On the even solutions to $\chi(k) = \chi(k+1)$

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Abstract

We consider even solutions $k > 2$ of the equation $\chi(n) = \chi(n+1)$, where χ denotes the alternating-sum-of-divisors function. We show that each k satisfies at least one of the following congruences: $k \equiv 24 \pmod{30}$, $k \equiv 8 \ (\mathrm{mod} 12)$, or $k \equiv 20 \ (\mathrm{mod} 30)$.

1 Notation

Let $\tau(n)$ denote the number of divisors of n. Let $P^{-}(n)$ denote the smallest prime factor of the integer $n > 1$. Let $p_2(n)$ denote the least factor of composite n, which is strictly greater than $P^{-}(n)$. Let $\chi(n)$ denote the alternating-sum-of-divisors function. We note that $\chi(n)$ is the sequence A071324.

Definition. Let the divisors of n be written as $1 = d_1 < d_2 < \ldots < d_t = n$, where $t = \tau(n)$. The *alternating sum-of-divisors* function $\chi(n)$ is defined as:

$$\chi(n) = \sum_{0 \le i < t} (-1)^i d_{t-i}.$$

Throughout this paper, the letter k will denote even integers greater than 2, with the property that $\chi(k) = \chi(k+1)$.

2 Introduction and Main Results

In this paper, we will analyze even solutions to the equation:

$$\chi(n) = \chi(n+1).$$

Similar equations for other arithmetic functions like $\sigma(n)$ (the sum of divisors function), or $\varphi(n)$ (the Euler totient function) have been well studied and analyzed in the literature. The following theorem is the main result of this paper.

Theorem 1. *Each k satisfies at least one of the following congruences:*

$$k \equiv 24 \pmod{30} \quad k \equiv 8 \ (\mathrm{mod} 12) \quad k \equiv 20 \ (\mathrm{mod} 30).$$

The main application of Theorem 1 is in computational searches for k. Note that the sequence A333261 contains solutions to $\chi(n) = \chi(n+1)$ as a sequence. Hence, we are essentially studying even terms of A333261. Looking into the terms of A333261, one finds that only 37 k values exist up to 6818712836. This fact heuristically showcases the sparsity of k. Here, Theorem 1 has considerable importance, since it allows elimination of a significant portion of possible values of k, thus speeding up computational searches considerably by avoiding costly $\chi(n)$ computations.

For proving Theorem 1, we first prove some preliminary bounds on $\chi(n)$, which will be subsequently used. Through these bounds, we restrict values of $P^-(k+1)$ and $p_2(k)$. This restriction on values of $P^-(k+1)$ and $p_2(k)$ allows us to obtain the result that there can be only 6 distinct combinations of $P^-(k+1)$ and $p_2(k)$ values. Further analysis allows us to eliminate 3 combinations, thus giving only 3 permissible pairs of $P^-(k+1)$ and $p_2(k)$ values. Through these permissible pairs, we obtain factors of k and $k+1$ for each permissible pair case, which we then translate to corresponding congruences, which will finally prove Theorem 1.

The paper is organized as follows. The first section of this note will discuss some preliminary bounds regarding $\chi(n)$, and the subsequent sections will continue the analysis on $P^-(k+1)$ and $p_2(k)$. The final main section will present a proof for Theorem 1. A simple result from a minor computational search for k using Theorem 1 is also included in the paper. An appendix will also be provided, which will contain a table consisting the values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$; where $\sigma(n)$ is the sum-of-divisors function, and $\omega(n)$ is the number of distinct prime factors of n. These values will be derived from the terms provided in the sequence A333261.

3 Preliminary Theorems: Bounds on $\chi(n)$

This section will discuss and prove some elementary bounds concerning $\chi(n)$.

Theorem 2. *For $n > 1$,*

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)}.$$

Proof. For $n > 1$, $d_{t-1} = n/P^-(n)$. We note:

$$\chi(n) = \sum_{0 \leq i < t} (-1)^i d_{t-i} = n - \frac{n}{P^-(n)} + \sum_{2 \leq i < t} (-1)^i d_{t-i}.$$

But if we group the elements of the sum: $\sum_{2 \leq i < t} (-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, so:

$$\chi(n) \geq n - \frac{n}{P^-(n)} = n\left(1 - \frac{1}{P^-(n)}\right),$$

and thus we conclude:

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^{-}(n)},$$

for $n > 1$. $\square$

Theorem 2 allows us to obtain a lower bound on $\chi(n)/n$ for odd n.

