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Complex Analysis

* **Definition:-** $\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}, i=\sqrt{-1}\}$.

$\mathbb{R} \subseteq \mathbb{C} \rightarrow$ if $y=0$. $x \in \mathbb{R}$.

$\cancel{\mathbb{C} \subseteq \mathbb{R}}$

* **Complex Function:-** Let $D \subseteq \mathbb{C}$ and $f: D \rightarrow \mathbb{C}$ be a function that assigns a unique number $w \in \mathbb{C}$ to each number $z \in D$ is called complex fnc.

$\downarrow$ complex Valued fnc

D stands for Domain.

Eg:- 1) $f(z) = 1+i$; $z \in \mathbb{C}$ (constant fnc)

2) polynomial fnc

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

a_i's $\in \mathbb{C}$, $a_n \neq 0$

3) Rational fnc's

$f(z) = \dfrac{p(z)}{q(z)}$; $q(z) \neq 0$.

Eg:- $f(z) = \dfrac{z-1}{z^2}$; $z \neq 0$, $D = \mathbb{C} - \{0\}$

Range: $\{f(z) \mid z \in D\}$.

$f(z) = z^3$.

$f(x+iy) = (x+iy)^3$

$= x^3 - iy^3 + 3x^2 iy - 3xy^2$

$= \underbrace{(x^3 - 3xy^2)}_{\downarrow} + (3x^2 y - y^3) i$

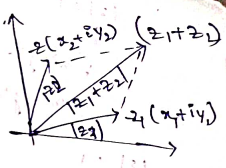
$u(x,y) + i v(x,y)$.

* $u: D (\subseteq \mathbb{R}^2) \rightarrow \mathbb{R}$
$v: D' (\subseteq \mathbb{R}^2) \rightarrow \mathbb{R}$ $\Big\} \rightarrow$ Real Valued fnc.

* **Triangle Inequality**

$|z_1 + z_2| \leq |z_1| + |z_2|$

for any $z_1, z_2 \in \mathbb{C}$



$|z_1 + z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + (w + \bar{w})$

$w = z_1\bar{z}_2$

$\bar{w} = \overline{z_1\bar{z}_2} = \bar{z}_1 z_2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(w)$

$\leq (|z_1|^2 + |z_2|^2) + 2|z_1||z_2|$

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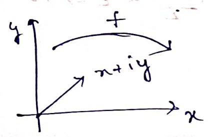
$f(z) = \dfrac{az+b}{cz+d}$ c & d are not zero simultaneously.

$D = \begin{cases} \mathbb{C} ; c=0 \\ \mathbb{C} - \{-\frac{d}{c}\} \; c \neq 0 \end{cases}$

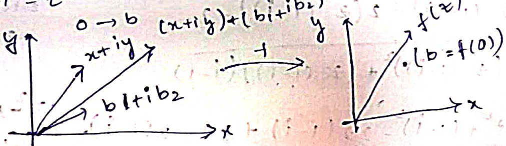
Visualization of complex function:-

$f(z) = z + b \longrightarrow$ b is a fixed complex no.



(i) $f(z) = z + b \rightarrow$ b is a fixed complex no.

$0 \rightarrow b$ $(x+iy) + (b_1 + ib_2)$

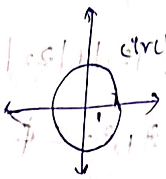


$\to |z| = 1$

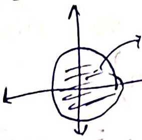
$\downarrow$

$\sqrt{x^2+y^2} = 1$

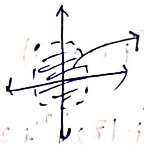
$x^2+y^2 = 1 \longrightarrow$



$|z| \le 1$



$|z| < 1$

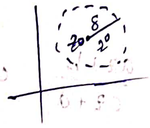


Limit:

A function $f(z)$ is said to have the limit 'l' as $z \to z_0$ i.e., ~~limit~~

$\lim_{z \to z_0} f(z) = l$.

if f is defined in same nbd of z_0,



$\forall \epsilon > 0, \exists \delta > 0$ s.t.

$|f(z) - l| < \epsilon$, whenever $0 < |z - z_0| < \delta$.

