COMPLEX NUMBERS

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<u>Def^n</u> :- $\mathbb{C} = \{x+iy \mid x,y \in \mathbb{R} \; ; \; i = \sqrt{-1}\}$

$\{x+i0 \; ; \; x \in \mathbb{R}\} = \mathbb{R}$

$\mathbb{R} \subseteq \mathbb{C}$

$z = x+iy$

$z\bar{z} = |z|^2$

Complex Function :-

Let $D \subseteq \mathbb{C}$ and $f : D \longrightarrow \mathbb{C}$ be a function that assigns a unique member $w \in \mathbb{C}$ to each ~~no~~ member $z \in D$, is called complex function
D stands for domain. (complex valued function)

Eg ① $f(z) = 1+i$, $z \in \mathbb{C}$ (constant function)

② Polynomial function $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 + a_0$

a_i's $\in \mathbb{C}$

$a_n \neq 0$

③ Rational function :- $f(z) = \dfrac{p(z)}{q(z)}$

$f(z) = \dfrac{z-1}{z^2}$; $z \neq 0$

$D = \mathbb{C} - \{0\}$

Range : $\{f(z) \mid z \in D\}$

$f(z) = z^3$

$f(1+i) = (1+i)^3$

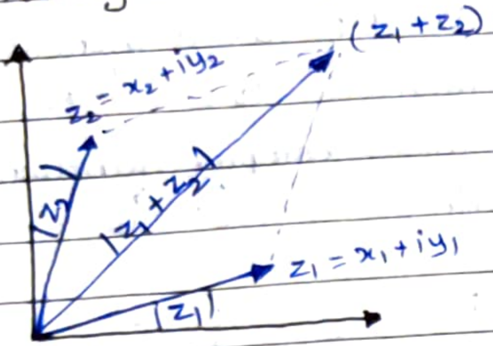
$= 1 + i^3 + 3i^2 + 3i$

$= -2 + 2i$ $\quad = -2 + 2i$

$$f(x+iy) = (x+iy)^3$$
$$= x^3 - iy^3 + 3xi^2y^2 + 3x^2iy$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$
$$= U(x,y) + i\,V(x,y)$$

$U : D(\subset \mathbb{R}^2) \longrightarrow \mathbb{R}$
$V : D'(\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}$
} Real valued function

Triangle inequality



$|z_1+z_2| \leq |z_1| + |z_2|$
for any $z_1, z_2 \in \mathbb{C}$

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$$
$$= z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_2}$$
$$= |z_1|^2 + |z_2|^2 + \omega + \overline{\omega}$$
$$= |z_1|^2 + |z_2|^2 + 2Re(\omega)$$

$\omega = z_1\overline{z_2}$
$\overline{\omega} = \overline{z_1}z_2$

$$2|z_1||z_2| = 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$
$$= 2\sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2}$$

$$2Re(\omega) \Rightarrow Re[(x_1+iy_1)(x_2-iy_2)]$$
$$= 2(x_1x_2 + y_1y_2)$$

$$(x_1x_2+y_1y_2)^2 = x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2$$

$$x_1^2y_2^2 + x_2^2y_1^2 \geq 2x_1x_2y_1y_2$$
$$x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 \geq x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2$$

✓

$\mathrm{Re}(\omega) \geq |z_1||z_2|$

$$\sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} \geq \sqrt{(x_1x_2 + y_1y_2)^2}$$

$$\sqrt{|z_1||z_2|} \geq \mathrm{Re}(\omega)$$

$$2|z_1||z_2| \geq 2\,\mathrm{Re}(\omega).$$

$$|z_1|^2 + |z_2|^2 + 2|z_1||z_2| \geq |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(\omega)$$

$$(|z_1| + |z_2|)^2 \geq |z_1 + z_2|^2$$

$$|z_1| + |z_2| \geq |z_1 + z_2|.$$

METHOD 2

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$$

$$\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$|z_1 + z_2|\cancel{^2} \leq |z_1| + |z_2|.$$

