IIT KANPUR

Undergraduate Project

under the guidance of Prof. Debasis Kundu (Dept. of MTH) and
Prof. Raghu Nandan Sen Gupta (Dept. of IME)

Shreyansh Sinha
200955
Junior Undergraduate
Mathematics and Scientific Computing

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Shreyansh Sinha

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Estimation of Parameters of Kumaraswamay Exponential and Gamma-Gompertz Distribution

Shreyansh Sinha

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1 Introduction

1.1 Reliability Theory

Survival analysis or Reliability analysis is a branch of Statistics for analysing the expected duration of time until one event occurs, such as death in biological organisms and failure in mechanical systems.

Various different distributions are used to model the survival data of different systems. Some of the most common ones used are Weilbull, Gamma, Log-Logistic, Exponential, etc. Various extensions of the above distribution are also introduced to better fit the distribution on the data. Examples include Inverse weibull, Gamma Extended Weibull etc.

In this report, we discuss about two distributions, namely Kumaraswamy Exponential distribution and Gamma-Gompertz Distribution and methods of parameter estimation for them.

1.2 Competing Risk Setup

A **competing-risk model** refers to a situation where a system (or organism) is exposed to two or more causes of failure (or death) but its eventual failure (or death) can be attributed to exactly one of the causes of failure. The basic information available in the competing risk model is the time of failure (T) of the system, and the corresponding cause of failure (δ)

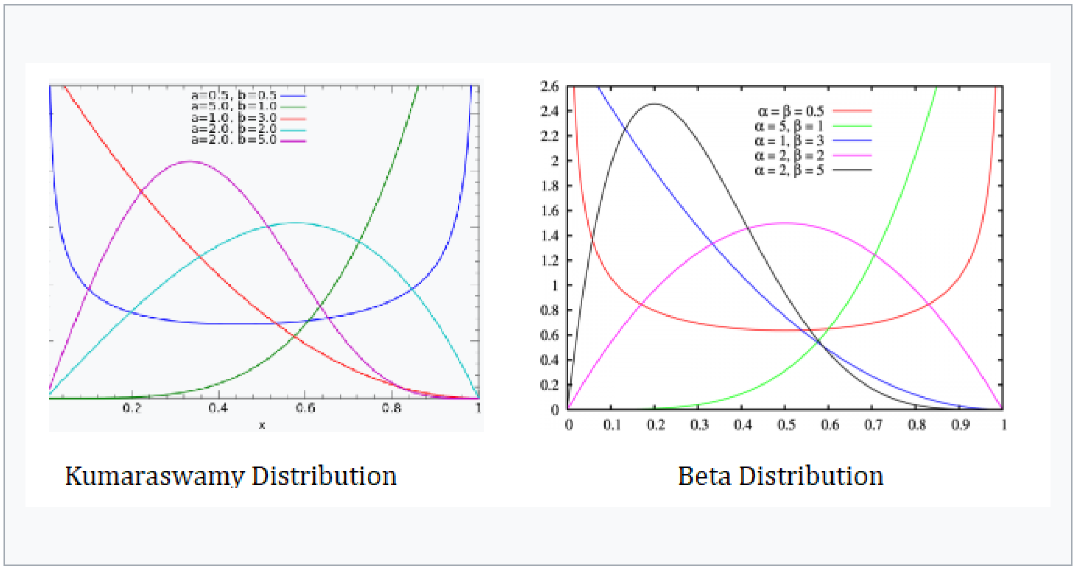

Figure 1: Comparing the Two Distributions

2 Distributions

2.1 Kumaraswamy Exponential Distribution

2.1.1 kumaraswamy distribution

The Kumaraswamy distribution is a two-variable family of distributions that is bounded at one and zero. Its CDF and PDF respectively are:-

$$F(x) = 1 - (1 - X^a)^b$$

$$f(x) = abX^{a-1}(1 - X^a)^{b-1}$$

The Kumaraswamy distribution is very similar to the **beta distribution** and has the same basic shape. So similar in fact, it's often referred to as a "Beta-like" distribution. However, the Kumaraswamy distribution is simpler to use and in some situations more tractable. It is often preferred over the Beta in part because the CDF has the closed form.

