# Iterative Multiplication Algorithm



$$
\begin{array}{r}
1\ 2\ 3\ 4 \\
\times\ 4\ 3\ 2\ 1 \\
\hline
1\ 2\ 3\ 4 \\
+\ 2\ 4\ 6\ 8\ \ \\
+\ 3\ 7\ 0\ 2\ \ \ \ \\
+\ 4\ 9\ 3\ 6\ \ \ \ \ \ \\
\hline
5\ 3\ 3\ 2\ 1\ 1\ 4
\end{array}
$$

$1234 \times 4321$

$= 1234 \times (4000 + 300 + 20 + 1)$

$=$

$1234 \times 1$

$+ \; 1234 \times 20$

$+ \; 1234 \times 300$

$+ \; 1234 \times 4000$

$= \; 1000 \times 1 + 200 \times 1 + 30 \times 1 + 4 \times 1$

$+ \; 1000 \times 20 + 200 \times 20 + 30 \times 20 + 4 \times 20$

$+ \; 1000 \times 300 + 200 \times 300 + 30 \times 300 + 4 \times 300$

$+ \; 1000 \times 4000 + 200 \times 4000 + 30 \times 4000 + 4 \times 4000$

$= \; 1 \times 1 \times 1000 + 2 \times 1 \times 100 + 3 \times 1 \times 10 + 4 \times 1$

$+ \; 1 \times 2 \times 10^4 + 2 \times 2 \times 1000 + 3 \times 2 \times 100 + 4 \times 2 \times 10$

$+ \; 1 \times 3 \times 10^5 + 2 \times 3 \times 10^4 + 3 \times 3 \times 1000 + 4 \times 3 \times 100$

$+ \; 1 \times 4 \times 10^6 + 2 \times 4 \times 10^5 + 3 \times 4 \times 10^4 + 4 \times 4 \times 1000$

# Iterative-Multiply $(x, y)$

```
Let x = x_n x_{n-1} ... x_1
Let y = y_n y_{n-1} ... y_1
s ← 0
p ← 1
for i = 2 to n+1
        j ← 1
        k ← i-1
        for l = 1 to i-1
                s ← s + x_j y_k p
                j ← j+1
                k ← k-1
        p ← 10p
for i = n+2 to 2n
        j ← i-n
        k ← n
        for l = 1 to 2n+1-i
                s ← s + x_j y_k p
                j ← j+1
                k ← k-1
        p ← 10p
return s
```

# Time Complexity of Iterative-Multiply



Lines 3 and 4 contribute: 2

Total contribution of lines 6, 7, and 12 is: $3(n-1)$

Total contribution of lines 14, 15, and 20 is: $3(n-1)$

Total contribution of lines 9, 10, and 11 is:

$3(1+2+\cdots+(n-1))$

Total contribution of lines 17, 18, and 19 is:

$3(1+2+\cdots+(n-1))$

Line 21 contributes: 1

Total Time Complexity of Iterative-Multiply:

$$T(n) = 2 + 3(n-1) + 3(n-1) + 3(1+2+\cdots+(n-1))$$

$$+\ 3(1+2+\cdots+(n-1)) + 1$$

$$= 3\left(1 + 2(n-1) + \frac{(n-1)n}{2} \times 2\right)$$

$$= 3(n^2 - n + 2n - 2 + 1) = 3(n^2 + n - 1) = \Theta(n^2)$$

# A Divide and Conquer Multiplication Algorithm

$1234 \times 4321 = (1200 + 34) \times (4300 + 21)$

$= 12 \times 43 \times 10000 + (12 \times 21 + 34 \times 43) \times 100 + 34 \times 21$

