CS F364: Design & Analysis of Algorithm

04 Matrix Multiplication Polynomial Evaluation

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Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1r} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & a_{p3} & \dots & a_{pr} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1q} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2q} \\ \dots & \dots & \dots & \dots & \dots \\ b_{r1} & b_{r2} & b_{r3} & \dots & b_{rq} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1q} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2q} \\ \dots & \dots & \dots & \dots & \dots \\ c_{p1} & c_{p2} & c_{p3} & \dots & c_{pq} \end{bmatrix}$$

Very popular computation step

```
MatrixMultiply(A, B, C)
  for i = 1 to p
    for j = 1 to q
      C[i,j] = 0
      for k = 1 to r
        C[i,j] += A[i,k]*B[k,j]
```

$$C_{ij} = \sum_{k=1}^{r} A_{ik} \times B_{kj}$$

Number of operations $\Theta(p \times q \times r) = \Theta(n^3)$

Can we do better?

Matrix Multiplication, Divide and Conquer

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Where

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$
$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$
$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$
$$C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

$$T(n) = 8T(n/2) + \Theta(n^2)$$

That is

$$T(n) = \Theta(n^3)$$

Still no help...

Strassen's Matrix Multiplication

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Where

$$P_1 = A_{11} \times (B_{12} - A_{22})$$
$$P_2 = (A_{11} + A_{12}) \times B_{22}$$
$$P_3 = (A_{21} + A_{22}) \times B_{11}$$
$$P_4 = A_{22} \times (B_{21} - B_{11})$$
$$P_5 = (A_{11} + A_{22}) \times (B_{11} + B_{22})$$
$$P_6 = (A_{12} - A_{22}) \times (B_{21} + B_{22})$$
$$P_7 = (A_{11} - A_{21}) \times (B_{11} + B_{12})$$

$$C_{11} = P_5 + P_4 - P_2 + P_6$$
$$C_{12} = P_1 + P_2$$
$$C_{21} = P_3 + P_4$$
$$C_{22} = P_5 + P_1 - P_3 - P_7$$

$$T(n) = 7T(n/2) + \Theta(n^2)$$
$$T(n) = \Theta(n^{\log_2 7}) = \Theta(n^{2.81})$$

18 additions/subtractions, 7 multiplications. $\Theta(n^{2.81})$

Polynomial Representations

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{n-1}x^{n-1}$$

- **Coefficient form:** $(a_0, a_1, a_2, a_3, \dots, a_{n-1})$
  Example: $2 + x + 7x^2 = (2, 1, 7)$
  Addition: $(2, 1, 7) + (2, -3, 1) = (?, ?, ?)$
  Multiplication: $(2, 1, 7) \times (2, -3, 1) = ?$
- **Point Value form:** $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_{n-1}, f(x_{n-1}))$ where all x_i are different
  Example: $2 + x + 7x^2 = (0, 2), (1, 10), (2, 32)$
  Addition: $(2, 1, 7) + (2, -3, 1) = ?$
  $2 - 3x + x^2 = (0, 2), (1, 0), (2, 0)$
  $(2, 1, 7) + (2, -3, 1) = (0, 4), (1, 10), (2, 32)$
  Multiplication: $(2, 1, 7) \times (2, -3, 1) = (0, 4), (1, 0), (2, 0)$ more points are needed.

Polynomial Evaluation

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{n-1}x^{n-1}$$

- How much time is needed to evaluate for a given x? $\Theta(n^2)$
- Horners' Rule consider the polynomial as

$$a_0 + x(a_1 + x(a_2 + x(a_3 + x(\dots + x(a_{n-2} + x(a_{n-1}))) \dots)))$$

- Time needed is $O(n)$
- Time needed to convert
- A polynomial could be converted to point value form by evaluating it at n different values. It is $O(n^2)$

Interpolation using Gaussian Elimination

When we want our polynomial back from the point value form

- Apply Gaussian Elimination that is a divide and conquer approach

Thank You!

Thank you very much for your attention! (Reference[1])

Queries ?

[1] [1] Book - *Introduction to Algorithm*, By THOMAS H. CORMEN, CHARLES E. LEISERSON, RONALD L. RIVEST, CLIFFORD STEIN