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$\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$ $\quad f(z) = \frac{iz}{2}$ $\quad D = \{z \in \mathbb{C} \mid |z| \le 1\}$

$|f(z) - l| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \left|\frac{i(z-1)}{2}\right| = \frac{|z-1|}{2} < \epsilon \Rightarrow |z-1| < 2\epsilon$

$|z - z_0| = |z - 1| < \delta = 2\epsilon$.

$\lim_{z \to 1-i} \frac{z^2 + 2z - 2}{z^2 - z + iz}$

$\lim_{z-(1-i) \to 0} \frac{(z-(1-i))^2 + 2(z-(1-i)) - 2}{(z-(1-i))^2 + (z-(1-i))(i-1)}$

$z^2 + (1-i)^2 - 2z(1-i)$

$-z + (1-i) + i(z) -$

$i($

$z^2 - z(1-i)$

$(z-(1-i))^2$

$= \frac{\cancel{z^2} + (1-i)^2 - 2\cancel{z}(1-i) + 2\cancel{z} - 2(1-i) - 2}{\cancel{z^2} + (1-i)^2 - \cancel{3z}(1-i) + (i^2-1)^2}$ $\quad z^2 + (1-i)^2 - 3z(1-i)$

$= \frac{z^2 + 2iz + \dots}{\cancel{z^2} + (-4i) \cancel{- 3z} + 3zi}$

$= \frac{\cancel{1} - \cancel{1} - \cancel{2i} - 4 + 2i}{2(1 - 1 - 2i)}$

$= \frac{\cancel{-}2}{\cancel{-}2i} = \frac{1}{i} = \underline{\underline{-i}}$

$= \frac{2i - 4}{-4i} \times \frac{4i}{4i} = \frac{2(-1) - 16i}{\cancel{+16}}$

$= \frac{-8 - 16i}{16} = -\left(i + \frac{1}{2}\right)$

* **Uniqueness**

$\forall \epsilon > 0 ; \exists \delta_1 > 0$.

$\lim_{z \to z_0} f(z) = w_0$; $|f(z) - w_0| < \epsilon$, whenever $|z - z_0| < \delta_1$

$\lim_{z \to z_0} f(z) = w_1$; $|f(z) - w_1| < \epsilon$, whenever $|z - z_0| < \delta_2$

$|w_0 - w_1| = |f(z) - w_0) - (f(z) - w_0)|$

$\le |f(z) - w_0| + |f(z) - w_1|$

$< \epsilon + \epsilon = 2\epsilon$

$\Rightarrow |w_0 - w_1| \to 0$

$\Rightarrow \boxed{w_0 = w_1}$

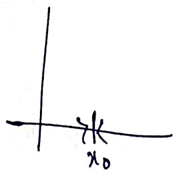
* Limit of the given fnc does not exists.

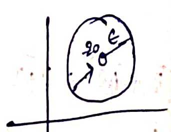
Eg:- $f(z) = \frac{z}{\bar{z}}$

$z = x + i0$ $\quad z - z_0$ ~~f(z)~~ $\quad f(z) = \frac{x}{x} = 1$

$z = iy$

$f(z) = \frac{iy}{-iy} = -1$





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Continuity :- A function $f: D (\subseteq \mathbb{C})$ is said to be continous at $z_0 \in D$ if $\lim_{z \to z_0} f(z)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$

In other words f is said to be continous at $z_0 \in D$ if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t

$$|f(z) - f(z_0)| < \varepsilon, \text{ Whenever } |z - z_0| < \delta.$$

* **Differentiablility** :-

Function f is said to be differentiable at z_0, if $\lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists, and is denoted as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z} \qquad \Delta z = z - z_0$$
$$z = \Delta z + z_0$$

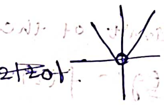
$\downarrow$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Q) Let $f(z) = z^2$ prove that $f'(z_0) = 2z_0$; $z_0 \in \mathbb{C}$

$$f'(z_0) = \lim_{\Delta z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{\Delta z \to z_0} (z_0 + z) = 2z_0$$

Q) check the differentiability of $f(z) = |z|^2$.



