Tackling Chaotic Noise in Continuous-Variable Distributed Quantum Network: supplemental document

This is a supplemental document with a detailed explanation of the math and construction of the quantum photonic network.

1. SQUEEZED STATES IN QUANTUM MECHANICS

The Heisenberg uncertainty principle sets a lower bound for the product of uncertainties in conjugate variables, such as position and momentum. However, it does not restrict individual uncertainties from being arbitrarily small. This concept leads to the idea of squeezed states, where one uncertainty is reduced at the expense of increasing the other [1].

Consider the ground state wave function of a harmonic oscillator:

$$\psi_0(x) \propto \exp\left(-\frac{x^2}{2}\right) \tag{S1}$$

We can introduce a squeezing parameter λ to obtain a squeezed wave function:

$$\psi_\lambda(x) \propto \frac{1}{(\lambda)^{1/4}} \exp\left(-\frac{x^2}{2\lambda}\right) \tag{S2}$$

For this squeezed state, the position and momentum uncertainties are given by:

$$\Delta x = \sqrt{\frac{\lambda}{2}}, \quad \Delta p = \frac{1}{\sqrt{2\lambda}} \tag{S3}$$

Note that the product $\Delta x \Delta p = \frac{1}{2}$ remains constant, independent of λ.

The squeezing process can be described by a unitary operator:

$$S(\xi) = \exp\left(\frac{\xi}{2}(a^2 - (a^\dagger)^2)\right) \tag{S4}$$

where a and $a^\dagger$ are the annihilation and creation operators, respectively.

The squeezing operator transforms the position and momentum operators as follows:

$$S^\dagger(\xi) x S(\xi) = x e^{-\xi}, \quad S^\dagger(\xi) p S(\xi) = p e^{\xi} \tag{S5}$$

These transformations lead to the following relations for expectation values:

$$\langle\psi|S^\dagger x S|\psi\rangle = \langle\psi|x|\psi\rangle e^{-\xi} \tag{S6}$$

$$\langle\psi|S^\dagger p S|\psi\rangle = \langle\psi|p|\psi\rangle e^{\xi} \tag{S7}$$

The time evolution of a squeezed state can be expressed as:

$$e^{-iHt/\hbar} S(\xi)|0\rangle = \exp\left(\frac{1}{2}(\xi e^{2i\omega t} a^2 - \xi^* e^{-2i\omega t}(a^\dagger)^2)\right)|0\rangle \tag{S8}$$

where H is the Hamiltonian and ω is the angular frequency of the harmonic oscillator.

A. Using Optical Parametric Oscillator for Squeezing

Using a non-linear medium that can perform a paramatric down conversion is placed inside of a optical cavity, this is to ideal way of squeezing as the interaction of the light with the non-linear and cavity is weak [2]. We consider a cavity build with one or two end mirrors with one of the end mirrors partially transmitting.

We derive the Heisenberg-Langenvin equation for the field operators a and $a^\dagger$:

$$a = -\Omega_p a^\dagger - \frac{\vartheta}{2}a + F(t) \quad \text{(S9)}$$

$$a^\dagger = -\Omega_p a - \frac{\vartheta}{2}a^\dagger + F^\dagger(t) \quad \text{(S10)}$$

Where ϑ is the cavity decay while $F(t)$ is the noise associated with the operator, it also assumed that the pump phase is = $\pi/2$. Under ideal conditions, the parametric conversions are the expectation value of the noise operator $\langle(F(t)\rangle = 0$.