2.1.2 KUMARASWAMY-G Distributions

Jones (2009) [2] highlighted several advantages of the Kumaraswamy distribution over the beta distribution: the normalizing constant is very simple, simple explicit formulae for the distribution and quantile functions which do not involve any special functions and a simple formula for random variate generation.

If G denotes the cdf of a random variable, Cordeiro and de Castro (2011) [3] defined the Kum-G distribution as :-

$$F(t) = 1 - [1 - G^a(t)]^b, t > 0$$

where a and b are positve shape parameters and the pdf is

$$f(t) = abg(t)G^{a-1}(t)[1 - G^a(t)]^{b-1}$$

2.1.3 Physical Interpretation

Consider a system is formed by b independent components and that each component is made up of a independent sub-components. Suppose the system fails if any of the b components fails and that each component fails if all of the a sub-components fail. Let $T_{j1}, ..., T_{ja}$ denote the lifetimes of the sub components within the j^{th} component, j = 1, . . . , b, having a common cdf G(t). Let T_j denote the lifetime of the jth component, for $j = 1, ..., b$ and let T denote the lifetime of the entire system. Then, the cdf of T is

$$Pr(T \leq t) = 1 - Pr(T_1 > t, ..., T_b > t)$$

$$Pr(T \leq t) = 1 - Pr^b(T_1 > t) = 1 - (1 - Pr(T_1 \leq t))^b$$

$$Pr(T \leq t) = 1 - (1 - Pr(T_{11} \leq t, ..., T_{1a} \leq t))^b = 1 - (1 - Pr^a(T_{11} \leq t))^b$$

$$Pr(T \leq t) = 1 - (1 - G^a(t))^b$$

2.1.4 Kumaraswamy Exponential Distribution

The cdf of the Kumaraswamy exponential distribution are:-

$$F_{EK}(X; \alpha, \beta, \gamma) = [1 - (1 - X^\alpha)^\beta]^\gamma, x \in (0, 1)$$

where α, β, γ are positive shape parameters (see fig 2)

2.2 Gamma-Gompertz Distribution

2.2.1 Gompertz Distribution

The Gompertz distribution is a probability distribution commonly used in survival analysis. Its probability density function (PDF) of the Gompertz distribution is given by:

$$f(x) = \alpha\beta e^{\alpha x} e^{-\beta(e^{\alpha x} - 1)}$$

Gompertz distribution (abbreviated as Go) plays an important role in many real life applications, especially for right and left skewed data. This is one of the most well-known distribution based on mortality laws. The Go has only monotone

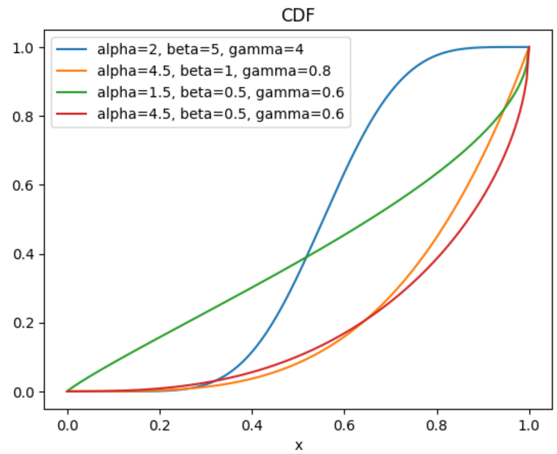

Figure 2: CDF of Kumaraswamy distribution

hazard function, but in applied sciences non-monotonic hazard function may be taken into account such as bathtub shaped hazard function. For this reason, several generalisation of Go have been constructed. We introduce the GGo distribution as one such generalisation:-

2.2.2 Gamma-Gompertz Distribution:-

A random variable X follows GGo distribution if the cdf and pdf are defined as:-

$$G(x) = \frac{\gamma(\theta, \lambda(e^{\alpha x} - 1)/\alpha)}{\Gamma(\theta)}$$

and,

$$g(x) = \frac{\lambda}{\Gamma(\theta)} e^{\alpha x - \frac{\lambda}{\alpha}(e^{\alpha x - 1})} (\frac{\lambda}{\alpha}((e^{\alpha x} - 1))^{\theta - 1}; \theta \geq 0, \alpha, \lambda, \theta > 0,$$

where $\lambda = \delta\beta$

The hazard function of the GGo distribution is bathtub shaped if $\theta < 1$ and increasing if $\theta \geq 1$ for all values of α, λ and θ. The proposed model can be considered as a good alternative model for fitting the positive data with a longer right tail and it also contains some well-known distribution.