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$x \leq \sqrt{x^2 + y^2}$

$\mathrm{Re}(z) \leq |z|$

$|z_1z_2| = |z_1||z_2|$

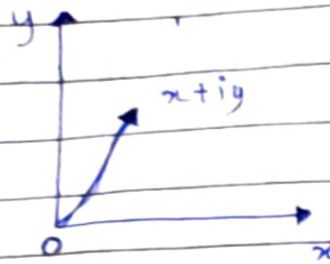
$|\bar{z}| = |z|$

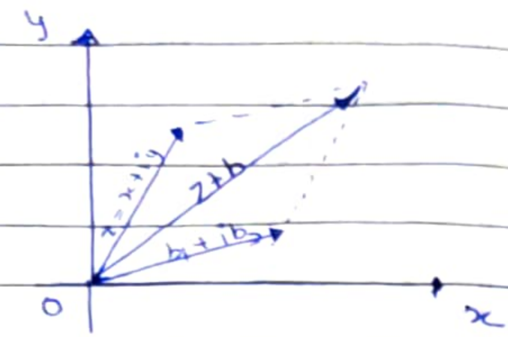
Visualization of Complex Function.

$f(z) = z + b$ b is a fixed complex No.

$f(z) = z$ $x + iy \longrightarrow x + iy + b_1 + ib_2$

$z + b$

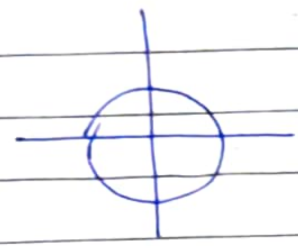




$|z| = 1$

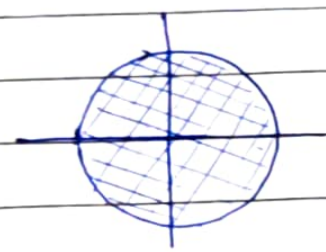
$\sqrt{x^2 + y^2} = 1$

$x^2 + y^2 = 1$



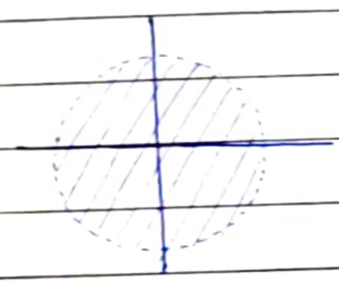
circle

$|z| \leq 1$



disc

$|z| < 1$



Open disc

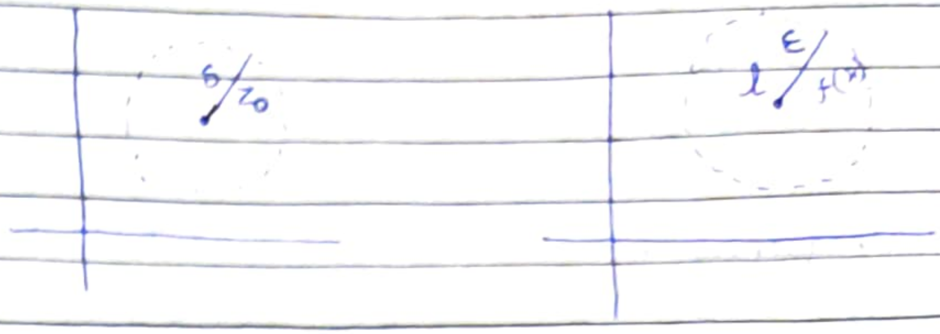
Limit :- A function $f(z)$ is said to have the limit 'l' as $z \to z_0$, i.e.,

$$\lim_{z \to z_0} f(z) = l$$

if f is defined in some neighbourhood of z_0

$\forall \epsilon > 0, \exists \delta > 0$

$|f(z) - l| < \epsilon$;
whenever $0 < |z - z_0| < \delta$



Q. Prove that $\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$.

Given $\epsilon > 0$.