$12 \times 43 = (10 + 2) \times (40 + 3)$

$= 1 \times 4 \times 100 + (1 \times 3 + 2 \times 4) \times 10 + 2 \times 3$

$= 400 + 110 + 6 = 516$

$12 \times 21 = (10 + 2) \times (20 + 1)$

$= 1 \times 2 \times 100 + (1 \times 1 + 2 \times 2) \times 10 + 2 \times 1$

$= 200 + 50 + 2 = 252$

$34 \times 43 = (30 + 4) \times (40 + 3)$

$= 3 \times 4 \times 100 + (3 \times 3 + 4 \times 4) \times 10 + 4 \times 3$

$= 1200 + 250 + 12 = 1462$

$34 \times 21 = (30 + 4) \times (20 + 1)$

$= 3 \times 2 \times 100 + (3 \times 1 + 4 \times 2) \times 10 + 4 \times 1$

$= 600 + 110 + 4 = 714$

$1234 \times 4321 = 5160000 + (252 + 1462) \times 100 + 714$

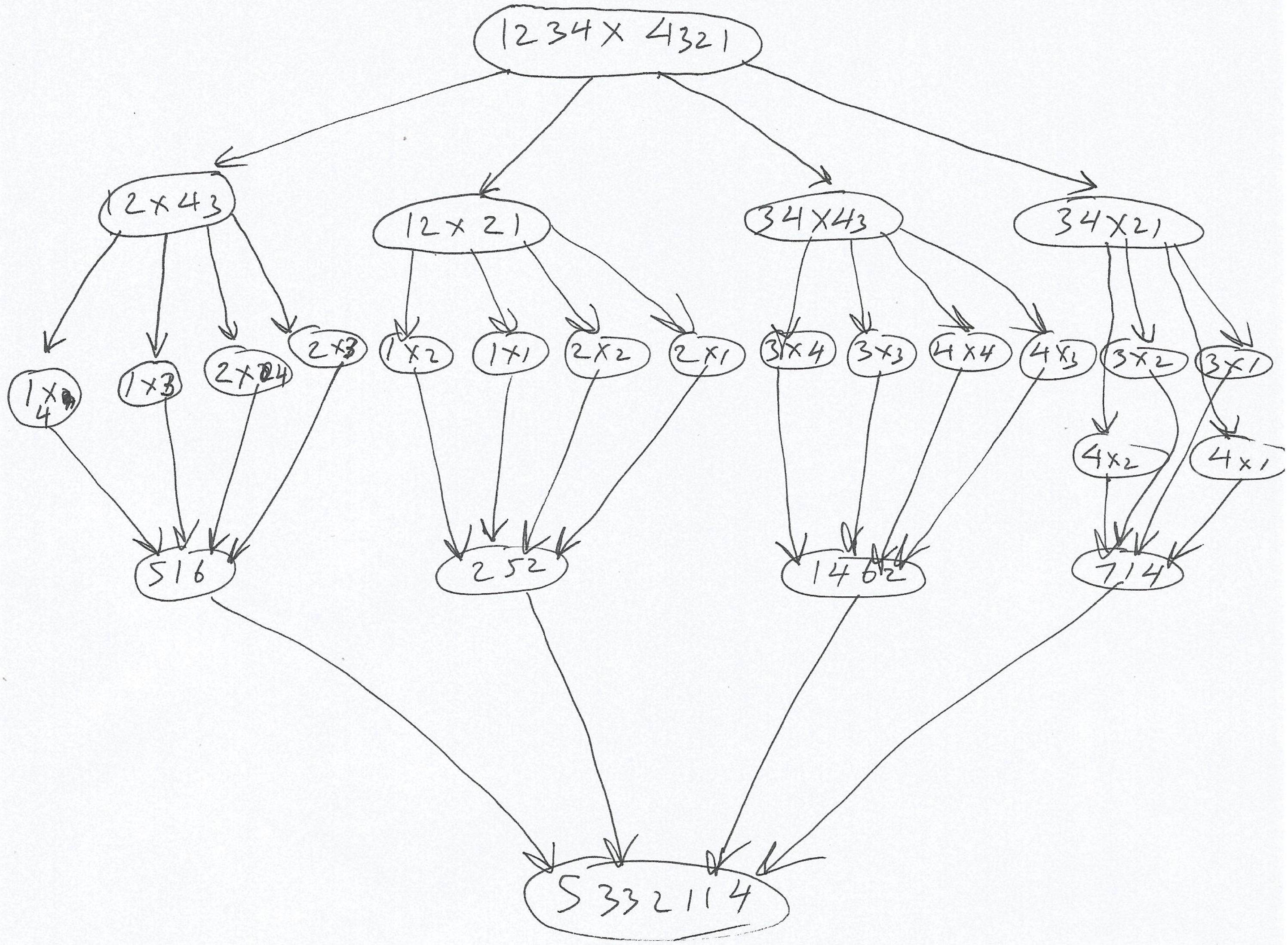
$= 5160000 + 171400 + 714 = 5332114$

Here, we are making use of the identity

$(H_1 \times 10^{n/2} + L_1) \times (H_2 \times 10^{n/2} + L_2)$

$= H_1 \times H_2 \times 10^n + (H_1 \times L_2 + 4 \times H_2) \times 10^{n/2} \; L_1 \times L_2$

