ELEMENTARY BOUNDS ON DIGITAL SUMS OF POWERS, FACTORIALS, AND LCMS

DAVID G RADCLIFFE

ABSTRACT. We prove that the sum of the base-b digits of a^n grows at least logarithmically in n if $\log(d)/\log(b)$ is irrational, where d is the smallest factor of a such that $\gcd(a/d, b) = 1$. Our approach uses only elementary number theory and applies to a wide class of sequences, including factorials and $\Lambda(n) = \operatorname{lcm}(1, 2, \ldots, n)$.

1. INTRODUCTION

This article was inspired by the following problem, which was posed and solved by Wacław Sierpiński[5, Problem 209]:

> *Prove that the sum of digits of the number 2^n (in decimal system) increases to infinity with n.*

The reader is urged to try to solve this problem on their own before proceeding. Note that it is not enough to prove that the sum of digits of 2^n is unbounded, since the sequence is not monotonic.

Consider the sequence of powers of 2 (sequence A000079 in the OEIS):

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

This sequence grows very rapidly. Now, let us define another sequence, by adding the decimal digits of each power of 2. For example, 16 becomes $1 + 6 = 7$, and 32 becomes $3 + 2 = 5$. The first few terms of this new sequence (A001370) are listed below.

$$1, 2, 4, 8, 7, 5, 10, 11, 13, 8, 7, \ldots.$$

Our new sequence grows much more slowly, and it is not monotonic. Nevertheless, it is reasonable to conjecture that it increases to infinity. Indeed, one might guess that the sum of the decimal digits of 2^n is asymptotic to $4.5 n \log_{10} 2$, since 2^n has

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$\lfloor n \log_{10} 2 \rfloor + 1$ decimal digits, and the digits seem to be approximately uniformly distributed among $0, 1, 2, \ldots, 9$. However, this stronger conjecture remains to be proved.

For an integer $b \geq 2$, we write $s_b(n)$ for the sum of the base-b digits of n, and $c_b(n)$ for the number of nonzero digits in that expansion. These functions are asymptotic to each other, since $c_b(n) \leq s_b(n) \leq (b-1)c_b(n)$ for all n and b; so we will restrict our attention to $c_b(n)$. For a prime p, $\nu_p(n)$ denotes the exponent of p in the prime factorization of n.

2. Digital sums of powers of two

We will present an informal proof that $\lim_{n \to \infty} c_{10}(2^n) = \infty$. The key idea, which might seem too obvious to state, is that a positive multiple of a number cannot be smaller than that number.

Let n be a positive integer, and write the decimal expansion of 2^n as $2^n = \sum_{i=0}^{\infty} d_i 10^i$, where $d_i \in \{0, \ldots, 9\}$ and $d_i = 0$ for all but finitely many terms. Note that the final digit d_0 of 2^n cannot be zero, since 2^n is not divisible by 10.

If $2^n > 10$, then 2^n is divisible by 16; so $2^n \bmod 10^4$, the number formed by the last four digits of 2^n, is also divisible by 16. If the first three of these digits were zero, then $2^n \bmod 10^4$ would be less than 10, which is impossible. So at least one of the digits d_1, d_2, d_3 is nonzero.

If $2^n > 10^4$, then 2^n is divisible by 2^{14}; so $2^n \bmod 10^{14}$, the number formed by the last 14 digits of 2^n, is also divisible by 2^{14}. If the first 10 of these digits were zero, then $2^n \bmod 10^{14}$ would be less than 10^4, which is impossible. So at least one of the digits $d_4, d_5, \ldots, d_{13}$ is nonzero.

We can continue in this way, finding longer and longer non-overlapping blocks of digits, each containing at least one nonzero digit, as illustrated in Figure 1. This shows that 2^n increases to infinity with n.

$$2^{103} = \boxed{101412048018258352} \; \boxed{1197362564} \; \boxed{300} \; \boxed{8}$$

FIGURE 1. Each block contains at least one nonzero digit.

Let us formalize this argument.

Theorem 1. *Let $(\varepsilon(k))_{k \geq 1}$ be a sequence of integers such that $\varepsilon(1) \geq 1$ and $2^{\varepsilon(k)} > 10^{\varepsilon(k-1)}$ for all $k \geq 2$. If n is a positive integer that is divisible by $2^{\varepsilon(k)}$ but not divisible by 10, then $c_{10}(n) \geq k$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial, so let us assume that $k \geq 2$. By the division algorithm, there exist integers $q \geq 0$ and $0 \leq r < 10^{\varepsilon(k-1)}$ such that $n = 10^{\varepsilon(k-1)}q + r$.