$$\cancel{f'(z_0) = \lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = |z| + |z_0| = 2|z_0|}$$

$$f'(z_0) = \lim_{z \to z_0} \frac{z \cdot \bar{z} - z_0 \cdot \bar{z}_0}{z - z_0} = \frac{|z|^2 - |z_0|^2}{z - z_0} = 2z_0$$

$$= \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z \cdot \bar{z}}{\Delta z}$$

$$= \frac{\cancel{z \cdot \bar{z}} + \Delta z \cdot \bar{z} + z \cdot \overline{\Delta z} + \Delta z \cdot \overline{\Delta z} - \cancel{z\bar{z}}}{\Delta z}$$

$$= \bar{z} + z\left(\frac{\overline{\Delta z}}{\Delta z}\right) + \overline{\Delta z}$$

$$= \lim_{z \to 0} \frac{\bar{z}}{z} \rightarrow \text{doesnot exists.}$$

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$$f(z) = |z|^2.$$

$z \to 0$ $\qquad \lim_{\substack{x \to 0 \\ y \to 0}} \bar{z} = 0.$

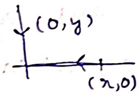
$x + iy \to 0 + i0$

$x \to 0$

$y \to 0$

$f(z) = \bar{z}$

$$\cancel{f'(z_0) = \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0}} = \lim_{z \to 0} \cancel{x - iy}$$



$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} \frac{x - iy}{x + iy} = 1 \text{ (along x-axis)}$$

$$\lim_{z \to 0} \frac{x - iy}{x + iy} = \lim_{z \to 0} \frac{-iy}{iy} = -1 \text{ (along y-axis)}$$

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{\overline{z_0 + \Delta z} - \bar{z}}{\Delta z}$$

$$= \frac{\bar{z}_0 + \overline{\Delta z} - \bar{z}_0}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \begin{cases} 1 & \to x\text{-axis} \\ -1 & \to y\text{-axis} \end{cases}$$

* **Analytic fnc** :-

A function $f(z)$ is said to be analytic at z_0 if it has a derivative in some nbd of z_0 including the point z_0 itsef

Eg:- $\dfrac{1}{z}$, $\dfrac{zi}{z}$, $z^2 + z + 1$

Entire Function :-

The fnc which are analytic throughout the complex plane are called entire fnc.

Eg :- e^z, polynomial fnc.

How to test differentiability?

$f(z) = u(x,y) + iv(x,y)$

Cauchy - Rieman Equation :-

The set of equations $u_x = v_y$ and $u_y = -v_x$ are called CR-eqn's

$u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$

$v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial u}{\partial y}$

Theorem :- Let $f(z)$ be a diff fnc at $z = z_0$. Then the partial derivatives u_x, v_x, v_y exists at $z = z_0$ & the following CR-eqn's

$$u_x = v_y, \quad u_y = -v_x$$

satisfy

At, $z = z_0$

$$f'(z_0) = u_x|_{z_0} + iv_x|_{z_0}$$

Note :- Converse of the above statement is not true.

$f(z) = z^2$

$f(z) = |z|^2, \bar{z}$

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$f(z) = u + iv$

$z = re^{i\theta} \Rightarrow x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x} \cdot (\cos\theta) + \frac{\partial u}{\partial y} (\sin\theta)$$

$$u_r = u_x\cos\theta + u_y\sin\theta \longrightarrow (i)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$v_r = v_x\cos\theta + v_y\sin\theta \longrightarrow (ii)$$

$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta \longrightarrow (iii)$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta \longrightarrow (iv)$$

$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \rightarrow$

$u_r = u_x\cos\theta + u_y\sin\theta$

$v_r = -u_y\cos\theta + u_x\sin\theta$

$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$

$v_\theta = u_y r\sin\theta + u_x r\cos\theta$

$\boxed{v_\theta = r u_r}$ $\boxed{u_\theta = -r v_r}$

Harmonic function :-

A real valued function H on two variables x and y is said to be harmonic if its partial derivatives exists and are continuous and satisfies Laplace eqn

$$H_{xx} + H_{yy} = 0$$

i.e., $\nabla^2 H = 0$

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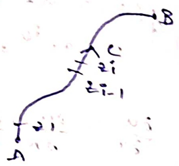
Complex Integration