$$|f(z) - l| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \left|i\left(\frac{z-1}{2}\right)\right|$$

$$= \frac{|z-1|}{2} < \epsilon \Rightarrow |z-1| < 2\epsilon$$

$$|z - z_0| = |z - 1| < \delta = 2\epsilon$$

$\forall \epsilon > 0, \exists \delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $|z - z_0| < \delta$

Q. $\lim_{z \to 1-i} \frac{z^2 + 2z - 2}{z^2 - z + iz} = ?$

$$f(z+\delta) = \frac{(1-i+\delta)^2 + 2(1-i+\delta) - 2}{(1-i+\delta)\delta}$$

$$\frac{\cancel{1} - \cancel{1} - 2i + \delta^2 + 2\delta - 2\delta i + \cancel{2} - 2i + 2\delta - \cancel{2}}{(1-i+\delta)\delta}$$

$$f(z+\delta) = \frac{\delta^2 + 4\delta - 2\delta i - 4i}{(1-i+\delta)\delta}$$

Limit ~~of~~ the given function doesnot not exist.

$f(z) = \frac{z}{\bar{z}}$ $z = x + i0$ $f(z) = 1$

$z = iy$ $f(z) = -1$

As $z \to z_0$

Limit dne.

Uniqueness :-

$\forall\ \epsilon > 0, \exists\ \delta > 0.$

$\lim_{z \to z_0} f(z) = \omega_0$ $|f(z) - \omega_0| < \epsilon$, whenever $|z - z_0| < \delta_1$

$\lim_{z \to z_0} f(z) = \omega_1$

$|f(z) - \omega_1| < \epsilon$, whenever $|z - z_0| < \delta_2$.

$|\omega_0 - \omega_1| = |\ f(z) - \omega_1) - (f(z) - \omega_0)\ |$

$\leq |f(z) - \omega_1| + |f(z) - \omega_0|$

$|\omega_0 - \omega_1| \to 0$.

$\omega_0 = \omega_1$.

$< \epsilon + \epsilon$

$< 2\epsilon$

Continuity :- A function $f : D(\subseteq \mathbb{C}) \to \mathbb{C}$ is said to continuous at $z_0 \in D$ if $\lim_{z \to z_0} f(z)$ exists and

$\lim_{z \to z_0} f(z) = f(z_0)$.

In other words, f is said to be continuous at $z_0 \in D$ if $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that

$|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$

Differentiability :-

A function f is said to be differentiable at z_0 if $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exist and is

denoted as $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$\Delta z = z - z_0$ $\quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$z = z_0 + \Delta z$

Let $f(z) = z^2$ Prove that

$f'(z_0) = 2z_0 \quad z_0 \in \mathbb{C}$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^2} + \Delta z^2 + 2z_0\Delta z - \cancel{z_0^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \Delta z + 2z_0$$

$$= 0 + 2z_0$$

$$= 2z_0$$

IMP

Check the differentiability of $f(z) = |z|^2$.

At $z_0 = 0$

$f(z_0) = 0$

Left continuouity :- $\lim_{h \to 0^+} \cancel{\frac{f(z_0+h) - f(z_0)}{}}$

$= |0+h|^2$

$= h^2 = 0$

Right continuity :- $\lim_{h \to 0^-} |0-h|^2 = h^2 = 0$

Differentiability :- $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$\lim_{\Delta z \to 0} \frac{|0+\Delta z|^2 - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

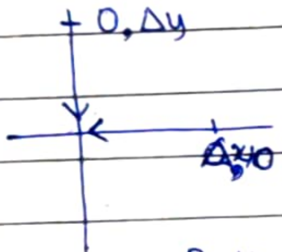
$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\bar{z} + \Delta z\bar{z} + \overline{\Delta z}z + (\Delta z)(\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \bar{z} + z\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}$$

limit of $\frac{\overline{\Delta z}}{\Delta z}$ dne.