Since $n \geq 2^{\varepsilon(k)} > 10^{\varepsilon(k-1)}$, it follows that $q \geq 1$.

Since n and $10^{\varepsilon(k-1)}$ are divisible by $2^{\varepsilon(k-1)}$, r is also divisible by $2^{\varepsilon(k-1)}$. But r is not divisible by 10, so $c_{10}(r) \geq k - 1$, by the induction hypothesis.

Note that $c_{10}(n) = c_{10}(q) + c_{10}(r)$, since the digit expansion of n is the concatenation of the digit expansions of q and r, possibly with leading zeros.

Therefore, $c_{10}(n) \geq 1 + (k - 1) = k$. $\square$

Corollary 1. *Let a be a positive integer that is divisible by 2 but not divisible by 10. Then $c_{10}(a^n) \geq \log_4(n)$ for all $n \geq 1$. In particular, $\lim_{n \to \infty} c_{10}(a^n) = \infty$.*

Proof. Let $\varepsilon(k) = 4^{k-1}$ for $k \geq 1$. This sequence satisfies $\varepsilon(1) \geq 1$ and $2^{\varepsilon(k)} > 10^{\varepsilon(k-1)}$ for all $k \geq 2$.

Let $n \geq 4$ be a positive integer, and let $k = \lceil \log_4 n \rceil$, so that $4^{k-1} < n \leq 4^k$. Then a^n is divisible by 2^n, so a^n is also divisible by $2^{\varepsilon(k)}$. But a^n is not divisible by 10.

Therefore, $c_{10}(a^n) \geq k \geq \log_4(n)$, by Theorem 1. $\square$

3. Generalizing to other bases

Our proofs rely only on divisibility properties and therefore extend naturally to other bases.

Theorem 2. *Let $b \geq 2$ be an integer that is not a power of a prime, and let p be a prime divisor of b. Let $(\varepsilon(k))_{k \geq 1}$ be a sequence of integers such that $\varepsilon(1) \geq 1$ and $p^{\varepsilon(k)} > b^{\varepsilon(k-1)}$ for all $k \geq 2$. If $\nu_p(n) \geq \varepsilon(k)$ and $b \nmid n$ then $c_b(n) \geq k$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial, so let us assume that $k \geq 2$. By the division algorithm, there exist integers $q \geq 0$ and $0 \leq r < b^{\varepsilon(k-1)}$ such that $n = b^{\varepsilon(k-1)}q + r$.

Since $n \geq p^{\varepsilon(k)} > b^{\varepsilon(k-1)}$, it follows that $q \geq 1$.

Since n and $b^{\varepsilon(k-1)}$ are divisible by $p^{\varepsilon(k-1)}$, r is also divisible by $p^{\varepsilon(k-1)}$. But r is not divisible by b, so $c_b(r) \geq k - 1$, by the induction hypothesis.

Note that $c_b(n) = c_b(q) + c_b(r)$, since the digit expansion of n is the concatenation of the digit expansions of q and r, possibly with leading zeros.

Therefore, $c_b(n) \geq 1 + (k-1) = k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Notation. Given an integer $b \geq 2$ with distinct prime divisors p and q, define a function $\xi = \xi_{b,p,q}$ as follows:

$$\xi(n) = \nu_p(n) - \nu_q(n)\frac{\nu_p(b)}{\nu_q(b)}.$$

It is easy to verify that $\xi(b^r m) = \xi(m)$ for all $r \geq 0$ and $m \geq 1$. Intuitively, ξ is a modified ν_p that ignores trailing zeros in the base-b representation of its argument.

Theorem 3. *Let $b \geq 2$ be an integer that is not a power of a prime, let p and q be distinct prime divisors of b, and let $(\varepsilon(k))_{k\geq 1}$ be defined as in Theorem 2. If $\xi(n) \geq \varepsilon(k)$, then $c_b(n) \geq k$. In particular, $\lim_{n\to\infty} c_b(a_n) = \infty$ for any sequence of positive integers that satisfies $\lim_{n\to\infty} \xi(a_n) = \infty$.*

Proof. Write n as $b^r m$, where m is not divisible by b. Note that $\xi(n) = \xi(m)$. Since $\nu_p(m) \geq \xi(m) \geq \varepsilon(k)$ and $b \nmid m$, Theorem 2 implies that $c_b(m) \geq k$. But $c_b(m) = c_b(n)$, since m and n have the same digits in base b, apart from trailing zeros. Therefore, $c_b(n) \geq k$. $\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4. *Let $a \geq 2$ and $b \geq 2$ be integers. Let d be the smallest factor of a such that $\gcd(a/d, b) = 1$. If $\log(d)/\log(b)$ is irrational, then $c_b(a^n) \geq \log_r n$ for all $n \geq 1$, where $r \geq 2$ is an integer depending only on a and b. In particular, $\lim_{n\to\infty} c_b(a^n) = \infty$.*

