Stochastic Ordered Empirical Risk Minimization

Joint Statistical Meetings 2022

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Team

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Motivation: Average-Case ⟶ Worst-Case

- **Current learning paradigm:** optimize average loss across training examples.
- Worst-case performance can often be more important than average-case.

‘I’m the Operator’: The Aftermath of a Self-Driving Tragedy

In 2018, an Uber autonomous vehicle fatally struck a pedestrian. In a WIRED exclusive, the human behind the wheel finally speaks.

2 Killed in Driverless Tesla Car Crash, Officials Say

“No one was driving the vehicle” when the car crashed and burst into flames, killing two men, a constable said.

A Tesla driver is charged in a crash involving Autopilot that killed 2 people

January 18, 2022 · 3:00 PM ET

Usual Setting

- $\ell_i(w) :=$ loss on example i with parameters/weights $w \in \mathbb{R}^d$.
- $\ell_{(i)}(w) :=$ i-th order statistic of $\ell(w) := \big(\ell_1(w), \ldots, \ell_n(w)\big)$.

Empirical Risk Minimization (ERM):

$$\min_w \sum_{i=1}^{n} \frac{1}{n} \ell_i(w)$$

Our Setting

- $\ell_i(w) :=$ loss on example i with parameters/weights $w \in \mathbb{R}^d$.
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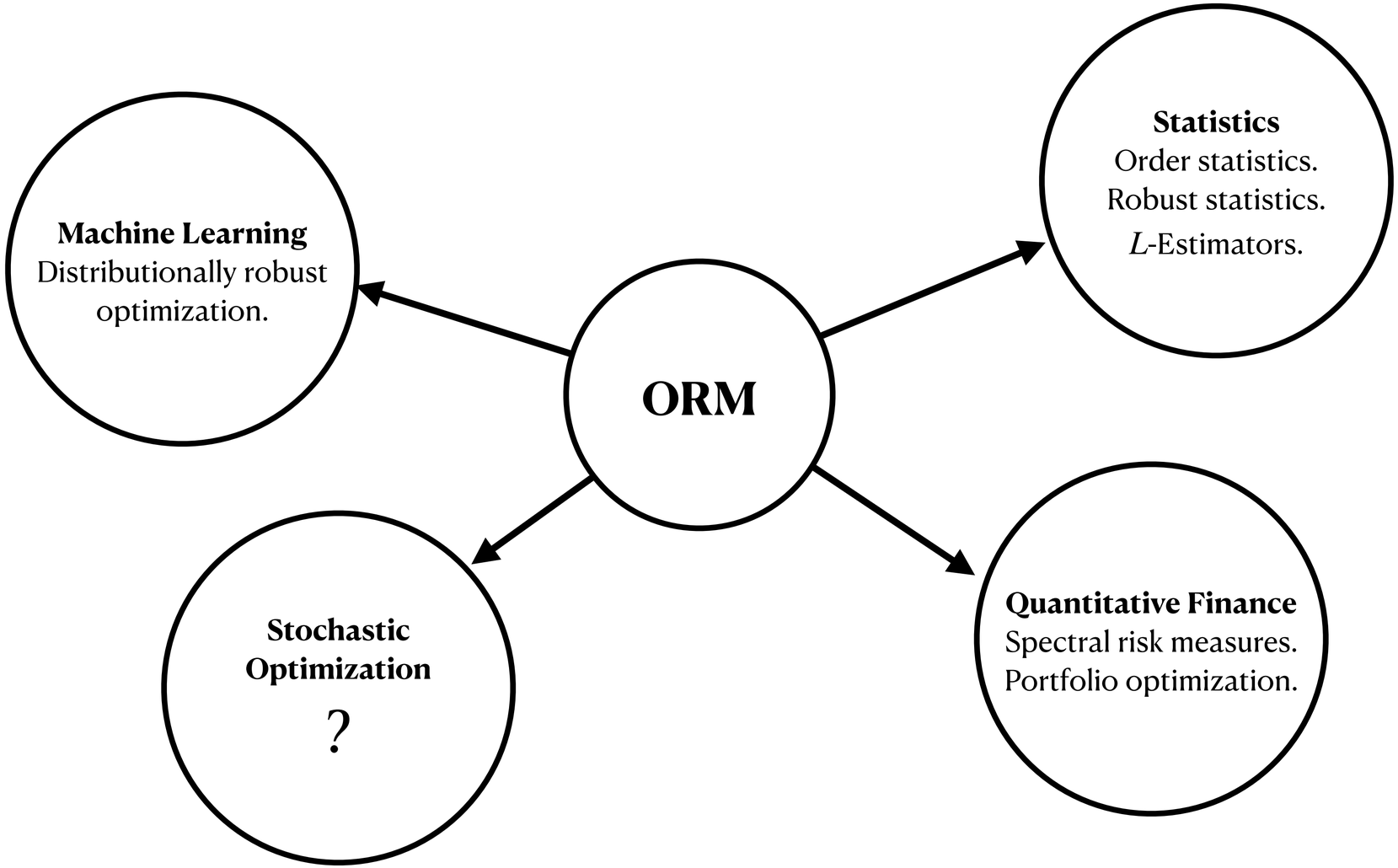
Ordered Empirical Risk Minimization (ORM):

$$\min_w \sum_{i=1}^n \frac{1}{n} \ell_i(w) \quad \longrightarrow \quad \min_w \sum_{i=1}^n \alpha_i \ell_{(i)}(w)$$

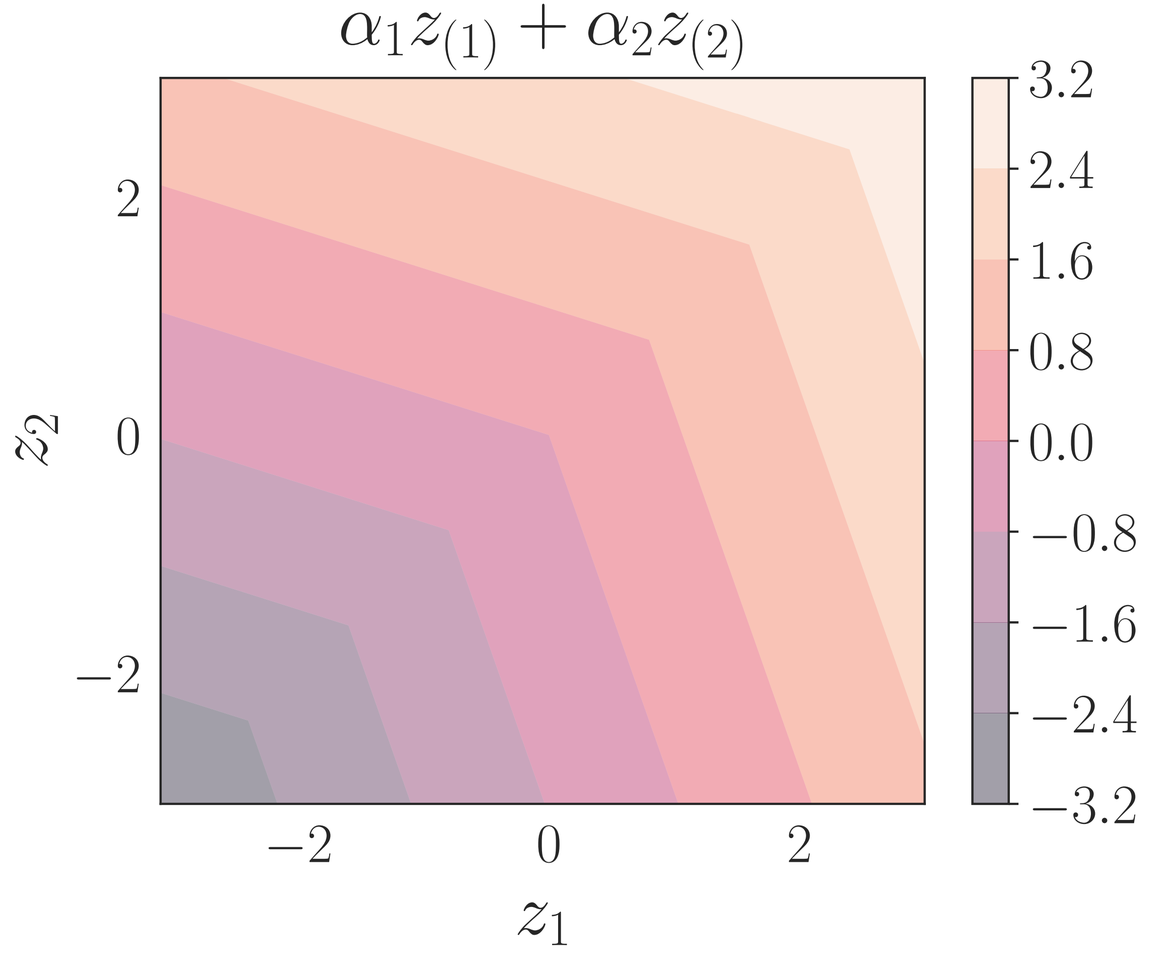
$$0 \le \alpha_1 \le \ldots \le \alpha_n, \sum_{i=1}^n \alpha_i = 1$$

