OVERDETERMINED PROBLEMS WITH HOMOGENEOUS WEIGHTS IN THE EUCLIDEAN PLANE

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Abstract. In this article, we study Serrin's overdetermined problems in the weighted setting on planar domains. In particular, using integral methods we show that if the homogeneous weighted overdetermined problem admits a solution in a domain that is symmetric in one direction, then the domain must be a disc. We also construct a simple example of a homogeneous weight for which the overdetermined problem cannot admit the ball as a possible domain of the equation, showing that our main result does not hold in general false if one does not impose any condition on the weight. Finally, we explain the connection between the isoperimetric inequality and Serrin's overdetermined problem.

1. Introduction

In the celebrated article [23], Serrin proved the following result. Let Ω be a smooth, connected, open, and bounded subset of $\mathbb{R}^n$ with external unit normal ν. If u solves

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for a constant $c \in \mathbb{R}$, then

$$u(x) = \frac{R^2 - |x|^2}{2n}$$

up to translation and Ω is a ball of radius R in $\mathbb{R}^n$. In other words, Serrin's theorem establishes a connection between solutions to (1.1) and the shape of the domain Ω, showing that the ball is the only domain supporting such a function as a solution of (1.1).

Serrin's proof relied on the Alexandrov moving plane method, which takes advantage of the invariance of the Laplacian under reflection. Shortly later, Weinberger in [25] came up with an alternative proof based on the analysis of the function

$$P = |\nabla u|^2 + \frac{2}{n} u.$$

By demonstrating that P is a subharmonic function, Weinberger applied the strong maximum principle along with a Pohozaev identity to show that P must be constant on Ω. This result entails the radial symmetry of the solution and leads to the conclusion that Ω is a ball. These two methods have been instrumental in generalizing Serrin's theorem to various settings and nonlinearities.

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In particular, the moving plane method applies to a more general class of nonlinearities and, in fact, Serrin proved that if u solves

$$\begin{cases} a(u,|\nabla u|)\Delta u + b(u,|\nabla u|)u_i u_j u_{ij} = f(u,|\nabla u|) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases}$$

with a, b and f continuously differentiable in each variable, then Ω is a ball provided that u is positive. Thus in particular if we consider the problem

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases}$$

when λ is the first non-zero eigenvalue of the domain, then Ω is a ball.
Note that when λ is any other eigenvalue then the moving planes method cannot be applied since the eigenfunction changes sign. Furthermore, extending this statement for all eigenvalues with reverse boundary conditions (i.e., $\frac{\partial u}{\partial \nu} = 0$ and $u = c$ on $\partial\Omega$) is equivalent to proving Schiffer's conjecture.
The P-function method was adapted in [13] to generalize Serrin's result to the case of p-Laplacian operator. See [20] for an extensive survey on the known proof techniques for Serrin's problem along with its generalizations.
In terms of motivations,solutions to the overdetermined problem arise when studying the torsion functional associated with a domain Ω. Let u be a solution of

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and denote the torsion of the domain Ω by

$$T(\Omega) := \int_\Omega u(x)\,dx.$$

Then, the well-known Saint-Venant inequality states that the torsional functional is maximized by a ball B of the same volume as Ω, i.e.

$$T(\Omega) \le T(B)$$

such that $|B| = |\Omega|$. By viewing T as a function of Ω and computing its domain derivative using the Hadamard formula, one deduces that the norm of the gradient of u must be constant at the boundary, which exactly falls into the category of Serrin's overdetermined problem.
Our motivations for studying the weighted problem are two-fold. First, we would like to explore methods that do not depend on the sign of the solution since such methods could potentially be used to develop tools to resolve problems such as Schiffer's conjecture. For instance, the proof of Theorem 1.1 below relies on integral identities that are independent of the sign of the solution. Furthermore, while studying the weighted problem we also discovered a new proof of Serrin's result which follows from a result of Talenti in [24]. See Section 4 for more details.
Second, following the development of weighted isoperimetric inequalities (see for instance [19]) leading up to the log-convex conjecture resolved in [7], we believe that similar investigations could provide interesting information about the torsional rigidity with implications to the Serrin type overdetermined problems.
For related quantitative rigidity results for overdetermined problems see e.g. [1, 3, 8–11, 15–18] and the references therein. See also [12, 14] and the literature cited therein for other delicate issues related to symmetry problems in general.

1.1. **Main Result.** The setting that we consider in this paper is as follows. Let w denote the weight and set

$$\Delta_w u := \frac{1}{w} \operatorname{div}(w \nabla u). \tag{1.2}$$

Let u be a solution to the following problem

$$\begin{cases} \Delta_w u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u_\nu = c & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain (i.e., open and connected set) in $\mathbb{R}^n$ with the boundary of class $C^{1,\gamma}$, for some $\gamma \in (0,1)$, and $u_\nu := \frac{\partial u}{\partial \nu}$ denotes the normal derivative.
We consider the case of homogeneous radial weights in the two-dimensional Euclidean plane and show under an additional assumption on the domain that the solution to (1.3) is radial, and thus the domain is a ball.