Corollary 1. *For odd n, $\chi(n)/n \geq 2/3$.*

Proof. We note, $\chi(1)/1 = 1 > 2/3$. For odd $n > 1$, $P^{-}(n) \geq 3$. Hence, by Theorem 2, we obtain:

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^{-}(n)} \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

$\square$

We will now prove an upper bound for $\chi(n)/n$:

Theorem 3. *For even $n > 2$,*

$$\frac{\chi(n)}{n} \leq \frac{1}{2} + \frac{1}{p_2(n)}.$$

Proof. For even $n > 2$, $d_{t-1} = n/2$ and $d_{t-2} = n/p_2(n)$. Hence:

$$\chi(n) = \sum_{0 \leq i < t} (-1)^i d_{t-i} = n - \frac{n}{2} + \frac{n}{p_2(n)} + \sum_{3 \leq i < t} (-1)^i d_{t-i}.$$

If we group the elements of the sum $\sum_{3 \leq i < t} (-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, so: $\chi(n) \leq n/2 + n/p_2(n)$, which gives $\chi(n)/n \leq 1/2 + 1/p_2(n)$ as desired. $\square$

Theorem 3 allows us to obtain an upper bound on $\chi(n)/n$ for even n.

Corollary 2. *For even n, $\chi(n)/n \leq 5/6$.*

Proof. We note, $\chi(2)/2 = 1/2 < 5/6$. Since, for even $n > 2$, $p_2(n) \geq 3$, we obtain:

$$\frac{\chi(n)}{n} \leq \frac{1}{2} + \frac{1}{p_2(n)} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

$\square$

4 Restricting $P^{-}(k+1)$ and $p_2(k)$ values

In this section, we present key theorems that restrict the values of $P^{-}(k+1)$ and $p_2(k)$, which will be crucial for forming the required permissible pairs of $P^{-}(k+1)$, $p_2(k)$. We begin by noting a simple relation.

Lemma 1.

$$\frac{5}{6} \geq \frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1} \geq \frac{2}{3}.$$

Proof.

$$\frac{5}{6} \geq \frac{\chi(k)}{k}$$

is a direct consequence of Corollary 2, as k is even. Since $\chi(k) = \chi(k+1)$, we note that

$$\frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1}.$$

Finally, by Corollary 1, we obtain

$$\frac{\chi(k+1)}{k+1} \geq \frac{2}{3}.$$

Combining these inequalities leads us to the desired result. □

The direct consequence of Lemma 1 is:

Proposition 1. *$\chi(k)/k > 2/3$ and $5/6 > \chi(k+1)/(k+1)$.*

We now move to the first main theorem of this section, which restricts the values of $p_2(k)$.

Theorem 4.

$$p_2(k) \in \{3, 4, 5\}.$$

Proof. By Theorem 3 and Proposition 1, we have:

$$\frac{1}{2} + \frac{1}{p_2(k)} \geq \frac{\chi(k)}{k} > \frac{2}{3},$$

which implies that:

$$\frac{1}{2} + \frac{1}{p_2(k)} > \frac{2}{3}.$$

This implies:

$$6 > p_2(k).$$

As k is even, hence, $6 > p_2(k) > 2$, which proves that $p_2(k)$ can be equal to 3,4 or 5 only. □

We now move to the second theorem of this section, which restricts values of $P^{-}(k+1)$.

Theorem 5.

$$P^-(k+1) \in \{3,5\}.$$

Proof. By Theorem 2 and Proposition 1, we obtain:

$$\frac{5}{6} > \frac{\chi(k+1)}{k+1} \geq 1 - \frac{1}{P^-(k+1)},$$

which implies that

$$\frac{5}{6} > 1 - \frac{1}{P^-(k+1)}.$$

This inequality implies that, $6 > P^-(k+1)$. As $k+1$ is odd, we conclude that $6 > P^-(k+1) > 2$. Hence, $P^-(k+1)$ can only be 3 or 5. $\square$

5 Eliminating combinations of $P^-(k+1), p_2(k)$

Theorem 4 and Theorem 5 restrict $p_2(k)$ and $P^-(k+1)$ values to the sets $\{3,4,5\}$ and $\{3,5\}$ respectively. Thus, there can only be 6 combinations of $P^-(k+1)$, $p_2(k)$. In this section, eliminate some combinations of $P^-(k+1)$, $p_2(k)$. We begin by eliminating the most immediate combinations.