For a steady state operation for squeezing, we find the expectation values of $\langle a\rangle$, $\langle a^2\rangle$ and $\langle a^\dagger a\rangle$. From Eqs. S9 and S10 with the assumption $\langle(F(t)\rangle = 0$

$$\frac{d}{dt}\langle a\rangle = -|\Omega_p\langle a^\dagger\rangle - \frac{\vartheta}{2}\langle a\rangle \quad \text{(S11)}$$

$$\frac{d}{dt}\langle a^\dagger\rangle = -|\Omega_p\langle a\rangle - \frac{\vartheta}{2}\langle a^\dagger\rangle \quad \text{(S12)}$$

The solution is written as:

$$\langle a\rangle_t = [\langle a_0\rangle cosh(\Omega_p t) - \langle a^\dagger\rangle_0 sinh(\Omega_p t)]e^{-\vartheta t/2} \quad \text{(S13)}$$

$$\langle a^\dagger\rangle_t = [\langle a_0^\dagger\rangle cosh(\Omega_p t) - \langle a\rangle_0 sinh(\Omega_p t)]e^{-\vartheta t/2} \quad \text{(S14)}$$

Now to describe the noise operators F and $F^\dagger$, we will use the Eqs. S9 and S10 in matrix representation [3]

$$\forall = -M\forall + \forall\tau \quad \text{(S15)}$$

$$\forall = \begin{bmatrix} a \\ a^\dagger \end{bmatrix} \quad \text{(S16)}$$

$$M = \begin{bmatrix} \vartheta/2 & \Omega_p \\ \Omega_p & \vartheta/2 \end{bmatrix} \quad \text{(S17)}$$

$$\tau = \begin{bmatrix} F \\ F^\dagger \end{bmatrix} \quad \text{(S18)}$$

This can also be written as

$$\forall(t) = e^{-Mt}\forall(0) + \int_o^t e^{-M(t-t')}\tau(t')dt' \quad \text{(S19)}$$

By multiplying Eq. S19 with $\tau^\dagger(t)$ while assuming that the field operators during initialization is $t = 0$ will be independent of the fluctuations $\langle a(0)F(t)\rangle = 0$

$$\tau^\dagger(t)\forall(t) = \begin{pmatrix} \langle F^\dagger a\rangle & \langle Fa\rangle \\ \langle F^\dagger a^\dagger\rangle & \langle Fa^\dagger\rangle \end{pmatrix}$$

$$\tau^\dagger(t)\forall(t) = \vartheta/2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(S20)}$$

Similarly

$$langle\forall^\dagger(t)\tau(t)\rangle = \begin{pmatrix} \langle a^\dagger F\rangle & \langle a^\dagger F^\dagger\rangle \\ \langle aF\rangle & \langle aF^\dagger\rangle \end{pmatrix}$$

$$\langle\forall^\dagger(t)\tau(t)\rangle = \vartheta/2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(S21)}$$

It can implied from the above equations that the correlation relations including the noise-operators will become zero apart from $\langle Fa^\dagger\rangle = \langle aF^\dagger\rangle = \vartheta/2$

The squeezing can also be described using Hermitian operators

$$X_1 = 1/2(ae^{-}i\phi/2 + a^{\dagger}e^{i\phi/2} \tag{S22}$$

$$X_2 = 1/2i(ae^{-}i\phi/2 - a^{\dagger}e^{i\phi/2} \tag{S23}$$

We compute the variances of the given operators in the case of steady state

$$(\Delta X_1)^2_{ss} = \frac{1}{8}\frac{\vartheta}{\vartheta/2 + \Omega_p} \tag{S24}$$

we assume at $\phi = 0$, the optimal squeezing can be achieved on the oscillation ($\Omega_p = \vartheta/2$) which makes the Eq S24

$$(\Delta X_1)^2_{ss} = \frac{1}{8} \tag{S25}$$

2. HOMODYNE DETECTION

An optical homodyne detector is a highly sensitive instrument designed to measure the quadrature of an optical field. This device employs optical interferometry between two fields of identical frequency, allowing it to detect minute fluctuations in an optical field, even those below the level of optical vacuum fluctuations. The typical implementation of this detector involves a 50:50 beam-splitter and two photodiodes that measure optical intensity. This setup is often referred to as 'balanced homodyne detection' [4].