Theorem 1.1. *Let Ω be a C^2 bounded, open and connected subset of $\mathbb{R}^2$ such that Ω has reflection symmetry in at least one direction and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution to problem (1.3) with $w = |x|^\alpha$ for any $\alpha > 0$.*
Then, up to a translation, we have that

$$u(x) = \frac{R^2 - |x|^2}{2(2+\alpha)}$$

and Ω is a ball of radius R.

1.2. **Outline.** Section 2 addresses the proof of Theorem 1.1. We then construct in Section 3 a simple example of a weight for which problem (1.3) does not admit the ball as its solution, showing that Theorem 1.1 cannot hold for general weights, also showing that it is sharp in its formulation.
In Section 4 we reprove Serrin's Theorem using Talenti's Comparision Theorem 4.1 and finally, in Section 5 we conclude by discussing some open problems.

1.3. **Proof Sketch and challenges.** Recall from the introduction that the two prominent ways to prove radial symmetry for overdetermined problems are the moving plane and the P-function method. We first observe that both these approaches fail in the context of Theorem 1.1.
To prove Serrin's result for (1.1), one considers the reflected solution $v(x) := u(\Sigma(x))$ where $\Sigma(\cdot)$ denotes the reflection about some appropriate plane H_ν with outward unit normal ν. The observation then is that the difference between the solution u and the reflected function v satisfies

$$\begin{cases} \Delta(u-v) = 0 & \text{in } \Sigma\left(\Omega_\nu\right), \\ u - v = 0 & \text{on } \partial\Sigma\left(\Omega_\nu\right) \cap \partial H_\nu, \\ u - v \geq 0, & \text{on } \partial\Sigma\left(\Omega_\nu\right) \setminus \partial H_\nu. \end{cases}$$

where $\Omega_\nu := \Omega \cap H_\nu$ and $\Sigma(\Omega_\nu)$ is the reflection of the cap Ω_ν about ∂H_ν which is obtained by moving the plane H_ν in the direction ν until $\Sigma(\Omega_\nu) \subset \Omega$. Then one can apply the strong maximum principle to deduce that either $u > v$ in $\Sigma(\Omega_\nu)$ or $u \equiv v$ in $\Sigma(\Omega_\nu)$. The latter case implies that Ω is symmetric about the plane H_ν so we want to exclude the former case and for this one needs a refined Hopf Lemma. See more details in the proof of Theorem 1.1 in [20].

In the weighted case, if u solves (1.3) and one defines $v(x) := u(x-y)$ for some translation $y \in \mathbb{R}^n$ and $f := u - v$, then

$$\begin{aligned} \Delta_w f(x) &= -1 - \Delta_w v(x) \\ &= -1 - (\Delta u)(x-y) - \nabla \log w(x) \cdot (\nabla u)(x-y) \\ &= (\nabla \log w \cdot \nabla u)(x-y) - \nabla \log w(x) \cdot (\nabla) u(x-y) \\ &= \nabla v(x-y) \cdot (\nabla \log w(x-y) - \nabla \log w(x)) \,. \end{aligned}$$

From the above expression, it is not clear how to apply the maximum principle in this case since the right-hand side of the above equation does not have a clear sign in general.

The proof using the P-function method for (1.1) proceeds instead by observing that the function

$$P = |\nabla u|^2 + \frac{2}{n} u$$

satisfies

$$\Delta \left(|\nabla u|^2 + \frac{2}{n} u \right) = 2 \sum_{i,j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 - \frac{2}{n} \geq 0.$$

Then by the strong maximum principle either $P \equiv c^2$ or $P < c^2$ in Ω. Using standard Pohozaev identities one can rule out the latter case and thus P must be constant which in turn implies that u is radially symmetric and thus Ω is a ball.

The motivation to study this function stems from the fact that the radial solution to (1.1) is of the form

$$u(x) = \frac{R^2 - |x|^2}{2n}$$

where $R > 0$ denotes the radius of the ball centered at the origin such that $\Omega = B_R(0)$. Thus the P function, in this case, is indeed

$$P = \frac{R^2}{n^2}$$

which is constant on Ω.

By a similar analogy when the weight is given by $w(x) = |x|^\alpha$ then the radial solutions to problem (1.3) take the form

$$u(x) = \frac{R^2 - |x|^2}{2(n+\alpha)}$$

which implies that the modified P function

$$P = |\nabla u|^2 + \frac{2}{n+\alpha} u = \frac{R^2}{(n+\alpha)^2}$$

is constant on Ω.