Related Work



Objective is Piecewise Linear

$$f(z_1, z_2) = 0.3z_{(1)} + 0.7z_{(2)}$$



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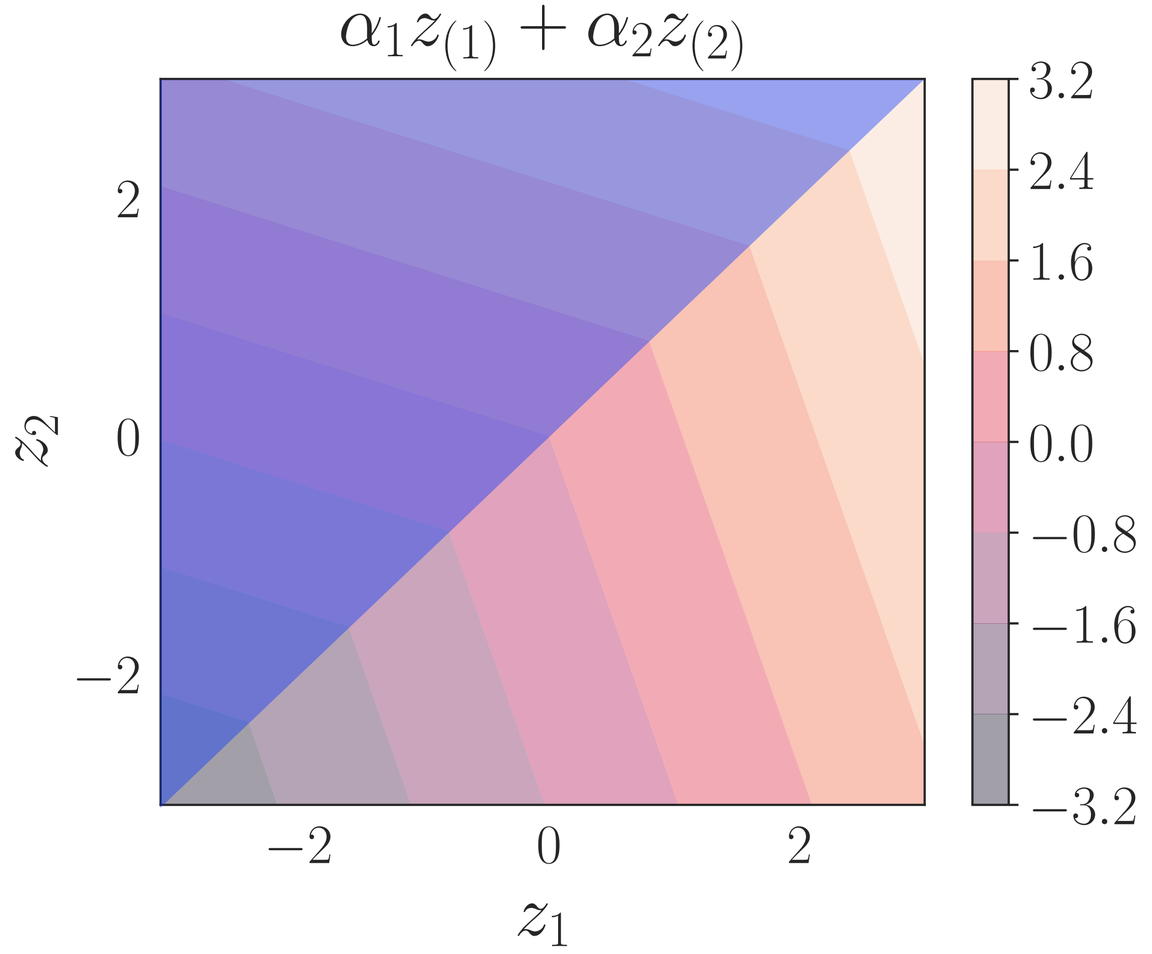

