On the Even Solutions of $\chi(n) = \chi(n+1)$

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Abstract

We consider even solutions of the equation $\chi(n) = \chi(n+1)$, where χ is the alternating-sum-of-divisors function. We show that each even solution satisfies at least one of three specific congruences.

1 Introduction

In number theory, a well-known and extensively examined problem type is the analysis of solutions of equations of the form:

$$f(n) = f(n+1),$$

where f is typically used to represent a specific arithmetic function. For example, the sequence A002961 is a list of solutions of $\sigma(n) = \sigma(n+1)$, where $\sigma(n)$ is the sum of positive divisors of n. Another commonly asked question is to analyze solutions of: $\varphi(n) = \varphi(n+1)$, where $\varphi(n)$ is the Euler totient function. The sequence A001274 is a list of solutions of this equation. In our paper, we analyze even solutions of the equation

$$\chi(n) = \chi(n+1), \tag{1}$$

where $\chi(n)$ is the alternating-sum-of-divisors function. The function $\chi(n)$ is also present in the OEIS as A071324. The solutions of Equation 1 (OEIS A333261) begins with

$$1, 5, 51, 68, 87, 116, 171, 176, 591, 2108, \ldots$$

In our paper, we prove that each even solution of Equation 1 satisfies at least one of three specific congruences.

1.1 Notation

We fix some notation. Let $\tau(n)$ be the number of divisors of n. Let $P^{-}(n)$ refer to the smallest prime factor of the integer n greater than one. Allow $p_2(n)$ to be the least factor of composite n, which is strictly greater than $P^-(n)$. Throughout the paper, the letter k will refer to even solutions of Equation 1. Let $\omega(n)$ be the number of distinct prime factors of n. We will now define $\chi(n)$.

Definition 1. Let the divisors of n be written as $1 = d_1 < d_2 < \ldots < d_t = n$, where $t = \tau(n)$. The *alternating sum-of-divisors* function $\chi(n)$ is defined as

$$\chi(n) = \sum_{0 \leq i < t} (-1)^i d_{t-i}.$$

1.2 Main results

The theorem presented below is the primary result of the paper.

Theorem 2. *Each k satisfies at least one of the following congruences:*

$$k \equiv 24 \pmod{30} \quad k \equiv 8 \pmod{12} \quad k \equiv 20 \pmod{30}.$$

The main application of Theorem 2 is in computer searches for k. Upon examining the terms of A333261, we find only 37 values of k up to 6818712836. This observation heuristically demonstrates the sparsity of k.Here, Theorem 2 has considerable importance, since it allows a reduction of possible values of k, therefore accelerating computer searches considerably by avoiding costly $\chi(n)$ calculations.

1.3 Proof strategy for Theorem 2

For proving Theorem 2, we first prove some preliminary bounds for $\chi(n)$, which will be subsequently used. Through these bounds, we restrict values of $P^-(k+1)$ and $p_2(k)$. The restriction on values of $P^-(k+1)$ and $p_2(k)$ allows us to obtain the result that there can be only six distinct combinations of $P^-(k+1)$ and $p_2(k)$ values. Further analysis allows us to eliminate three combinations, thus giving only three permissible pairs of $P^-(k+1)$, $p_2(k)$. Through these permissible pairs, we obtain factors of k and $k+1$ for each permissible pair case, which we then translate to corresponding congruences.

1.4 Organization of the paper

The structure of the paper is as follows. Section 2 presents some preliminary bounds regarding $\chi(n)$. Section 3 will discuss restrictions on $P^-(k+1)$ and $p_2(k)$ values. Section 4 showcases the elimination of some combinations of $P^-(k+1), p_2(k)$, and Section 5 will put forward the final permissible pairs of $P^-(k+1), p_2(k)$. Finally, Section 6 will present a proof for Theorem 2. In the end of the paper, an appendix will also be provided, containing a table with the values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$. The values presented in Table 2 are calculated based on the known terms of the sequence A333261.

2 Preliminary theorems regarding bounds for $\chi(n)$

This section will establish some fundamental bounds related to $\chi(n)$.