Thus one might expect that studying the P function might lead to useful information. However, from the computations in Section 2 it will be clear that $\Delta_w P$ does not have a clear sign and therefore it is not obvious how to make use of the P function method. This makes the weighted overdetermined problem challenging since we need to adapt the existing tools in a novel way.

We approach this problem by using integral identities since the identities satisfied by the homogeneous weight are analogous to the ones obtained for the usual Laplacian. We first isolate the problematic term that has no clear sign and then observe that by performing integration by parts several times we can reduce this term to a quantity that vanishes provided the domain is symmetric in at least one direction.

Regarding the counterexample, our idea is to start with a radial solution u to (1.3) corresponding to an annular domain. In particular, we consider a function u of the form

$$\begin{aligned} u(r) &= \frac{1}{2}(a-r)(r-b) \\ &= \frac{1}{2}\left(-r^2+(a+b)r-ab\right), \end{aligned}$$

where geometrically we think of the parameters $0 < a < b$ corresponding to the inner and outer radius of an annular domain. Then assuming that the weight w is radial one can write down the ODE satisfied by u, but this time we solve for the weight w instead of u. As a result, we find an example of a singular weight for which (1.3) does not have the ball as its solution.

2. Homogeneous weights and proof of Theorem 1.1

In this section, we consider weights of the type $w(x) = |x|^\alpha$ for some $\alpha > 0$ and prove a weighted analog of Serrin's theorem in the two-dimensional setting under the assumption that the domain has reflection symmetry in at least one direction, namely we establish Theorem 1.1.

As discussed earlier, the moving plane and the P-function method do not seem to work in this case, since the weighted operator lacks reflection symmetry and the P-function is not sub-harmonic. Thus, we make use of integral identities. Note that a similar strategy is carried out in [22] where the authors proved Theorem 1.1 under the assumption that the homogeneous weight satisfies

$$D^2 \log w(\nabla u, \nabla u) + \frac{(\nabla \log w \cdot \nabla u)^2}{\alpha} \leq 0 \quad \text{in } \Omega. \tag{2.1}$$

As we are now going to remark, assuming this condition in our framework would be problematic, since, on the one hand, in general, it imposes a rather "abstract" condition on the solution for which one does not have an immediate intuition, and, on the other hand, in the specific case of the weigths treated here, the condition boils down to another one which entails by itself the required symmetry of the problem (in this sense, a condition of this type in our setting would give the desired resul without even using all the structure of the partial differential equation and the boundary conditions, which would be clearly an overkilling).

Indeed, when $w(x) = |x|^\alpha$ the condition (2.1) reduces to

$$\frac{\alpha|\nabla u|^2}{|x|^2} - \frac{\alpha(x\cdot\nabla u)^2}{|x|^4} \leq 0, \tag{2.2}$$

since

$$\begin{aligned} D^2 \log w(\nabla u, \nabla u) &= \frac{D^2 w(\nabla u, \nabla u)}{w} - \frac{(\nabla u \cdot \nabla w)^2}{w^2} \\ &= \frac{\alpha|\nabla u|^2}{|x|^2} - \frac{2\alpha(x\cdot\nabla u)^2}{|x|^4} \\ \text{and} \qquad \frac{(\nabla \log w \cdot \nabla u)^2}{\alpha} &= \frac{\alpha}{|x|^4}(x\cdot\nabla u)^2. \end{aligned} \tag{2.3}$$

However, by the Cauchy-Schwarz inequality we have that

$$\frac{\alpha|\nabla u|^2}{|x|^2} - \frac{\alpha(x\cdot\nabla u)^2}{|x|^4} \geq 0.$$

Hence, assuming (2.2) in this case would lead to

$$\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4} = 0,$$

which directly implies that u is a radial. Hence, condition (2.1) in the presence of homogeneous weights of the type $w(x) = |x|^\alpha$ trivializes the problem, since it gives the expected results without even using all the assumptions of the problem. Furthermore, as mentioned, condition (2.1) has the disadvantage that it involves the solution u itself, making this assumption somewhat not natural, therefore it would be desirable to weaken or drop it: in fact, here, we will recover Serrin's result in the homogeneous weighted setting of interest, without imposing further conditions such as (2.1).

Remark 2.1. We point out that if u is a solution of problem (1.3), then one can check that $u \geq 0$ in Ω, by testing the equation against the negative part of u.

We now prove an L^1 estimate using the weighted Bochner identity.

Proposition 2.2. *Let Ω be a C^2 bounded, open and connected subset of $\mathbb{R}^2$ such that Ω has reflection symmetry in at least one direction and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution to problem (1.3) with $w = |x|^\alpha$ for some $\alpha > 0$.*
Then,

$$\int_\Omega u\, d\mu \leq \frac{2+\alpha}{2+\alpha+2} c^2 |\Omega|_w, \tag{2.4}$$

where

$$d\mu := w(x)\, dx = |x|^\alpha\, dx$$
$$\text{and} \qquad |\Omega|_w := \int_\Omega w(x)\, dx.$$

To prove this result, we will make use of the following lemma, which we state and prove in any dimension for the sake of completeness.

