DAA Tutorial 10 Solution

2: To prove that Vertex Cover $\in$ NP-C, we have to prove that Vertex Cover $\in$ NP and Vertex Cover $\in$ NP-H.

Vertex Cover $\in$ NP : On input (G, k), NTM N uses non deterministic moves to guess a subset of vertices S, and then it deterministically verifies for each pair of vertices (i, j) if $\overline{ij} \in G$ then at least one of i or j is in S. If $|S| \leq k$, and S is a vertex cover of G, then N will accept the input. Assuming G to be the adjacency matrix representation of the input graph, on a RAM-NTM this can be done in $O(|G| + |k|)$ time and on a multitape NTM this can be done in $O((|G| + |k|)^2)$ time which is polynomial in input size. (5)

Vertex Cover $\in$ NP-H : We will reduce the Independent Set Problem (IndSet), which is an NP-C problem, to the Vertex Cover problem. The Independent Set Problem is defined as:

IndSet = { (G, k) / G is the adjacency matrix of an undirected graph having a subgraph of at least k vertices having no edges between them }

The linear time reduction f from IndSet to VertexCover is defined as:

$$f((G,k)) = (G, n-k)$$

where n is the number of vertices in G. (5)

$(G,k) \in$ IndSet $\iff$ $(G, n-k) \in$ VertexCover

Consider an independent set S of G of size at least k. Consider any edge $\overline{ij}$ in G. We can have only two possibilities for $\overline{ij}$: either both of i and j are not in S or exactly one of i or j is not in S (third possibility that both i and j are in S is ruled out because S is an independent set). This implies that the complement of the set S in G is a vertex cover of size at most $n-k$. (10)

3:/ To prove that HamPath $\in$ NP-C, we have to prove that HamPath $\in$ NP and HamPath $\in$ NP-H. (3)

<u>HamPath $\in$ NP</u> : On input $G=(V,E)$, NTM N uses nondeterministic moves to guess a path of length $|V|$ (having $|V|$ vertices), and then it deterministically verifies that the path is a Hamiltonian path of G. If this is the case, then it will accept, otherwise it will reject. Assuming G to be the adjacency matrix representati of the input graph, on a RAM-NTM this can be done in $O(|G|)$ time and on a multitape NTM this can be done in $O(|G|^2)$ time which is polynomial in input size. (5)

<u>HamPath $\in$ NP-H</u> : We will reduce the Directed Hamiltonian Path (DHamPath) problem, which is an NP-C problem, to the HamPath Problem.

The Directed Hamiltonian Path problem is defined as:

DHamPath = { G | G is the adjacency matrix of a directed graph having a directed Hamiltonian path }

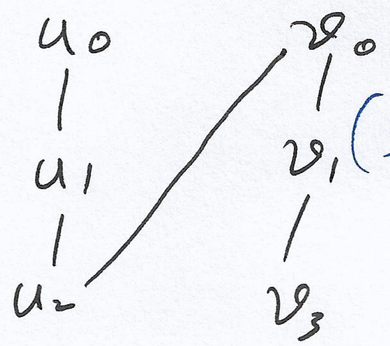
The linear time reduction f from DHamPath to HamPath is defined as:

$$f(\underbrace{V, E}_{G}) = (\underbrace{V', E'}_{G'})$$

where $V' = V \times \{0, 1, 2\}$ and

$$E' = \{\{\langle u,0\rangle, \langle u,1\rangle\}, \{\langle u,1\rangle, \langle u,2\rangle\} \mid u \in V\} \cup \{\{\langle u,2\rangle, \langle v,0\rangle\} \mid \langle u,v\rangle \in E\}$$

$u \longrightarrow v$ is converted to



(5)

Suppose a Hamiltonial path $(u_1, u_2, \dots u_n)$ exists in G. Then the corresponding Hamiltonian path in G' will be $(u_{10}, u_{11}, u_{12}, u_{20}, u_{21}, u_{22}, \dots u_{(n-1)0}, u_{(n-1)1}, u_{(n-1)2}, u_{n0}, u_{n1}, u_{n2})$ (5)

Suppose a Hamiltonian path exists in G'. Then it has to start from some u_0 (if it starts from some v_3, then simply reverse the path). After this, the only way to proceed is to visit u_1 and then u_3. After u_3, the only way to proceed is some v_0 such that $(u, \underline{v}) \in E$. These transitions will gives us a directed Hamiltonian path in G. (5)