On the even solutions to $\chi(k) = \chi(k+1)$

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Abstract

We consider even solutions $k > 2$ of the equation $\chi(n) = \chi(n+1)$, where χ denotes the alternating-sum-of-divisors function. We show that each k satisfies at least one of the following congruences: $k \equiv 24 \pmod{30}$, $k \equiv 8\ (\mathrm{mod}12)$, or $k \equiv 20\ (\mathrm{mod}30)$.

1 Notation

Let $\tau(n)$ denote the number of divisors of n, $P^{-}(n)$ denote the smallest prime factor of the integer $n > 1$, $p_2(n)$ denote the least factor of composite n, which is greater than $P^{-}(n)$. Let $\chi(n)$ denote the alternating-sum-of-divisors function. We note that $\chi(n)$ is the sequence A071324 in the OEIS.

Definition. Let the divisors of n be written as $1 = d_1 < d_2 < \ldots < d_t = n$, where $t = \tau(n)$. The *alternating sum-of-divisors* function $\chi(n)$ is defined as: $\chi(n) = \sum_{0 \le i < t} (-1)^i d_{t-i}$.

Throughout this paper, the letter k will be used to denote even integers greater than 2, such that $\chi(k) = \chi(k+1)$.

2 Introduction and Main Results

In this note, we will analyze even solutions to the equation: $\chi(n) = \chi(n+1)$. It is worth noting that, the sequence A333261 contains all solutions to $\chi(n) = \chi(n+1)$ as a sequence. Hence, we are essentially studying even terms of A333261.The following theorem is the main result of this paper.

Theorem 1. *Each k satisfies at least one of the following congruences: $k \equiv 24 \pmod{30}$, $k \equiv 8\ (\mathrm{mod}12)$, or $k \equiv 20\ (\mathrm{mod}30)$.*

The main application of Theorem 1 is in computational searches for k. The proof makes use of the analysis of permissible pairs of $P^{-}(k+1)$ and $p_2(k)$, then translating these pairs to congruence relations for k. The first section of this note will discuss some preliminary bounds regarding $\chi(n)$, and the subsequent sections will continue the analysis on $P^{-}(k+1)$ and $p_2(k)$. The final section will present a proof for Theorem 1.

3 Preliminary Theorems: Bounds on $\chi(n)$

This section will discuss and prove some elementary bounds concerning $\chi(n)$.

Theorem 2. *For $n > 1$, $\chi(n)/n \geq 1 - 1/P^{-}(n)$.*

Proof. For $n > 1$, $d_{t-1} = n/P^{-}(n)$. We note: $\chi(n) = \sum_{0 \leq i < t}(-1)^i d_{t-i} = n - n/P^{-}(n) + \sum_{2 \leq i < t}(-1)^i d_{t-i}$. But if we group the elements of the sum $\sum_{2 \leq i < t}(-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, so: $\chi(n) \geq n - n/P^{-}(n) = n(1 - 1/P^{-}(n))$, and thus we conclude, $\chi(n)/n \geq 1 - 1/P^{-}(n)$ for $n > 1$. □

Theorem 2 allows us to obtain a lower bound on $\chi(n)/n$ for odd n.

Corollary 1. *For odd n, $\chi(n)/n \geq 2/3$.*

Proof. We note, $\chi(1)/1 = 1 > 2/3$. For odd $n > 1$, $P^{-}(n) \geq 3$. Hence, by Theorem 2, we obtain: $\chi(n)/n \geq 1 - 1/P^{-}(n) \geq 1 - 1/3 = 2/3$. □

We will now prove an upper bound for $\chi(n)/n$:

Theorem 3. *For even $n > 2$, $\chi(n)/n \leq 1/2 + 1/p_2(n)$.*

Proof. For even $n > 2$, $d_{t-1} = n/2$ and $d_{t-2} = n/p_2(n)$. Hence: $\chi(m) = \sum_{0 \leq i < t}(-1)^i d_{t-i} = n - n/2 + n/p_2(n) + \sum_{3 \leq i < t}(-1)^i d_{t-i}$. If we group the elements of the sum $\sum_{3 \leq i < t}(-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, so: $\chi(n) \leq n/2 + n/p_2(n)$, which gives $\chi(n)/n \leq 1/2 + 1/p_2(n)$ as desired. □

Theorem 3 allows us to obtain an upper bound on $\chi(n)/n$ for even n.

Corollary 2. *For even n, $\chi(n)/n \leq 5/6$.*

Proof. We note, $\chi(2)/2 = 1/2 < 5/6$. Since, for even $n > 2$, $p_2(n) \geq 3$, we obtain: $\chi(n)/n \leq 1/2 + 1/p_2(n) \leq 1/2 + 1/3 = 5/6$, for even $n > 2$. □

This corollary concludes this section.

4 Analysis on $P^{-}(k+1)$ and $p_2(k)$ values

In this section, we present key theorems that restrict the values of $P^{-}(k+1)$ and $p_2(k)$, which will be crucial for forming the required permissible pairs of $P^{-}(k+1)$, $p_2(k)$. We begin by noting a simple relation.

Lemma 1. $5/6 \geq \chi(k)/k > \chi(k+1)/(k+1) \geq 2/3$.

