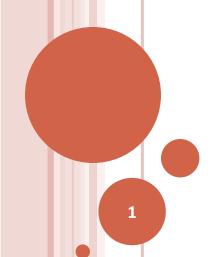
CS F364 Design & Analysis of Algorithms

ALGORITHM DESIGN TECHNIQUES - GREEDY

Matroids – A Theoretical Framework for Greedy Algorithms

- Properties of Matroids
 - Maximal Independent Subsets
- Weighted Matroids
 - Definition
 - Optimal Subsets
 - Example: Minimum Spanning Tree problem
- Greedy Algorithm for Weighted Matroid
 - Template Algorithm
 - Correctness and Efficiency



MATROIDS: EXTENSIONS TO SUBSETS

- Definition (Extension):
 - Given a matroid G = <S,I>, an element x not in A is said to be an extension of A is in I if (AU{ x }) is in I:
 - oi.e. if addition of x to A preserves independence.
- Example:
 - Consider a graphic matroid M_G:
 - Let A be independent set of edges.
 - Then edge e is an extension of A iff
 - e is not in A and
 - adding e does not induce a cycle

MATROIDS: MAXIMAL INDEPENDENT SUBSET

- Definition (MIS):
 - Given a matroid G = <S,I>, an independent subset A is in I is said to be maximal if it has no extensions:
 - oi.e. if A is not a subset of any B in I.
- Example:
 - Consider a graphic matroid M_G:
 - What would be an MIS for M_G?

MATROIDS: PROPERTY OF MISS

- Theorem (Size of MISs):
 - All maximal independent subsets of a matroid are of equal size.
- Proof (by contradiction):
 - Let A be an MIS of a given matroid M.
 - Suppose there is another independent subset B of M that is maximal and |B|>|A|
 - Then by the exchange property
 - othere exists an x in B-A such that (A U {x}) is in I
 - i.e. A is not maximal.

QED

WEIGHTED MATROIDS

- Definition (Weighted Matroids):
 - A matroid M = <S,I> is weighted if there is a weight function w: S --> Z⁺
- Definition (Weight of independent sets):
 - The weight function w can be extended to the members of I:
 - For any A is in I, $w(A) = \Box_{x \text{ in } A} w(x)$

WEIGHTED MATROIDS - OPTIMAL SUBSETS

- Given a weighted matroid M = <S, I, w>, an independent subset A with maximum possible weight is said to be optimal.
 - Claim:

Since the weight function is positive on elements of S, an optimal subset is always a maximal independent subset.

- Terminology:
 - To avoid confusion, we may refer to optimal subsets as maximum weight subset.

WEIGHTED MATROIDS - OPTIMAL SUBSETS

Recall:

Given, an undirected graph G=(V,E), define $M_G = \langle S_G, I_G \rangle$ as:

$$\circ S_G = E$$

- olf $A \subseteq S$, then A is in I_G iff A is acyclic
- Then an MIS of M_G is a spanning tree of G
- Consider the *minimum spanning tree* problem:
 - Given a weighed graph G = (V,E,w) extend M_G above by the weight function w':

$$ow'(e) = w_m - w(e)$$
 where $w_m >= max_{e in E}(w(e))$

 Then an optimal subset of the weighted matroid is a minimum spanning tree:

$$\mathbf{o} \mathbf{w}'(A) = \Sigma_{e \text{ in A}} \mathbf{w}'(e) = (|V|-1)^* \mathbf{w}_0 - \Sigma_{e \text{ in A}} \mathbf{w}(e)$$

= $(|V|-1)^* \mathbf{w}_0 - \mathbf{w}(A)$ for any MIS.

WEIGHTED MATROIDS — GREEDY ALGORITHM

- GreedyWM(M) // M is a weighted matroid: <S,I,w>
 - 1. $A = \{\}$ // A is the optimal subset being constructed
 - 2. let N=|M.S|
 - 3. sort elements of M.S in decreasing order by weight w
 - 4. for i = 1 to N
 if A U { M.S[i] } in M.I then A = A U { M.S[i] }
 - 5. return A
- Theorem:
 - Given a weighted matroid M, GreedyWM(M) returns an optimal subset.
- Complexity:
 - Time taken = Time for sorting + Time for N iterations= O(N*logN + N*f(N))

where f(N) is time taken for testing whether a subset is independent.

CORRECTNESS OF GREEDYWM: GREEDY CHOICE

- Lemma (MAT_CHOICE):
 - Let M = <S,I,w> be a weighted matroid with S sorted in decreasing order by weight.
 - Let x be the first element of S such that {x} is in I, if it exists:
 - then there is an optimal subset A of S such that x is in A.
- Proof (by cases):
 - If no such x exists then we are done.
 - else, let B be a nonempty optimal subset.
 - o If x is in B, then let A = B; we are done.
 - \circ else then for any element y of B, $w(y) \le w(x)$. Why?
 - o Let A = { x }
 - Using the exchange property find a y in B and add it to A until |A| = |B| i.e. A = B { y } U { x } for some y in B.
 - w(A) >= w(B).
 - But B is optimal i.e. A must also be optimal.

CORRECTNESS OF GREEDYWM: ORDER OF CHOICE

- o Lemma (CHOICE_ORDER):
 - Let M = <S,I> be a matroid.
 - If x is in S and x is an extension of some A in I,
 then x is also an extension of {}

• Proof:

- Since x is an extension of A, A U {x} is in I.
- Since I is hereditary, {x} must be in I.

• Corollary:

- Let M = <S,I> be a matroid.
- If x is in S and x is not an extension of {},
 then x is not an extension of any A in I.

CORRECTNESS OF GREEDYWM: OPTIMAL SUBSTRUCTURE

- Given a weighted matroid M = <S, I, w>, and an element x in S such that { x } is in I, , define a contraction of M by x as the weighted matroid M' = <S', I', w'>:
 - S' = { y in S | { x, y } is in I }
 - I' = { B subset of S { x } | B U { x } is in I }
 - w' is w restricted to S'.
- Theorem (OPTIMAL_SUBSTRUCTURE):
 - Let x be the first element of S chosen by GreedyWM for the weighted matroid M = <S,I,w>.
 - The remaining problem of finding a maximum-weight independent subset containing x reduces to
 - ofinding the maximum-weight independent subset of M', the contraction of M by x.
- Proof: (omitted).