Theorem 3. *For n greater than one, we have*

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)}.$$

Proof. For n greater than one, we have $d_{t-1} = n/P^-(n)$. Hence, we note that

$$\chi(n) = \sum_{0 \leq i < t} (-1)^i d_{t-i} = n - \frac{n}{P^-(n)} + \sum_{2 \leq i < t} (-1)^i d_{t-i}.$$

But if we group the elements of the sum: $\sum_{2 \leq i < t} (-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, so:

$$\chi(n) \geq n - \frac{n}{P^-(n)} = n\left(1 - \frac{1}{P^-(n)}\right). \tag{2}$$

Inequality 2 allows us to conclude that the inequality:

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)}$$

holds for $n > 1$. □

Theorem 3 provides a lower bound for $\chi(n)/n$ when n is odd.

Corollary 4. *For odd n, we have $\chi(n)/n \geq 2/3$.*

Proof. We note that $\chi(1)/1 = 1$, which is greater than $2/3$. For odd n greater than one, $P^-(n)$ is at least three. Hence, by Theorem 3, the following can be concluded:

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)} \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

□

We will now demonstrate an upper bound for $\chi(n)/n$.

Theorem 5. *For even n greater than two, we have*

$$\frac{\chi(n)}{n} \leq \frac{1}{2} + \frac{1}{p_2(n)}.$$

Proof. For even n greater than two, we note that $d_{t-1} = n/2$ and $d_{t-2} = n/p_2(n)$. As a result:

$$\chi(n) = \sum_{0 \le i < t} (-1)^i d_{t-i} = n - \frac{n}{2} + \frac{n}{p_2(n)} + \sum_{3 \le i < t} (-1)^i d_{t-i}.$$

If we group the elements of the sum $\sum_{3 \le i < t} (-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, hence we conclude that

$$\chi(n) \le \frac{n}{2} + \frac{n}{p_2(n)},$$

which implies

$$\frac{\chi(n)}{n} \le \frac{1}{2} + \frac{1}{p_2(n)},$$

as desired. □

Theorem 5 provides an upper bound for $\chi(n)/n$ for even n.

Corollary 6. *For even n, we have $\chi(n)/n \le 5/6$.*

Proof. We note that $\chi(2)/2 = 1/2$, which is less than $5/6$. Since for even n greater than two, $p_2(n)$ is at least three, therefore we conclude that

$$\frac{\chi(n)}{n} \le \frac{1}{2} + \frac{1}{p_2(n)} \le \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

□

3 Restricting $P^-(k+1)$ and $p_2(k)$ values

In this section, we outline important theorems that restrict the values of $P^-(k+1)$ and $p_2(k)$, which will be crucial for establishing the required permissible pairs of $P^-(k+1)$, $p_2(k)$. We begin by noting a simple relation.

Lemma 7. *The following inequality holds for each k:*

$$\frac{5}{6} \ge \frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1} \ge \frac{2}{3}.$$

Proof. The inequality:

$$\frac{5}{6} \ge \frac{\chi(k)}{k}$$

is a direct consequence of Corollary 6, as k is even. Since $\chi(k) = \chi(k+1)$, we note that

$$\frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1}.$$

Finally, by Corollary 4, we obtain

$$\frac{\chi(k+1)}{k+1} \geq \frac{2}{3}.$$

Combining these inequalities leads us to the desired result. □

The direct implication of Lemma 7 is as follows:

Proposition 8. *For each k, the ratio $\chi(k)/k$ is greater than $2/3$ and the ratio $\chi(k+1)/(k+1)$ is less than $5/6$.*

We now move on to the first key theorem of this section, which restricts the values of $p_2(k)$.

Theorem 9. *For each k, we have*

$$p_2(k) \in \{3, 4, 5\}.$$

Proof. By Theorem 5 and Proposition 8, we have

$$\frac{1}{2} + \frac{1}{p_2(k)} \geq \frac{\chi(k)}{k} > \frac{2}{3},$$

which implies that:

$$\frac{1}{2} + \frac{1}{p_2(k)} > \frac{2}{3}.$$

As a result:

$$6 > p_2(k).$$

As k is even, hence, $6 > p_2(k) > 2$, which proves that $p_2(k)$ can be equal to three, four, or five only. □

We now discuss the second theorem of this section, which places restrictions on the possible values of $P^-(k+1)$.