Lemma 2.3. *Let Ω be a C^2 bounded, open and connected subset of $\mathbb{R}^n$ and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of (1.3) with $w = |x|^\alpha$ for some $\alpha > 0$.*
Then,

$$\begin{aligned}\int_\Omega u \left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x \cdot \nabla u)^2}{|x|^4} \right) d\mu& \\ = (n-2) \int_\Omega \frac{u^2}{|x|^4} d\mu + \sum_{i \neq j} \int_\Omega \frac{u^2}{2|x|^4} x_i x_j \partial_{ij} u\, d\mu.&\end{aligned} \tag{2.5}$$

Proof. We point out that, integrating by parts and using the equation in (1.3),

$$\begin{aligned}\int_\Omega u \frac{|\nabla u|^2}{|x|^2}\, d\mu &= \int_\Omega \nabla \left(\frac{u^2}{2} \right) \cdot \frac{w \nabla u}{|x|^2}\, dx \\ &= -\int_\Omega \frac{u^2}{2|x|^2} \operatorname{div}(w \nabla u)\, dx - \int_\Omega \frac{u^2}{2} w \nabla u \cdot \nabla \left(\frac{1}{|x|^2} \right) dx \\ &= \int_\Omega \left(\frac{u^2}{2|x|^2} + \frac{u^2}{|x|^4} (x \cdot \nabla u) \right) d\mu.\end{aligned} \tag{2.6}$$

Moreover, for all i, $j \in \{1, \ldots, n\}$,

$$\partial_i \left(\frac{x_i x_j}{|x|^4} \right) = \frac{x_j + \delta_{ij} x_i}{|x|^4} - \frac{4 x_i^2 x_j}{|x|^6}.$$

Therefore, integrating by parts once again, we see that

$$\begin{aligned}
&\int_\Omega u\frac{(x\cdot\nabla u)^2}{|x|^4}\,d\mu\\
&=\sum_{i,j=1}^n\int_\Omega(u\partial_i u)(w\partial_j u)\frac{x_ix_j}{|x|^4}\,dx\\
&=\sum_{i,j=1}^n\int_\Omega\partial_i\left(\frac{u^2}{2}\right)w\partial_j u\frac{x_ix_j}{|x|^4}\,dx\\
&=-\sum_{i,j=1}^n\int_\Omega\frac{u^2}{2}\partial_i(w\partial_j u)\frac{x_ix_j}{|x|^4}\,dx-\sum_{i,j=1}^n\int_\Omega\frac{u^2}{2}w\partial_j u\,\partial_i\left(\frac{x_ix_j}{|x|^4}\right)dx\\
&=-\sum_{i,j=1}^n\int_\Omega\frac{u^2}{2}\left(\partial_i w\partial_j u+w\partial_{ij}u\right)\frac{x_ix_j}{|x|^4}\,dx\\
&\qquad-\sum_{i,j=1}^n\int_\Omega\frac{u^2}{2}w\partial_j u\left(\frac{x_j+\delta_{ij}x_i}{|x|^4}-\frac{4x_i^2x_j}{|x|^6}\right)dx\\
&=-\sum_{i,j=1}^n\int_\Omega\frac{u^2}{2}\left(\partial_i w\partial_j u+w\partial_{ij}u\right)\frac{x_ix_j}{|x|^4}\,dx\\
&\qquad-(n-3)\sum_{j=1}^n\int_\Omega\frac{u^2}{2}w\partial_j u\frac{x_j}{|x|^4}\,dx.
\end{aligned}\tag{2.7}$$

Now we note that

$$x\cdot\nabla w=\alpha w\tag{2.8}$$

by the homogeneity of the weight. From this observation and (2.7) we infer that

$$\begin{aligned}
&\int_\Omega u\frac{(x\cdot\nabla u)^2}{|x|^4}\,d\mu\\
&\quad=-\int_\Omega\left[\frac{u^2}{2|x|^4}\Big(\alpha(x\cdot\nabla u)+\sum_{i,j=1}^n x_ix_j\partial_{ij}u\Big)+(n-3)\frac{u^2}{|x|^4}(x\cdot\nabla u)\right]d\mu.
\end{aligned}\tag{2.9}$$

Moreover,

$$\Delta_w u=\frac{\nabla w\cdot\nabla u}{w}+\Delta u=\nabla\log w\cdot\nabla u+\Delta u,\tag{2.10}$$

and therefore

$$-1=\frac{\nabla|x|^\alpha\cdot\nabla u}{|x|^\alpha}+\Delta u=\frac{\alpha x\cdot\nabla u}{|x|^2}+\Delta u.$$

