50 marks	CS 218 Quiz 4	8:30-9:25, 8/4/16

Problem 1:[15 marks] In the Hamiltonian path problem (HP), the input is an undirected graph G, and two vertices u, v of G. The goal is to decide if G contains a path from u to v which passes through every vertex exactly once. Show that $HP \leq_K ILP$. It is sufficient if you specify what different inequalities/equalities you will assert, you do not need to put them into a matrix form.

Hint: Consider defining variables x_{wj} which must take the value 1 if w is at distance j from u along the path, and 0 otherwise, where $1 \leq j \leq n-2$. Suppose (w, z) is not an edge; then for any j, the vertices w, z should not be the jth and $j+1$th vertices in the path respectively. How do you express this constraint? What other kinds of constraints do you need to put on x_{wj}?

Consecutive vertices on the path must have an edge between them:

$\forall w, z\ s.t.\ (w,z) \notin E: \quad x_{wj} + x_{z,j+1} \leq 1.$ 5 marks

Each vertex must be at a unique distance:

$\forall w: \quad \sum_j x_{wj} = 1.$ 3 marks

At any distance there must be a unique vertex

$\forall j: \quad \sum_w x_{wj} = 1$ 3 marks

Unless there is an edge (u, w), w cannot be at distance 1

$\forall w\ s.t.\ (u,w) \notin E : x_{w1} = 0$ 2 marks

Unless there is an edge (w, v) w cannot be at distance $n-2$

$\forall w : (u,w) \notin E : x_{w,n-2} = 0$ 2 marks

$\forall w, j : x_{wj} \in \{0, 1\}$

Given an assignment to x_{wj} we can decide which vertices must be at what distance. Likewise if a path exists, we can set $x_{wj} = 1$ if w is placed at distance j – this will satisfy inequalities/equations above. And the reduction can clearly be done in polytime.

Problem 2:[13 marks] The construction version (CHP) of the Hamiltonian path problem is same as HP, except that it returns the said path (i.e. a path that passes through each vertex exactly once) if it exists, and "False" otherwise. Show that $CHP \leq_C HP$.

CHP(G,u,v){

If G consists of a single vertex v, return the empty path // base case

For each edge (u,w) in G

G' = G - remove u and incident edges.

If HP(G',w,v) return w || CHP(G',w,v)

end for

return "False"

} 7 marks

CHP calls itself on graphs with fewer vertices, and so at most $|V|$ times. Over all the calls, HP is called at most once per edge, so $|E|$ times, i.e. polynomial in input size. Additional work per recursive call is removal of a vertex, clearly polynomial. 6 marks.

You may lose upto 3 marks if you do not clearly estimate the number of times HP gets called.

Problem 3: The advertisement layout problem AL, takes as inputs (a) integers p,n, the number of pages and the number of advertisements, (b) h,w, the height and width of each page, (c) H[1..n],W[1..n], where the ith advertisement has height H[i], width W[i]. The goal is to decide if the given advertisements (rectangles of the given dimensions) can be layed out in p pages. Specifically, each advertisement must be assigned a position inside some page such that no two advertisements overlap.

(a)[7 marks] Show that AL $\in$ NP.

Verify(x[1..n], y[1..n], p[1..n] = top left corner coordinates and page no for each advertisement){

check that the advertisements to not overlap and stay on the page.

Return true if all checks succeed, and 0 otherwise.

} 4 marks

runs in polytime. 1 mark

If x has solution y will exist and Verify will return true. 1 mark

If verify returns true for some y, then y must be a solution. 1 mark

(b)[15 marks] Let ALi denote AL with p=i pages. Show AL1 is NP-hard. Partial credit for showing that AL2 is NP-hard.

Reduction from Partition.

IM($a_1, \ldots, a_n$){

for i=1..n H[i]=a_i, $W[i] = 1$

$h = \sum_i a_i/2, w = 2$

} 10 marks

Instance map runs in polytime. 1 mark

AL1 $\Rightarrow$ items stacked in two columns $\Rightarrow$ heights in column i in ith partition. 1 mark

implication is 2 way. 1 mark

overall structure: reduction from something, IM arguments: 2 marks

If you reduce to AL2 you get upto 10 marks.