Let's examine the transformation of operators through this optical circuit using the Heisenberg picture

The fields are transformed through the beam-splitter as follows:

$$a' = \frac{1}{\sqrt{2}}(a + ib) \tag{S26}$$

$$b' = \frac{1}{\sqrt{2}}(b + ia) \tag{S27}$$

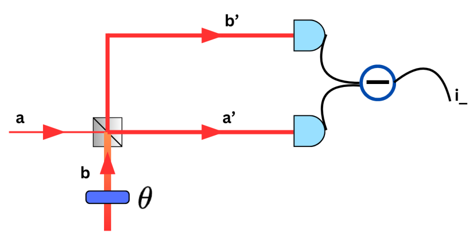

Fig. S1. Diagram for a balanced optical homodyne detection: a is the mode that we will measure which will be mixed through a beam-splitter along with b is the mode that is with a strong coherent state α_{LO}

The number operator in the beam-splitter outputs models what the photodiodes observe:

$$a'^{\dagger}a' = \frac{1}{2}(a^{\dagger}a + ia^{\dagger}b - iab^{\dagger} + b^{\dagger}b) \tag{S28}$$

$$b'^{\dagger}b' = \frac{1}{2}(a^{\dagger}a - ia^{\dagger}b + iab^{\dagger} + b^{\dagger}b) \tag{S29}$$

In a balanced detector, the photocurrents from the two photodiodes are subtracted, yielding the 'difference current':

$$i_{-} \propto iab^{\dagger} - ia^{\dagger}b \tag{S30}$$

This formulation cancels out contributions proportional to $a^\dagger a$ and $b^\dagger b$, potentially suppressing unwanted intensity noise on mode b. Considering mode b as a strong coherent state (local oscillator) compared to mode a, we apply the transformation Figure S1:

$$b \to \alpha_{LO} = |\alpha_{LO}|e^{i\pi/2}e^{i\theta} \tag{S31}$$

This transforms the difference current to:

$$i_- \propto |\alpha_{LO}|(ae^{-i\theta} + a^\dagger e^{i\theta}) \tag{S32}$$

The 'rotated' quadrature operator is defined as:

$$X_\theta = \frac{1}{\sqrt{2}}(ae^{-i\theta} + a^\dagger e^{i\theta}) = X\cos\theta + P\sin\theta \tag{S33}$$

Thus, the difference current can be expressed as:

$$i_- \propto \sqrt{2}|\alpha_{LO}|X_\theta \tag{S34}$$

There have been techiques thorugh which we can measure the X and P Quadratures simultaneously, proposed in Leonhardt Chapter 6[4], Figure S2.

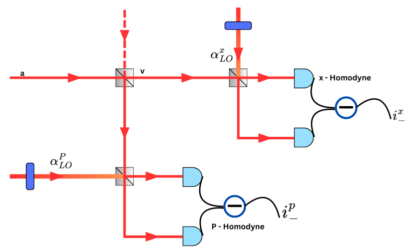

Fig. S2. A possible diagram for a dual homodyne detection: Using two separate local oscillator and choosing each phase such that one can measure the position and other momentum quadratures

3. QUANTUM NOISE AND VQE ENHANCED QUANTUM MACHINE LEARNING

Quantum channels, denoted as $\mathcal{N}$, are trace-preserving completely positive maps that can be described using Kraus operators. For an input state ρ, the action of a quantum channel can be expressed as:

$$\mathcal{N}(\rho) = \sum_j K_j \rho K_j^\dagger \tag{S35}$$

where K_j are Kraus operators satisfying $\sum_j K_j^\dagger K_j = I$, with I being the identity matrix.