Thus, recalling (2.6) and (2.9),

$$\begin{aligned}
&\int_\Omega u\left(\frac{|\nabla u|^2}{|x|^2}-\frac{(x\cdot\nabla u)^2}{|x|^4}\right)d\mu\\
&\quad=\int_\Omega\left[\frac{u^2}{2|x|^2}+\frac{u^2}{|x|^4}(x\cdot\nabla u)\right]d\mu\\
&\quad\quad+\int_\Omega\left[\frac{u^2}{2|x|^4}\Big(\alpha(x\cdot\nabla u)+\sum_{i,j=1}^n x_ix_j\partial_{ij}u\Big)+(n-3)\frac{u^2}{|x|^4}(x\cdot\nabla u)\right]d\mu\\
&\quad=\int_\Omega\frac{u^2}{2|x|^2}\,d\mu+(n-2)\int_\Omega\frac{u^2}{|x|^4}(x\cdot\nabla u)\,d\mu\\
&\quad\quad+\int_\Omega\frac{u^2}{2|x|^4}\Big(\alpha(x\cdot\nabla u)+\sum_{i,j=1}^n x_ix_j\partial_{ij}u\Big)\,d\mu\\
&\quad=\int_\Omega\frac{u^2}{2|x|^2}\,d\mu+(n-2)\int_\Omega\frac{u^2}{|x|^4}(x\cdot\nabla u)\,d\mu\\
&\quad\quad+\int_\Omega\frac{u^2}{2|x|^4}\Big(-|x|^2-|x|^2\Delta u+\sum_{i,j=1}^n x_ix_j\partial_{ij}u\Big)\,d\mu\\
&\quad=(n-2)\int_\Omega\frac{u^2}{|x|^4}(x\cdot\nabla u)\,d\mu+\sum_{i\neq j}\int_\Omega\frac{u^2}{2|x|^4}x_ix_j\partial_{ij}u\,d\mu\\
&\quad=(n-2)\int_\Omega(x|x|^{\alpha-4})\cdot\nabla\left(\frac{u^3}{3}\right)dx+\sum_{i\neq j}\int_\Omega\frac{u^2}{2|x|^4}x_ix_j\partial_{ij}u\,d\mu
\end{aligned}$$

which establishes the desired result. □

We this preliminary work, we can prove Proposition 2.2 when $n=2$ for any $\alpha>0$ since this condition causes the first term in the identity (2.5) to vanish.

Proof of Proposition 2.2. For the sake of completeness, we will perform our computations in any dimension n and then specify when the conditions $\alpha>0$ and $n=2$ are needed.

Integrating by parts, we have that

$$\int_\Omega u\,d\mu=\int_\Omega wu\,dx=-\int_\Omega u\ \mathrm{div}(w\nabla u)\,dx=\int_\Omega w|\nabla u|^2\,dx=\int_\Omega|\nabla u|^2\,d\mu.$$

Next, we recall the notation in (1.2) and we use the weighted Bochner formula to obtain that[1]

$$\begin{aligned}
\Delta_w|\nabla u|^2&=\frac{1}{w}\,\mathrm{div}(w\nabla|\nabla u|^2)\\
&=2\left(\left|D^2u\right|^2+\nabla\Delta_w u\cdot\nabla u-D^2\log w(\nabla u,\nabla u)\right)\\
&=2\left|D^2u\right|^2-2D^2\log w(\nabla u,\nabla u).
\end{aligned}$$

[1]As usual, if M is a square matrix of order n and V, W are vectors in $\mathbb{R}^n$, we use the notation

$$M(V,W):=\sum_{i,j=1}^n M_{ij}V_iW_j.$$

Thus, using the notation $d\sigma_w := w\, d\sigma$ for the weighted boundary measure, we have that

$$
\begin{aligned}
\int_\Omega u\, d\mu &= \int_\Omega |\nabla u|^2(-\Delta_w u)\, d\mu \\
&= -\int_\Omega \operatorname{div}(w\nabla u)|\nabla u|^2\, dx \\
&= -\int_\Omega \operatorname{div}(w|\nabla u|^2\nabla u)\, dx + \int_\Omega w\nabla u\cdot\nabla|\nabla u|^2\, dx \\
&= -c^2\int_{\partial\Omega} u_\nu d\sigma_w + \int_\Omega w\nabla u\cdot\nabla|\nabla u|^2\, dx \\
&= -c^2\int_\Omega \operatorname{div}(w\nabla u)\, dx - \int_\Omega u\Delta_w(|\nabla u|^2)\, d\mu \\
&= c^2\int_\Omega w\, dx - 2\int_\Omega u\left(\left|D^2u\right|^2 - D^2\log w(\nabla u,\nabla u)\right)\, d\mu \\
&= c^2|\Omega|_w - 2\int_\Omega u\left(\left|D^2u\right|^2 - D^2\log w(\nabla u,\nabla u)\right)\, d\mu.
\end{aligned}
\tag{2.11}
$$

