Quicksort and Selection

Abhiram Ranade

January 18, 2016

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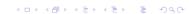x_{ij} is a 0-1 variable $\Rightarrow E[x_{ij}] = p(x_{ij} = 1)$

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- Quicksort is among the fastest sorting algorithms.
- Make sure you understand what we proved: Expected time for any instance is $O(n \log n)$. Hence also the expected time for the worst instance.

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Can be done in $O(n)$ time.   Next.

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Lemma: Using recursion tree $T(n = O(n)$ Proof soon.

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We stop recursion when $n < 50$ $\Rightarrow$ In the recurrence $n \geq 50$

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- ▶ Exercise: Will the algorithm work if we took medians of 3-tuples or of 7-tuples?
- ▶ We can improve constants, e.g. 24/25. Key emphasis in this course: do not worry about the constant, but get a simple clean argument.

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$o(g) \subset O(g), \quad \omega(g) \subset \Omega(g)$