$\Delta z = \Delta x + i\Delta y$



when $\Delta y \to 0$

$= z + \bar{z}$

when $\Delta x \to 0$

$= -z + \bar{z}$

Both must be equal

$$z + \bar{z} = -z + \bar{z}$$

$$2z = 0$$

$$z = 0$$

Hence differentiable only at origin

At $z_0 = 0$, $\lim_{z \to 0} \frac{f(z) - f(z_0)}{z - z_0}$

$$= \lim_{z \to 0} \frac{|z|^2 - f(0)}{z - 0} = \lim_{z \to 0} \frac{z\bar{z} - 0}{z}$$

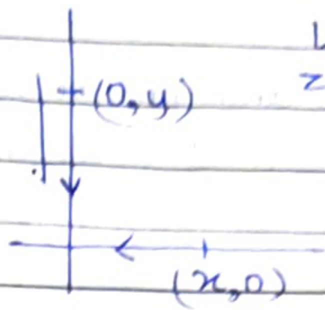
$$= \lim_{z \to 0} \bar{z}$$

IMP

$$f(z) = \bar{z}$$

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x - iy}{x + iy} = 1 \text{ (along x-axis)}$$

$$\lim_{z \to 0} \frac{x - iy}{x + iy} = -1 \text{ (along y axis)}$$



$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

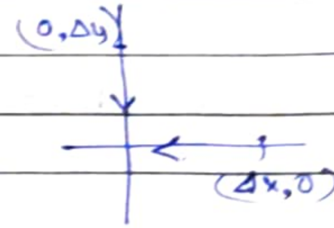
$$= \lim_{\Delta z \to 0} \frac{f(x_0 + iy_0 + \Delta x + i\Delta y) - f(x_0 + iy_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{x_0 + \Delta x - iy_0 - i\Delta y - x_0 + iy_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$\Delta y \to 0$ $\quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = 1$



$\Delta x \to 0$ $\quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{z} = -1$

So, not differentiable at any point

Analytic function :-

If $f(z)$ is differentiable at z_0 and all the neighbourhood points of z_0.

A function $f(z)$ is said to be analytic at z_0 if it has a derivative in some neighbourhood of z_0 including the point z_0 itself.

Eg :- $\frac{1}{z}$, $\frac{zi}{2}$, z^2+z+1, z^2.

$\downarrow$

for non zero point

Entire function :-

The functions which are analytic throughout the complex plane are called entire function.

Eg :- e^z, polynomial function.

How to test differentiability?

$f(z) = u(x,y) + iv(x,y)$.

Cauchy - Riemann Equation :-

The set of equations $u_x = v_y$ and $u_y = -v_x$.

are called CR - eqn^s.

$u_x = \frac{\partial u}{\partial x}$ $\quad u_y = \frac{\partial u}{\partial y}$ $\quad v_x = \frac{\partial v}{\partial x}$ $\quad v_y = \frac{\partial v}{\partial y}$

Theorem :- Let $f(z) = u + iv$ be a differentiable function at $z = z_0$. Then the partial derivatives u_x, u_y, v_x, v_y exists at $z = z_0$ and the following CR-eqn^s.

$$u_x = v_y, \quad u_y = -v_x.$$

Satisfy

At $z = z_0$,

$$f'(z_0) = v_x\Big|_{z_0} + i v_x\Big|_{z_0}.$$

<u>NOTE</u> :- Converse of above statement is not true.

Q/E $f(z) = z^2$ $\quad f(z) = |z|^2, \bar{z}$.

If it is analytic then it need not be differentiable.

$f(z) = u + iv$.