We point out that the computations performed so far have not used the particular form of the weight w. From now on, we will instead use the fact that $w(x)=|x|^\alpha$. Our goal is to estimate the quantity $\left|D^2u\right|^2 - D^2\log w(\nabla u,\nabla u)$ from below. For this, using (2.3), we arrive at

$$
D^2\log w(\nabla u,\nabla u) + \frac{1}{\alpha}\left(\nabla\log w\cdot\nabla u\right)^2 = \alpha\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right). \tag{2.12}
$$

Note that, in polar coordinates in $\mathbb{R}^2$,

$$
\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4} = \frac{1}{r^2}\left(u_r^2 + \frac{u_\theta^2}{r^2}\right) - \frac{u_r^2}{r^2} = \frac{u_\theta^2}{r^4},
$$

where u_r and u_θ denote the radial and angular derivatives respectively. Accordingly, the quantity $\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}$ can be heuristically seen as a term measuring the obstruction for u to be radial.

Using (2.12) and the Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
&-\left(\left|D^2u\right|^2 - D^2\log w(\nabla u,\nabla u)\right) \\
&\quad\le -\left(\frac{(\Delta u)^2}{n} - \alpha\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right) + \frac{1}{\alpha}(\nabla\log w\cdot\nabla u)^2\right),
\end{aligned}
\tag{2.13}
$$

which, together with (2.11), implies that

$$
\begin{aligned}
\int_\Omega u\, d\mu \le c^2|\Omega|_w &- 2\int_\Omega u\left(\frac{(\Delta u)^2}{n} + \frac{(\nabla\log w\cdot\nabla u)^2}{\alpha}\right)\, d\mu \\
&+ 2\alpha\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right)\, d\mu.
\end{aligned}
\tag{2.14}
$$

Next, using that

$$
\frac{(x+y)^2}{n+\alpha} = \frac{x^2+y^2+2\left(\sqrt{\frac{\alpha}{n}}\,x\right)\left(\sqrt{\frac{n}{\alpha}}\,y\right)}{n+\alpha} \le \frac{x^2+y^2+\frac{\alpha}{n}x^2+\frac{n}{\alpha}y^2}{n+\alpha} = \frac{x^2}{n} + \frac{y^2}{\alpha},
$$

and recalling (2.10), we deduce from (2.14) that

$$
\begin{aligned}
\int_\Omega u d\mu &\le c^2|\Omega|_w - \int_\Omega \frac{2u\left(\Delta_w u\right)^2}{n+\alpha} d\mu + 2\alpha\int_\Omega u\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right)\, d\mu \\
&= c^2|\Omega|_w - \int_\Omega \frac{2u}{n+\alpha} d\mu + 2\alpha\int_\Omega u\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right)\, d\mu.
\end{aligned}
\tag{2.15}
$$

Thus, moving one term to the left-hand side, we get that

$$\int_\Omega u\,d\mu \le \frac{n+\alpha}{n+\alpha+2}\left(c^2|\Omega|_w + 2\alpha\int_\Omega u\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right)d\mu\right). \tag{2.16}$$

We now claim that

$$\int_\Omega u\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right)d\mu = 0. \tag{2.17}$$

To establish this claim we will need to exploit the additional assumption, $\alpha > 0$ and $n = 2$. More precisely, we recall Lemma 2.3, then the above conditions imply that

$$\int_\Omega u\left(\frac{|\nabla u|^2}{|x|^2} - \frac{(x\cdot\nabla u)^2}{|x|^4}\right)d\mu = \sum_{i\neq j}\int_\Omega \frac{u^2}{2|x|^4}x_i x_j \partial_{ij}u\,d\mu.$$

We also use that Ω is symmetric, say, with respect to x_n, and therefore the above integral vanishes, which completes the proof of (2.17).
In light of (2.16) and (2.17), we obtain (2.4), as desired. □

Next, using the dilation invariance of the weighted Laplacian for homogeneous weights, we can deduce that equality holds true in (2.4).

Lemma 2.4. *Let u be a solution to* (1.3) *with $w \in C^1(\mathbb{R}^n \setminus \{0\})$. Assume that w is homogeneous of degree $\alpha > 0$. Then,*

$$\int_\Omega u\,d\mu = \frac{n+\alpha}{n+\alpha+2}c^2|\Omega|_w.$$

Proof. Recalling (2.8), the proof follows as in the usual Pohozaev identity, multiplying by $x\cdot\nabla u$ both sides of the first equation in (1.3) and integrating by parts. See for instance equations (2.4), (2.5), (2.6), and (2.7) in [22] for details. □