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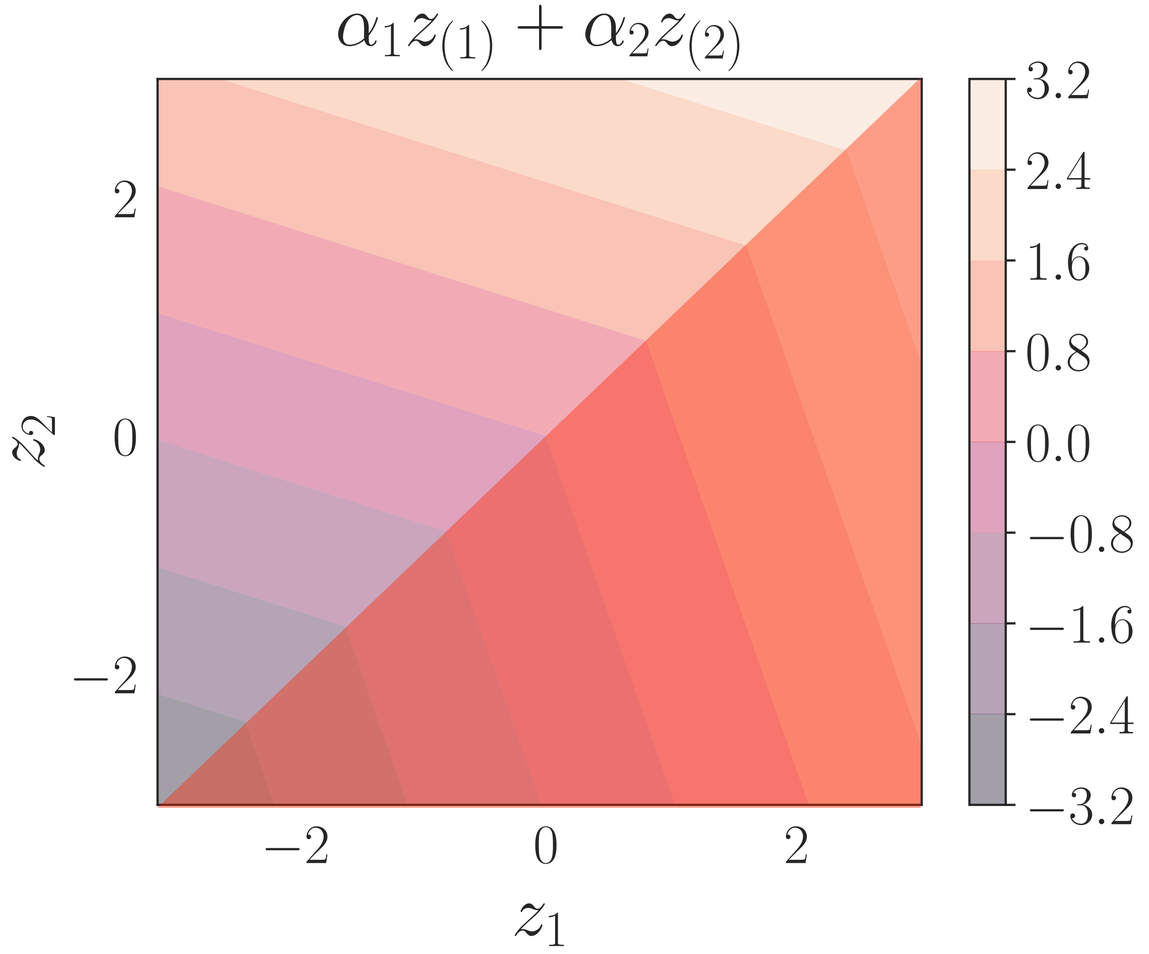
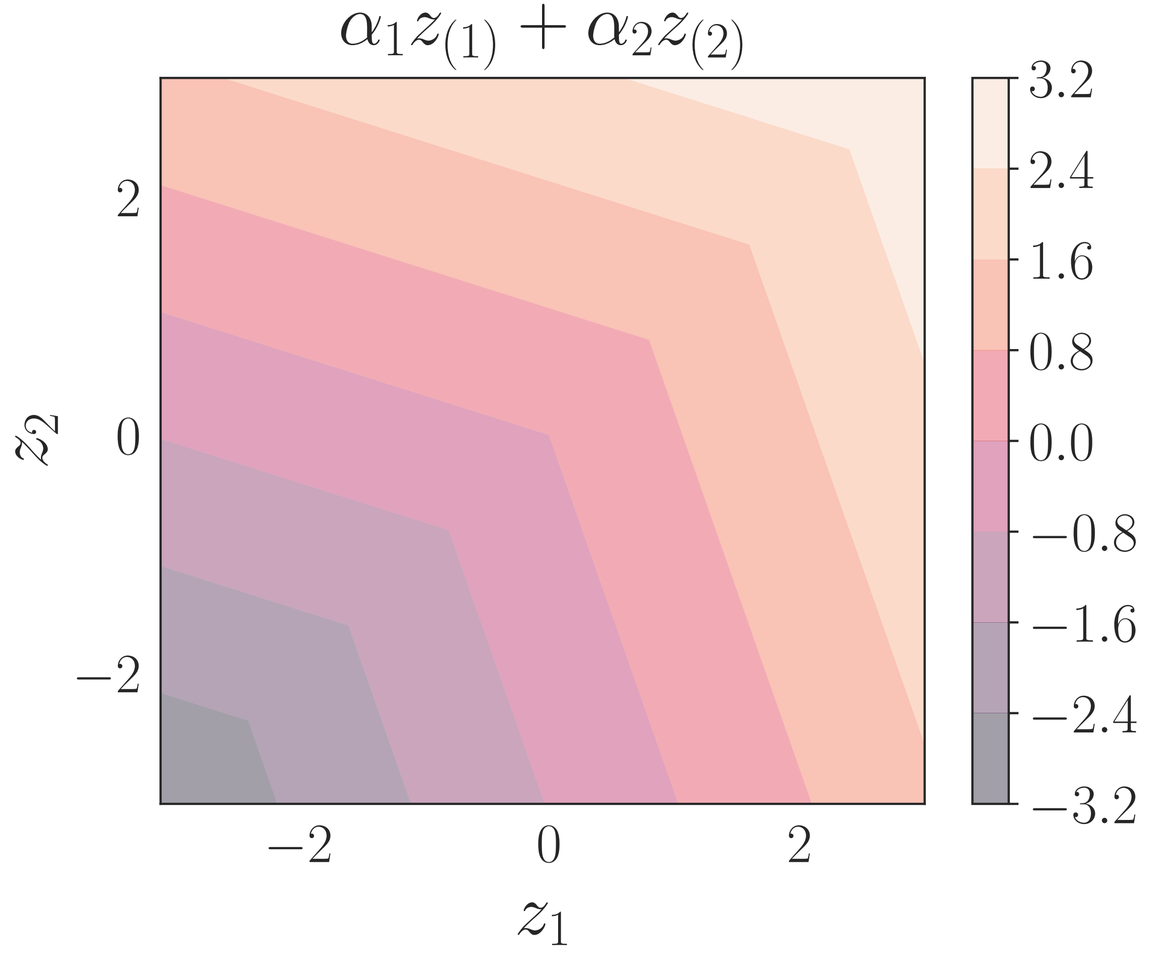


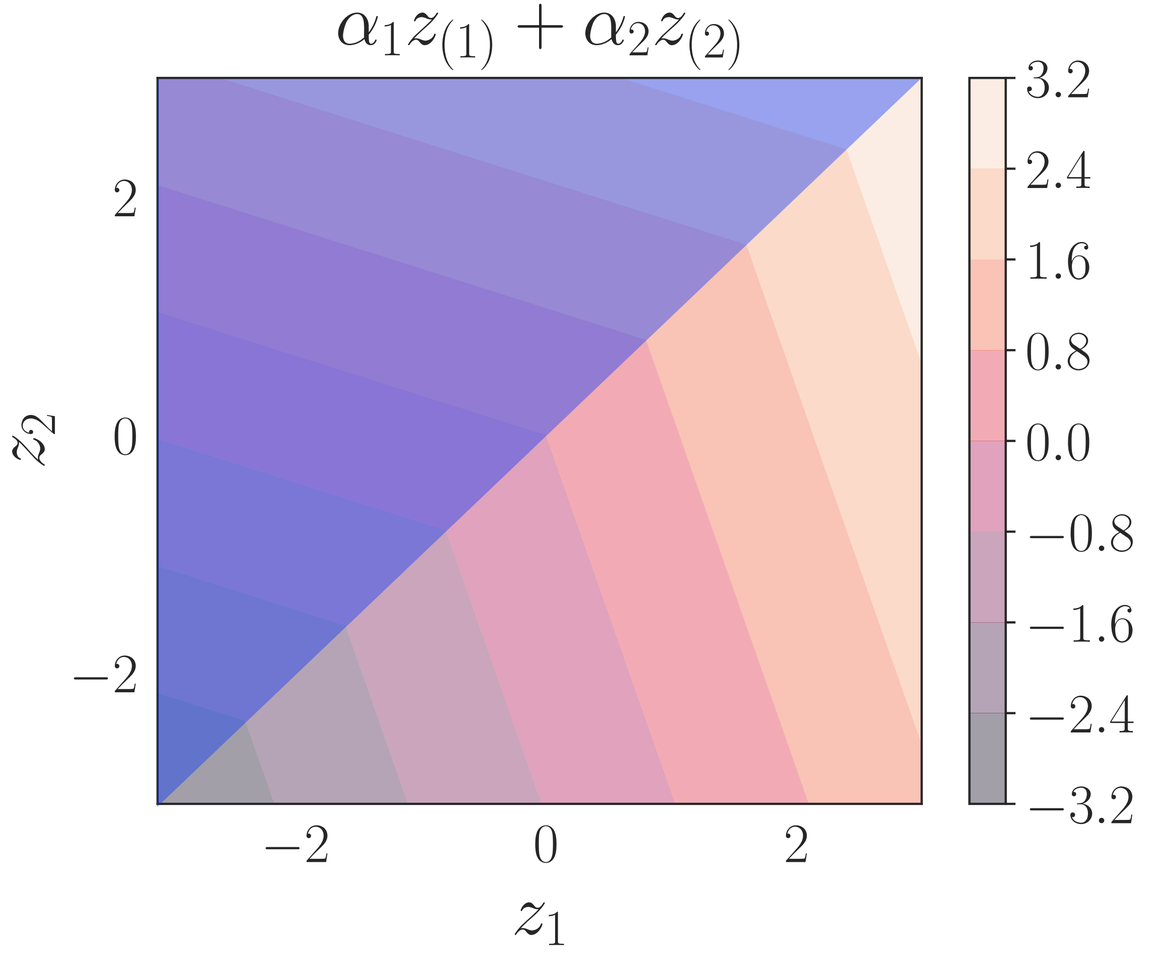
Objective is Piecewise Linear

$$f(z_1, z_2) = \max_{\pi} 0.3z_{\pi(1)} + 0.7z_{\pi(2)}$$



Objective is Piecewise Linear

$$f(z_1, z_2) = \max_{\pi} 0.3 z_{\pi(1)} + 0.7 z_{\pi(2)}$$
$$= 0.3 z_{\pi^*(1)} + 0.7 z_{\pi^*(2)}$$
$$\pi^*(1) = 1$$
$$\pi^*(2) = 2$$