# Divide and Conquer Graph




DAC-Multiply ( x, y )

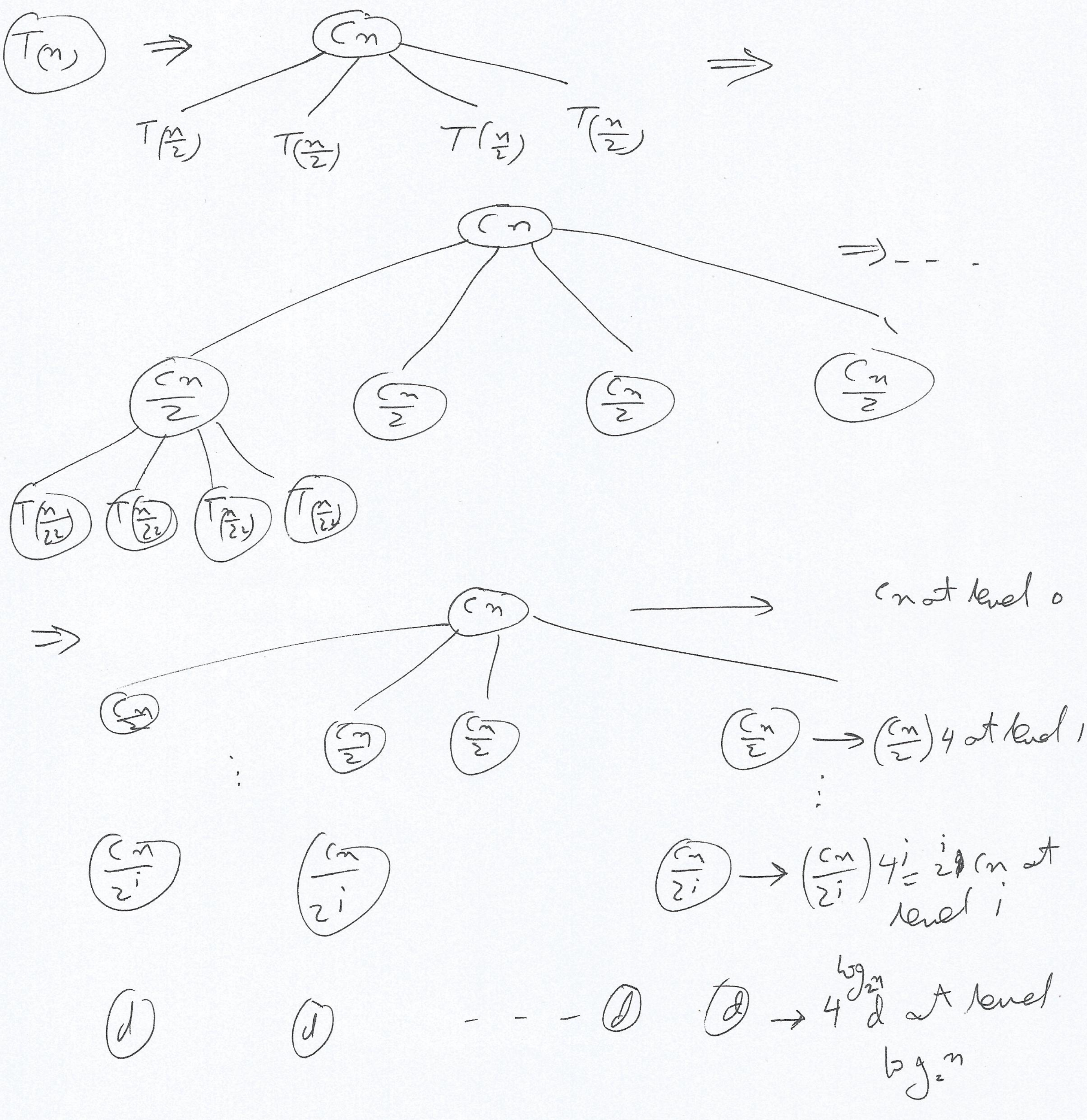
Let $x = H_1 \times 10^{n/2} + L_1$ } Divide Step
Let $y = H_2 \times 10^{n/2} + L_2$
if $n = 1$ } Base Case
return $xy$
$A \leftarrow$ DAC-Multiply ( $H_1$, $H_2$ ) } Recursive Step
$B \leftarrow$ DAC-Multiply ( $H_1$, $L_2$ )
$C \leftarrow$ DAC-Multiply ( $L_1$, $H_2$ )
$D \leftarrow$ DAC-Multiply ( $L_1$, $L_2$ )
return $A \times 10^n + (B+C) \times 10^{n/2} + D$ } Conquer Step

