ELEMENTARY BOUNDS ON DIGITAL SUMS OF POWERS, FACTORIALS, AND LCMS

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Abstract. We prove lower bounds on digital sums of powers, multiples of powers, factorials, and the least common multiple of $\{1, \ldots, n\}$, using only elementary number theory. We conclude with an expository proof of the previously known result that the sum of the base-b digits of a^n tends to infinity with n if and only if $\log(a)/\log(b)$ is irrational.

1. Introduction

This expository article establishes lower bounds on digital sums of powers, multiples of powers, factorials, and the least common multiple of $\{1, \ldots, n\}$, using only elementary number theory.

We were inspired by the following problem, which was posed and solved by Wacław Sierpiński [8, Problem 209]:

> *Prove that the sum of digits of the number 2^n (in decimal system) increases to infinity with n.*

The reader is urged to attempt this problem on their own before proceeding. Note that it is not enough to prove that the sum of digits of 2^n is unbounded, since the sequence is not monotonic.

Consider the sequence of powers of 2 (sequence A000079 in the OEIS):

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots.$$

This sequence grows very rapidly. Now, let us define another sequence, by adding the decimal digits of each power of 2. For example, 16 becomes $1 + 6 = 7$, and 32 becomes $3 + 2 = 5$. The first few terms of this new sequence (A001370) are listed below.

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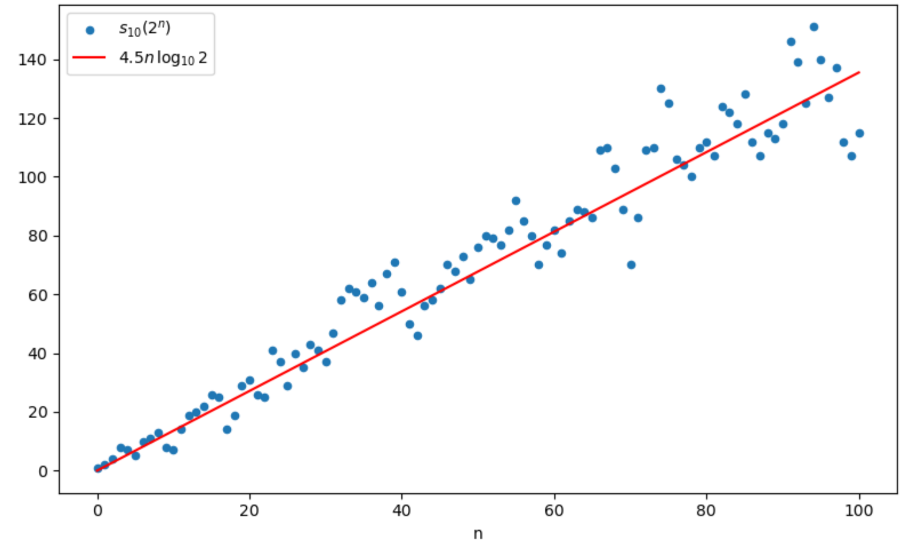

FIGURE 1. Scatter plot of the digital sum of 2^n for $n \le 100$ together with the conjectured linear approximation.

$$1, 2, 4, 8, 7, 5, 10, 11, 13, 8, 7, \dots.$$

This new sequence grows much more slowly, and it is not monotonic. Nevertheless, it is reasonable to conjecture that it tends to infinity. Indeed, one might guess that the sum of the decimal digits of 2^n is approximately equal to $4.5n\log_{10} 2$, since 2^n has $\lfloor n\log_{10} 2\rfloor + 1$ decimal digits, and the digits seem to be approximately uniformly distributed among $0, 1, 2, \dots, 9$. However, this stronger conjecture remains to be proved. See Figure 1.

We prove in Section 3 that the digital sum of 2^n is greater than $\log_4 n$ for all $n \ge 1$. But first, we introduce some notation.

2. NOTATION

For an integer $b \ge 2$, we write $s_b(n)$ for the sum of the base-b digits of n, and $c_b(n)$ for the number of nonzero digits in that expansion. These functions are equivalent up to a constant factor, since $c_b(n) \le s_b(n) \le (b-1)c_b(n)$ for all n and b; so we focus on $c_b(n)$.

The function s_b is *subadditive*: $s_b(m + n) \le s_b(m) + s_b(n)$ for all positive integers m and n. Equality holds if no carries occur in the digitwise addition of m and n.

Otherwise, each carry reduces the digital sum by $b-1$. The function c_b is likewise subadditive.

For a prime p, $\nu_p(n)$ denotes the exponent of p in the prime factorization of n. If p does not divide n then $\nu_p(n) = 0$.