Theorem 10. *For each k, we have*

$$P^-(k+1) \in \{3, 5\}.$$

Proof. By Theorem 3 and Proposition 8, we get

$$\frac{5}{6} > \frac{\chi(k+1)}{k+1} \geq 1 - \frac{1}{P^-(k+1)},$$

which implies that

$$\frac{5}{6} > 1 - \frac{1}{P^-(k+1)}. \tag{3}$$

As a result of Inequality 3, we obtain that $P^-(k+1)$ is less than six. As $k+1$ is odd, we conclude that $6 > P^-(k+1) > 2$. Hence, $P^-(k+1)$ can only be three or five. □

4 Eliminating combinations of $P^-(k+1), p_2(k)$

Theorem 9 and Theorem 10 restrict the values of $p_2(k)$ and $P^-(k+1)$ to the sets $\{3,4,5\}$ and $\{3,5\}$ respectively. Hence, there can only be six possible combinations of $P^-(k+1)$, $p_2(k)$. In this section, we eliminate some combinations of $P^-(k+1)$, $p_2(k)$. We begin by eliminating the most immediate combinations.

Proposition 11. *For each k, the following conditions hold.*

- *If $p_2(k) = 3$, then, $P^-(k+1) \neq 3$.*
- *If $p_2(k) = 5$, then, $P^-(k+1) \neq 5$.*

Proof. The proof is quite simple. If $p_2(k) = 3$, then, 3 is a factor of k, hence $k+1$ cannot have 3 as a factor. Hence, $P^-(k+1) \neq 3$. A similar argument can be applied to the case of $p_2(k) = 5$. □

We will now eliminate one additional combination.

Proposition 12. *For each k, if $p_2(k) = 4$, then, $P^-(k+1) \neq 5$.*

Proof. Combining Lemma 7, Theorem 3 and Theorem 5, we obtain that

$$\frac{1}{2} + \frac{1}{p_2(k)} \geq \frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1} \geq 1 - \frac{1}{P^-(k+1)}.$$

As a result, the inequality:

$$\frac{1}{2} + \frac{1}{p_2(k)} > 1 - \frac{1}{P^-(k+1)}, \tag{4}$$

holds for any permissible pair of $p_2(k)$, $P^-(k+1)$ values. It is easy to check that the combination of $p_2(k) = 4$ and $P^-(k+1) = 5$ does not satisfy Inequality 4. It should be noted that, apart from the combination of $p_2(k) = 4$ and $P^-(k+1) = 5$, the combination of $p_2(k) = 5$ and $P^-(k+1) = 5$ is also ruled out by Inequality 4. All other combinations satisfied Inequality 4. □

5 Permissible pairs of $P^-(k+1), p_2(k)$

In this section, we apply Theorem 9, Theorem 10, Proposition 11 and Proposition 12 to obtain all the permissible pairs of $p_2(k)$, $P^-(k+1)$. We present the following theorem as our principal result.

Theorem 13. *For each k, all the possible permissible pairs of $p_2(k)$,$P^-(k+1)$ are as follows:*

- *If $p_2(k) = 3$, then $P^-(k+1) = 5$.*

- *If $p_2(k) = 4$, then $P^{-}(k+1) = 3$.*
- *If $p_2(k) = 5$, then $P^{-}(k+1) = 3$.*

Proof. We note that through Theorem 9 and Theorem 10, one obtains the following possible combinations for $p_2(k)$,$P^{-}(k+1)$:

Index	$p_2(k)$	$P^{-}(k+1)$
1	3	3
2	4	3
3	5	3
4	3	5
5	4	5
6	5	5

Table 1: All the possible combinations of $p_2(k)$ with corresponding $P^{-}(k+1)$, according to Theorem 9 and Theorem 10.

Through Proposition 11, the combinations with index numbers 1 and 6 in Table 1 are eliminated. Similarly, by applying Proposition 12, we can eliminate the combination with the index number 5. Therefore, we are left with an exhaustive list of all the permissible pairs. □

6 Proof of Theorem 2

In this section, we present the proof of Theorem 2.

Proof. Through Theorem 13, we can obtain factors of k and $k+1$ for each permissible pair case. We convert these factors into congruence relations for k and $k+1$, and then we apply the Chinese remainder theorem to obtain the corresponding congruence for k. Since every possible k satisfies at least one of the permissible pair cases, it is guaranteed that each k will satisfy at least one congruence, since each permissible pair has a corresponding congruence. In the subsequent section, we will analyze each permissible pair case in detail.