$f(z) \to$ continous

$\xi_i = z_{i-1} \, z_i$

$z_i - z_{i-1} = \delta z_i$

$f(\xi_i)(z_i - z_{i-1})$

$\sum_{i=0}^{n} f(\xi_i)\, \delta z_i$

$z = x + iy$

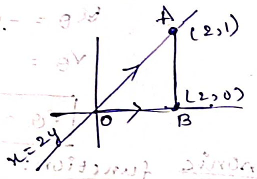
$dz = dx + i\,dy$

$f(z) = u + iv$

$\int_C f(z) \cdot dz = \int_C (u+iv)(dx + i\,dy)$

$= \int_C (u\,dx - v\,dy) + i \int v\,dx + u\,dy$

1) $\int_0^{2+i} (\bar{z})^2 \cdot dz$



i) along OA $x = 2y$

$z = x + iy$

$z = 2y + iy$

$dz = (2+i)\,dy$

$f(z) = (\bar{z})^2 = (x - iy)^2 = (2-i)^2 y^2$

$\int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (2-i)^2 y^2 \cdot (2+i)\,dy$

$= \frac{(2-i)^2 (2+i)\, y^3}{3} \Big|_0^1$

$= \frac{1}{3}(2-i)^2(2+i)$

ii) along OB $y = 0$

$z = x$

$dz = dx$

$f(z) = (\bar{z})^2 = x^2$

$\int_0^2 x^2 \cdot dx = \frac{1}{3}(8) = \frac{8}{3}$

iii) along BA $x = 2$

$z = 2 + iy$

$dz = i\,dy$

$f(z) = (2 - iy)^2$

$\int_0^1 (2 - iy)^2 \cdot i\,dy$

$= \int_0^1 (4 - y^2)\, i\,dy + \int_0^1 +4y\,d$

$= 4i - \frac{1}{3}(1)i + 2$

$= \left(\frac{11}{3}\right)i + 2$

2) $\int_C \frac{1}{z-a} \cdot dz$; $C : |z-a| = r$

$z - a = r \cdot e^{i\theta}$; $0 \le \theta \le 2\pi$

$dz = i r \cdot e^{i\theta} \cdot d\theta$

$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{\cancel{r \cdot e^{i\theta}}\, i\, d\theta}{\cancel{r \cdot e^{i\theta}}}$

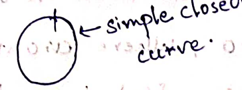
$= i\theta \Big|_0^{2\pi}$

$= 2\pi i$

* Cauchy's Theorem

If $f(z)$ is analytic on a closed curve C and within with $f'(z)$ as continous curve C, then.

$\int_C f(z) \cdot dz = 0$



Eg:- ~~$\int f(z)\,dz = 0$~~

1) $\int_C e^z \, dz \;;\; |z| = 1$

$\hookrightarrow = 0$ irrespective of c. by cauchy's theorem.

2) $\int \sec z \, dz \,,\; c : |z| < 1$

$\cos z = 0 \longrightarrow$ with in the $\underline{c}$

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$.

3) $\int_C \frac{1}{z^2} \, dz \;;\; c : |z| < 1$

$z = re^{i\theta} \;;\; 0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \frac{\cancel{r} \cdot e^{i\theta} \cdot i \cdot d\theta}{r^{\cancel{2}} \cdot \cancel{e^{2i\theta}}}$$

$$= \int_0^{2\pi} \frac{i \cdot d\theta}{r \cdot e^{i\theta}}$$

$$= \frac{\cancel{i}}{r} \frac{e^{-i\theta}}{-\cancel{i}} \Big|_0^{2\pi} = +\frac{1}{r}\left[1 - e^{-2\pi i}\right]$$

$e^{-2\pi i} = \cos(-2\pi) + i\sin(-2\pi) = 1$

$= 0$

$\frac{1}{z^2}$ is not analytic.

converse is not true.

* If analytic its ans is '0' if ans is '0' then it may not be analytic.