We use Bachmann-Landau notation [3] to describe the approximate size of functions. Let f and g be real-valued functions defined on a domain D, usually the set of positive integers.

One writes
$$f(n) = O(g(n))$$
if there exists a positive real number C such that
$$|f(n)| \leq Cg(n) \qquad \text{for all } n \in D.$$

The notation
$$f(n) = o(g(n))$$
means that
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

In particular, $O(1)$ denotes a bounded function, and $o(1)$ denotes a function that tends to 0 as $n \to \infty$.

Finally,
$$f(n) = \Theta(g(n))$$
means that there exist positive real numbers C and C' such that
$$Cg(n) < |f(n)| < C'g(n) \qquad \text{for all } n \in D.$$

3. Digital sums of powers of two

We present an informal proof that $c_{10}(2^n)$ tends to infinity as $n \to \infty$. See [5] for an alternative approach.

Let n be a positive integer, and write the decimal expansion of 2^n as $2^n = \sum_{i=0}^{\infty} d_i 10^i$, where $d_i \in \{0, \dots, 9\}$ and $d_i = 0$ for all but finitely many terms. Note that the final digit d_0 of 2^n cannot be zero, since 2^n is not divisible by 10.

If $2^n > 10$, then 2^n is divisible by 16; so $2^n \bmod 10^4$, the number formed by the last four digits of 2^n, is also divisible by 16. If the first three of these digits were zero, then $2^n \bmod 10^4$ would be less than 10, which is impossible. So at least one of the digits d_1, d_2, d_3 is nonzero.

$$
\begin{array}{lll}
2^{0} & = & 1 \\
2^{4} & = & 1\ 6 \\
2^{14} & = & 1\ 638\ 4 \\
2^{47} & = & 1\ 4073748835\ 532\ 8 \\
2^{157} & = & 1\ 826877046663628647754606040895353\ 7745699156\ 787\ 2
\end{array}
$$

FIGURE 2. Digits of 2^n subdivided into blocks. Each block contains at least one nonzero digit.

If $2^n > 10^4$, then 2^n is divisible by 2^{14}; so $2^n \bmod 10^{14}$, the number formed by the last 14 digits of 2^n, is also divisible by 2^{14}. If the first 10 of these digits were zero, then $2^n \bmod 10^{14}$ would be less than 10^4, which is impossible. So at least one of the digits $d_4, d_5, \ldots, d_{13}$ is nonzero.

We can continue in this way, finding longer and longer non-overlapping blocks of digits, each containing at least one nonzero digit, as illustrated in Figure 2. This proves that $c_{10}(2^n)$ tends to infinity as $n \to \infty$.

Let us formalize this argument.

Theorem 1. *Let $(e(k))_{k\geq 1}$ be a sequence of integers such that $e(1) \geq 1$ and $2^{e(k)} > 10^{e(k-1)}$ for all $k \geq 2$. If n is a positive integer that is divisible by $2^{e(k)}$ but not divisible by 10, then $c_{10}(n) \geq k$.*

Proof. We argue by induction on k. The case $k = 1$ is immediate, since any positive integer has at least one nonzero digit.

Assume now that $k \geq 2$, and that the statement holds for $k - 1$. Let n be a positive integer divisible by $2^{e(k)}$ but not by 10. Apply the division algorithm to write

$$n = 10^{e(k-1)}q + r, \qquad 0 \leq r < 10^{e(k-1)},$$

for integers q, r.

Because $n \geq 2^{e(k)} > 10^{e(k-1)}$ by hypothesis, the quotient satisfies $q \geq 1$.

Next, both n and $10^{e(k-1)}q$ are divisible by $2^{e(k-1)}$, hence their difference

$$r = n - 10^{e(k-1)}q$$

is also divisible by $2^{e(k-1)}$.

Moreover, r is not divisible by 10, since n is not divisible by 10.

Therefore, $c_{10}(r) \geq k - 1$ by the induction hypothesis.

Finally, the decimal expansion of n is obtained by concatenating the decimal expansion of q with the (possibly zero-padded) expansion of r. Thus,

$$c_{10}(n) = c_{10}(q) + c_{10}(r) \geq 1 + (k-1) = k.$$

This completes the proof.

□

Corollary 1. *Let a be a positive integer that is divisible by 2 but not divisible by 10. Then $c_{10}(a^n) \geq \log_4(n)$ for all $n \geq 1$. In particular, $\lim_{n\to\infty} c_{10}(a^n) = \infty$.*

Proof. Let $e(k) = 4^{k-1}$ for $k \geq 1$. This sequence satisfies $e(1) \geq 1$ and

$$2^{e(k)} > 10^{e(k-1)}$$

for all $k \geq 2$.