# Complexity of DAC-Multiply



Divide and Conquer steps take linear time $cn$. There are four subproblems of size $n/2$ in recursion step.

$T(n) = 4T(\frac{n}{2}) + cn$ for $n > 1$, and $T(1) = d$



$$T(n) = cn + 2cn + \cdots + 2^i cn + \cdots + 2^{\log_2 n - 1} cn + 4^{\log n} d$$

$$= cn(1 + 2 + \cdots + 2^{\log_2 n - 1}) + (2^{\log_2 n})^2 d$$

$$= \frac{cn(2^{\log_2 n} - 1)}{2 - 1} + n^2 d$$

$$= cn(n-1) + n^2 d$$

$$= (c+d)n^2 - cn = \Theta(n^2)$$

## Karatsuba's Divide and Conquer Multiplication Algorithm

Here, we make use of the identity

$$(H_1 \times 10^{n/2} + L_1) \times (H_2 \times 10^{n/2} + L_2)$$

$$= H_1 \times H_2 \times 10^n + (H_1 \times L_2 + L_1 \times H_2) \times 10^{n/2} + L_1 \times L_2$$

$$= H_1 \times H_2 \times 10^n + ((H_1 + L_1)(H_2 + L_2) - (H_1 H_2 + L_1 L_2)) \times 10^{n/2}$$
$$+ L_1 \times L_2$$

Now, we have only three subproblems:

$H_1 \times H_2$

$L_1 \times L_2$

and $(H_1 + L_1) \times (H_2 + L_2)$

$1234 \times 4321 = 12 \times 43 \times 10000 + ((12+34) \times (43+21) - (12 \times 43 + 34 \times 21)) \times 100 + 34 \times 21$

$= 12 \times 43 \times 10000 + (46 \times 64 - (12 \times 43 + 34 \times 21)) \times 100 + 34 \times 21$

$12 \times 43 = 1 \times 4 \times 100 + ((1+2) \times (4+3) - (1 \times 4 + 2 \times 3)) \times 10 + 2 \times 3$

$= 400 + (3 \times 7 - (4+6)) \times 10 + 6$

$= 400 + 110 + 6 = 516$

$34 \times 21 = 3 \times 2 \times 100 + ((3+4) \times (2+1) - (3 \times 2 + 4 \times 1)) \times 10 + 4 \times 1$

$= 600 + (7 \times 3 - (6+4)) \times 10 + 4$

$= 600 + 110 + 4 = 714$

$46 \times 64 = 4 \times 6 \times 100 + ((4+6) \times (6+4) - (4 \times 6 + 6 \times 4)) \times 10 + 6 \times 4$

$= 2400 + (10 \times 10 - (24+24)) \times 10 + 24$

$= 2400 + 520 + 24 = 2944$

$(10 \times 10 = 1 \times 1 \times 100 + ((1+0) \times (1+0) - (1 \times 1 + 0 \times 0)) \times 10 + 0 \times 0 = 100)$