$z = re^{i\theta}$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

$$u_r = u_x\cos\theta + u_y\sin\theta. \longrightarrow (1)$$

$$\frac{\partial v}{\partial r} = v_x\cos\theta + v_y\sin\theta$$

$$= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$v_r = v_x\cos\theta + v_y\sin\theta. \longrightarrow (2).$$

$$u_\theta = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta) \longrightarrow (3).$$

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta). \longrightarrow (4).$$

$u_x = v_y \qquad u_y = -v_x$

$$\boxed{v_\theta = r u_r}$$

$$v_\theta = r\left[(-v_x \sin\theta) + v_y \cos\theta\right]$$

$$r\left[u_y \sin\theta + u_x \cos\theta\right]$$

$$V_\theta = r u_r.$$

$$u_\theta = r\left(-u_x \sin\theta + u_y \cos\theta\right]$$

$$u_\theta = r\left(-v_y \sin\theta + -v_x \cos\theta\right]$$

$$u_\theta = -r\left(v_y \sin\theta + v_x \cos\theta\right).$$

$$\boxed{u_\theta = -r v_r}$$

Harmonic function : $\qquad H = u + iv.$

A real valued function H on two variables x and y is said to be harmonic if it's partial derivatives exists and are continuous and satisfies Laplace eqn.

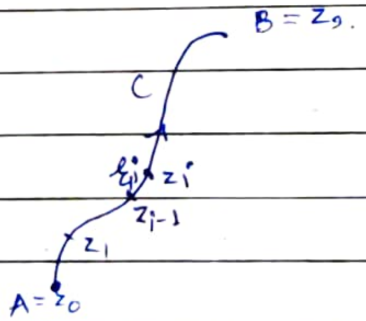
$$H_{xx} + H_{yy} = 0$$

$$\text{i.e.} \quad \nabla^2 H = 0.$$

Complex Integration :-

$f(z) \rightarrow$ continuous

$\xi_i \rightarrow z_{i-1}\, z_i$



$$z_i - z_{i-1} = \delta z_i$$

$$f(\xi_i)(z_i - z_{i-1}).$$

$$\sum_{i=0}^{n} f(\xi_{i})\delta z_{i}$$

$z = x + iy$

$dz = dx + idy \qquad f(z) = u + iv$

$$\int_c f(z)\,dz = \int_c (u + iv)(dx + idy)$$

$$= \int_c u\,dx - v\,dy + i\int_c v\,dx + u\,dy$$

Q. $\displaystyle\int_0^{2+i} (\bar{z})^2 dz$

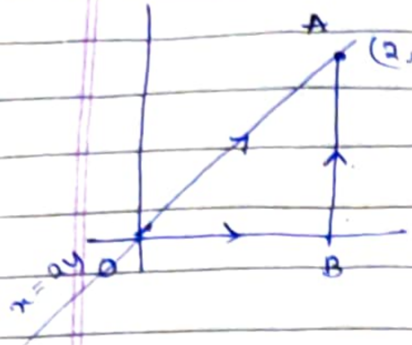
$\displaystyle\int_0^{2+i} (x - iy)^2 dz$

$\displaystyle\int_0 (x^2 - y^2 + 2xiy)(dx + idy)$

$$= \int_0^{2+i} x^2 dx - y^2 dx - 2xiy\,dx + i x^2 dy - iy^2 dy + 2xy\,dy$$

$$= \left[\frac{x^3}{3} - \frac{iy^3}{3}\right]_0^{2+i} + \int_0^{2+i} i(x^2 dy - 2xy\,dx) + \int_0^{2+i} 2xy\,dy - y^2 dx$$