Let $n \geq 4$ be a positive integer, and let $k = \lceil \log_4 n \rceil$, so that $4^{k-1} < n \leq 4^k$. Then a^n is divisible by 2^n, so a^n is also divisible by $2^{e(k)}$. But a^n is not divisible by 10.

Therefore, $c_{10}(a^n) \geq k \geq \log_4(n)$, by Theorem 1. □

4. Generalizing to other bases

Our proofs rely only on divisibility properties and therefore extend naturally to other bases.

Theorem 2. *Let $b \geq 2$ be an integer that is not a power of a prime, and let p be a prime divisor of b. Let $(e(k))_{k\geq 1}$ be a sequence of integers such that $e(1) \geq 1$ and $p^{e(k)} > b^{e(k-1)}$ for all $k \geq 2$. If $\nu_p(n) \geq e(k)$ and $b \nmid n$, then $c_b(n) \geq k$.*

Proof. Follow the proof of Theorem 1, but replace 2 with p where appropriate, and replace 10 with b. □

It is desirable to relax the condition that b does not divide n, since multiplying by a factor of the form b^r does not change the number of nonzero digits of n in base b. To facilitate this, we introduce the following notation. Let $b \geq 2$ be an integer with distinct prime divisors p and q. Define the function $\xi = \xi_{b,p,q}$ by

$$\xi(n) = \nu_p(n) - \nu_q(n)\frac{\nu_p(b)}{\nu_q(b)}. \tag{1}$$

One easily verifies that $\xi(b^r u) = \xi(u)$ for all integers $r \geq 0$ and $u \geq 1$. Intuitively, ξ is a modified version of ν_p that is insensitive to trailing zeros in the base-b expansion of its argument.

Theorem 3. *Let $b \geq 2$ be an integer that is not a power of a prime, let p and q be distinct prime divisors of b, and let $(e(k))_{k\geq 1}$ be defined as in Theorem 2. If $\xi(n) \geq e(k)$, then $c_b(n) \geq k$.*

Proof. Write n as $b^r u$, where u is not divisible by b. Note that $\xi(n) = \xi(u)$. Since $\nu_p(u) \geq \xi(u) \geq e(k)$ and $b \nmid u$, Theorem 2 implies that $c_b(u) \geq k$. But $c_b(u) = c_b(n)$, since u and n have the same digits in base b, apart from trailing zeros. Therefore, $c_b(n) \geq k$. $\square$

Theorem 4. *Let $a \geq 2$ and $b \geq 2$ be integers. Let d be the smallest factor of a such that $\gcd(a/d, b) = 1$, and suppose that $\log(d)/\log(b)$ is irrational. Then $c_b(a^n) > C \log n$ for all $n \geq 1$, where $C > 0$ depends only on a and b.*

Proof. Let $p_1^{e_1} \cdots p_t^{e_t}$ be the prime factorization of b. Then

$$d = p_1^{f_1} \cdots p_t^{f_t},$$

where $f_i = \nu_{p_i}(a)$. Note that some of the f_i may be zero. If

$$\frac{f_1}{e_1} = \cdots = \frac{f_t}{e_t}$$

then $\log(d)/\log(b) = f_1/e_1$, which is rational. Therefore, if $\log(d)/\log(b)$ is irrational, then the ratios f_i/e_i are not all equal, which implies that b has two prime factors $p = p_i$ and $q = p_j$ such that

$$\frac{f_i}{e_i} > \frac{f_j}{e_j},$$

and so $\xi(a) > 0$.

Let $r = \lceil \log_p b \rceil$, and let $e(k) = r^{k-1}$ for $k \geq 1$. This sequence satisfies $e(1) \geq 1$ and

$$p^{e(k)} > b^{e(k-1)}$$

for all $k \geq 2$.

Let n be a positive integer, and let $k = \lceil \log_r \xi(a^n) \rceil$, so that $r^{k-1} < \xi(a^n) \leq r^k$. Then $\xi(a^n) > e(k)$, so Theorem 3 implies that $c_b(a^n) \geq k$. But $\xi(a^n) = n\xi(a)$, hence $k = \Theta(\log n)$, and the conclusion follows. $\square$

It is straightforward to determine an explicit value for C, although we have not done so here. One can show that if $a, b \geq 2$ are integers, a divides b, and if b has at least one prime divisor that does not divide a, then

$$c_b(a^n) > \frac{\log n}{\log\left(2 \log b \,/\, \log a\right)}$$

for n sufficiently large.

