50 marks **CS 218 Quiz 2** **11:00-12:00, 24/2/16**

There will be no clarifications. In case of doubts please make the most reasonable assumptions about what the intent of the question is, state them, and proceed.

Use the first two pages for question 1, page 3 for question 2, and page 4 for rough work. Really only one page should suffice for each.

Problem 1:[30 marks] Suppose you are travelling from Mumbai to Delhi in an electric car which has a battery that can hold one Coulomb of charge on which it can run 100 km. You start with your car fully charged. You are given numbers $D[1..n]$ and $P[1..n]$ where $D[i]$ denotes the distance of the ith charging station from Mumbai, and $P[i]$ the charging price per Coulomb at the ith charging station. If your battery currently holds x coulombs, you may buy any amount of charge between 0 and $1 - x$ coulombs.

Give an algorithm to decide where you will buy the charge, and how much you will buy so as to minimize your total expense. You can assume that $D[i]$ are given in increasing order, and that there are sufficiently many charging stations that it is possible to make the journey without running out of charge.

You must structure your answer as: (a) Greedy choice lemma, (b) clear description of the residual problem(s) resulting after the greedy choice is made, (c) Proof of optimal substructure.

Solution 1: The problem instance for this is: (i,x) : least expensive way to go from station i to the end with initial charge = x.

(a) Let j = first station after i with price no more than p[i] within the next 100 km. If such a j exists, then at the current station, i, fill so as to barely reach j if necessary. 10 marks

If no such j exists, then fill to capacity or just enough to reach end. j=i+1, x = charge at station i+1. 5 marks

(b) New instance for the recursion: (j,x), with j, x as determined above.

5 marks, -2 if no charge.

(c) Proof of correctness:

Case j exists: If opt charges more than greedy at i, then greedy will do no worse by making up the deficit at j. If opt charges less then opt will need to charge more before j, at a higher rate. Greedy can then fill up as much at j so as to match opt.

Case j does not exist: If OPT charges less, it will need to recharge at some station j' in the next 100km. At j' greedy can top up to match opt at that point and greedy will have gained.

10 marks

You cannot just say that greedy does better than opt at the first station, or in the first few stations. You have to ensure that throughout the journey greedy will do no worse. The argument has to be like the cache replacement problem: if you take a greedy step now, it is possible to take steps later on so that the cost is no more than opt and at some point your have the same amount of charge as opt and you can then on follow opt. You lose 5 marks if you did not write this – it is especially needed for the case j does not exist.

————————Another solution————————————————

(a) Let i = least expensive station of all n stations. Ensure that you have 0 charge when you enter i, unless you are forced to enter i with charge filled at the beginning. In any case leave i with charge at full capacity

10 marks

or less if you can reach the destination without using up full capacity.

3 marks

Suppose OPT arrives at i with charge filled at some previous station j. Then you can improve upon OPT by replacing that charge by filling at i. Suppose OPT leaves i with x less charge. At the next point that OPT fills it could have saved money by instead having filled at i.

5 marks

(b) This will split the problem into 2 subproblems: $1..i$ and $i..n$. Note that in both we have full charge at the beginning, so they are truly instances of our original problem.

8 marks

(c) Full solution S = solutions S1, S2 to 2 instances with full charging at i.

Consider optimal solution S*. This must have full charging at i because of (a). S1' = chargings before i, S2' = chargings after i. S1' cannot be less expensive than S1 because S1 is optimal for 1..i. Likewise S2'.

5 marks

Problem 2:[20 marks] An independent set in a graph $G = (V, E)$ is a set of vertices $V' \subseteq V$ s.t. $u, v \in V' \Rightarrow (u, v) \notin E$. Suppose G is a forest, i.e. a collection of trees.

You are to prove the following to get an algorithm to find a maximum independent set, i.e. an independent set of largest size.

(a) Let u be any leaf in G. Show that there must exist a maximum independent set that contains u.

(a) Consider an optimal solution. If u is not in it, attempt to add u. If the parent of u is there remove the parent. The result must be an independent set of the same size. So we now have a set which is also a maximum independent set.

8 marks

(b) State the residual instance after using (a) to make a greedy choice.

G after removing the parent and u.

7 marks

It does not make sense to say "remove all leaves": if G consists of isolated edges, then all vertices are leaves and you will remove all. Furthermore, you were specifically told that the greedy choice was to put u, and only u, in the independent set.

(c) Prove optimal substructure.

Let I' = optimal subset for residual instance. Then I=I'+ u must be optimal for original.

Suppose some I* is optimal for original instance. Wlog it must contain u. Consider I" = I* - u. This cannot contain the parent of u. Thus this is an IS for the residual instance.

$|I*| = |I''| + 1 \leq |I'| + 1$ by optimality of I' for residual instance.

But this is $|I|$. Thus $|I|$ is at least as large as the optimal, and is clearly an IS.

5 marks