By weak duality theorem, for any feasible dual solution $y$:

$$\sum_{i=1}^{n} y_i \le Z^*_{LP} \le OPT$$

Let $y^*$ be an optimal solution to the dual LP, and consider the solution in which we choose all subsets for which the corresponding dual inequality is tight; that is, the inequality is met with equality for subset $S_j$: $\sum_{i: e_i \in S_j} y_i^* = w_j$. Let $I'$ denote the indices of the subsets in this solution. We will prove that this algorithm also is an $f$-approximation algorithm for the set cover problem.

The collection of subsets $S_j$, $j \in I'$, is a set cover: Suppose that there exists some uncovered element $e_k$. Then for each subset $S_j$ containing $e_k$, it must be the case that

$$\sum_{i: e_i \in S_j} y_i^* < w_j.$$

Let $\epsilon$ be the smallest difference between the RHS and LHS of all constraints involving $e_k$; that is,

$$\epsilon = \min_{j: e_k \in S_j} \left( w_j - \sum_{i: e_i \in S_j} y_i^* \right).$$ We have $\epsilon > 0$.

Consider now a new dual solution $y'$ in which $y'_k = y^*_k + \epsilon$ and every other component $y'$ is the same as in $y^*$. Then $y'$ is a dual feasible solution since for each $j$ such that $e_k \in S_j$:

$$\sum_{i: e_i \in S_j} y'_i = \sum_{i: e_i \in S_j} y^*_i + \epsilon \le w_j,$$ by the definition of $\epsilon$.

For each $j$ such that $e_k \notin S_j$:

$$\sum_{i: e_i \in S_j} y_i' = \sum_{i: e_i \in S_j} y_i^* \leq w_j, \text{ as before.}$$

Furthermore, $\sum_{i=1}^{n} y_i' > \sum_{i=1}^{n} y_i^*$, which contradicts the optimality of $y^*$. Thus, it must be the case that all elements are covered and $I'$ is a set cover.

The dual rounding algorithm is an $f$-approximation algorithm for the set cover problem.

$$j \in I' \Leftrightarrow w_j = \sum_{i: e_i \in S_j} y_i^*$$

$$\begin{aligned}
\sum_{j \in I'} w_j &= \sum_{j \in I'} \sum_{i: e_i \in S_j} y_i^* \\
&= \sum_{i=1}^{n} |\{j \in I' : e_i \in S_j\}| \cdot y_i^* \\
&\leq \sum_{i=1}^{n} f_i y_i^* \\
&\leq f \sum_{i=1}^{n} y_i^* \\
&\leq f \cdot OPT
\end{aligned}$$

$$\Rightarrow \left(\sum_{j \in I'} w_j\right) / OPT \leq f$$

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This algorithm is called <u>dual rounding</u> algorithm for set cover.

The Primal-Dual Algorithm for Set Cover: The previous two algorithms for Set Cover (Primal Rounding and Dual Rounding) require solving a LP. In the dual rounding algorithm, the optimal dual solution is a lower bound for OPT. Any feasible dual solution is also a lower bound for OPT: $\sum_{i=1}^{n} y_i \le OPT$ for any feasible dual solution.

If we construct our Set Cover using the method of dual rounding algorithm: $j \in I' \iff \sum_{i: e_i \in S_j} y_i = w_j$ and $I'$ is a Set Cover, then we can easily show that this will again give an f-approximation algorithm for Set Cover (using the same set of inequalities as in the case of dual rounding algorithm). This is example of a primal-dual algorithm in which we are not required to solve the dual LP. Primal-dual algorithms start with a dual feasible solution, and use dual information to infer a primal, possible infeasible, solution. If the primal solution is indeed infeasible, the dual solution is modified to increase the value of the dual objective function.

Primal-dual Algorithm:

$y \leftarrow 0$
$I \leftarrow \phi$
while there exists $e_i \notin \bigcup_{j \in I} S_j$ do
    Increase the dual variable $y_i$ until there is some $l$ with $e_i \in S_l$ such that $\sum_{j: e_j \in S_l} y_j = w_l$

    $I \leftarrow I \cup \{l\}$