In 1973, Senge and Straus [7, Theorem 3] proved that if $a \geq 1$ and $b \geq 2$ are positive integers, then $\lim_{n\to\infty} c_b(a^n) = \infty$ if and only if $\log(a)/\log(b)$ is irrational. However, they did not demonstrate a lower bound. In 1980, Stewart [9, Theorem 2] proved that if $\log(a)/\log(b)$ is irrational, then

$$c_b(a^n) > \frac{\log n}{\log\log n + C} - 1$$

for $n > 4$, where C depends on a and b alone. Theorem 4 achieves a stronger bound, but with stricter conditions on a and b. We replicate Stewart's result in Section 6.

5. Digital sums of factorials and LCMs

In this section, we demonstrate a logarithmic lower bound for digital sums of factorials, and also for the digital sum of $\Lambda_n := \operatorname{lcm}(1, 2, \ldots, n)$.

We need the following elementary result, which is due to Stolarsky [10] for base two, and generalized to all bases by Balog and Dartyge [1].

Theorem 5. *Let $m, r \geq 1$ and $b \geq 2$ be integers. If m is divisible by $b^r - 1$, then $s_b(m) \geq (b-1)r$.*

Proof. Write the base-b expansion of m as a concatenation of r-digit blocks, so that

$$m = \sum_{i=0}^{k-1} B_i b^{ri}, \qquad 0 \leq B_i < b^r, \qquad B_{k-1} \geq 1.$$

Define the *block-sum* operator G by

$$G(m) = \sum_{i=0}^{k-1} B_i.$$

Observe that $G(m) \equiv m \pmod{b^r - 1}$, since $b^r \equiv 1 \pmod{b^r - 1}$. Also, $G(m) < m$ for $m \geq b^r$, and $G(m) = m$ for $0 \leq m < b^r$.

Iterate G on m: define $m_0 = m$ and $m_{t+1} = G(m_t)$. By the observations above, (m_t) is a sequence of positive multiples of $b^r - 1$, which is strictly decreasing while its terms exceed $b^r - 1$. Therefore, the sequence must eventually reach $b^r - 1$, which is the unique positive multiple of $b^r - 1$ that is less than b^r.

Since s_b is subadditive,

$$s_b(G(m)) \leq \sum_{i=0}^{k-1} s_b(B_i) = s_b(m).$$

Therefore,

$$s_b(m) \geq s_b(b^r - 1) = (b-1)r.$$

□

This result implies lower bounds of the form $s_b(n!) > C \log n$ and $s_b(\Lambda_n) > C \log n$ for some $C > 0$, depending only on b. For if $n \geq b^r - 1$, then $n!$ and Λ_n are divisible by $b^r - 1$, hence $s_b(n!) \geq (b-1)r$ and $s_b(\Lambda_n) \geq (b-1)r$.

Stronger bounds are known. In 2015, Sanna [6] proved that

$$s_b(n!) > C \log n \log \log \log n$$

for every integer $n > e^e$ and every $b \geq 2$, where C is a constant depending only on b, and they established the same lower bound for $s_b(\Lambda_n)$.

6. Digital sums of powers: the general case

In this final section, we prove that the number of nonzero digits in the base-b expansion of a^n tends to infinity as $n \to \infty$, provided that $\log(b)/\log(a)$ is irrational. This result appears in earlier work of Senge and Straus [7] and Stewart [9], but we present an argument that we hope is more accessible.

The irrationality condition is necessary. Indeed, if $\log(a)/\log(b) = r/s \in \mathbb{Q}$, then

$$a^{ns} = b^{nr}$$

for every integer n, so a^{ns} has only one nonzero digit in base b.

Recall that a nonzero algebraic number $\alpha \in \mathbb{C}$ is a root of a unique irreducible integer polynomial P with positive leading coefficient and coprime coefficients. The *degree* of α is equal to the degree of P, and the *height* of α is the maximum of the absolute values of the cofficients of P. A rational integer has degree 1 and height equal to its absolute value.

Our proof relies on a version of Baker's theorem on linear forms in logarithms, which we state without proof.

Theorem 6. [2, p. 23] *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers with degrees at most d. Suppose that $\alpha_1, \dots, \alpha_{n-1}$ and α_n have heights at most $A' \geq 4$ and $A \geq 4$ respectively. Let $\beta_1, \dots, \beta_n$ be integers of absolute value at most $B \geq 2$. If*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \alpha_n \neq 0,$$

there exists $C > 1$, depending only on n, d, and A', such that

$$|\Lambda| > C^{-\log A \log B}.$$

See [4] *for an explicit lower bound.*

Baker's theorem gives effective lower bounds on linear forms in logarithms of algebraic numbers. In our context, it guarantees that the expression in (3) below cannot be too small, which limits how well powers of a can be approximated by powers of b.