Proposition 2.

- If $p_2(k) = 3$, then, $P^-(k+1) \neq 3$.
- If $p_2(k) = 5$, then, $P^-(k+1) \neq 5$.

Proof. The proof is quite simple. If $p_2(k) = 3$, then, 3 is a factor of k, hence $k+1$ cannot have 3 as a factor. Hence, $P^-(k+1) \neq 3$. A similar argument can be applied to the case of $p_2(k) = 5$. $\square$

We eliminate another combination:

Proposition 3. *If $p_2(k) = 4$, then, $P^-(k+1) \neq 5$.*

Proof. Combining Lemma 1, Theorem 2 and Theorem 3, we obtain:

$$\frac{1}{2} + \frac{1}{p_2(k)} \geq \frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1} \geq 1 - \frac{1}{P^-(k+1)}.$$

This implies that the inequality:

$$\frac{1}{2} + \frac{1}{p_2(k)} > 1 - \frac{1}{P^-(k+1)},$$

holds for any permissible pair of $p_2(k)$, $P^-(k+1)$ values. It is easy to check that the combination $p_2(k) = 4$ and $P^-(k+1) = 5$ does not satisfy the required condition. It should be noted that, apart from the combination of $p_2(k) = 4$ and $P^-(k+1) = 5$, the combination $p_2(k) = 5$ and $P^-(k+1) = 5$ is also ruled out by this condition. All other combinations satisfied this inequality. $\square$

6 Permissible pairs of $P^{-}(k+1), p_2(k)$

In this section, we apply Theorem 4, Theorem 5, Proposition 2 and Proposition 3 to obtain all the permissible pairs of $p_2(k)$, $P^{-}(k+1)$. The following theorem is our main result.

Theorem 6.

- If $p_2(k) = 3$, then $P^{-}(k+1) = 5$.
- If $p_2(k) = 4$, then $P^{-}(k+1) = 3$.
- If $p_2(k) = 5$, then $P^{-}(k+1) = 3$.

Proof. We note that through Theorem 4 and Theorem 5, one obtains the following possible combinations for $p_2(k)$,$P^{-}(k+1)$:

Index	$p_2(k)$	$P^{-}(k+1)$
1	3	3
2	4	3
3	5	3
4	3	5
5	4	5
6	5	5

Table 1: All possible combinations of $p_2(k)$ with corresponding $P^{-}(k+1)$, according to Theorem 4 and Theorem 5.

Through Proposition 2, the combinations with index numbers 1 and 6 in Table 1 are eliminated. Similarly, by applying Proposition 3, we can eliminate the combination with the index number 5. Hence, we are left with an exhaustive list of all the permissible pairs. This concludes our proof. □

7 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof. Through Theorem 6, we can obtain factors of k and $k+1$ for each permissible pair case. We convert these factors into congruence relations for k and $k+1$, and then we apply the Chinese Remainder Theorem to obtain congruences for k. Since every possible k satisfies at least one of the permissible pair cases, it is guaranteed that k will satisfy at least one congruence, since each permissible pair has a corresponding congruence. In the following part, we go through each permissible pair case.

- If $p_2(k) = 3$, then $P^-(k+1) = 5$. Since k is even, this pair implies that 6 is a factor of k, and 5 is a factor of $k+1$. Hence, $k \equiv 0 \pmod 6$, and $k+1 \equiv 0 \pmod 5$. Thus, after solving, we obtain that $k \equiv 24 \pmod{30}$.
- If $p_2(k) = 4$, then $P^-(k+1) = 3$. This pair implies that 4 is a factor of k and 3 is a factor of $k+1$. Hence $k \equiv 0 \pmod 4$ and $k+1 \equiv 0 \pmod 3$. After solving, we obtain that $k \equiv 8 \pmod{12}$.
- If $p_2(k) = 5$, then $P^-(k+1) = 3$; implies that 10 is a factor of k and 3 is a factor of $k+1$. Hence, we obtain that, $k \equiv 0 \pmod{10}$ and $k+1 \equiv 0 \pmod 3$. After solving, we obtain that $k \equiv 20 \pmod{30}$.