$$= \frac{8}{3} - i\left(\frac{i^3}{3}\right) + i\int$$



(i) Along OA $\quad x=2y$

$z = x+iy$

$z = 2y+iy = (2+i)y$

$dz = (2+i)dy$

$f(z) = (\bar{z})^2 = (x-iy)^2$

$= (2y-iy)^2$

$= (2-i)^2 y^2$

$$\int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2 y^2 (2+i)\, dy$$

$$= (3-4i)(2+i)\int_0^1 y^2\, dy$$

$$= \left(\frac{10-5i}{3}\right)$$

(ii) Along OB

$y=0$

$z = x \qquad dz = dx$

$(\bar{z})^2 = x^2$

$$\int_0^2 x^2\, dx = \frac{8}{3}$$

(iii) Along AB

$x = 2 \qquad z = 2+iy$

$dz = i\,dy$

Q. $\int_C \frac{1}{z-a}\,dz \quad ; \; C : |z-a| = r$

$z-a = re^{i\theta} \quad ; 0<\theta \le 2\pi$

$dz = re^{i\theta}\, i\, d\theta$

$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{\cancel{re^{i\theta}}\, i\, d\theta}{\cancel{re^{i\theta}}}$

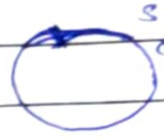
$= i\,[\theta]_0^{2\pi}$

$= 2\pi i$

Cauchy's Theorem :-

If $f(z)$ is analytic on and within a closed curve C with $f'(z)$ as continuous on curve C, then

$\int_C f(z)\,dz = 0$



simple closed curve

$\int_C e^z\,dz \quad ; \; |z| = 1$

e^z is analytic.

e^z is continuous

$\therefore \int_C e^z\,dz$ is $= 0$.

eg:- $\int_C \sec z\, dx \quad ; \; C : |z| < 1$

$\sec z = \frac{1}{\cos z}$.

For $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \qquad \cos z = 0$.

$\frac{\pi}{2} = 1.57$.

For $|z|<1$, $\sec z$ is analytic as $\cos z \neq 0$.

$\therefore \int_C \sec z\, dz = 0$.

Q. $\int_c \frac{1}{z^2} dz$: $c : |z| < 1$

such curves are called simple connected curves

$z = re^{i\theta}$; $0 \leq \theta \leq 2\pi$.

$dz = re^{i\theta} i \, d\theta$

$$\int_0^{2\pi} \frac{i \, d\theta}{re^{i\theta}}$$

$$= \frac{i}{r} \int_0^{2\pi} e^{-i\theta} d\theta$$

$$= \frac{i}{-ir} \left[e^{-i\theta} \right]_0^{2\pi}$$

$$= -\frac{1}{r} \left[e^{-2\pi i} - 1 \right]$$

$$= 0$$

$$= -\frac{1}{r} \int_0^{2\pi} (\cos\theta - i\sin\theta) d\theta$$

$$= -\frac{1}{r} \left[\sin\theta + i\cos\theta \right]_0^{2\pi}$$

Converse of Cauchy's theorem is not true as $\frac{1}{z^2}$ is not analytic.

$\int_0 \cos\theta - i\sin\theta$

Morera's Theorem :-

If $f(z)$ is continuous in a region D and $\int_C f(z) dz = 0$ ~~where C~~ around every closed curve C in D, then $f(z)$ is analytic.

Cauchy's Integral formula

If $f(z)$ is analytic on and within a simple closed curve C and 'a' is a point inside the closed curve C then.

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

In general

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

$f^{(n)}(a) \longrightarrow n^{th}$ derivative at point a

Q $\int_C \frac{z^2-z+1}{z-1} dz$; $C ; |z| = 1$

$C : |z| = \frac{1}{2}$

~~For~~ $|z| = \frac{1}{2}$, $\int_C \frac{z^2-z+1}{z-1} dz = 0$ $\because$ it is analytic.

Comparing it to $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

$\frac{f(a)}{2\pi i} = \int_C \frac{f(z)}{z-1} dz$ $a = 1$ $|z| = 1$ $2\pi i$

Q $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$ $C : |z| = 3$

$\Rightarrow 4\pi i$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$= \int \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} - \int \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}$

$2\pi i(+1) - (-2\pi i)$

$= 4\pi i$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} \qquad C: |z| = \frac{3}{2}.$$

$$= \int \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}}{z-1}$$

$\because \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$ is analytic for $|z| = \frac{3}{2}$.

for $|z| = \frac{1}{2}$ $\Rightarrow 0$.

