DAA Tutorial 2 Solution

3:

DAC-Matrix (A, B)

$n \leftarrow$ A.rows

Let C be a new $n \times n$ matrix

if $n = 1$

$c_{11} = a_{11} b_{11}$ } Base case

else Partition A as
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Partition B as
$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Partition C as
$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Some steps may be required for computing the partial sums

Divide Step

Use k applications of DAC-Matrix on $\frac{n}{3} \times \frac{n}{3}$ matrices to compute the partial products } Recursion Step

Compute the matrix C using the partial products } Conquer Step

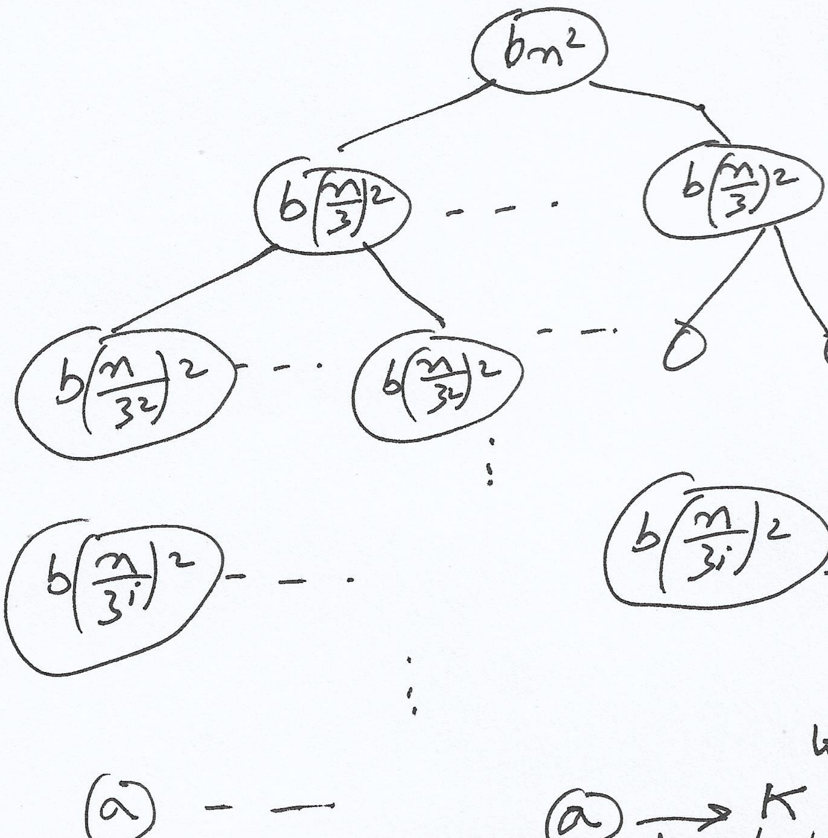
Return C (5)

Assuming that Divide and Conquer steps take bn^2 time, we get the following recurrence:

$$T(1) = a \qquad (5)$$

$$T(n) = k\,T\left(\frac{n}{3}\right) + bn^2 \quad \text{for } n > 1$$

where $T(n)$ is the time complexity of DAC-Matrix $(A_{n\times n}, B_{n\times n})$. We use the recursion tree method:



$\rightarrow bn^2$ at level 0

$\rightarrow k\,b\left(\frac{n}{3}\right)^2 = \left(\frac{k}{9}\right) bn^2$ at level 1

$\rightarrow k^2 b\left(\frac{n}{3^2}\right)^2 = \left(\frac{k}{9}\right)^2 bn^2$ at level 2

$\vdots$

$\rightarrow k^i b\left(\frac{n}{3^i}\right)^2 = \left(\frac{k}{9}\right)^i bn^2$ at level i

$\rightarrow k^{\log_3 n} \cdot a = \left(3^{\log_3 k}\right)^{\log_3 n} a = \left(3^{\log_3 n}\right)^{\log_3 k} a = n^{\log_3 k}\, a$ at level $\log_3 n$

$$T(n) = bn^2 + \left(\frac{k}{9}\right)bn^2 + \left(\frac{k}{9}\right)^2 bn^2 + \cdots + \left(\frac{k}{9}\right)^{\log_3 n - 1} bn^2 + n^{\log_3 k} a$$

$$= bn^2\left(1 + \left(\frac{k}{9}\right) + \left(\frac{k}{9}\right)^2 + \cdots + \left(\frac{k}{9}\right)^{\log_3 n - 1}\right) + a\, n^{\log_3 k}$$

$$= bn^2\left(\frac{\left(\frac{k}{9}\right)^{\log_3 n} - 1}{\frac{k}{9} - 1}\right) + a \cdot n^{\log_3 k}$$

$$= \left(\frac{k^{\log_3 n}}{9^{\log_3 n}} - 1\right)\left(\frac{bn^2}{\frac{k}{9} - 1}\right) + a \cdot n^{\log_3 k}$$