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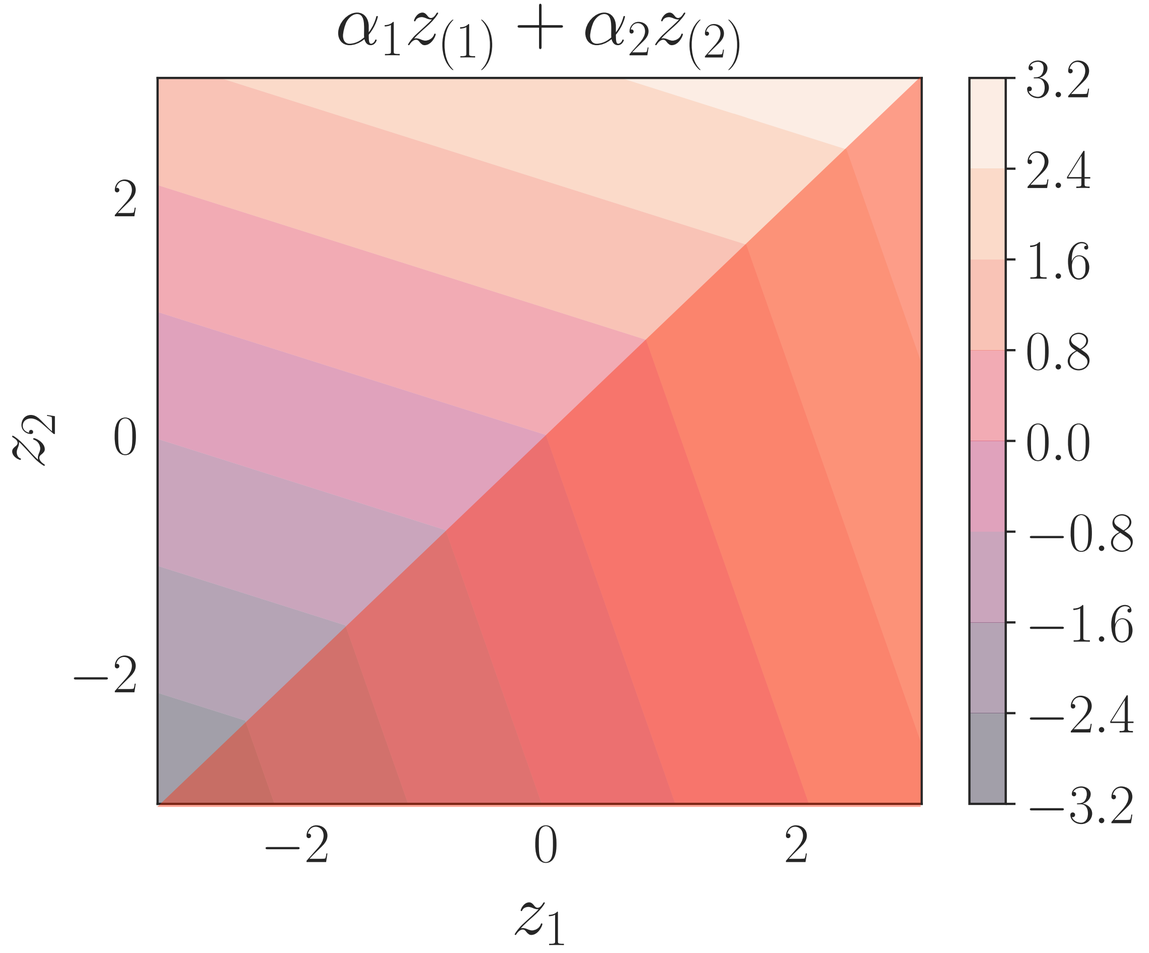

Optimization Properties

$$\sum_{i=1}^{n} \alpha_i \cdot \ell_{(i)}(w) = \max_{\pi} \sum_{i=1}^{n} \alpha_i \cdot \ell_{\pi(i)}(w)$$

- Convex if the ℓ_i's are.

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- Non-smooth.

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- Convex if the ℓ_i's are.
- Non-smooth.
- Gradient: $\sum_{i=1}^{n} \alpha_i \nabla \ell_{(i)}(w)$.

Optimization Properties

$$\sum_{i=1}^{n} \alpha_i \cdot \ell_{(i)}(w) = \max_{\pi} \sum_{i=1}^{n} \alpha_i \cdot \ell_{\pi(i)}(w)$$

- Convex if the ℓ_i's are.
- Non-smooth.
- ~~Gradient~~ Subdifferential: $\operatorname{conv}\left\{ \sum_{i=1}^{n} \alpha_i \nabla \ell_{\pi(i)}(w) : \pi \text{ achieves max} \right\}$.

Algorithms

- **Brute Force:** (batch) gradient descent: $w_{t+1} \leftarrow w_t - \eta_t \sum_{i=1}^{n} \alpha_i \nabla \ell_{(i)}(w_t)$, where η_t is a learning rate.

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- **Problem:** $O(n \log n)$ per-iteration time complexity. Need to know α_i weights for unbiased mini-batch estimate of gradient.

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- **Brute Force:** (batch) gradient descent: $w_{t+1} \leftarrow w_t - \eta_t \sum_{i=1}^{n} \alpha_i \nabla \ell_{(i)}(w_t)$, where η_t is a learning rate.
- **Problem:** $O(n \log n)$ per-iteration time complexity. Need to know α_i weights for unbiased mini-batch estimate of gradient.
- **Contributions:** stochastic optimization algorithms for ORM objective with $O(1)$ gradient calls per-iteration and extremely fast convergence empirically!

Stochastic Gradient Descent (SGD)

- **Initialization:** Set $w_0 = 0 \in \mathbb{R}^d$.
- **Update:**
  - Sample i_t uniformly.
  - Compute $g_t = \nabla \ell_{i_t}(w_t)$.
  - Set $w_{t+1} = w_t - \eta_t g_t$.

Stochastic Average Gradient Accelerated (SAGA)

- **Initialization:** Set $w_0 = 0 \in \mathbb{R}^d$, store table of gradients $g^{(i)} = \nabla \ell_i(w_0)$ for $i = 1,\ldots,n$, and compute $\bar{g}_0 = \sum_{i=1}^{n} \frac{1}{n} g^{(i)}$.
- **Update:**
  - Sample i_t uniformly.
  - Compute $v_t = \nabla \ell_{i_t}(w_t) - g^{(i_t)} + \bar{g}_t$.
  - Set $w_{t+1} = w_t - \eta_t v_t$.