Proof of Theorem 1.1. Using Proposition 2.2 and Lemma 2.4 we deduce that equality must be attained in (2.13) and (2.15). Thus

$$D^2u = \frac{\Delta u}{n}\,\mathrm{Id}$$
$$\text{and}\qquad \frac{\Delta u}{n} = \frac{\nabla\log w\cdot\nabla u}{\alpha}.$$

We also recall from (2.10) that

$$\Delta_w u = \frac{\nabla w\cdot\nabla u}{w} + \Delta u = \nabla\log w\cdot\nabla u + \frac{n}{\alpha}\nabla\log w\cdot\nabla u = \frac{n+\alpha}{\alpha}\nabla\log w\cdot\nabla u,$$

and thus

$$\frac{\Delta u}{n} = \frac{\Delta_w u}{n+\alpha} = -\frac{1}{n+\alpha}.$$

Therefore

$$D^2u = \frac{\Delta u}{n}\,\mathrm{Id} = -\frac{1}{n+\alpha}\,\mathrm{Id}$$

which implies that $u(x) = \frac{R^2-|x-x_0|^2}{2(n+\alpha)}$ for some $x_0 \in \mathbb{R}$ and $R > 0$.
Finally we observe that

$$x_0 = 0. \tag{2.18}$$

Indeed

$$
\begin{aligned}
-1 &= \Delta_w u \\
&= \Delta u + \alpha \frac{x \cdot \nabla u}{|x|^2} \\
&= -\frac{n}{n+\alpha} - \frac{\alpha}{n+\alpha}\left(1 - \frac{x \cdot x_0}{|x|^2}\right) \\
&= -1 + \frac{\alpha}{n+\alpha}\frac{x \cdot x_0}{|x|^2}.
\end{aligned}
$$

Therefore, for all $x \in \Omega$, we find that $\frac{\alpha}{n+\alpha}\frac{x \cdot x_0}{|x|^2} = 0$. This establishes (2.18) and completes the proof of the desired result. □

3. Counterexamples for the weighted Serrin Problem

In this section, we construct explicit examples of radial weights for which one cannot expect the overdetermined problem to have the ball as the solution. Consider $w(x) = |x|^{1-n}$ for $n \geq 2$. In this setting, if u is a radial solution to the equation

$$\Delta_w u = -1,$$

then it must solve the following ordinary differential equation in the variable $r = |x|$:

$$-1 = u'' + \frac{n-1}{r}u' + \frac{w'u'}{w} = u''.$$

This implies that u is of the form

$$
\begin{aligned}
u(r) &= \frac{1}{2}(r-a)(b-r) \\
&= \frac{1}{2}\left(-r^2 + (a+b)r - ab\right)
\end{aligned}
$$

where $0 < a < b$ are constants that correspond to the inner and outer radius of an annular domain. Note that $w(r)$ does not blow up in the annulus $a < |x| < b$.
Observe that this is analogous to the weighted isoperimetric problem. For instance, in $\mathbb{R}^2$ with radial density r^{-1}, an annulus has a smaller perimeter than a ball, and, in this case, minimizers to the isoperimetric problem do not exist. See Proposition 4.2 in [5] for more details.

4. Connection between Isoperimetric Inequality and Overdetermined Problem

Since the domains supporting solutions to overdetermined problems are very often expected to be balls, it seems that these problems are a manifestation of some underlying isoperimetric principle. However, apriori it is not clear where this is coming from.
As discussed earlier, the two main steps required to prove Serrin's result for (1.1) were an L^1 inequality

$$\int_\Omega u dx \leq \frac{n}{n+2}c^2|\Omega|$$

and a Pohozaev identity

$$\int_\Omega u dx = \frac{n}{n+2}c^2|\Omega|. \tag{4.1}$$

The latter equality follows from the dilation invariance of the Laplace operator and therefore tells us nothing about the geometry of the solution. However, the first estimate can be seen as a consequence of the isoperimetric inequality applied to the level sets of the solution u, where u solves (1.1). To see this, first define the distribution function and the isoperimetric profile

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|$$
$$I(s) = \inf \{\mathrm{Per}(A) : A \subset \mathbb{R}^n, A \text{ of locally finite perimeter}, |A| = s\}.$$

Then by the co-area formula and Hölder's inequality we have

$$-\frac{d}{dt}\int_{|u|>t} |\nabla u| df \leq \left(-\frac{d}{dt}\int_{|u|>t} |\nabla u|^2 df\right)^{1/2} (-\mu_u'(t))^{1/2}.$$

From the isoperimetric inequality we have

$$-\frac{d}{dt}\int_{|u|>t} |\nabla u| dx = \mathrm{Per}(\{|u| > t\}) \geq I(\mu_u(t)).$$

which implies

$$I(\mu_u(t)) \leq \left(-\frac{d}{dt}\int_{|u|>t} |\nabla u|^2 df\right)^{1/2} (-\mu_u'(t))^{1/2}. \tag{4.2}$$