Morera's Theorem:-

If $f(z)$ is continous in a region D and $\int_C f(z)\,dz = 0$, ~~where~~ around every closed curve c in D, then $f(z)$ is analytic.

Cauchy's Integral Formula:-

If $f(z)$ is analytic on and within a simple closed curve c and 'a' is a point inside the closed curve c then.

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \, dz.$$

In general,

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \, dz.$$

1) $\int \frac{z^2 - z + 1}{z - 1} \, dz$. $\quad C : |z| = 1$

$C : |z| = \frac{1}{2}$

$\int \cancel{z\,dz} + \cancel{\frac{1}{z-1}\,dz}$. } at $|z| = \frac{1}{2} \rightarrow$ analytic.

$f(z) = z^2 - z + 1$.

$a = 1$

$$f(1) = \frac{1}{2\pi i} \int \frac{z^2 - z + 1}{z - 1} \, dz.$$

$2\pi i\, f(1)$

$$\underline{2\pi i} = \int \frac{z^2 - z + 1}{z - 1}$$

2) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} \, dz \,.\; |z| = 3$ $\quad f(z)$

$$\oint_C \frac{f(z)}{z-2} - \frac{f(z)}{z-1} \, dz.$$

$f(2)(2\pi i) - f(1)\, 2\pi i$

$2\pi i (1 + 1) = 4\pi i$

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$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} \qquad C: |z| = \frac{3}{2}$$

as $1 < |z| < 2$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \cdot dz$$

$$f(1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \cdot dz$$

$= -1$

$$-1 \cdot 2\pi i = \int_C \frac{f(z)}{z-a} \cdot dz$$

1) $f(z) = u + iv$

$u = x^2 - y^2$.

$f(z) \rightarrow$ Analytic.

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \qquad v = 2xy + f(x)$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad v = +2xy + g(y)$$

$\cancel{\frac{\partial v}{\partial x} = 2y}$

$f(x) = g(y) = c$.

$v = 2xy + c$

$f(z) = (x^2 - y^2) + i(2xy + c)$

* **Taylor's Series**

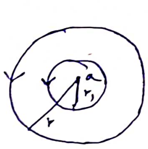
$f(z) \rightarrow$ Analytic

Domain $D = C$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

Laurent's Series:-

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} \cdots$$



Singularities:

isolated singularity.

$f(z) = \frac{1}{z}$; $z = 0$.

$f(z) = \frac{1}{\tan \frac{\pi}{z}}$; $z = 1, \frac{1}{2}, \frac{1}{3}$

Poles:

$$f(z) = a_0 + a_1(z-a) + \cdots$$
$$a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots + a_{-m}(z-a)^{-m}.$$

$m=1 \rightarrow$ Simple pole.

Residues:- (a_{-1})

$\lim_{z \to a} (z-a)^m f(z) = a_{-1} \rightarrow$ Residue at $z=a$ for $m=1$

$(z-a)^m \frac{d^{m-1}}{dx^{m-1}} f(z) \rightarrow$ Residue at $z=a$ for m.

$= a$

Residues: (a_{-1})

Cauchy Residue Theorem

$f(z) \rightarrow$ analytic inside except at $a_1 \cdots a_n$.

$$\oint_C f(z)\,dz = 2\pi i \times (\mathrm{Res}\,f(a_1) + \mathrm{Res}\,f(a_2) + \cdots + \mathrm{Res}\,f(a_n)).$$

Eg:- $\oint_C \frac{e^z}{z(z-1)^2} \cdot dz \qquad C: |z| = 2$

$2\pi i \times (\mathrm{Res}f(0) + \mathrm{Res}f(1))$

$2\pi i \times (1 + e)$

$\mathrm{Res}f(0) = \lim_{z \to 0} (\cancel{z} - 0) \frac{e^z}{\cancel{z}(z-1)^2} = 1$

$\mathrm{Res}f(1) = (\cancel{z-1})^2 \frac{d}{d} \left(\frac{e^z}{z(z-1)^2} \right).$

$$= \lim_{z \to 0} \frac{e^z z(z-1)^2 - e^z \left(2(z-1)z - (z-1)^2\right)}{z^2 (z-1)^4}$$