave eq^n can be written as

$XT'' - c^2 X''T = 0$

$\frac{X''}{X} = \frac{T''}{c^2 T} = k$

so, we have

$X'' - kX = 0$ & $T'' - kc^2 T = 0$

case 1:- $k > 0$ & $k = p^2$ $X'' - p^2 X = 0$

$X = c_1 e^{px} + c_2 e^{-px}$

Also $T'' - p^2 c^2 T = 0$

$T = c_3 e^{pct} + c_4 e^{-pct}$

$u = XT$

$= (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})$

case 2:- Let $k = -p^2$ & $k < 0$ $X'' + p^2 X = 0$

$X = c_5 \cos(px) + c_6 \sin(px)$

also, $T'' + p^2 c^2 T = 0$

$T = c_7 \cos(pct) + c_8 \sin(pct)$

$\therefore u = (c_5 \cos(px) + c_6 \sin(px))(c_7 \cos(pct) + c_8 \sin(pct))$

case 3:- Let $k = 0$ $X'' = 0$ & $T'' = 0$

so, $X = c_9 x + c_{10}$

$T = c_{11} t + c_{12}$

$u = (c_9 x + c_{10})(c_{11} t + c_{12})$

so, the sol^n of wave eq^n is

$u = (c_5 \cos px + c_6 \sin px)(c_7 \cos(pct) + c_8 \sin(pct))$

Ex:- a string is stretched and fastened to two points l apart motion is started by displacing the string on the form $y = a\sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at time t is

$$y(x,t) = a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi c t}{l}\right)$$

$y(0,t) = 0$, $y(l,t) = 0$.

Since the initial transverse velocity of any point of the string is zero, so $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$y(x,0) = a\sin\frac{\pi x}{l}$

We have

$$u = (A\cos px + B\sin px)(A_1\cos pct + B_1\sin pct)$$

$u(0,t) = 0$

$\Rightarrow A(A_1\cos pct + B_1\sin pct) = 0$

$\Rightarrow A = 0$

$u(x,t) = B\sin px\,(A_1\cos pct + B_1\sin pct)$

$u(l,t) = 0$

$\Rightarrow B\sin pl\,(A_1\cos pct + B_1\sin pct) = 0$

$$\frac{\partial u}{\partial t} = B\sin px\,(A_1 pc\sin pct + B_1 pc\cos pct).$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}$$

$\Rightarrow B B_1 pc\sin px = 0$ So, $BB_1pc = 0$

If $B=0$ then $u=0$

So, $B_1 = 0$

$\therefore u(x,t) = B\sin px\, A_1\cos pct.$

$u(x,t) = B\sin pl\, A_1\cos pct = 0.$

Since B & $A_1 \neq 0$, we have $\sin pl = 0$

$\Rightarrow pl = n\pi$.

$\Rightarrow \boxed{p = \frac{n\pi}{l}}$

$u(x,t) = BA_1\sin\left(\frac{x n\pi}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$

$y = a\sin\left(\frac{\pi x}{l}\right)$

But $u(x,0) = BA_1\sin\frac{n\pi x}{l}$

$\therefore BA_1\sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l}$

So, $BA_1 = a$ & $n = 1$

Hence $u(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$