Stochastic Average Gradient Accelerated (SAGA)

- **Initialization:** Set $w_0 = 0 \in \mathbb{R}^d$, store table of gradients $g^{(i)} = \nabla \ell_i(w_0)$ for $i = 1,\ldots,n$, and compute $\bar{g}_0 = \sum_{i=1}^{n} \frac{1}{n} g^{(i)}$.
- **Update:**
  - Sample i_t uniformly
  - Compute $v_t = \nabla \ell_{i_t}(w_t) - g^{(i_t)} + \bar{g}_t$.
  - Set $w_{t+1} = w_t - \eta_t v_t$.

v_t is an unbiased estimate of the gradient!

$$\mathbb{E}_{i_t|w_t}\left[v_t\right] = \mathbb{E}\left[\nabla \ell_{i_t}(w_t)\right] + \mathbb{E}\left[-g^{(i_t)} + \bar{g}_t\right] = \sum_{i=1}^{n} \frac{1}{n} \nabla \ell_i(w_t)$$

Stochastic Average Gradient Accelerated (SAGA)

- **Initialization:** Set $w_0 = 0 \in \mathbb{R}^d$, store table of gradients $g^{(i)} = \nabla \ell_i(w_0)$ for $i = 1,\ldots,n$, and compute $\bar{g}_0 = \sum_{i=1}^{n} \frac{1}{n} g^{(i)}$.
- **Update:**
  - Sample i_t uniformly.
  - Compute $v_t = \nabla \ell_{i_t}(w_t) - g^{(i_t)} + \bar{g}_t$.
  - Set $w_{t+1} = w_t - \eta_t v_t$.
  - Sample j_t uniformly and update $\bar{g}_{t+1} = \bar{g}_t - \frac{1}{n} g^{(j_t)} + \frac{1}{n} \nabla \ell_{j_t}(w_{t+1})$ and $g^{(j_t)} = \nabla \ell_{j_i}(w_{t+1})$.

Ordered-SAGA (O-SAGA)

- **Initialization:** Set $w_0 = 0 \in \mathbb{R}^d$, store table of losses $\ell^{(i)} = \ell_i(w_0)$ and gradients $g^{(i)} = \nabla \ell_i(w_0)$ for $i = 1,\ldots,n$. Compute $\pi =$ argsort of $\ell^{(i)}$'s and store $\bar{g}_0 = \sum_{i=1}^{n} \alpha_i g^{(\pi(i))}$.
- **Update:**
  - Sample i_t uniformly.
  - Compute $v_t = n\alpha_{i_t} \nabla \ell_{\pi(i_t)}(w_t) - n\alpha_{i_t} g^{(\pi(i_t))} + \bar{g}_t$.
  - Set $w_{t+1} = w_t - \eta_t v_t$.
  - Sample j_t uniformly and update $\ell^{(j_t)} = \ell_{j_t}(w_{t+1})$, $g^{(j_t)} = \nabla \ell_{j_t}(w_{t+1})$, sorting vector π and $\bar{g}_{t+1}$ simultaneously.

O-SAGA Update Direction is Biased

$$\mathbb{E}_{i_t|w_t}\left[v_t\right] = \mathbb{E}\left[n\alpha_{i_t}\nabla\ell_{\pi(i_t)}(w_t) - n\alpha_{i_t}g^{(\pi(i_t))} + \bar{g}_t\right]$$

$$= \sum_{i=1}^{n}\alpha_i\nabla\ell_{\pi(i)}(w_t) - \sum_{i=1}^{n}\alpha_i g^{(\pi(i))} + \bar{g}_t(w_t)$$

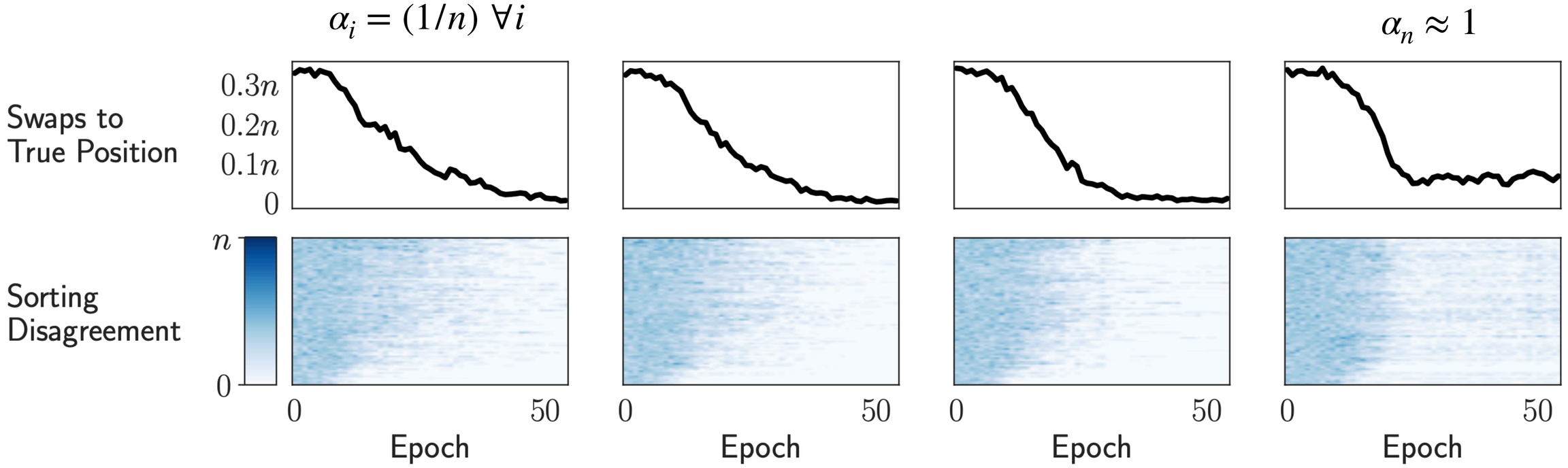
$$= \sum_{i=1}^{n}\alpha_i\nabla\ell_{\pi(i)}(w_t)$$ Sorting from running table (could be stale).

$$\neq \sum_{i=1}^{n}\alpha_i\nabla\ell_{(i)}(w_t)$$ Sorting from $\ell_1(w_t), \ldots, \ell_n(w_t)$ (true gradient).

Bias Decreases with Iteration Number

Error between π and true argsort stabilizes quickly,

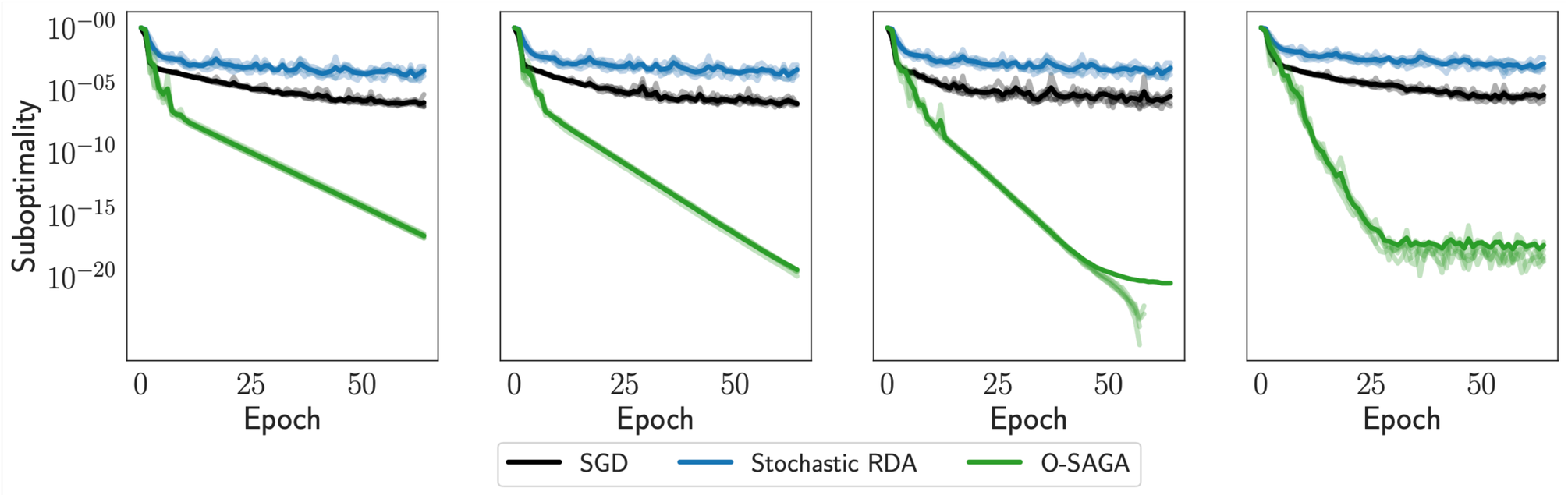
so using the "wrong" argsort still works.