- (Case 1): If $p_2(k) = 3$, then $P^{-}(k+1) = 5$. Since k is even, this pair implies that 6 is a factor of k, and 5 is a factor of $k+1$. Therefore, $k \equiv 0 \pmod{6}$, and $k+1 \equiv 0 \pmod{5}$. As a result, we conclude that $k \equiv 24 \pmod{30}$.
- (Case 2): If $p_2(k) = 4$, then $P^{-}(k+1) = 3$. This pair implies that 4 is a factor of k and 3 is a factor of $k+1$. Hence $k \equiv 0 \pmod{4}$ and $k+1 \equiv 0 \pmod{3}$. After solving, we obtain that $k \equiv 8 \pmod{12}$.
- (Case 3): If $p_2(k) = 5$, then $P^{-}(k+1) = 3$. This pair implies that 10 is a factor of k and 3 is a factor of $k+1$. Therefore, we obtain that, $k \equiv 0 \pmod{10}$ and $k+1 \equiv 0 \pmod{3}$. As a result, we conclude that $k \equiv 20 \pmod{30}$.

These arguments conclude the proof of Theorem 2. □

7 Acknowledgments

8 Appendix

We present Table 2 consisting of the values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$ for k up to 6818712836.

References

[1] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, Published electronically at https://oeis.org, 2025.

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(Concerned with sequences A071324 and A333261.)

Index	k	$\chi(k)$	**Congruence**	$\tau(k)$	$\sigma(k)$	$\omega(k)$
1	68	48	$k \equiv 8 \pmod{12}$	6	126	2
2	116	84	$k \equiv 8 \pmod{12}$	6	210	2
3	176	120	$k \equiv 8 \pmod{12}$	10	372	2
4	2108	1480	$k \equiv 8 \pmod{12}$	12	4032	3
5	9308	6480	$k \equiv 8 \pmod{12}$	12	17640	3
6	18548	13908	$k \equiv 8 \pmod{12}$	6	32466	2
7	37928	25920	$k \equiv 8 \pmod{12}$	16	77760	3
8	180548	135408	$k \equiv 8 \pmod{12}$	6	315966	2
9	192428	142560	$k \equiv 8 \pmod{12}$	12	341880	3
10	200996	149688	$k \equiv 8 \pmod{12}$	12	355740	3
11	3960896	2646960	$k \equiv 8 \pmod{12}$	28	7924800	3
12	8198156	6048000	$k \equiv 8 \pmod{12}$	12	14582400	3
13	9670748	7153920	$k \equiv 8 \pmod{12}$	12	17156160	3
14	11892512	7938000	$k \equiv 8 \pmod{12}$	24	23814000	3
15	16585748	12402720	$k \equiv 8 \pmod{12}$	12	29115072	3
16	25367396	18900000	$k \equiv 8 \pmod{12}$	12	44688000	3
17	25643012	18823200	$k \equiv 8 \pmod{12}$	12	45830400	3
18	29768312	20487168	$k \equiv 8 \pmod{12}$	32	64696320	4
19	61735352	42484608	$k \equiv 8 \pmod{12}$	32	134161920	4
20	68571248	46949760	$k \equiv 8 \pmod{12}$	20	133415568	3
21	101346368	67647600	$k \equiv 8 \pmod{12}$	28	202460352	3
22	102132290	68124672	$k \equiv 20 \pmod{30}$	32	204374016	5
23	114246470	76204800	$k \equiv 20 \pmod{30}$	32	228614400	5
24	166123268	124592448	$k \equiv 8 \pmod{12}$	6	290715726	2
25	228081452	162086400	$k \equiv 8 \pmod{12}$	24	425476800	4
26	250391552	166927704	$k \equiv 8 \pmod{12}$	44	502579440	3
27	514531676	375701760	$k \equiv 8 \pmod{12}$	24	935079936	4
28	804078968	555024960	$k \equiv 8 \pmod{12}$	64	1930521600	5
29	1010223896	691891200	$k \equiv 8 \pmod{12}$	32	2075673600	4
30	1153706948	857623200	$k \equiv 8 \pmod{12}$	12	2036855100	3
31	1338817292	953557920	$k \equiv 8 \pmod{12}$	24	2472187200	4
32	2005484096	1354872960	$k \equiv 8 \pmod{12}$	112	5133075840	5
33	2676172592	1839166560	$k \equiv 8 \pmod{12}$	20	5187219336	3
34	3386945432	2288563200	$k \equiv 8 \pmod{12}$	32	6865689600	4
35	3840293552	2596920480	$k \equiv 8 \pmod{12}$	20	7562547216	3
36	6616978928	4680178688	$k \equiv 8 \pmod{12}$	40	15288926208	4
37	6818712836	4800660480	$k \equiv 8 \pmod{12}$	24	13068464640	4

Table 2: Table of k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$.