Tutorial 7, Design and Analysis of Algorithms, 2019

1. Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on each edge e, a designated source $s \in V$, and a designated sink $t \in V$. You are also given an integer maximum $s - t$ flow in G, defined by a flow value f_e on each edge e. Now suppose we pick a specific edge $e \in E$ and increase its capacity by one unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

2. (a) List all the minimum $s - t$ cuts in the flow network pictured in Figure 1. The capacity of each edge appears as a label next to the edge.

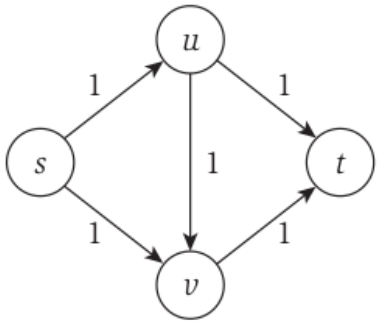

Figure 1: Flow graph for question 2(a). What are the minimum $s - t$ cuts in this flow network?

(b) What is the minimum capacity of an $s - t$ cut in the flow network in Figure 2? Again, the capacity of each edge appears as a label next to the edge.

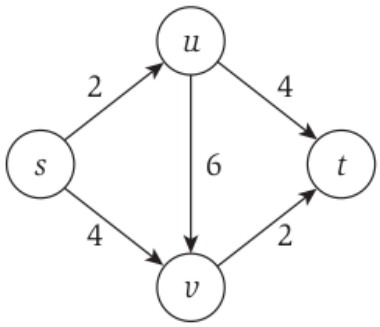

Figure 2: Flow graph for question 2(b). What is the minimum capacity of an $s - t$ cut in this flow network?

3. Figure 3 shows a flow network on which an $s-t$ flow has been computed. The capacity of each edge appears as a label next to the edge, and the numbers in boxes give the amount of flow sent on each edge. (Edges without boxed numbers−specifically, the four edges of capacity 3− have no flow being sent on them.)

   (a) What is the value of this flow? Is this a maximum (s, t) flow in this graph?

   (b) Find a minimum $s - t$ cut in the flow network pictured in Figure 3, and also say what its capacity is.

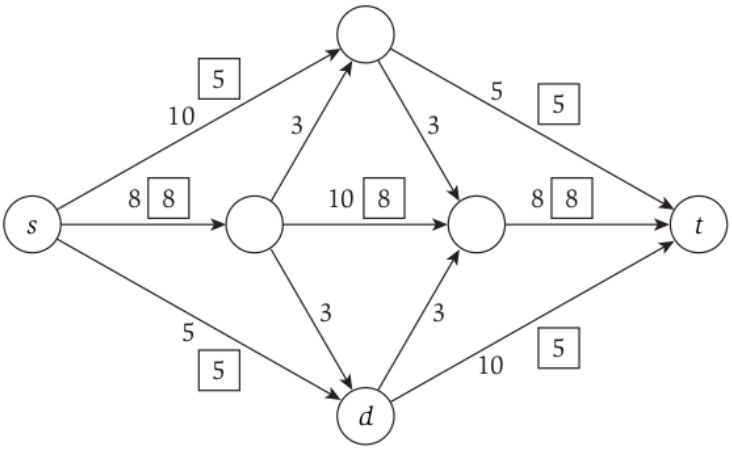

Figure 3: Flow graph for question 3. What is the value of the depicted flow? Is it a maximum flow? What is the minimum cut?

4. Figure 4 shows a flow network on which an $s - t$ flow has been computed. The capacity of each edge appears as a label next to the edge, and the numbers in boxes give the amount of flow sent on each edge. (Edges without boxed numbers have no flow being sent on them.)

   (a) What is the value of this flow? Is this a maximum (s, t) flow in this graph?

   (b) Find a minimum $s - t$ cut in the flow network pictured in Figure 4, and also say what its capacity is.

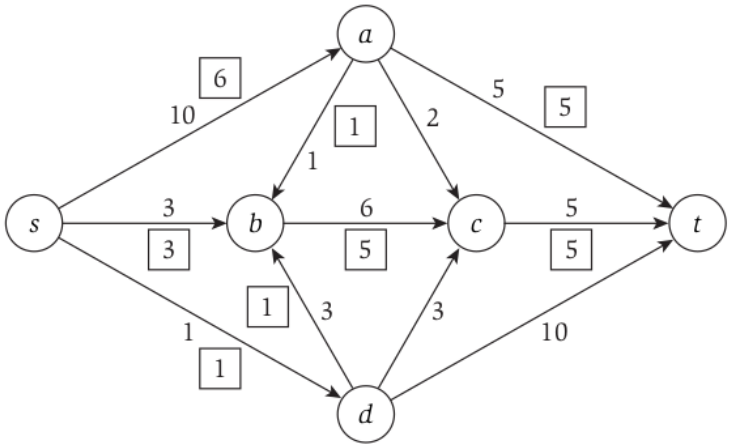

Figure 4: Flow graph for question 4. What is the value of the depicted flow? Is it a maximum flow? What is the minimum cut?

5. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.
   Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e. If f is a maximum $s - t$ flow in G, then f saturates every edge out of s with flow (i.e., for all edges e out of s, we have $f(e) = c_e$).

6. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.
Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a mimimum $s-t$ cut with respect to these capacities $c_e : e \in E$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum $s-t$ cut with respect to these new capacities $1 + c_e : e \in E$.

7. Consider the flow graph in Figure 5. Capacities of the edges are shown along the edges.

   (a) Using the Ford-Fulkerson algorithm, find the maximum s-t flow, showing all the steps involved. What is the value of max s-t flow?

   (b) Using your solution in (a), find the minimum s-t cut, giving an explanation for the conversion. What is the value of min s-t cut?

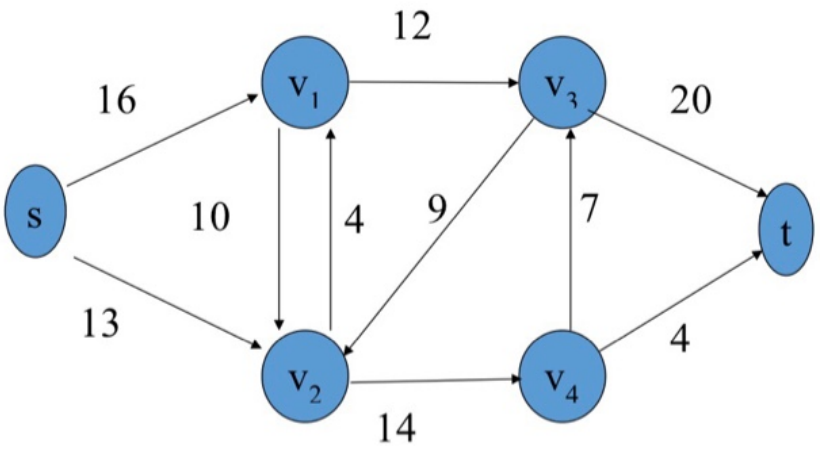

Figure 5: Flow graph for question 7.