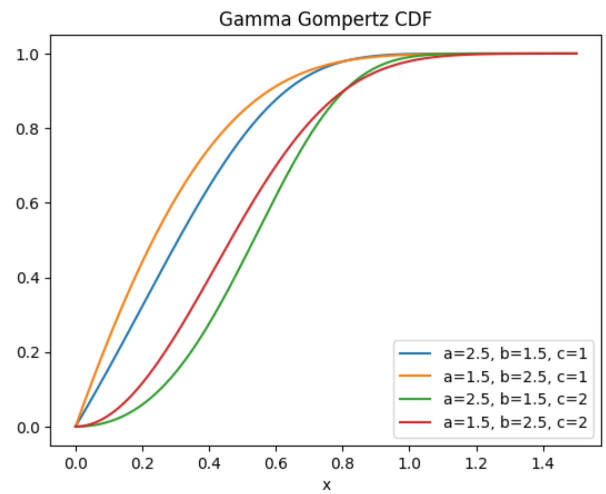

Figure 3: CDF of Gamma-Gompertz Distribution

3 Methodologies :

3.1 Methods for Parameter Estimation

3.1.1 Maximum Likelihood Estimator(MLE)

Maximum Likelihood Estimation (MLE) is a commonly used method in statistical inference to estimate the parameters of a statistical model. It is based on the principle of finding the parameter values that maximize the likelihood of observing the given data. The likelihood function is defined as the probability of observing the data, given the model parameters.

Suppose we have a random sample $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ from a probability distribution with an unknown parameter vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$. The likelihood function $L(\theta; \mathbf{X})$ is defined as the joint probability density function (PDF) or probability mass function (PMF) of the observed data, evaluated at the observed data points $\mathbf{X}$. The goal of MLE is to find the parameter values $\hat{\theta}$ that maximize the likelihood function:

$$\hat{\theta} = \arg\max_{\theta} L(\theta; \mathbf{X}) \tag{1}$$

3.1.2 Bayesian Estimation

Bayesian estimation is a statistical approach that combines data with prior knowledge or beliefs to make inferences about unknown parameters. It is based on Bayes' theorem, which allows updating of prior beliefs with observed data to obtain posterior probability distributions of the parameters of interest.

Suppose we have a random sample $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ from a probability distribution with an unknown parameter vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$. Bayesian estimation involves specifying a prior probability distribution, denoted as $p(\theta)$, which represents our subjective beliefs or knowledge about the parameters before observing the data. The likelihood function $L(\theta; \mathbf{X})$ is the joint probability density function (PDF) or probability mass function (PMF) of the observed data, evaluated at the observed data points $\mathbf{X}$.

The key idea in Bayesian estimation is to update the prior beliefs with the observed data using Bayes' theorem, which gives the posterior distribution of the parameters as:

$$p(\theta|\mathbf{X}) = \frac{L(\theta; \mathbf{X}) \cdot p(\theta)}{p(\mathbf{X})} \tag{2}$$

where $p(\theta|\mathbf{X})$ is the posterior distribution, $L(\theta; \mathbf{X})$ is the likelihood function, $p(\theta)$ is the prior distribution, and $p(\mathbf{X})$ is the marginal likelihood, which serves as a normalizing constant.

3.2 Sampling Techniques

3.2.1 Inverse Transform

Let $F(x)$ be the cumulative distribution function (CDF) of a random variable X. The inverse transform method works by generating random numbers from a uniform distribution on the interval $[0, 1]$, denoted as U, and then applying the inverse of the CDF, denoted as $F^{-1}(U)$, to obtain samples from the target distribution. Specifically, for a given random number u from the uniform distribution, we can compute $F^{-1}(u)$ using the inverse of the CDF, and this value represents a random sample from the distribution.

The inverse transform method is particularly useful for generating random samples from distributions with known closed-form expressions for their CDFs and when the inverse of the CDF can be computed efficiently. It is a simple and widely used method for generating random samples from a variety of continuous and discrete distributions.