$1234 \times 4321$

$= 5160000 + (2944 - (516 + 714)) \times 100 + 714$

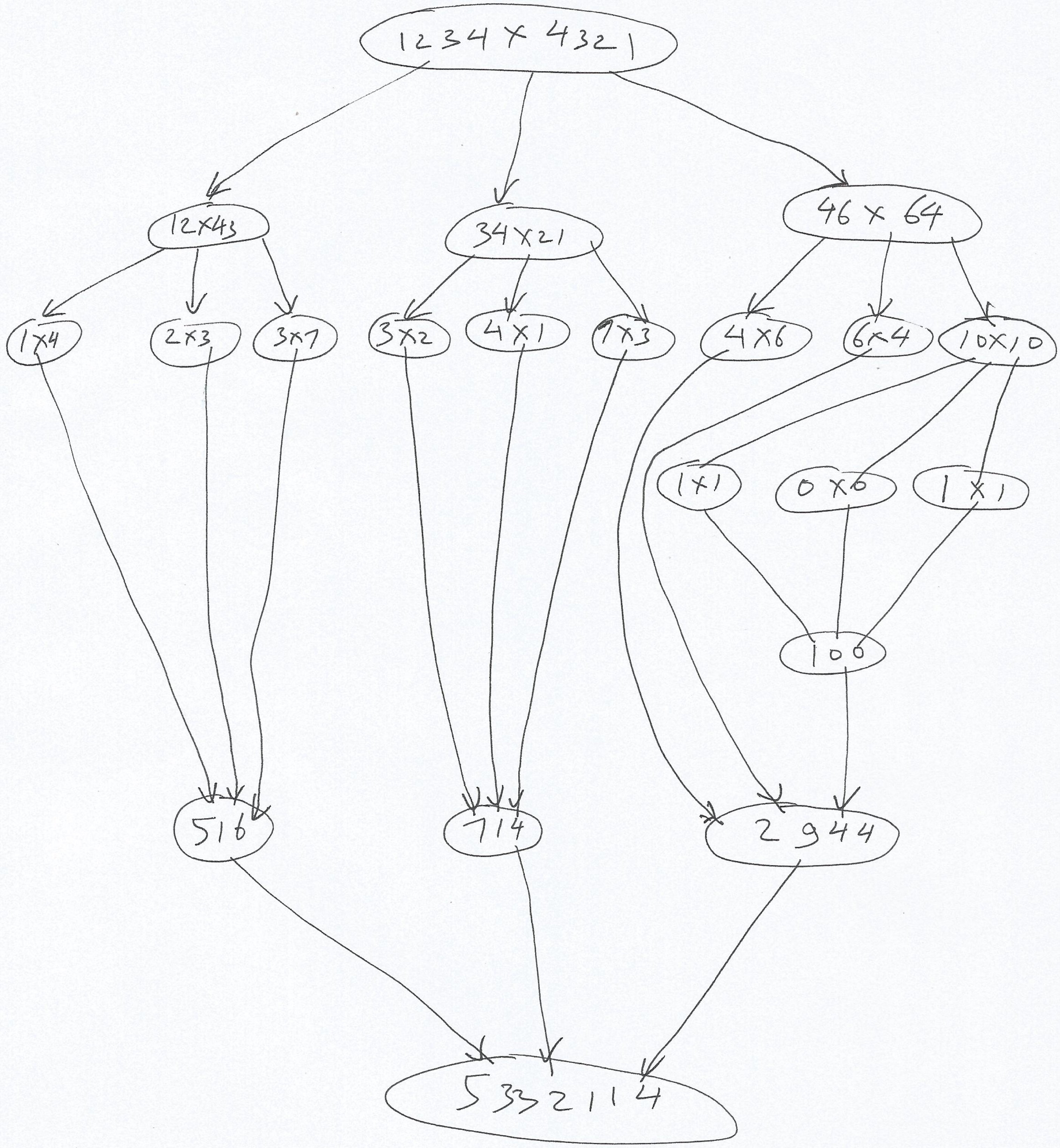
$= 5160000 + (2944 - 1230) \times 100 + 714$

$= 5160000 + 171400 + 714$

$= 5332114$

# Divide and Conquer Graph





Karatsuba-Multiply $(x, y)$

```
Let x = H1 × 10^(n/2) + L1      } Divide Step
Let y = H2 × 10^(n/2) + L2      }

if n = 1                        } Base case
    return x y                  }

A ← Karatsuba-Multiply (H1, H2)               } Recursive
B ← Karatsuba-Multiply (L1, L2)               } Step
C ← Karatsuba-Multiply (H1+L1, H2+L2)         }
return A × 10^n + (C − (A+B)) × 10^(n/2) + B  } Conquer Step
```