Q $f(z) = u + iv$.

$u = x^2 - y^2$

$f(z)$ is analytic.

Find $f(z)$.

$u_x = v_y$

$u_y = -v_x$

$v_y = 2x$

$v_x = +2y$.

~~$\partial v = \frac{\partial v}{\partial x} \partial x + \partial$~~

$v = 2xy + \phi(x)$

$v_x = 2y + \phi'(x)$.

$2y = 2y + \phi'(x)$

$\phi'(x) = 0$

$\phi(x) = c$ c is an arbitrary const.

$v = 2xy + c$.

$f(z) = (x^2 - y^2) + i(2xy + c)$.

Taylor's Series :-

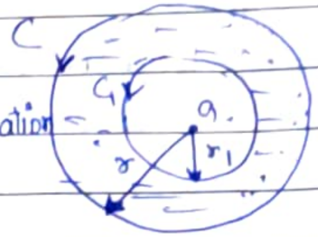
$f(z) \longrightarrow$ analytic

Domain $D = C$ (on and inside of c).

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

Laurent's Series:-

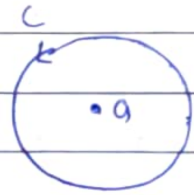
function is only analytic within $C - C_1$ and we don't have any information about within C_1.



Anular disc

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

Singularities :-



Isolated singularity

$f(z) = \frac{1}{z}$; $z = 0$.

$f(z) = \frac{1}{\tan \pi/z}$; $z = 1, 1/2, 1/3$

At just some certain points the function is not analytic otherwise it is differentiable in the neighbourhood points.

Poles of order m :-

$$f(z) = a_0 + a_1(z-a) + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots + a_{-m}(z-a)^{-m}$$

There are finite number of negative powers. (m).

m=1 $\longrightarrow$ simple pole

$$f(z) = a_0 + a_1(z-a) + \dots + a_{-1}(z-a)^{-1}$$

Residues :-

The coefficient of $(z-a)^{-1} \Rightarrow a_{-1}^{*}$

of laurent series expansion.

$$\lim_{z \to a} (z-a) f(z) = a_{-1}$$

For simple pole.
Residue at z = a. (m=1)

GENERAL FORMULA

$$\lim_{z \to a} \left[(z-a)^m \frac{d^{m-1}}{dz^{m-1}} f(z) \right]$$

Residue of z = a for m.

Cauchy Residue Theorem :-

$$\oint_c f(z) dz = 2\pi i \times \left(\text{Res at } f(a_1) + \text{Res } f(a_2) + \dots + \text{Res } f(a_n) \right)$$

Function f(z) has finite number of singularities.

$f(z) \rightarrow$ Analytic inside C except at $a_1 \dots a_n$).

Q. $$\oint_c \frac{e^z}{z(z-1)^2} dz \qquad c : |z| = 2$$

~~Res~~. z=0 and z=1 are singularities.

$$\oint_c \frac{e^z}{z(z-1)^2} dz = 2\pi i \times \left(\text{Res } f(0) + \text{Res } f(1) \right)$$

Analytic fn → CR eqn^s

complex integration → Cauchy integral formula

└→ Residue Theorem

Residue $f(0) = \lim_{z \to 0} z \dfrac{e^z}{(z-1)^2} = 1$

Residue $f(1) = \lim_{z \to 1} (z-1)^2 \left[\dfrac{d}{dz} \dfrac{e^z}{z(z-1)^2} \right]$

$= \lim_{z \to 1} (z-1)^2 \left[\dfrac{z(z-1)^2 e^z - \left[(z-1)^2 + 2(z-1)z\right] e^z}{z^2 (z-1)^4} \right.$