The steps for generating random samples using the inverse transform method are as follows:

1. Generate a random number u from a uniform distribution on the interval $[0, 1]$.
2. Compute $F^{-1}(u)$ using the inverse of the CDF, where $F^{-1}(u)$ is the random sample from the target distribution.

3.2.2 Metropolis Hasting Algorithm

The Metropolis-Hastings algorithm is a Markov chain-based method that generates samples iteratively, where each sample is proposed based on the current state of the Markov chain. The algorithm works by defining a proposal distribution, which is a distribution from which a new sample is proposed. The proposal distribution can be any distribution, and it does not have to be similar to the target distribution. The proposed sample is then accepted or rejected based on a probability, determined by both the proposal distribution and the target distribution.

The basic steps of the Metropolis-Hastings algorithm are as follows:

1. **Initialization:**Choose an initial state for the Markov chain, denoted as x_0.

2. **Proposal:** At each iteration t, propose a new sample x' from the proposal distribution $q(x'|x_t)$, where x_t is the current state of the Markov chain.

3. **Acceptance probability:** Compute the acceptance probability, denoted as α, which is the probability of accepting the proposed sample x'. The acceptance probability is defined as:

$$\alpha = min(1, \frac{p(x')}{p(x_t)} . \frac{q(x_t/x')}{q(x'/x_t)})$$

   where $p(x')$ and $p(x_t)$ are the probabilities of the proposed sample and the current state, respectively, and $q(x_t|x')$ and $q(x'|x_t)$ are the probabilities of proposing x_t given x' and proposing x' given x_t, respectively.

4. **Acceptance/rejection step:** Generate a random number u from a uniform distribution between 0 and 1. If $u \leq \alpha$, accept the proposed sample x' as the next state of the Markov chain, i.e., set $x_{t+1} = x'$. Otherwise, reject the proposed sample and set $x_{t+1} = x_t$.

5. **Repeat:** Repeat the proposal, acceptance probability, and acceptance/rejection steps for a desired number of iterations or until a convergence criterion is met.

4 Proposed Models

4.1 2-Competing Risk Kumaraswamy Exponential Distribution

In this section we discuss about the estimation of parameters of 2-competing risk model where both the risks follows Kumaraswamy distribution.

4.1.1 Setting up the 2-competing risk model:-

Suppose we have a two component system connected in series. The failure distribution of component 1 and 2 are given by $f_1(t)$ and $f_2(t)$ respectively. We assume that α and β are same for both the distribution. Let the other parameters be γ_1 and γ_2.

Consider we have 'n' such systems which are identical to each other. We assume $X'_{ij}s$ are independent random variables for i=1,2,...,n and j=1,2 distributed according to Kumaraswamy Exponential distribution. Let X_{i1} and X_{i2} are the time to failure for the i^th system due to component 1 and component 2 respectively and $\delta_1, \delta_2, ..., \delta_n$ denote the cause of failure for the i^{th} system. Then, T be the time of failure of i^{th} system will be minimum of the two i.e. $T = min(T_1, T_2)$.

4.1.2 Using the MLE approach

Suppose of the n systems n_1 fail due to cause 1 and n_2 fail due to cause 2. Then the likelihood function for observed data $(x_1, \delta_1), (x_2, \delta_2), ..., (x_n, \delta_n)$ is given by

L=$\prod_{i=1}^{n}[f_1(x_i)\bar{F_2}(x_i)]^{I(\delta_i=1)}[f_2(x_i)\bar{F_1}(x_i)]^{I(\delta_i=2)}$
$L = \prod_{i=1}^{n}[\alpha\beta\gamma_1 x_i^{(\alpha-1)}(1-x_i^{\alpha})^{\beta-1}(1-(1-x_i^{\alpha})^{\beta})^{(\gamma_1-1)}[1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_2}]]^{I(\delta)=1}[\alpha\beta\gamma_2 x_i^{(\alpha-1)}(1-x_i^{\alpha})^{\beta-1}(1-(1-x_i^{\alpha})^{\beta})^{(\gamma_2-1)}[1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_1}]]^{I(\delta)=2}$