$$= \left(\frac{\left(3^{\log_3 k}\right)^{\log_3 n}}{\left(3^2\right)^{\log_3 n}} - 1\right)\left(\frac{bn^2}{\frac{k}{9} - 1}\right) + a \cdot n^{\log_3 k}$$

$$= \left[\frac{\left(3^{\log_3 n}\right)^{\log_3 k}}{\left(3^{\log_3 n}\right)^2} - 1\right]\left(\frac{bn^2}{\frac{k}{9} - 1}\right) + a \cdot n^{\log_3 k}$$

$$= \left[\frac{n^{\log_3 k}}{n^2} - 1\right]\left(\frac{bn^2}{\frac{k}{9} - 1}\right) + a \cdot n^{\log_3 k} = \Theta\left(n^{\log_3 k}\right)$$

(5)

$$\Theta(n^{\log_3 k}) = o(n^{\log_2 7}) \Rightarrow \log_3 k < \log_2 7$$

$$\Rightarrow k < 3^{\log_2 7} = 21.85 \Rightarrow \boxed{k = 21} \quad (5)$$

6: Let the two databases be A and B. We compare the $\frac{n}{2}$th smallest values from A and B. Suppose that $A[\frac{n}{2}] < B[\frac{n}{2}]$.

Then the median cannot be in $A[1:\frac{n}{2}]$ and $B[\frac{n}{2}+1:n]$.

Proof: ① median $\notin A[1:\frac{n}{2}]$

We assume that median $\in A[1:\frac{n}{2}]$

$\Rightarrow$ median $= A[\frac{n}{2}-i]$ for some $0 \leq i < \frac{n}{2}$.

But $B(\frac{n}{2}) > A[\frac{n}{2}] \geq A[\frac{n}{2}-i] =$ median.

$\Rightarrow$ median is greater than $(\frac{n}{2}-i-1)+(\frac{n}{2}-j)$

$= n-1-(i+j)$ numbers $= n-1 \Rightarrow i+j=0$ (5)

$\Rightarrow i=j=0 \Rightarrow$ median $> B(\frac{n}{2})$ a contradiction

2 median $\notin B[\frac{n}{2}+1 : n]$

We assume that median $\in B[\frac{n}{2}+1 : n]$

$\Rightarrow$ median $= B[\frac{n}{2}+j]$ for some $1 \le j \le \frac{n}{2}$.

But $A[\frac{n}{2}] < B[\frac{n}{2}] < B[\frac{n}{2}+j] =$ median.

$\Rightarrow$ median is greater than $(\frac{n}{2}) + (\frac{n}{2}+j-1)$ (5)

$=$ $n-1+j$ numbers $= n-1 \Rightarrow j = 0$ (a contradiction).

```
Median(A, B, la, ha, lb, hb)
    if la = ha AND lb = hb
        x ← query A[la]
        y ← query B[lb]
        if x < y
            return x
        else
            return y

    else
        x ← query A[(la+ha-1)/2]

        y ← query B[(lb+hb-1)/2]
```

```
if x < y
        Median(A, B, (la+ha+1)/2, ha, lb, (lb+hb-1)/2)
else
        Median(A, B, la, (la+ha-1)/2, (lb+hb+1)/2, hb)
```

(5)

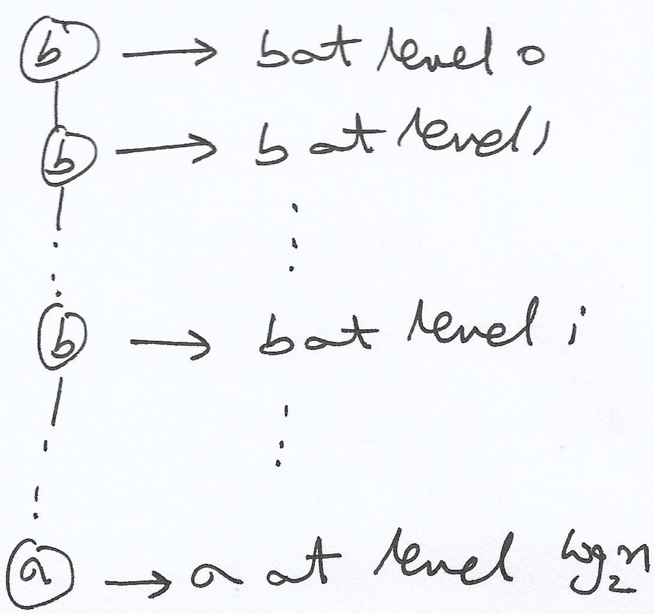
We call the function Median $(A, B, 1, n, 1, n)$.

Let $T(n)$ denote the time complexity of the above function. We get the following recursion:

$$T(1) = a$$

$$T(n) = T\left(\frac{n}{2}\right) + b \quad \text{for } n > 1$$

Using recursion tree method:



$$\Rightarrow T(n) = b(\log_2 n - 1) + a = \Theta(\log_2 n) = \Theta(\log n)$$

(5)