QNT 402 Quantum Computation Homework 2
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Problem 1. *Optical Bloch Equations*

The optical Bloch equations describe the time evolution of a two-level quantum system driven by external fields. In this case, we have a qubit with the Hamiltonian:

$$H = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$$

where Δ is the detuning, Ω is the Rabi frequency, and σ_x and σ_z are Pauli matrices. Additionally, the qubit can spontaneously emit a photon from state $|0\rangle$ to $|1\rangle$ at a rate Γ.

The density matrix ρ for the system can be written as:

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$$

The time evolution of the density matrix is given by the Lindblad master equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

where L_k are the Lindblad operators. In this case, we have one Lindblad operator corresponding to the spontaneous emission process: $L = \sqrt{\Gamma}\sigma_{01}$.

Now, we can write down the Optical Bloch Equations by expressing the time derivatives of the density matrix elements in terms of the elements themselves:

$$\frac{d\rho_{00}}{dt} = -i\left(\frac{\Delta}{2}\rho_{01} - \frac{\Omega}{2}\rho_{10}\right) + \Gamma\rho_{11}$$

$$\frac{d\rho_{01}}{dt} = -i\left(\frac{\Delta}{2}\rho_{00} - \frac{\Omega}{2}\rho_{11}\right) - \frac{\Gamma}{2}\rho_{01}$$

$$\frac{d\rho_{10}}{dt} = -i\left(-\frac{\Delta}{2}\rho_{11} - \frac{\Omega}{2}\rho_{00}\right) - \frac{\Gamma}{2}\rho_{10}$$

$$\frac{d\rho_{11}}{dt} = i\left(\frac{\Delta}{2}\rho_{10} + \frac{\Omega}{2}\rho_{01}\right) - \Gamma\rho_{11}$$

These equations represent the time evolution of the density matrix elements for a qubit undergoing spontaneous emission.

Let's denote the density matrix elements as ρ_{00}, ρ_{01}, and ρ_{11} for the diagonal elements and ρ_{10} (which is the complex conjugate of ρ_{01} for the off-diagonal element. The Hamiltonian in the rotating frame is given by:

$$H = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$$

where Δ is the detuning and Ω is the Rabi frequency. The spontaneous emission term can be represented by a Lindblad superoperator with a decay rate :

$$L[\rho] = \frac{\Gamma}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-)$$

Here, σ_- and σ_+ are the lowering and raising operators, respectively.
The master equation for the density matrix ρ is given by:

$$\dot{\rho} = -i[H, \rho] + L[\rho]$$

Now, substituting the expressions for H and $L[\rho]$, and using the commutation relations for Pauli matrices, we can derive the Optical Bloch Equations:

$$\dot{\rho}_{00} = -\Gamma\rho_{00} + i\frac{\Omega}{2}(\rho_{01} - \rho_{10})$$

$$\dot{\rho}_{01} = -\left(\frac{\Gamma}{2} + i\frac{\Delta}{2}\right)\rho_{01} - i\frac{\Omega}{2}(\rho_{00} - \rho_{11})$$

$$\dot{\rho}_{11} = \Gamma\rho_{00} - i\frac{\Omega}{2}(\rho_{01} - \rho_{10})$$

These equations describe the time evolution of the density matrix elements for the qubit system including the effects of spontaneous emission.

Hamiltonian and Schrödinger Equation:

The Hamiltonian for the system is given by:

$$H = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$$

where Δ is the detuning, Ω is the Rabi frequency, and σ_x and σ_z are Pauli matrices.
The time evolution of a quantum system is described by the Schrödinger equation:

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

Let's assume the state $|\psi\rangle$ can be written as a column vector:

$$|\psi\rangle = \begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix}$$

Now, we can write the Schrödinger equation as a system of differential equations:

$$i\hbar\frac{d}{dt}\begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix} = \left(-\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x\right)\begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix}$$

Lindblad Master Equation:
To include the spontaneous emission process, we use the Lindblad master equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

In this case, we have one Lindblad operator corresponding to the spontaneous emission process: $L = \sqrt{\Gamma}\sigma_{01}$.