$L = \prod_{i=1}^{n}[\alpha\beta x_i^{(\alpha-1)}(1-x_i^{\alpha})^{\beta-1}[\gamma_1(1-(1-x_i^{\alpha})^{\beta})^{(\gamma_1-1)}[1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_2}]]^{I(\delta)=1}[\gamma_2(1-(1-x_i^{\alpha})^{\beta})^{(\gamma_2-1)}[1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_1}]]^{I(\delta)=2}$

or

$L = (\prod_{i=1}^{n}[\alpha\beta x_i^{(\alpha-1)}(1-x_i^{\alpha})^{\beta-1}])(\prod_{i=1}^{n_1}[\gamma_1(1-(1-x_i^{\alpha})^{\beta})^{(\gamma_1-1)}[1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_2}]])(\prod_{i=1}^{n_2}[\gamma_2(1-(1-x_i^{\alpha})^{\beta})^{(\gamma_2-1)}[1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_1}]])$

Then the log-Likelihood Equation (ℓ) is given by:-

$\ell = \sum_{i=1}^{n}[ln\alpha + ln\beta + (\alpha-1)lnx_i + (\beta-1)ln(1-x_i^{\alpha})] + n_1 ln\gamma_1 + n_2 ln\gamma_2 + \sum_{i=1}^{n_1}[(\gamma_1-1)ln((1-(1-x_i^{\alpha})^{\beta})) + ln(1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_2})] + \sum_{i=1}^{n_2}[(\gamma_2-1)ln((1-(1-x_i^{\alpha})^{\beta})) + ln(1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_1})]$

The Corresponding likelihood equations are obtained by taking the derivatives with respect to the parameters $(\alpha, \beta, \gamma_1, \gamma_2)$ i.e.

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=0}^{n}[\frac{1}{\alpha} + lnx_i - \frac{\beta-1}{1-x_i^{\alpha}}x_i^{\alpha}lnx_i] + \sum_{i=1}^{n_1}[\frac{(\gamma_1-1)}{1-(1-x_i^{\alpha})^{\beta}}\beta(1-x_i^{\alpha})^{(\beta-1)}x_i^{\alpha}lnx_i - \frac{\gamma_2(1-(1-x_i^{\alpha})^{\beta})^{\gamma_2-1}}{1-(1-(1-x_i^{\alpha})^{\beta})^{\gamma_2}}\beta(1-x_i^{\alpha})^{(\beta-1)}x_i^{\alpha}lnx_i] + \sum_{i=1}^{n_2}[\frac{(\gamma_2-1)}{1-(1-x_i^{\alpha})^{\beta}}\beta(1-x_i^{\alpha})^{(\beta-1)}x_i^{\alpha}lnx_i -$$

$$\frac{\gamma_1(1-(1-x_i^\alpha)^\beta)^{\gamma_1-1}}{1-(1-(1-x_i^\alpha)^\beta)^{\gamma_1}}\beta(1-x_i^\alpha)^{(\beta-1)}x_i^\alpha ln x_i]$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=0}^{n}[\frac{1}{\beta} + ln(1-x_i^\alpha)] + \sum_{i=1}^{n_1}[-\frac{\gamma_1-1}{1-(1-x_i^\alpha)^\beta}(1-x_i^\alpha)^\beta ln(1-x_i^\alpha)+$$

$$\frac{\gamma_2(1-(1-x_i^\alpha)^\beta)^{\gamma_2-1}}{1-(1-(1-x_i^\alpha)^\beta)^{\gamma_2}}(1-x_i^\alpha)^\beta ln(1-x_i^\alpha)]+\sum_{i=1}^{n_2}[-\frac{\gamma_2-1}{1-(1-x_i^\alpha)^\beta}(1-x_i^\alpha)^\beta ln(1-x_i^\alpha)+$$

$$\frac{\gamma_1(1-(1-x_i^\alpha)^\beta)^{\gamma_1-1}}{1-(1-(1-x_i^\alpha)^\beta)^{\gamma_1}}(1-x_i^\alpha)^\beta ln(1-x_i^\alpha)]$$

$$\frac{\partial \ell}{\partial \gamma_1} = \frac{n_1}{\gamma_1} + \sum_{i=0}^{n_1} ln(1-(1-x_i^\alpha)^\beta) - \sum_{i=0}^{n_2}\frac{(1-(1-x_i^\alpha)^\beta)^{\gamma_1} ln(1-(1-x_i^\alpha)^\beta)}{1-(1-(1-x_i^\alpha)^\beta)^{\gamma_1}}$$

$$\frac{\partial \ell}{\partial \gamma_2} = \frac{n_2}{\gamma_2} + \sum_{i=0}^{n_2} ln(1-(1-x_i^\alpha)^\beta) - \sum_{i=0}^{n_1}\frac{(1-(1-x_i^\alpha)^\beta)^{\gamma_2} ln(1-(1-x_i^\alpha)^\beta)}{1-(1-(1-x_i^\alpha)^\beta)^{\gamma_2}}$$