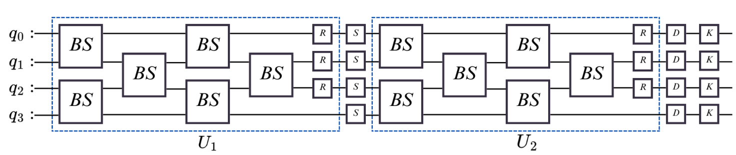

Fig. S3. The linear multimode interferometers U_1 and U_2 have been expanded. This decomposition results in a combination of two-mode beamsplitters (BSgate) without phase shifts and single-mode phase shifters (Rgate). A Kerr gate is used as a non-Gaussian non-linear activation gate

In photonic four-dimensional state experiments Figure S3, combining polarization and path degrees of freedom, common noise types like depolarization and dephasing can be modeled as Pauli channel noise [5]. A Pauli channel is a random unitary channel represented as:

$$\mathcal{N}(\rho) = \sum_j p_j K_j \rho K_j^\dagger \tag{S36}$$

Here, K_j are Pauli matrices, and p_j are probabilities satisfying $\sum_j p_j = 1$.

For systems with multiple degrees of freedom (a and b, representing polarization and path), the noise can be modeled as:

$$\mathcal{N}_{a,b}(\rho_{a,b}) = \sum_{j,k} p_{j,k} (K_j^a \otimes K_k^b) \rho_{a,b} (K_j^a \otimes K_k^b)^\dagger \tag{S37}$$

where K_j^a and K_k^b are Pauli operators acting on degrees of freedom a and b, respectively.

Measurement errors in quantum systems can be represented by stochastic matrices:

$$\mathbf{q} = \Lambda \mathbf{p} \tag{S38}$$

where $\mathbf{p}$ and $\mathbf{q}$ are ideal and noisy probability distributions, respectively, and Λ is a left stochastic matrix satisfying $\sum_j \Lambda_{j,k} = 1$.

When measuring Pauli observables on a state affected by Pauli noise [6], we can decompose the density matrix in the eigenbasis of the observable P:

$$\rho = \sum_j \alpha_{j,j} |\phi_j\rangle\langle\phi_j| + \sum_{j,k(j\neq k)} \alpha_{j,k} |\phi_j\rangle\langle\phi_k| \tag{S39}$$

The off-diagonal terms do not contribute to measurement outcomes, and the Pauli channel's effect is to randomly apply Pauli operators to the input state, preserving this property.

A. Influence of Noise Level on Half-wave plate

The noise level, denoted as ε, is related to the Hong-Ou-Mandel (HOM) visibility V by the equation $\varepsilon = 1 - V$. In an experimental setup, the visibility V is adjusted by rotating the polarization of one photon in a pair using a HWP. This section explores how $\varepsilon(\theta)$ depends on the angle θ of the HWP axis. Assuming that the photon source emits pairs with horizontal polarization, the HWP modifies the pair's state as follows, the entire process is summarized in the Figure S4:

$$a_H^1 a_H^2 |\mathrm{vac}\rangle \to (a_H^1 \cos(2\theta) + a_V^1 \sin(2\theta)) a_H^2 |\mathrm{vac}\rangle \tag{S40}$$

where a_P^i is the photon creation operator for mode i with polarization p, and $|\mathrm{vac}\rangle$ is the vacuum state. To determine HOM visibility, photons are directed to input ports of a symmetric directional coupler (DC), with input creation operators a expressed through output operators b:

$$a_P^1 = \frac{b_P^a + b_P^b}{\sqrt{2}}, \quad a_P^2 = \frac{b_P^a - b_P^b}{\sqrt{2}} \tag{S41}$$

Substituting these into the transformed state yields:

$$|\psi_{\mathrm{out}}(\theta)\rangle = \frac{(b_H^1)^2 - (b_H^2)^2}{2} \cos(2\theta) + (b_V^1 b_H^1 - b_V^2 b_H^2 + b_V^2 b_H^1 - b_V^1 b_H^2) \sin(2\theta) |\mathrm{vac}\rangle \tag{S42}$$

The coincidence probability between output modes 1 and 2, measured by detectors insensitive to polarization, is given by:

$$p(\theta) = \sum_{P_1, P_2 = H,V} |\langle \mathrm{vac}| (b_{P_1}^1)^\dagger (b_{P_2}^2)^\dagger |\psi_{\mathrm{out}}(\theta)\rangle|^2 \tag{S43}$$