## Complexity of Karatsuba-Multiply

Divide and Conquer steps take linear time $cn$.
There are two subproblems of size $\frac{n}{2}$ ($H_1 \times H_2$ and $L_1 \times L_2$), and one subproblem of size either $\frac{n}{2}$ or $\frac{n}{2}+1$ ($(H_1+L_1) \times (H_2+L_2)$) in the recursion step.

$$T(1) = d$$

$$T(n) = 2T\left(\frac{n}{2}\right) + T(n') + cn \quad \text{for } n > 1$$

where $n'$ can be $\frac{n}{2}$ or $\frac{n}{2}+1$ because $(H_1+L_1)$ or $(H_2+L_2)$ can have one extra digit

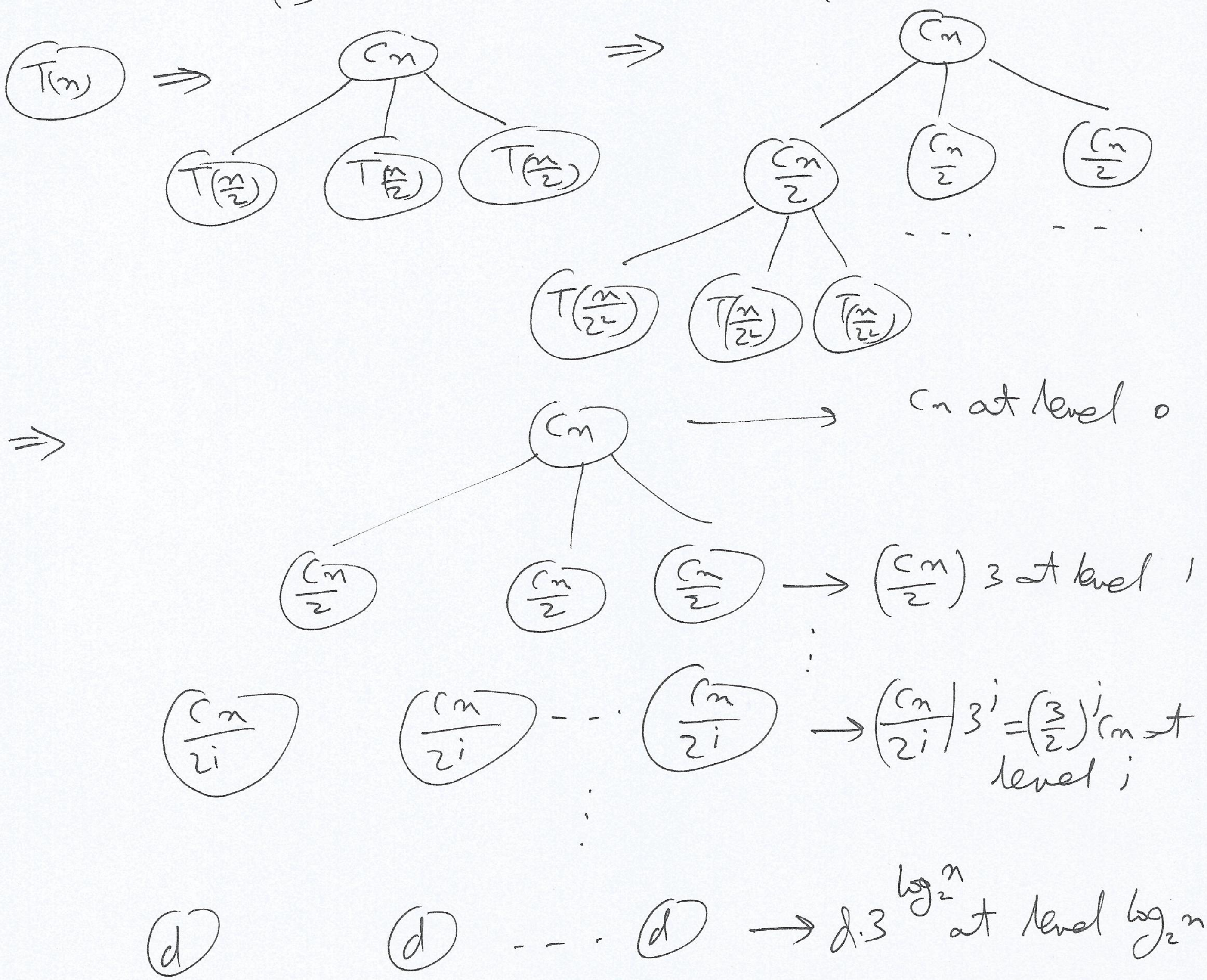
Example: In the $46 \times 64$ subproblem, we have $(4+6) \times (6+4) = 10 \times 10$ has one extra digit in each factor.