For evaluating the MLEs of $\alpha, \beta, \gamma_1, \gamma_2$ we have to simultaneously solve the set of equations given above. As they do not have a closed form solution, therefore we have to use some other numerical methods.

4.1.3 Algorithm:

1. Create a sample of N points (10^6, in this case) from the Kumaraswamy Exponential distribution for some parameter values.
2. We use Inverse Sampling technique, calculating the inverse quantile function and using it to generate samples from the distribution.
3. We employ Bootstrapping where smaller set of samples (of size n), is taken without replacement and the MLE is calculated for them. (say $\hat{}^{i}$)
4. We repeat the process for different smaller samples of size n for B times (here B represent the times we bootstrapped the data)
5. We have taken 3 different values for n namely 20, 50 and 100 and corresponding to them the values of are B= $5*10^4, 2*10^4$ and 10^4 respectively.
6. The average of each iteration's estimate is taken as the parameter value. ($\hat{\theta} = \sum \hat{\theta}_i / B$)
7. Variance or Error is calculated as

$$var(\hat{\theta}) = \sum (\hat{\theta} - \theta_i)^2 B/(B-1)$$

4.2 Bayesian estimation of Gamma-Gompertz Distribution

Suppose we have a random sample of n independent and identically distributed observations $x_1, x_2, \ldots, x_n$ from a Gamma-Gompertz distribution with unknown parameters α, λ and θ. Our goal is to estimate these parameters using Bayesian inference, assuming a prior distribution for the parameters.

Let $f(x_i|\alpha, \beta)$ denote the probability density function of the Gamma-Gompertz distribution for observation x_i, and let $p(\alpha)$, $p(\lambda)$ and $p(\theta)$ denote the prior distributions for the parameters respectively. The joint posterior distribution for α, λ and θ given the data is proportional to the product of the likelihood function and the prior distributions:

$$p(\alpha, \lambda, \theta|x_1, x_2, \ldots, x_n) \propto \prod_{i=1}^{n} f(x_i|\alpha, \lambda, \theta) \cdot p(\alpha) \cdot p(\lambda) \cdot p(\theta)$$

To obtain the posterior distribution, we can use a Markov Chain Monte Carlo (MCMC) method, such as the Metropolis-Hastings algorithm, to sample from the joint posterior distribution of parameters. The sampled values can then be used to estimate the parameters and calculate confidence intervals or other statistics.

4.2.1 Gamma Prior

Suppose each parameter follows a gamma distribution with $(\delta_i, 1)$. Then prior joint distribution is :

$$p(\alpha, \lambda, \theta) = \frac{1}{\Gamma(\delta_1)}\alpha^{\delta_1 - 1}e^{-\alpha}\frac{1}{\Gamma(\delta_2)}\lambda^{\delta_1 - 1}e^{-\lambda}\frac{1}{\Gamma(\delta_3)}\theta^{\delta_3 - 1}e^{-\theta}$$

4.2.2 Exponential Prior

Suppose each parameter follows a exponential distribution with $(\alpha, \lambda, \theta)$. Then prior joint distribution is :-

$$p(\alpha, \lambda, \theta) = e^{-(\alpha+\theta+\lambda)}$$

4.2.3 Uniform Prior

Suppose each parameter follows a exponential distribution with $(1/l_1, 1/l_2, 1/l_3)$. Then prior joint distribution is

$$p(\alpha, \lambda, \theta) = 1/(l_1 l_2 l_3)$$

where l_i is the length of respective interval of distribution

4.2.4 Algorithm:

1. Create a sample of N points (104, in this case) from the distribution for some parameter values. We Utilize the Metropolis Hasting algorithm to sample from the data, taking standard normal distribution as our proposal distribution.