Using canonical commutation relations, this simplifies to:

$$p(\theta) = \frac{\sin^2(2\theta)}{2} \tag{S44}$$

When $\theta = 0$, photons are indistinguishable, maximizing the HOM dip with zero probability ($p(0) = 0$) and visibility $V = 1$. For distinguishable photons, the coincidence probability is $1/2$, yielding $V = 0$. Thus, HOM visibility is:

$$V(\theta) = \frac{1/2 - p(\theta)}{1/2 + p(\theta)} \tag{S45}$$

Setting noise level as $\varepsilon = 1 - V$, we derive its dependence on angle:

$$\varepsilon(\theta) = \frac{2\sin^2(2\theta)}{1 + \sin^2(2\theta)} \tag{S46}$$

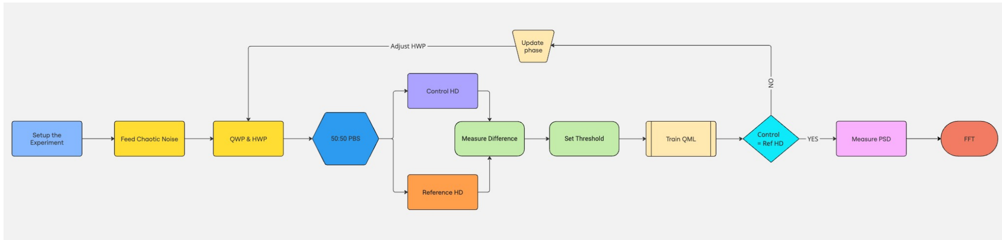

Fig. S4. We set up the experiment with the production of the squeezed photons using EOM and OPO. The squeezed light is then passed through a Beam Splitter Network creating 4 entangled probes. Chaotic noise is fed into each arm and making each probe chaotic and dephased.This noisy signal passes through a combination of $\lambda/4$ and $\lambda/2$ waveplates. The $\lambda/2$ is primarily used to control the de-phasing of the signal by updating itself. The 50:50 PBS unevenly divides the noisy signal, the Homodyne Detection is used to measure the phase and the photon number, based on the threshold difference set by Control Homodyne and Reference Homodyne. After each update the difference is measured between the two HDs and the threshold parameter is updated. We train a Quantum Machine Learning model to minimize the difference after each pass using regression. Once the difference between the two HDs is optimal, the final Power Spectral Density measurement is performed followed by a Fast Fourier Transformation in the post-processing

B. Dependence of Hamiltonian Expectation Value on Noise Level

This section demonstrates that the Hamiltonian expectation value, denoted as $\langle H \rangle$, depends approximately linearly on noise level ε, expressed as $\langle H \rangle \approx c_1 + c_2\varepsilon$ for small noise levels ($\varepsilon \ll 1$).

First, we calculate how any outcome probability $p(\theta)$ varies with HWP angle. By inverting the relation for noise level from Eq. (S11), we show that $p(\varepsilon) \equiv p(\theta(\varepsilon))$ is nearly linear in terms of noise. Since expectation value $\langle H \rangle$, given by Eq. (S4), is a linear combination of outcome probabilities $p(\varepsilon)$, it follows that:

$$p(\varepsilon) = |U_{11}U_{22} + U_{12}U_{21}|^2 - 2\,\mathrm{Re}(U_{11}U_{22}U_{12}^*U_{21}^*)/(2-\varepsilon) \tag{S47}$$

The expectation value then simplifies to:

$$\langle H \rangle = c_1 + 2\frac{c_2\epsilon}{2-\epsilon} \tag{S48}$$

For small noise levels ($\varepsilon \ll 1$), variations in the denominator are negligible, making expectation value approximately linear:

$$\langle H \rangle \approx c_1 + \epsilon c_2 \tag{S49}$$

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