Proof. $5/6 \geq \chi(k)/k$ is a direct consequence of Corollary 2, as k is even. $\chi(k)/k > \chi(k+1)/(k+1)$, since $\chi(k) = \chi(k+1)$. Finally, by Corollary 1, we obtain $\chi(k+1)/(k+1) \geq 2/3$, since $k+1$ is odd. Combining these inequalities leads us to the desired result. □

The direct consequence of Lemma 1 is:

Proposition 1. *$\chi(k)/k > 2/3$ and $5/6 > \chi(k+1)/(k+1)$.*

We now move to the first main theorem of this section, which restricts the values of $p_2(k)$.

Theorem 4. *$p_2(k)$ can be equal to 3,4, or 5 only.*

Proof. By Theorem 3 and Proposition 1, we have: $1/2 + 1/p_2(k) \geq \chi(k)/k > 2/3$, which implies that $1/2 + 1/p_2(k) > 2/3$. Solving this inequality, we obtain that $6 > p_2(k)$. As k is even, hence, $6 > p_2(k) > 2$, which proves that $p_2(k)$ can be equal to 3,4 or 5 only. □

We now move to the second theorem of this section, which restricts values of $P^-(k+1)$.

Theorem 5. *$P^-(k+1)$ can be 3 or 5 only.*

Proof. By Theorem 2 and Proposition 1, we obtain: $5/6 > \chi(k+1)/(k+1) \geq 1-1/P^-(k+1)$, which implies that $5/6 > 1 - 1/P^-(k+1)$. Solving this inequality, one obtains that $6 > P^-(k+1)$. As $k+1$ is odd, we obtain that $6 > P^-(k+1) > 2$. Hence, $P^-(k+1)$ can only be 3 or 5. □

5 Eliminating pairs of $P^-(k+1), p_2(k)$

Theorem 4 and Theorem 5 restrict $p_2(k)$ and $P^-(k+1)$ values to a few elements only. This implies that there can be only a few permissible pairs of $P^-(k+1)$, $p_2(k)$. In this section, we work to eliminate some combinations of $P^-(k+1)$, $p_2(k)$. We begin by eliminating the most immediate cases.

Proposition 2. *If $p_2(k) = 3$, then, $P^-(k+1) \neq 3$, and, if $p_2(k) = 5$, then, $P^-(k+1) \neq 5$.*

We leave the proof of this Proposition as an exercise to the reader. We now move to an interesting combination:

Proposition 3. *If $p_2(k) = 4$, then, $P^-(k+1) \neq 5$.*

Proof. By Lemma 1, Theorem 2 and Theorem 3, we obtain: $1/2 + 1/p_2(k) \geq \chi(k)/k > \chi(k+1)/(k+1) \geq 1 - 1/P^-(k+1)$. This implies that $1/2 + 1/p_2(k) > 1 - 1/P^-(k+1)$ holds for any permissible pair of $p_2(k)$, $P^-(k+1)$ values. It is easy to check that the pair $p_2(k) = 4$ and $P^-(k+1) = 5$ does not satisfy the required condition, hence cannot be a permissible pair. □

6 Final permissible pairs of $P^{-}(k+1), p_2(k)$

Theorem 4 and Theorem 5 restrict $p_2(k)$ and $P^{-}(k+1)$ values to a few elements only. Proposition 2 and Proposition 3 eliminate a few candidate pairs. Thus, applying Theorem 4, Theorem 5, Proposition 2 and Proposition 3, we obtain the final permissible pairs of $p_2(k)$, $P^{-}(k+1)$. The following theorem summarizes our results.

Theorem 6. *If $p_2(k) = 3$, then $P^{-}(k+1) = 5$. If $p_2(k) = 4$, then $P^{-}(k+1) = 3$. If $p_2(k) = 5$, then $P^{-}(k+1) = 3$.*

7 Proof of Theorem 1

Through Theorem 6, we can obtain factors of k and $k+1$ for each permissible pair case. We convert these factors into congruence relations for k, and then apply the Chinese Remainder Theorem to obtain congruence relations for k. We now begin the proof.

Proof. We know that if $p_2(k) = 3$, then $P^{-}(k+1) = 5$. Since k is even, this pair implies that, 6 is a factor of k, and 5 is a factor of $k+1$. Hence, $k \equiv 0 \pmod 6$, and $k+1 \equiv 0 \pmod 5$. Thus, after solving, we obtain that $k \equiv 24 \pmod{30}$. Similarly, we now take the case that if $p_2(k) = 4$, then $P^{-}(k+1) = 3$. This pair implies that 4 is a factor of k and 3 is a factor of $k+1$. Hence $k \equiv 0 \pmod 4$ and $k+1 \equiv 0 \pmod 3$. After solving, we obtain that $k \equiv 8 \pmod{12}$. Finally, the case that if $p_2(k) = 5$, then $P^{-}(k+1) = 3$; implies that 10 is a factor of k and 3 is a factor of $k+1$. Hence, we obtain that, $k \equiv 0 \pmod{10}$ and $k+1 \equiv 0 \pmod 3$. After solving, we obtain that, $k \equiv 20 \pmod{30}$. This concludes our proof. $\square$