2. Prior distribution for the parameters is selected from the three distributions (gamma, exponential and uniform) .

3. Posterior distribution is defined as the product of the prior and likelihood (or, log(posterior) = log(prior)+ log(likelihood). As the close form of the posterior don't exist, we use Metropolis Hasting Algorithm to sample from the posterior distribution.

4. Now by Central Limit Theorem, we can say that the sample mean tends towards the distribution mean which is our estimate for the parameters. we say $\sum x_i/n \to \bar{x}$ and $\bar{x}$ is our estimate value.(where $\bar{x}$ represents mean of distribution)

5. Variance of posterior distribution is calculated as well as Confidence Interval are calculated, to check the quality of our estimate.

5 Results

5.1 2-competing risk Kumaraswamy Exponential MLE:

n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma_1}$	$\hat{\gamma_2}$	$var(\hat{\alpha})$	$var(\hat{\beta})$	$var(\hat{\gamma_1})$	$var(\hat{\gamma_2})$
20	0.7452	0.4861	1.9748	2.881	0.2759	0.1996	0.2266	0.0741
50	0.8050	0.5734	2.0041	2.8968	0.2748	0.2474	0.24927	0.0590
100	0.8050	0.5734	2.0041	2.8968	0.2748	0.2474	0.2492	0.0590

Table 1: $\alpha = 0.5$ $\beta = 0.5$ $\gamma_1 = 2$ $\gamma_2 = 3$

n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma_1}$	$\hat{\gamma_2}$	$var(\hat{\alpha})$	$var(\hat{\beta})$	$var(\hat{\gamma_1})$	$var(\hat{\gamma_2})$
20	1.0304	1.2513	1.6878	2.5177	0.8672	3.6430	2.0415	2.9507
50	1.1076	1.5940	1.2634	1.8422	0.3877	2.5238	0.6407	0.8439
100	0.9078	1.3238	1.0014	1.7037	0.2164	1.9010	0.4874	0.4223

Table 2: $\alpha = 0.5$ $\beta = 0.5$ $\gamma_1 = 1$ $\gamma_2 = 2$

Here, n represents the number of points taken out of sample to calculate the MLE, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma_1}$ and $\hat{\gamma_2}$ denote the values of the MLE estimates, whereas $var(\hat{\alpha})$, $var(\hat{\beta})$, $var(\hat{\gamma_1})$ and $var(\hat{\gamma_2})$ denote the Error/ Variance of these parameters.

n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma_1}$	$\hat{\gamma_2}$	$var(\hat{\alpha})$	$var(\hat{\beta})$	$var(\hat{\gamma_1})$	$var(\hat{\gamma_2})$
20	4.1404	1.3628	2.7249	3.2163	6.0432	1.6581	1.4348	2.9123
50	4.7311	1.6785	2.4571	3.0213	9.0456	4.0155	1.0024	2.8375
100	4.4999	1.6997	2.6428	2.8228	5.0779	2.8464	1.8330	1.1250

Table 3: $\alpha = 5$ $\beta = 2$ $\gamma_1 = 3$ $\gamma_2 = 3$

We can see that the results are not quite good in case of MLE, because finding the Solution is a tedious task even with numerical methods and fast computers. We must other methods to find the roots or can try some other optimisation techniques.

5.2 Estimation using Bayesian Estimation for Gamma-Gompertz

5.2.1 Gamma Prior(shape=1,scale=1)

$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$Var(\alpha)$	$var(\lambda)$	$var(\theta)$	$CI_{\alpha,95}$	$CI_{\lambda,95}$	$CI_{\theta,95}$	$CI_{\alpha,99}$	$CI_{\lambda,99}$	$CI_{\theta,99}$
1.470	2.498	1.006	0.450	1.165	0.825	[1.344, 1.608]	[2.353, 2.696]	[0.979, 1.051]	[0, 1.637]	[0, 2.754]	[0, 1.062]

Table 4: $\alpha = 1.5$ $\lambda = 2.5$ $\theta = 1$

$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$Var(\alpha)$	$var(\lambda)$	$var(\theta)$	$CI_{\alpha,95}$	$CI_{\lambda,95}$	$CI_{\theta,95}$	$CI_{\alpha,99}$	$CI_{\lambda,99}$	$CI_{\theta,99}$
2.349	1.613	1.041	0.811	0.320	0.690	[2.217, 2.512]	[1.490, 1.769]	[1.008, 1.093]	[0, 2.547]	[0, 1.797]	[0, 1.104]

Table 5: $\alpha = 2.5$ $\lambda = 1.5$ $\theta = 1$

Here, $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\theta}$ denote the values of the mean of the Posterior distribution, whereas $var(\hat{\alpha})$, $var(\hat{\lambda})$ and $var(\hat{\theta})$ denote the Variance of these posterior. Also, $CI_{\alpha,95}$ denote the Confidence interval of 95% for α (similarly others are defined).