O-SAGA Converges Linearly and Dominates Baselines

$$\text{Suboptimality} = \frac{\text{current value} - \text{min value}}{\text{initial value} - \text{min value}}$$

$\alpha_i = (1/n)\ \forall i$ $\alpha_n \approx 1$



Summary

We present a stochastic algorithm to optimize non-smooth L-statistics of the empirical loss distribution, that

- finds an exact minimizer,
- has bias proportional to suboptimality,
- makes $O(1)$ gradient calls per update, and
- dominates out-of-the-box convex optimizers on synthetic and real data.

Thank you!

Appendix

Notation

- $\ell(w) = \big(\ell_1(w), \ldots, \ell_n(w)\big)$.
- $\alpha_i\big(\ell(w)\big) = j$ such that $\ell_i(w)$ gets multiplied by α_j in $\sum_{i=1}^{n} \alpha_i \ell_{(i)}(w)$, i.e.

$$\sum_{i=1}^{n} \alpha_i \ell_{(i)}(w) = \sum_{i=1}^{n} \alpha_i\big(\ell(w)\big)\, \ell_i(w)$$

- $\alpha\big(\ell(w)\big) = \Big(\alpha_1\big(\ell(w)\big), \ldots, \alpha_n\big(\ell(w)\big)\Big)$.

ORM Optimization

- **Attempt 3:** minibatch stochastic regularized dual averaging (SRDA): out-of-the-box convex optimization method.
- **Problem:** Non-vanishing bias for estimating $\bar{g}_t$, unlike ERM.

ORM Optimization

- **Attempt 2:** minibatch stochastic subgradient method (SGD): $w_{t+1} \leftarrow w_t - \eta_t g_t$, where $g_t = \sum_{j=1}^{m} \alpha'_j \nabla \ell_{(j)}(w_t)$ and $\alpha'_j = \left(j/m\right)^r - ((j-1)/m)^r$.
- **Problem:** Non-vanishing bias for estimating $\bar{g}_t$, unlike ERM.

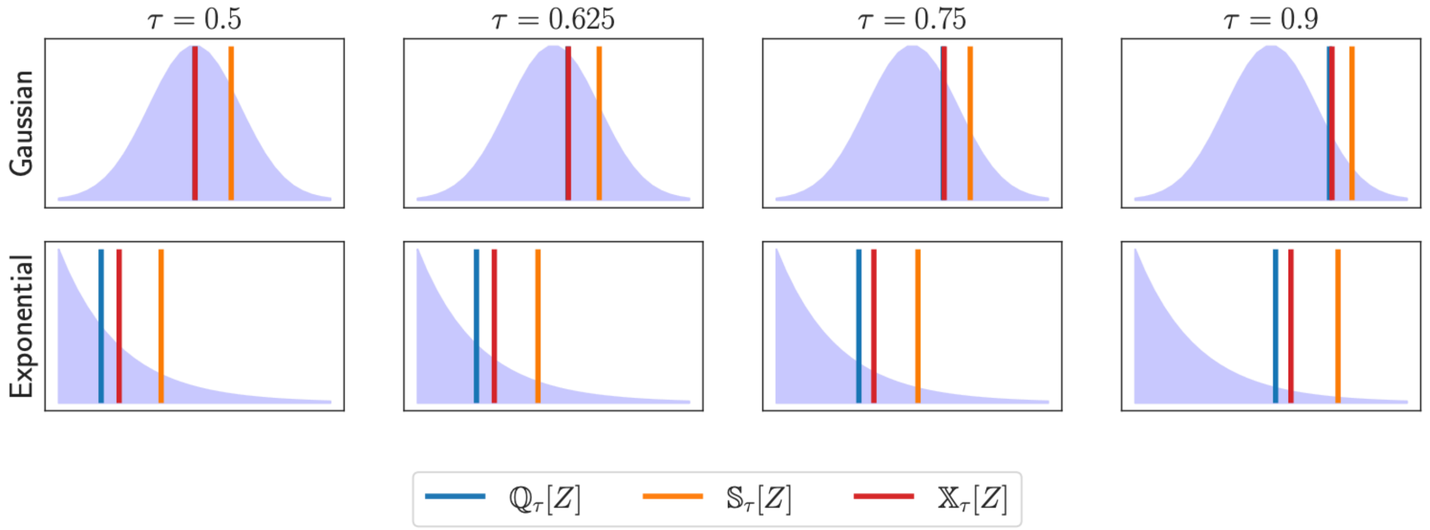
Background

- Let Z be an r.v. with CDF F and *quantile function* $F^{-1}(t) = \inf\{z : F(z) > t\}$.
- Let $a : [0,1] \to \mathbb{R}$ be a *weight function*.
- Population *L-functional* or *spectral risk measure*:

$$\mathbb{L}[Z] = \mathbb{L}[F] = \int F^{-1}(t) \cdot a(t)\ dt$$

Background

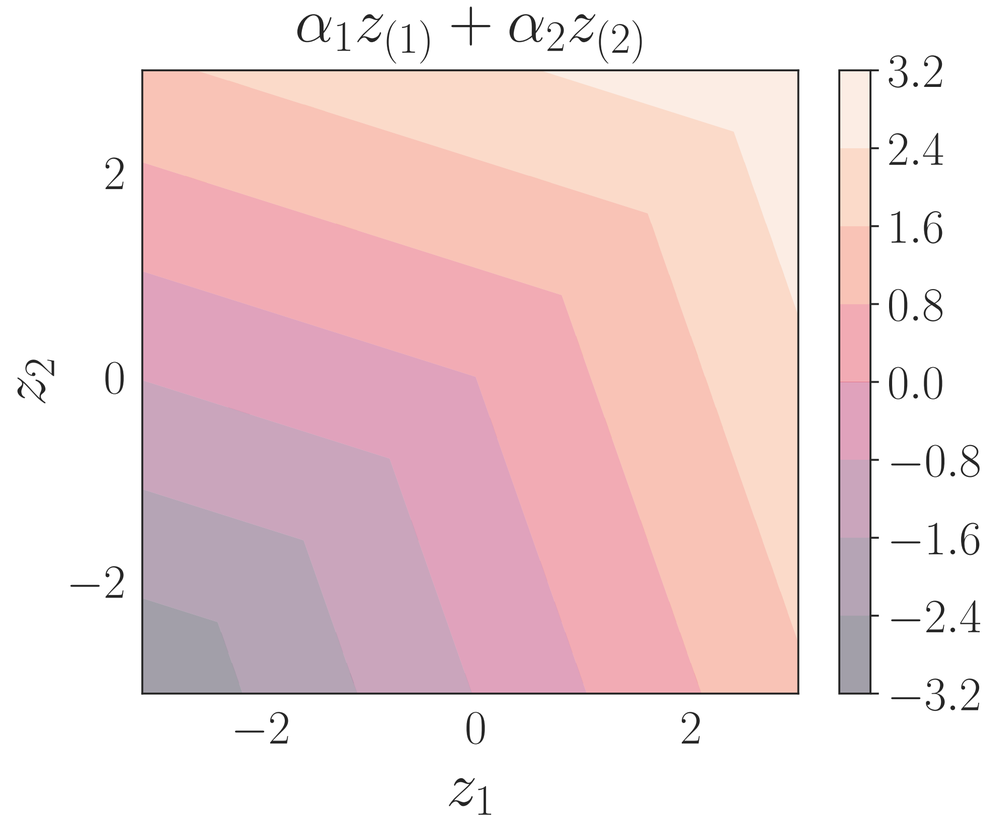
- τ-**Quantile:** $\mathbb{Q}_\tau[Z] = F^{-1}(\tau) \leftrightarrow a(t) =$ point mass at τ.
- τ-**Superquantile:** $\mathbb{S}_\tau[Z] = \mathbb{E}\left[Z \mid Z > \mathbb{Q}_\tau[Z]\right] \leftrightarrow a(t) = 1/(1-\tau) \cdot I_{[\tau,1]}(t)$.
- τ-**Extremile:** $\mathbb{X}_\tau[Z] = \mathbb{E}\left[\max\{Z_1, \ldots, Z_r\}\right] \leftrightarrow a(t) = t^r$ with $r = -(1/\log_2(\tau))$.