Due to this reason, finding exact formula for $T(n)$ is difficult. However, under the assumption that $T(n)$ is a non-decreasing function ( $T(n) \le T(n+1)$ ), we can find upper and lower bounds for $T(n)$ as follows:

$$T(1) = d$$

$$T(n) = 2T\left(\frac{n}{2}\right) + T(n') + cn \le 3T\left(\frac{n}{2}+1\right) + cn$$

$$T(n) = 2T\left(\frac{n}{2}\right) + T(n') + cn \ge 3T\left(\frac{n}{2}\right) + cn$$



$$T(n) \geq cn + \left(\tfrac{3}{2}\right)cn + \cdots + \left(\tfrac{3}{2}\right)^i cn + \cdots + \left(\tfrac{3}{2}\right)^{\log_2 n - 1} cn + 3^{\log_2 n} d$$

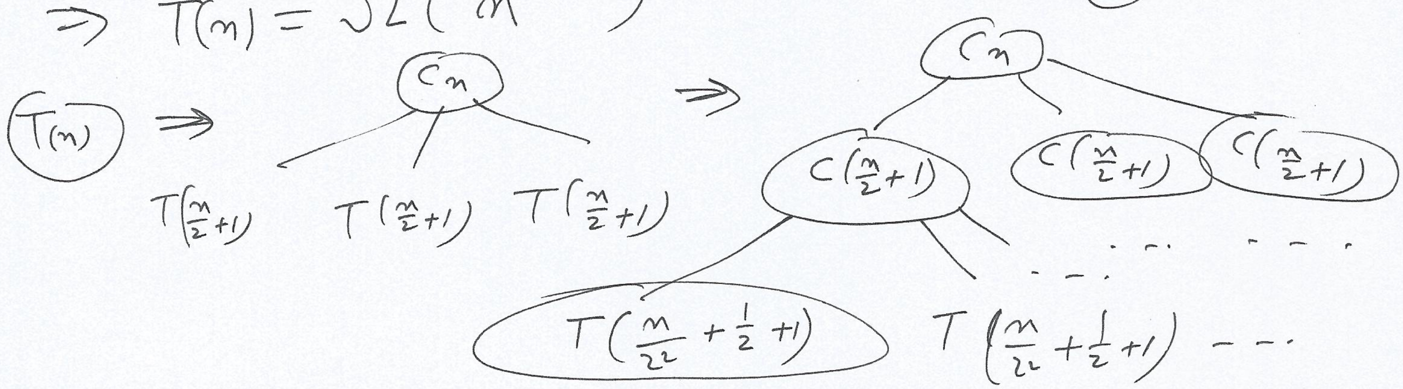
$$= cn\left(1 + \left(\tfrac{3}{2}\right) + \cdots + \left(\tfrac{3}{2}\right)^{\log_2 n - 1}\right) + 3^{\left(\frac{\log_3 n}{\log_3 2}\right)} d$$

