Online Problems and Online Algorithms

- Online Paging Problem:
  - Performance of Online Algorithms
  - Competitive Performance

Paging Algorithm – Miss Rates

- Online Paging Lemma:
  - For any deterministic online algorithm A there exist sequences of arbitrary length such that A misses on every request
    - i.e. $f_A([\rho_0 , \rho_1 , \ldots \rho_n]) = n$
  - Proof:
    - Consider an adversary who chooses the next input ρ_i to be a page that is not one of the k pages in M_0
      - Since $|M_1| = k+1$, there always exists one such page.
- Implication:
  - Worst case analysis is not useful in comparing these algorithms

Paging Algorithm – Competitive Analysis

- Definition:
  - A deterministic online page replacement algorithm A is said to be C-competitive if there exists a constant b such that on every sequence of requests $\rho = \rho_0, \rho_1, \ldots \rho_n$,
    - $f_A(\rho) - C * f_{OPT}(\rho) <= b$

  where the constant b must be independent of n but may depend on k.
  - The competitiveness coefficient of A, denoted C_A, is the smallest C such that A is C-competitive.
- Online Paging Competitiveness Theorem:
  - For any deterministic online algorithm A for paging, $C_A >= k$
  - Proof:
    - By GreedyPaging Lemma and Online Paging Lemma

Paging Algorithm – Competitive Analysis

- Claim:
  - $C_{LRU} = k$
  - Proof:
    - Partition the input sequence into rounds R_0 , R_1 ... R_t
      - such that each round R_j results in exactly k misses by LRU.
    - In each round R_j , all the k+1 pages must have been accessed.
      - Why?
    - So, the ratio of misses by LRU to optimal misses is at most k i.e. for any input sequence ρ
      - $f_{LRU}(\rho) / f_{OPT}(\rho) <= k$
      - i.e. $C_{LRU} <=k$
    - But by OPC Theorem: $C_{LRU} >= k$.

Paging Algorithm – Competitive Analysis

- Claim: $C_{FIFO} = k$
  - Proof: (similar to the proof for LRU: left as exercise)
- Claim: $C_{LFU} > k$
  - Proof:
    - Consider a sequence ρ where
      - ρ_0 , ρ_1 , ... ρ_j are k-1 distinct pages with 2 accesses each and
      - $\rho_{j+2*i-1}$, ρ_{j+2*i} are a pair of different pages repeated for each i = 1, 2, ...
        - and are different from ρ_0 , ρ_1 , ... ρ_j
    - $f_{LFU}(\rho) = n - 2*k + 1$ and $f_{OPT}(\rho) = 1$
    - Therefore the ratio $f_{LFU}(\rho) / f_{OPT}(\rho)$ is O(n)
      - i.e. not bounded