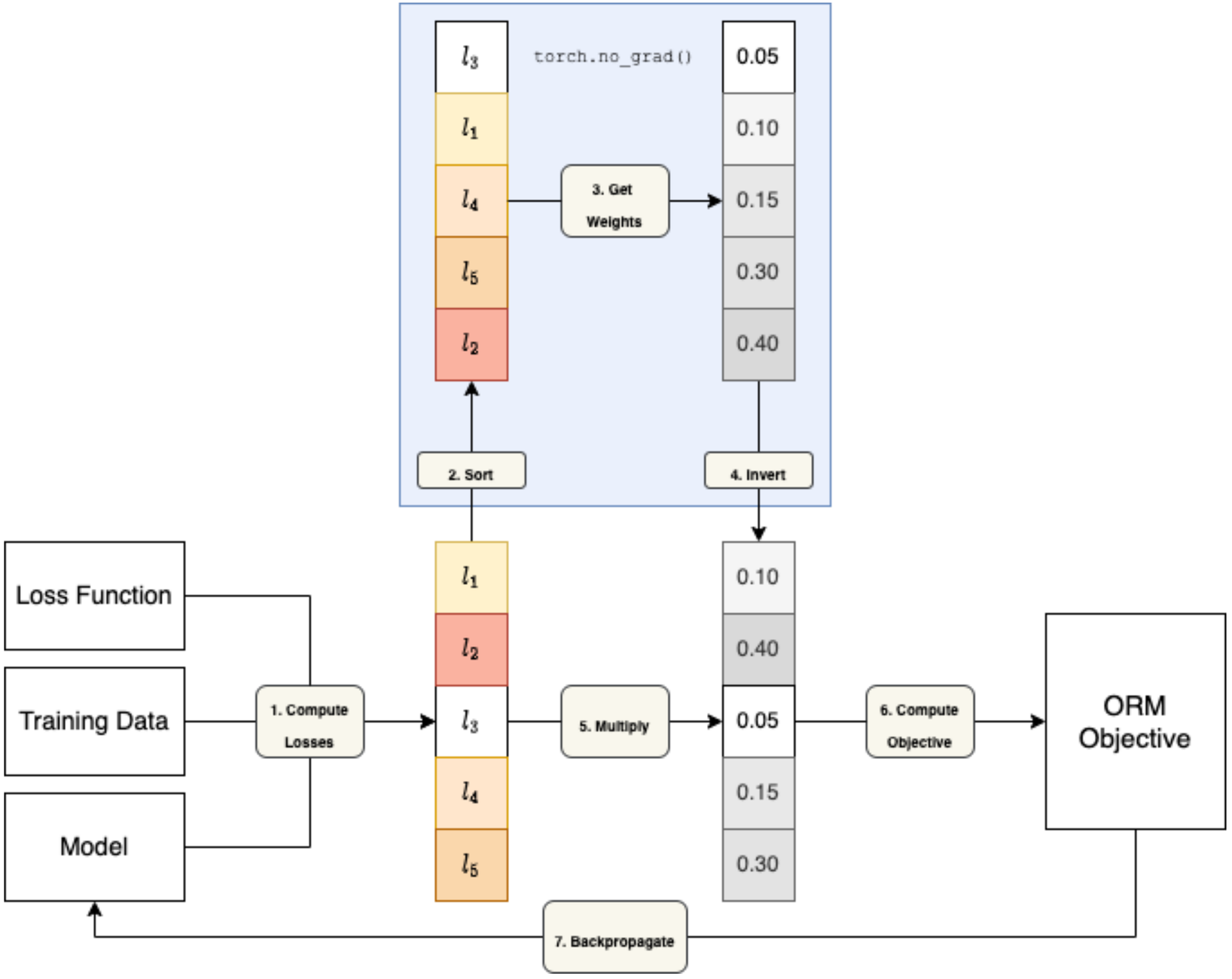


Background

- Consider sample $z_1, \ldots, z_n$ with empirical distribution F_n and *order statistics* $z_{(1)} \leq \ldots \leq z_{(n)}$.

$$
\begin{aligned}
\mathbb{L}[F_n] &= \int_0^1 F_n^{-1}(t) \cdot a(t)\ dt \\
&= \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z_{(i)} \cdot a(t)\ dt \\
&= \sum_{i=1}^{n} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} a(t)\ dt \right) z_{(i)} \\
&= \sum_{i=1}^{n} \alpha_i z_{(i)}
\end{aligned}
$$




torch.no_grad()
l_3
l_1
l_4
l_5
l_2
3. Get Weights
0.05
0.10
0.15
0.30
0.40
2. Sort
4. Invert
Loss Function
Training Data
Model
1. Compute Losses
l_1
l_2
l_3
l_4
l_5
5. Multiply
0.10
0.40
0.05
0.15
0.30
6. Compute Objective
ORM Objective
7. Backpropagate

ORM Objective

$$\sum_{i=1}^{n} \alpha_i \ell_{(i)}(w)$$

- Setting $\alpha_i = (1/n)$ recovers average, $\alpha_n = 1$ recovers maximum.
- Class of *superquantiles* $\alpha_i = 1/\big(n(1-\tau)\big)\, I_{i \geq \tau n}$ for $\tau \in [0,1)$.
- Class of *extremiles* $\alpha_i = (i/n)^r - ((i-1)/n)^r$ for $r \in [1,\infty)$.

Connecting Views

- Picture with combining densities and stuff

Setting: Population View

- Z is a random variable with iid copies $Z_1, \ldots, Z_n$ (e.g. training examples).
- Losses $\ell_i(w) = l(w, Z_i)$ are realizations of $l(w, Z) \sim F$.

Stochastic Optimization with Expectation:

$$\min_w \mathbb{E}\left[l(w, Z)\right]$$

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Stochastic Optimization with Tail Risk Measure:

$$\min_w \mathbb{E}\left[l(w, Z)\right] \longrightarrow \min_w \mathbb{T}\left[l(w, Z)\right]$$

Connecting Views

$$\sum_{i=1}^{n} x_i \cdot a\left(F_n(x_i)\right) \qquad \approx \qquad \int x \cdot a(F(x))\ dx$$

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$$\sum_{i=1}^{n} x_{(i)} \cdot \alpha_i$$

Connecting Views

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$$\sum_{i=1}^{n} x_{(i)} \cdot \alpha_i \quad \approx \quad \int_0^1 F^{-1}(t) \cdot a(t)\ dt$$

L-Statistics & Spectral Risk Measures

- LHS is an *L-statistic,* or linear combination of order statistics.
- RHS is a *spectral risk measure*, or weighted integral of a quantile function.