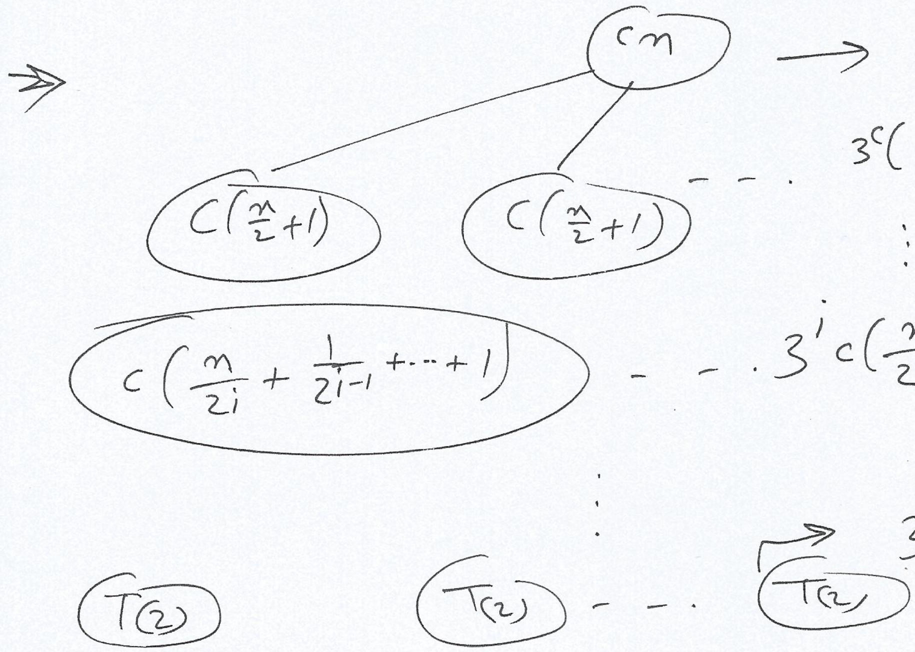
$$= \frac{cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)}{\frac{3}{2} - 1} + \left(3^{\log_3 n}\right)^{\log_2 3} d$$

$$= 2cn\left(\frac{n^{\log_2 3}}{n} - 1\right) + n^{\log_2 3} d$$

$$= (2c + d)\, n^{\log_2 3} - 2cn$$

$$\Rightarrow T(n) = \Omega\left(n^{\log_2 3}\right) \longrightarrow 1$$




$cn$ at level 0

$3c\left(\frac{n}{2}+1\right)$ at level 1

$\vdots$

$3^i c\left(\frac{n}{2^i} + \frac{1}{2^{i-1}} + \cdots + 1\right)$ at level $i$

$3^{\log_2 n} \cdot b$ at level $\log_2 n$

Assuming $T(2) = b \geq d$

2)

$$T(n) \le cn + 3c\left(\frac{n}{2}+1\right) + \dots + 3^i c\left(\frac{n}{2^i} + \frac{1}{2^{i-1}} + \dots + 1\right) + 3^{\log_2 n} b$$

$$= cn\left(1 + \left(\frac{3}{2}\right) + \dots + \left(\frac{3}{2}\right)^i + \dots + \left(\frac{3}{2}\right)^{\log_2 n - 1}\right)$$

$$+ c\left(1 + \frac{1-\frac{1}{2^2}}{1-\frac{1}{2}} + \frac{1-\frac{1}{2^3}}{1-\frac{1}{2}} + \dots + \frac{1-\frac{1}{2^i}}{1-\frac{1}{2}} + \dots + \frac{1-\frac{1}{2^{\log_2 n - 1}}}{1-\frac{1}{2}}\right)$$

$$+ 3^{\left(\frac{\log_3 n}{\log_3 2}\right)} b$$

$$= \frac{cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)}{\frac{3}{2} - 1} + 2c\left(\left(1-\frac{1}{2}\right) + \left(1-\frac{1}{2^2}\right) + \dots + \left(1-\frac{1}{2^{\log_2 n - 1}}\right)\right)$$

$$+ \left(3^{\log_3 n}\right)^{\log_2 3} b$$

$$= 2cn\left(\frac{n^{\log_2 3}}{n} - 1\right) + 2c\left(\log_2 n - 1\right)$$

$$- c\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{\log_2 n - 2}}\right) + n^{\log_2 3} b$$

$$= (2c+b)\, n^{\log_2 3} - 2cn + 2c\log_2 n - 2c - c\left(\frac{1 - \frac{1}{2^{\log_2 n - 1}}}{1-\frac{1}{2}}\right)$$

$$= (2c+b)\, n^{\log_2 3} - 2cn + 2c\log_2 n - 4c + \frac{c}{n} \Rightarrow$$

$T(n) = O(n^{\log_2 3}) \longrightarrow (2)$

From (1) and (2) we get: $\boxed{T(n) = \Theta(n^{\log_2 3})}$