5.2.2 Exponential Prior(Rate=1)

Here, $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\theta}$ denote the values of the mean of the Posterior distribution, whereas $var(\hat{\alpha})$, $var(\hat{\lambda})$ and $var(\hat{\theta})$ denote the Variance of these posterior. Also, $CI_{\alpha,95}$ denote the Confidence interval of 95% for α (similarly others are defined).

$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$Var(\alpha)$	$var(\lambda)$	$var(\theta)$	$CI_{\alpha,95}$	$CI_{\lambda,95}$	$CI_{\theta,95}$	$CI_{\alpha,99}$	$CI_{\lambda,99}$	$CI_{\theta,99}$
1.452	2.523	1.009	0.474	1.219	0.839	[1.311, 1.615]	[2.333, 2.739]	[0.977, 1.057]	[0, 1.662]	[0, 2.823]	[0, 1.073]

Table 6: $\alpha = 1.5$ $\lambda = 2.5$ $\theta = 1$

$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$Var(\alpha)$	$var(\lambda)$	$var(\theta)$	$CI_{\alpha,95}$	$CI_{\lambda,95}$	$CI_{\theta,95}$	$CI_{\alpha,99}$	$CI_{\lambda,99}$	$CI_{\theta,99}$
2.345	1.616	1.042	0.8046	0.3186	0.6875	[2.2164, 2.520]	[1.4763, 1.772]	[1.005, 1.094]	[0, 2.575]	[0, 1.8128]	[0, 1.108]

Table 7: $\alpha = 2.5$ $\lambda = 1.5$ $\theta = 1$

5.2.3 Uniform Prior([0,10])

$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$Var(\alpha)$	$var(\lambda)$	$var(\theta)$	$CI_{\alpha,95}$	$CI_{\lambda,95}$	$CI_{\theta,95}$	$CI_{\alpha,99}$	$CI_{\lambda,99}$	$CI_{\theta,99}$
1.452	2.524	1.010	0.475	1.222	0.840	[1.315, 1.611]	[2.329, 2.738]	[0.976, 1.0587]	[0, 1.663]	[0, 2.805]	[0, 1.072]

Table 8: $\alpha = 1.5$ $\lambda = 2.5$ $\theta = 1$

Here, $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\theta}$ denote the values of the mean of the Posterior distribution, whereas $var(\hat{\alpha})$, $var(\hat{\lambda})$ and $var(\hat{\theta})$ denote the Variance of these posterior. Also, $CI_{\alpha,95}$ denote the Confidence interval of 95% for α (similarly others are defined).

$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$Var(\alpha)$	$var(\lambda)$	$var(\theta)$	$CI_{\alpha,95}$	$CI_{\lambda,95}$	$CI_{\theta,95}$	$CI_{\alpha,99}$	$CI_{\lambda,99}$	$CI_{\theta,99}$
2.355	1.672	1.064	0.822	0.358	0.695	[1.910, 2.845]	[1.255, 2.148]	[0.936, 1.216]	[0, 2.938]	[0, 2.273]	[0, 1.254]

Table 9: $\alpha = 2.5$ $\lambda = 1.5$ $\theta = 1$

6 Scope Of Work:

6.1 Kumaraswamy Distribution

As the results for the MLE equations for Kumaraswamy are not very promising, we can use some other numerical methods to better find the estimators or can use some meta-heuristic methods to maximise the likelihood function.

Also, we can then remove the condition on α and β to find the parameter for a 6-parameter distribution, with a more complex likelihood function. Further, we can shift from MLE to Bayesian analysis for the same.

6.2 Gamma-Gompertz

For Gamma-Gompertz, we can shift from complete data to censored data with type-I or type-II censoring, and then use different methods to find the parameters.

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