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Statistical View of Objective

$$\sum_{i=1}^{n} \alpha_i \cdot \ell_{(i)}(w) = \max_{\pi} \sum_{i=1}^{n} \alpha_i \cdot \ell_{\pi(i)}(w)$$

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- LHS is an *L-statistic*, or linear combination of order statistics.
- Order statistic $\ell_{(i)}(w)$ is the (i/n)-th quantile of the empirical distribution of losses.

Optimization View of Objective

$$\sum_{i=1}^{n} \alpha_i \cdot \ell_{(i)}(w) = \max_{\pi} \sum_{i=1}^{n} \alpha_i \cdot \ell_{\pi(i)}(w)$$

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- Non-smooth.
- Gradient:

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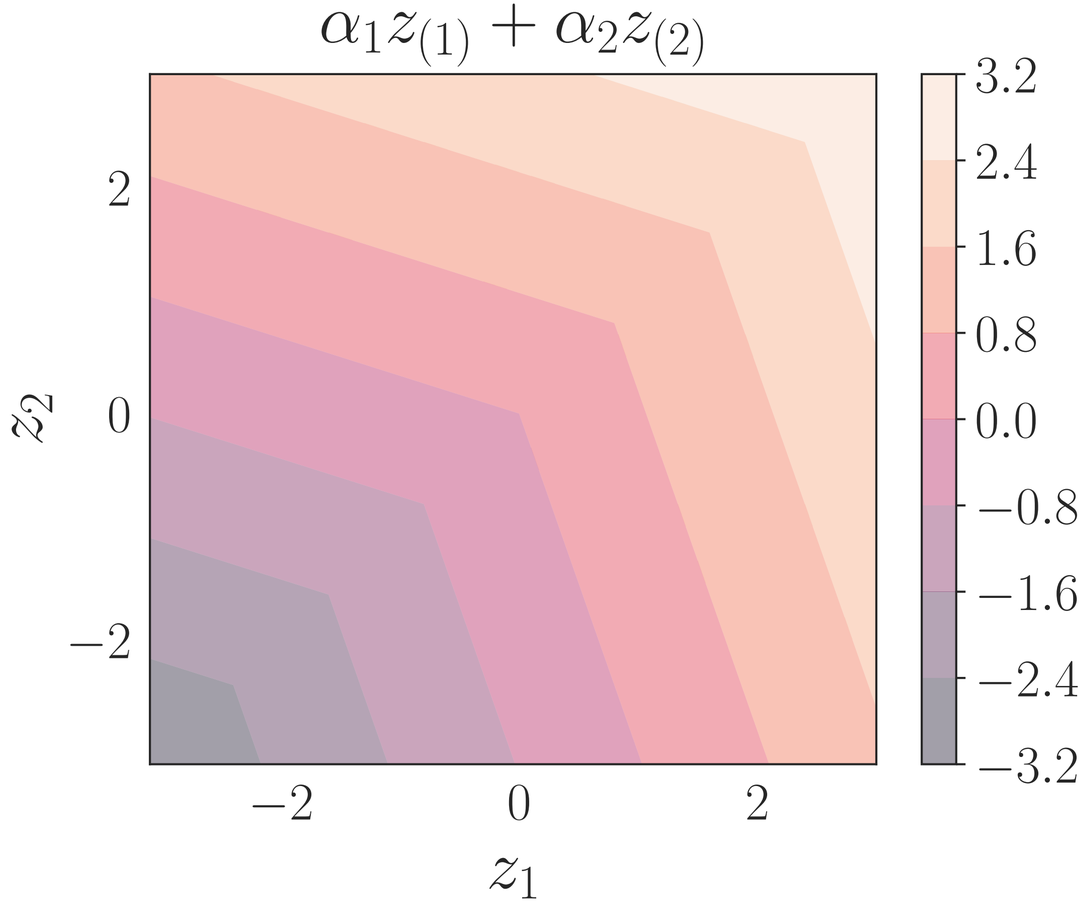

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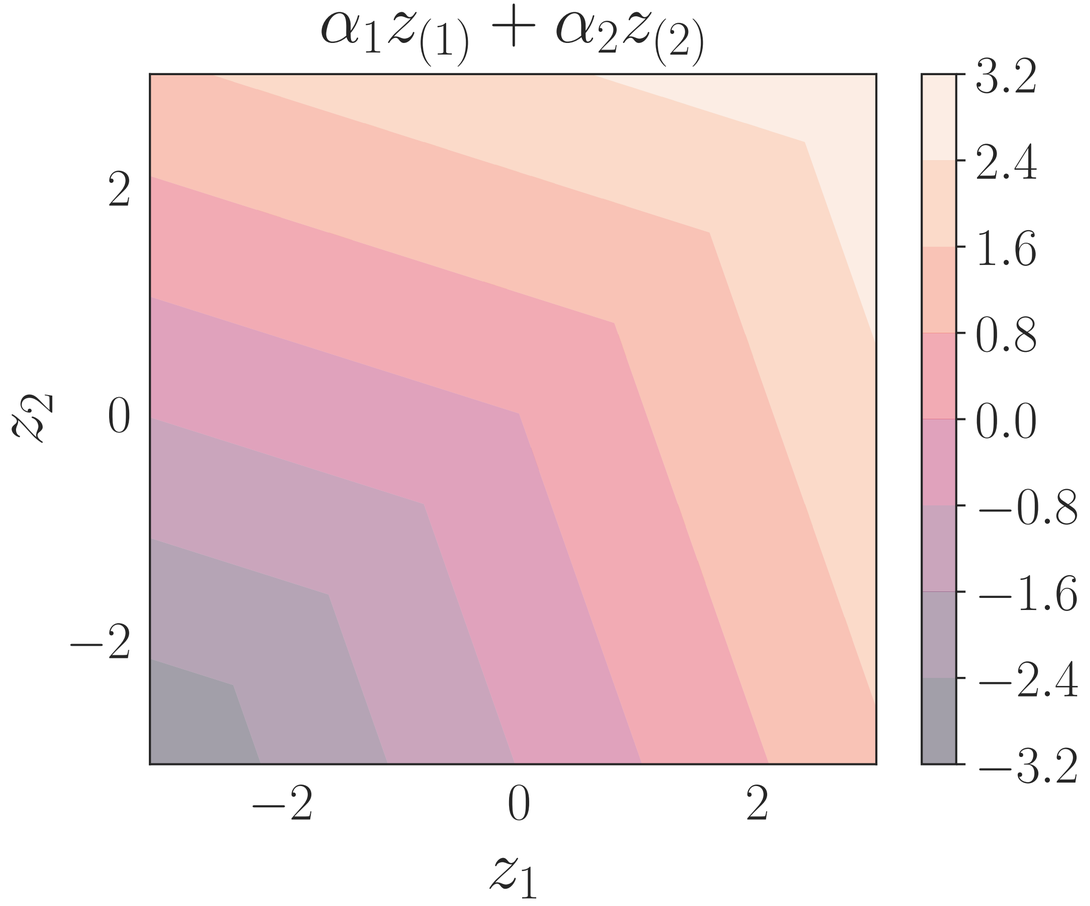

Statistical View of Objective

$$\sum_{i=1}^{n} \alpha_i \cdot \ell_{(i)}(w) = \max_{\pi} \sum_{i=1}^{n} \alpha_i \cdot \ell_{\pi(i)}(w)$$

- LHS is an *L-statistic*, or linear combination of order statistics.
- Order statistic $\ell_{(i)}(w)$ is the (i/n)-th quantile of the empirical distribution of losses.
- LHS also a *spectral risk measure*, or weighted average of the (empirical) quantile function.