Proof. Let $p_1^{e_1} \cdots p_t^{e_t}$ be the prime factorization of b. Then $d = p_1^{f_1} \cdots p_t^{f_t}$, where $f_i = \nu_{p_i}(a)$. Note that some of the f_i may be zero. If

$$\frac{f_1}{e_1} = \cdots = \frac{f_t}{e_t}$$

then $\log(d)/\log(b) = f_1/e_1$, which is rational. Therefore, if $\log(d)/\log(b)$ is irrational, then the ratios f_i/e_i are not all equal, which implies that b has two prime factors $p = p_i$ and $q = p_j$ such that $f_i/e_i > f_j/e_j$, and so $\xi(a) > 0$.

Let $r = \lceil \log(b)/\log(p) \rceil$, and let $\varepsilon(k) = r^{k-1}$ for $k \geq 1$. This sequence satisfies $\varepsilon(1) \geq 1$ and $p^{\varepsilon(k)} > b^{\varepsilon(k-1)}$ for all $k \geq 2$.

Let n be a positive integer, and let $k = \lceil \log_r n \rceil$, so that $r^{k-1} < n \leq r^k$. Then a^n is divisible by p^n, so a^n is also divisible by $p^{\varepsilon(k)}$.

Therefore, $c_b(a^n) \geq k \geq \log_r(n)$, by Theorem 3. $\qquad\qquad\qquad\qquad\square$

In 1973, Senge and Straus[4, Theorem 3] proved that if $a \geq 1$ and $b \geq 2$ are positive integers, then $\lim_{n \to \infty} c_b(a^n) = \infty$ if and only if $\log(a)/\log(b)$ is irrational. However, they did not demonstrate a lower bound. In 1980, Stewart[6, Theorem 2] proved that if $\log(a)/\log(b)$ is irrational, then

$$c_b(a^n) > \frac{\log n}{\log \log n + C} - 1$$

for $n > 4$, where C depends on a and b alone. Theorem 4 achieves a stronger bound, but with stricter conditions on a and b.

4. Related sequences

Theorem 3 can be applied to many other sequences. We will give two examples here. As before, let $b \geq 2$ be an integer, and let p and q be distinct prime divisors of b.

Factorials. By Legendre's formula[2, p. 263], $\nu_p(n!) = (n - s_p(n))/(p - 1)$. Thus,

$$\xi(n!) \approx n \left(\frac{1}{p - 1} - \frac{\nu_p(b)}{(q - 1)\nu_q(b)} \right)$$

whenever $(p - 1)\nu_p(b) \neq (q - 1)\nu_q(b)$, since $s_p(n) + s_q(n) \ll n$.

Therefore, $\lim_{n \to \infty} c_b(n!) = \infty$ for any b with distinct factors p and q satisfying

$$(p - 1)\nu_p(b) > (q - 1)\nu_q(b).$$

In particular, $\lim_{n \to \infty} c_{10}(n!) = \infty$.

Cumulative LCMs. Let $\Lambda_n = \mathrm{lcm}(1, 2, \ldots, n)$. This is sequence A003418 in the OEIS. It is easy to see that $\nu_p(\Lambda_n) = \lfloor \log_p(n) \rfloor$. Thus,

$$\xi(\Lambda_n) = \lfloor \log_p n \rfloor - \lfloor \log_q n \rfloor \cdot \frac{\nu_p(b)}{\nu_q(b)} \approx \log n \cdot \left(\frac{1}{\log p} - \frac{1}{\log q} \cdot \frac{\nu_p(b)}{\nu_q(b)} \right).$$

This quantity is nonzero, since $\log(p)/\log(q)$ is irrational; and we may assume that it is positive, else we can switch p and q. Therefore, $\lim_{n \to \infty} \xi(\Lambda_n) = \infty$, hence $\lim_{n \to \infty} c_b(\Lambda_n) = \infty$ for every base b that is not a prime power.

These arguments imply lower bounds of the form $c_b(n!) > C \log n$ and $c_b(\Lambda_n) > C \log \log n$, with certain restrictions on b. However, Sanna[3] proved that

$$s_b(n!) > C \log n \log \log \log n$$

for every integer $n > e^e$ and every $b \geq 2$, where C is a constant depending only on b, and he established the same inequality for $s_b(\Lambda_n)$.