Furthermore, testing the equation $-\Delta u = 1$ with an appropriate cut-off function (see the proof of Theorem 4.1 in [4] for more details) it can be shown that

$$-\frac{d}{dt}\int_{|u|>t} |\nabla u|^2 dx \leq \mu_u(t), \qquad \text{for a.e. } t > 0. \tag{4.3}$$

Combining (4.2) and (4.3) we get

$$\begin{aligned} I\left(\mu_u(t)\right)^2 &\leq \left(-\mu_u'(t)\right)\mu_u(t) \\ 1 &\leq I\left(\mu_u(t)\right)^{-2}\left(-\mu_u'(t)\right)\mu_u(t) \\ t &\leq \int_0^t I\left(\mu_u(s)\right)^{-2}\left(-\mu_u'(s)\right)\mu(s)ds \\ t &\leq \int_{\mu_u(t^-)}^{|\Omega|} I(s)^{-2} s ds. \end{aligned}$$

Integrating the last inequality yields the following more general result due to Talenti in [24],

Theorem 4.1 (Talenti Comparison)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $u \in W_0^{1,2}(\Omega)$ be a solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

with $f \in L^2(\Omega)$. Let f^* be the spherically symmetric rearrangement of f and denote $v \in W_0^{1,2}(\Omega^*)$ as the solution to the symmetrized problem

$$\begin{cases} -\Delta v &= f^\star & \text{in } \Omega^\star \\ v &= 0 & \text{on } \partial\Omega^\star \end{cases}.$$

where Ω^* is the ball (centered at the origin) with the same measure as Ω. Then

$$u^* \leq v$$

where $u^\star$ is the spherically symmetric rearrangement of u. Therefore,

$$\int_\Omega |u|^p dx \leq \int_{\Omega^\star} v^p dx$$

for every positive exponent p.

Applying the above theorem with $f \equiv 1$ we deduce that

Corollary 4.2. *For any solution of u of* (1.1) *we have*

$$\int_\Omega u dx \leq \frac{n}{n+2}c^2|\Omega|.$$

Proof. By Theorem 4.1 we know that

$$u^* \leq v$$

where v solves the radial ODE which has an explicit solution

$$v(x) = \frac{R^2 - |x|^2}{2n}$$

on $\Omega^* = B_R(0)$ is a ball centered at the origin and of radius $R > 0$ such that $|\Omega| = |\Omega^*|$. Then since u is positive by the maximum principle we have

$$\int_\Omega u(x)dx \leq \int_{\Omega^*} v(x)dx = \frac{n}{n+2}c^2|\Omega|.$$

□

The proof of Serrin's result now follows from the Pohozaev identity (4.1).
Regarding the weighted setting, an analogous result holds for radial log convex weights as shown in Theorem 4.1 in [4],

Theorem 4.3. *Let $w = e^{g(|x|)}$ be a radial increasing log-convex density on $\mathbb{R}^n$ and let Ω be an open subset of $\mathbb{R}^n$ with finite weighted volume $|\Omega|_w$ with respect to the weighted Lebesgue measure $d\mu = w(x)dx$. Let $h \in L^2(\Omega, d\mu)$. Let $u \in W_0^{1,2}(\Omega, d\mu)$ be a solution to*

$$\begin{cases} -\operatorname{div}(w(x)\nabla u) = w(x)h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

and let $v \in W_0^{1,2}(\Omega^*, d\mu)$ solve

$$\begin{cases} -\Delta u - \frac{g'(|x|)}{|x|} x \cdot \nabla u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Then

$$u^*(x) \leq v(x), \quad x \in \Omega^*.$$

Thus applying the above Theorem with $h \equiv 1$ we can deduce the following,

Corollary 4.4. *For any solution of u of* (1.3) *we have*

$$\int_\Omega u d\mu \leq \frac{n}{n+2}c^2|\Omega|_w. \tag{4.6}$$

However, proving a Pohozaev identity similar to (4.1) for log-convex weights remains a challenging open problem since the weighted operator has no dilation invariance.

5. Some open problems

The results presented in this article naturally give rise to several intriguing questions and problems for further investigation. These include:

(i) Case when $n > 2$ and $\alpha > 0$. Is it possible to prove Theorem 1.1 for all $n > 2$ and $\alpha > 0$ without any symmetry assumption on the domain?

(ii) Weights on unbounded domains. For which class of weights does the weighted analog of Serrin's theorem hold? For instance, what can be said about log-concave weights such as $\exp(-|x|^2/2)$? As we saw in Section 4 one only needs to establish a relevant Pohozaev identity for these weights since the weighted version of Talenti's

comparison theorem already gives the L^1 estimate (4.6).

(iii) Stability for the weighted problem. Can we establish stability under perturbations of the boundary conditions for the weighted problem? This issue has been extensively explored in the context of non-weighted problems. For an extensive survey of results in this direction, refer to the introduction of [21].

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