These arguments conclude our proof of Theorem 1. □

8 One k $11613999978 \geq k \geq 10^{10}$

A small search was done on a 1.00 GHz Intel Core i5-1035G1 CPU. The search took 16 CPU hours. We implemented the results from Theorem 1 in our algorithm to significantly cut down on possible candidate k values. The following is our result.

Theorem 7. *For $10^{10} \leq k \leq 11613999978$:*

$$k = 10519952096, \ \chi(10519952096) = 7050607200, \ \textit{and } 10519952096 \equiv 8 \pmod{12}.$$

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10 Appendix

We present a table consisting of the values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$ for a few k.

Index	k	$\chi(k)$	**Congruence**	$\tau(k)$	$\sigma(k)$	$\omega(k)$
1	68	48	$k \equiv 8 \pmod{12}$	6	126	2
2	116	84	$k \equiv 8 \pmod{12}$	6	210	2
3	176	120	$k \equiv 8 \pmod{12}$	10	372	2
4	2108	1480	$k \equiv 8 \pmod{12}$	12	4032	3
5	9308	6480	$k \equiv 8 \pmod{12}$	12	17640	3
6	18548	13908	$k \equiv 8 \pmod{12}$	6	32466	2
7	37928	25920	$k \equiv 8 \pmod{12}$	16	77760	3
8	180548	135408	$k \equiv 8 \pmod{12}$	6	315966	2
9	192428	142560	$k \equiv 8 \pmod{12}$	12	341880	3
10	200996	149688	$k \equiv 8 \pmod{12}$	12	355740	3
11	3960896	2646960	$k \equiv 8 \pmod{12}$	28	7924800	3
12	8198156	6048000	$k \equiv 8 \pmod{12}$	12	14582400	3
13	9670748	7153920	$k \equiv 8 \pmod{12}$	12	17156160	3
14	11892512	7938000	$k \equiv 8 \pmod{12}$	24	23814000	3
15	16585748	12402720	$k \equiv 8 \pmod{12}$	12	29115072	3
16	25367396	18900000	$k \equiv 8 \pmod{12}$	12	44688000	3
17	25643012	18823200	$k \equiv 8 \pmod{12}$	12	45830400	3
18	29768312	20487168	$k \equiv 8 \pmod{12}$	32	64696320	4
19	61735352	42484608	$k \equiv 8 \pmod{12}$	32	134161920	4
20	68571248	46949760	$k \equiv 8 \pmod{12}$	20	133415568	3
21	101346368	67647600	$k \equiv 8 \pmod{12}$	28	202460352	3
22	102132290	68124672	$k \equiv 20 \pmod{30}$	32	204374016	5
23	114246470	76204800	$k \equiv 20 \pmod{30}$	32	228614400	5
24	166123268	124592448	$k \equiv 8 \pmod{12}$	6	290715726	2
25	228081452	162086400	$k \equiv 8 \pmod{12}$	24	425476800	4
26	250391552	166927704	$k \equiv 8 \pmod{12}$	44	502579440	3
27	514531676	375701760	$k \equiv 8 \pmod{12}$	24	935079936	4
28	804078968	555024960	$k \equiv 8 \pmod{12}$	64	1930521600	5
29	1010223896	691891200	$k \equiv 8 \pmod{12}$	32	2075673600	4
30	1153706948	857623200	$k \equiv 8 \pmod{12}$	12	2036855100	3
31	1338817292	953557920	$k \equiv 8 \pmod{12}$	24	2472187200	4
32	2005484096	1354872960	$k \equiv 8 \pmod{12}$	112	5133075840	5
33	2676172592	1839166560	$k \equiv 8 \pmod{12}$	20	5187219336	3
34	3386945432	2288563200	$k \equiv 8 \pmod{12}$	32	6865689600	4
35	3840293552	2596920480	$k \equiv 8 \pmod{12}$	20	7562547216	3
36	6616978928	4680178688	$k \equiv 8 \pmod{12}$	40	15288926208	4
37	6818712836	4800660480	$k \equiv 8 \pmod{12}$	24	13068464640	4

Table 2: Values of values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$.

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(Concerned with sequences A071324 and A333261.)