Deriving OBEs:

Now, let's calculate the commutator $[H, \rho]$ and apply the Lindblad operator:

$$[H, \rho] = H\rho - \rho H$$

Substitute this into the Lindblad master equation and simplify:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}(H\rho - \rho H) + \Gamma \left(\sigma_{01} \rho \sigma_{01}^\dagger - \frac{1}{2}\{\sigma_{01}^\dagger \sigma_{01}, \rho\} \right)$$

Now, express the density matrix ρ in terms of its elements:

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$$

Apply the commutators and Lindblad operator, then express the time derivatives of the density matrix elements in terms of the elements themselves. This leads to the set of OBEs we provided earlier:

$$\frac{d\rho_{00}}{dt} = -i\left(\frac{\Delta}{2}\rho_{01} - \frac{\Omega}{2}\rho_{10}\right) + \Gamma\rho_{11}$$

$$\frac{d\rho_{01}}{dt} = -i\left(\frac{\Delta}{2}\rho_{00} - \frac{\Omega}{2}\rho_{11}\right) - \frac{\Gamma}{2}\rho_{01}$$

$$\frac{d\rho_{10}}{dt} = -i\left(-\frac{\Delta}{2}\rho_{11} - \frac{\Omega}{2}\rho_{00}\right) - \frac{\Gamma}{2}\rho_{10}$$

$$\frac{d\rho_{11}}{dt} = i\left(\frac{\Delta}{2}\rho_{10} + \frac{\Omega}{2}\rho_{01}\right) - \Gamma\rho_{11}$$

$$\frac{d\rho_{ij}}{dt} = 0$$

Let's solve for P_e in terms of the given parameters Δ, Γ, and s.

Now, let's solve these equations. We can simplify them further by expressing the density matrix elements in terms of the excited state population (P_e):

$$\rho_{00} = 1 - P_e$$

$$\rho_{11} = P_e$$

$$\rho_{01} = 0$$

$$\rho_{10} = 0$$

Substitute these into the OBEs:

$$0 = \frac{\Delta}{2}P_e - \frac{\Omega}{2}(1 - P_e) + \Gamma P_e$$

$$0 = \frac{\Delta}{2}(1 - P_e) - \frac{\Omega}{2}P_e - \frac{\Gamma}{2} \times 0$$

$$0 = -\frac{\Delta}{2}P_e - \frac{\Omega}{2}(1 - P_e) - \frac{\Gamma}{2} \times 0$$

$$0 = \frac{\Delta}{2}(1 - P_e) + \frac{\Omega}{2}P_e - \Gamma P_e$$

Now, solve these equations to find P_e. Once we have P_e, the total photon scattering rate (R_s) is given by:

$$R_s = \Gamma P_e$$

Now, let's express P_e in terms of Δ, Γ, and s:

$$0 = \frac{\Delta}{2}P_e - \frac{\Omega}{2}(1 - P_e) + \Gamma P_e$$

Solve for P_e:

$$P_e = \frac{\Omega^2}{\Omega^2 + \frac{\Delta^2}{4} + \frac{\Gamma^2}{4}}$$

Now, substitute P_e into the expression for R_s:

$$R_s = \frac{\Gamma\Omega^2}{\Omega^2 + \frac{\Delta^2}{4} + \frac{\Gamma^2}{4}}$$

Finally, express s in terms of Ω and Γ:

$$s = \frac{2\Omega^2}{\Gamma^2}$$

Substitute $\Omega^2 = \frac{s\Gamma^2}{2}$ into the expression for R_s:

$$R_s = \frac{\Gamma\frac{s\Gamma^2}{2}}{\frac{s\Gamma^2}{2} + \frac{\Delta^2}{4} + \frac{\Gamma^2}{4}}$$

Simplify further:

$$R_s = \frac{s\Gamma^3}{2s + \Delta^2 + \Gamma^2}$$

So, the total photon scattering rate R_s as a function of Δ, Γ, and s is:

$$R_s = \frac{s\Gamma^3}{2s + \Delta^2 + \Gamma^2}$$