The next lemma deals with the possibility that some power of a might be divisible by b.

Lemma 1. *Let $a, b \geq 2$ be integers with $\log(a)/\log(b)$ irrational. Then there exist C and C', depending only on a and b, such that whenever*

$$a^n = b^r u,$$

we have

$$Cn \leq \log u \leq C'n.$$

In other words, $\log u = \Theta(n)$.

Proof. Since $\log(a)/\log(b)$ is irrational, we can choose primes p, q such that

$$\nu_p(a)\nu_q(b) > \nu_q(a)\nu_p(b).$$

Hence $\xi(a) > 0$ for the function $\xi = \xi_{b,p,q}$ defined in Equation (1).

Because $\nu_p(u) \geq \xi(u) = \xi(a^n) = n\xi(a)$, the integer u is divisible by $p^{\lceil n\xi(a) \rceil}$. Therefore

$$\log u \geq Cn, \qquad C = \xi(a) \log p > 0.$$

On the other hand, $\log u \leq C'n$ for $C' = \log a$, since $u \leq a^n$. □

We now prove the main theorem.

Theorem 7. *Let $a, b \geq 2$ be integers, and suppose that $\log(a)/\log(b)$ is irrational. Then for all sufficiently large n,*

$$c_b(a^n) > \frac{\log n}{\log \log n + C}$$

for some $C > 0$ depending only on a and b.

Proof. Let

$$a^n = b^m \left(d_1 b^{-m(1)} + \cdots + d_k b^{-m(k)}\right),$$

where $m = \lceil \log_b a^n \rceil$, $k = c_b(a^n)$, $d_i \in \{1, \ldots, b-1\}$, and

$$1 = m(1) < \cdots < m(k) \leq m.$$

That is, $d_1, \ldots, d_k$ are the nonzero digits of a^n in base b, and $m(1), \ldots, m(k)$ are their positions when the digits are numbered from left (most significant) to right (least significant). Assume that $k \geq 2$.

Fix $i \in \{1, \ldots, k-1\}$. We estimate the ratio $m(i+1)/m(i)$, which will ultimately bound k.

Define

$$q = b^{m(i)}(d_1 b^{-m(1)} + \cdots + d_i b^{-m(i)}),$$
$$r = b^m(d_{i+1} b^{-m(i+1)} + \cdots + d_k b^{-m(k)}),$$

so that

$$a^n = b^{m-m(i)} q + r.$$

From the base-b expansion we obtain the bounds

$$b^{m-1} \leq a^n \leq b^m,$$
$$b^{m(i)-1} \leq q \leq b^{m(i)},$$
$$b^{m-m(i+1)} \leq r \leq b^{m-m(i+1)+1}.$$

These imply the approximation

$$\frac{-\log(a^{-n} r)}{\log(q)} = \frac{m(i+1)}{m(i)} + O(1). \tag{2}$$

Now set

$$\Lambda = \log(a^{-n} b^{m-m(i)} q) = -n \log a + (m - m(i)) \log b + \log q. \tag{3}$$

Then (since $a^{-n} r < 1/2$)

$$|\Lambda| = -\log(1 - a^{-n} r) < 2a^{-n} r.$$

By Theorem 6,

$$|\Lambda| > C^{-\log q \log n}$$

for some $C > 1$ depending only on a and b. Thus,

$$a^{-n} r > C^{-\log q \log n},$$

and hence, for n sufficiently large,

$$\frac{-\log(a^{-n}r)}{\log(q)} < C\log n \tag{4}$$

for some $C > 0$ depending only on a and b.

Combining (2) and (4) gives

$$\frac{m(i+1)}{m(i)} < C\log n.$$

Summing the logarithms of these ratios,

$$\log m(k) = \sum_{i=1}^{k-1} \log\left(\frac{m(i+1)}{m(i)}\right)$$

which yields

$$\log m(k) < (k-1)(\log\log n + C). \tag{5}$$

Write a^n as $b^{m-m(k)}u$, where

$$b^{m(k)-1} < u < b^{m(k)}.$$

Thus $m(k) = \Theta(\log(u))$, and $\log(u) = \Theta(n)$ by Lemma 1, so

$$\log m(k) = \log n + O(1). \tag{6}$$

Therefore, (5) and (6) imply that

$$k > \frac{\log n}{\log\log n + C}$$

for all sufficiently large n, as required. □

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