5. Digital sums of powers: the general case

In this final section, we will prove that the number of nonzero digits in the base-b expansion of a^n increases to infinity with n, provided that $\log(b)/\log(a)$.

This condition is necessary, because if $\log(a)/\log(b) = r/s \in \mathbb{Q}$, then $a^{ns} = b^{nr}$ for all n, so a^{ns} has only one nonzero digit in its base-b expansion.

The proof is not elementary, as it relies on a deep result in transcendental number theory, concerning linear forms in logarithms.

Theorem 5. *[1, p. 23] Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers with degrees at most d. Suppose that $\alpha_1, \ldots, \alpha_{n-1}$ and α_n have heights at most $A' \geq 4$ and $A \geq 4$ respectively. Let $\beta_1, \ldots, \beta_n$ be rational integers with heights (absolute values) at most $B \geq 2$. If $\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \alpha_n \neq 0$, there exists $C > 1$, depending only on A, A', B, n, and d, such that $|\Lambda| > C^{-\log A \log B}$.*

Recall that a nonzero algebraic number α is a root of an irreducible polynomial P with integer coefficients. We may assume that the greatest common divisor of the coefficients of P is 1, and the leading coefficient is positive. These requirements uniquely determine P. The *degree* of α is equal to the degree of P, and the *height* of α is the maximum of the absolute values of the cofficients of P. For a rational integer, the degree is 1 and the height is the absolute value.

Lemma 1. *Let $a, b \geq 2$ be integers such that $\log(a)/\log(b)$ is irrational. Then there exists a constant $C > 0$ so that if $a^n = b^r m$ and $m \nmid d$ then $\log m > Cn$.*

Proof. If $\log(a)/\log(b)$ is irrational then there exist primes p, q so that $\nu_p(a)\nu_q(b) > \nu_q(a)\nu_p(b)$. Consequently, $\xi(a) > 0$, where $\xi = \xi_{b,p,q}$ is defined as in Section 3.

Since $\nu_p(m) \geq \xi(m) = \xi(a^n) = n\xi(a)$, it follows that m is divisible by $p^{\lceil n\xi(a) \rceil}$, so $\log m > Cn$ where $C = \xi(a) \log p > 0$. $\square$

Theorem 6. *Let $a, b \geq 2$ be integers, and suppose that $\log(a)/\log(b)$ is irrational. Then $c_b(a^n)$ increases to infinity with n.*

Proof. Let

$$(1) \qquad\qquad a^n = b^m(d_i b^{-m_1} + \cdots + d_r b^{-m_r}),$$

where $m = \lfloor \log_b a \rfloor$, $r = c_b(a^n)$, $d_i \in \{1, \ldots, b-1\}$, and $1 = m_1 < \cdots < m_r \leq m$. Assume that $r \geq 2$.

Choose $i \in \{1, \ldots, r-1\}$. We seek an upper bound on m_{i+1}/m_i, which will imply an upper bound on n.

Let $q = b^{m_i}(d_1 b^{-m_1} + \cdots + d_i b^{-m_i})$ and $r = b^m(d_{i+1} b^{-m_{i+1}} + \cdots d_r b^{-m_r})$, so that $a^n = b^{m-m_i} q + r$.

Observe that the following inequalities hold:
$$b^{m-1} < a^n < b^m,$$
$$b^{m_i - 1} < q < b^{m_i},$$
$$b^{m-m_{i+1}} < r < b^{m-m_{i+1}+1}.$$

By combining these inequalities, we find that

(2) $$-\log(a^{-n} r) / \log(q) \approx m_{i+1}/m_i.$$

Let $\Lambda = n \log a + (m - m_i) \log b + \log q$. Then $|\Lambda| = \log(1 + a^{-n} r) < a^{-n} r$. But by Theorem 5, $|\Lambda| > C^{-\log n \log q}$ for some $C > 1$ depending only on a and b. Therefore, $a^{-n} r > C^{-\log n \log q}$, so

(3) $$-\log(a^{-n} r) / \log(q) < C' \log n.$$

Since $\log m_r = \sum_{i=1}^{r-1} \log(m_{i+1}/m_i)$, Equations (2) and (3) imply that
$$m_r < r(\log \log n + C).$$

But $m_r = \Theta(\log n)$ by Lemma 1, hence
$$r > \frac{\log n}{\log \log n + C}$$
for some $C > 0$ depending only on a and b, when n is sufficiently large. Since $r = c_b(a^n)$, this confirms that $\lim_{n \to \infty} c_b(a^n) = \infty$.

$\square$

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