CS F364: Design & Analysis of Algorithm

16 Network Flow Search Trees

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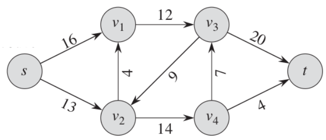
Flow Network

Flow network $G = (V, E)$ is a directed graph

- Every edge $(u, v) \in E$ has a non-negative **capacity** $\boxed{c(u, v) \geq 0}$
- There are two distinguished vertices: **source** s, and **sink** t
- For each vertex $v \in V$ the network has a path $s \rightsquigarrow v \rightsquigarrow t$
- Self loops are not allowed $(u, u) \notin E$
- **No reverse edge**, If $(u, v) \in E$ then $(v, u) \notin E$
- If $(u, v) \notin E$ then $c(u, v) = 0$,
- Graph is connected, so $|E| \geq |V| - 1$



Flow

Flow in the graph G is a real valued function $f : V \times V \rightarrow \mathbb{R}$

1. **Capacity Constraint:** $\forall u, v \in V$ we require $0 \leq f(u, v) \leq c(u, v)$
2. **Flow conservation:** $\forall u \in V - \{s, t\}$ we require $\boxed{\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)}$ here if $(u, v) \notin E$ then $f(u, v) = 0$,

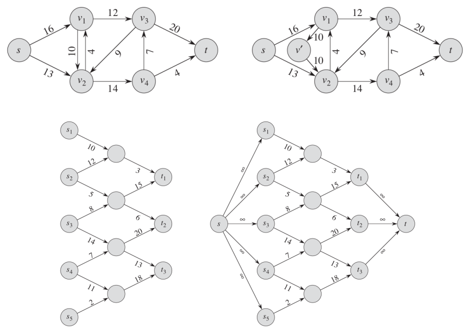
Network flow is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

Typically, no edge enters the source, so $\sum_{v \in V} f(v, s) = 0$

Maximum-flow problem: given a flow network G with source s and sink t; we wish to find a flow of maximum value.

Dealing Anti-Parallel Edges, Many Source/Sink (Supersource, Supersink)



Ford-Fulkerson method

Algorithm 1: FORD-FULKERSON-METHOD(G,s,t)

Initialize flow f to 0
while $\exists$ *an augmenting path p in the residual network* G_f **do**
  Augment the flow f along p
return f

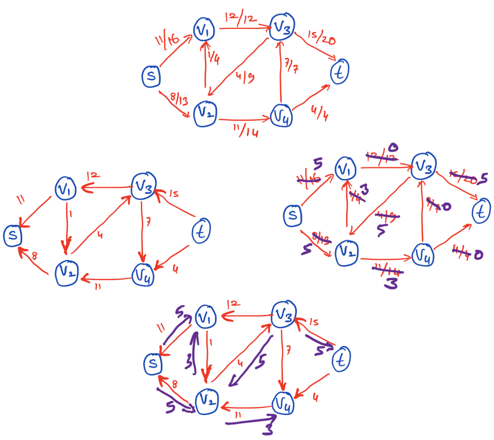
Given a flow network $G = (V, E)$ with source s, sink t, and flow f

Residual network G_f has two type of edges

1. Residue: $c_f(u, v) = c(u, v) - f(u, v)$
2. flow: $c_f(u, v) = f(v, u)$

Observe that $|E_f| \leq 2|E|$

Example: Residual Network



Augmentation $f \uparrow f'$

If f is a flow in G; and f' in its residual network G_f then

$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

$f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$

Capacity Constraints

As $f'(v,u) \le c_f(v,u) = f(u,v)$

$$\begin{aligned} (f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) \\ &\ge f(u,v) + f'(u,v) - f(u,v) \\ &= f'(u,v) \\ &\ge 0 \end{aligned}$$

Also

$$\begin{aligned} (f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) \\ &\le f(u,v) + f'(u,v) \\ &\le f(u,v) + c_f(u,v) \\ &= f(u,v) + c(u,v) - f(u,v) \\ &= c(u,v) \end{aligned}$$

Flow Conservation

For $u \in V - \{s,t\}$

$$\begin{aligned} \textstyle\sum_{v \in V} (f \uparrow f')(u,v) &= \textstyle\sum_{v \in V} (f(u,v) + f'(u,v) - f'(v,u)) \\ &= \textstyle\sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) - \sum_{v \in V} f'(v,u) \\ &= \textstyle\sum_{v \in V} f(v,u) + \sum_{v \in V} f'(v,u) - \sum_{v \in V} f'(u,v) \\ &= \textstyle\sum_{v \in V} (f(v,u) + f'(v,u) - f'(u,v)) \\ &= \textstyle\sum_{v \in V} (f \uparrow f')(v,u) \end{aligned}$$

Let's find $f \uparrow f'$

$$\begin{aligned} |f \uparrow f'| &= \sum_{v \in V} (f \uparrow f')(s,v) - \sum_{v \in V} (f \uparrow f')(v,s) \\ &= \sum_{v \in V_1} (f \uparrow f')(s,v) - \sum_{v \in V_2} (f \uparrow f')(v,s) \\ &= \sum_{v \in V_1} (f(s,v) + f'(s,v) - f'(v,s)) - \sum_{v \in V_2} (f(v,s) + f'(v,s) - f'(s,v)) \\ &= \sum_{v \in V_1} f(s,v) + \sum_{v \in V_1} f'(s,v) - \sum_{v \in V_1} f'(v,s) - \sum_{v \in V_2} f(v,s) - \sum_{v \in V_2} f'(v,s) + \sum_{v \in V_2} f'(s,v) \\ &= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) + \sum_{v \in V_1} f'(s,v) + \sum_{v \in V_2} f'(s,v) - \sum_{v \in V_1} f'(v,s) - \sum_{v \in V_2} f'(v,s) \\ &= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) + \sum_{v \in V_1 \cup V2} f'(s,v) - \sum_{v \in V_1 \cup V2} f'(v,s) \\ &= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{v \in V} f'(s,v) - \sum_{v \in V} f'(v,s) \\ &= |f| + |f'| \end{aligned}$$

where $V_1 = \{v : (s,v) \in E\}$ and $V_2 = \{v : (v,s) \in E\}$

Augmenting Paths

- **Augmenting path** p is a simple path from s to t in residual n/w G_f of a flow network $G = (V, E)$ having source s and sink t
- **Residual capacity** $c_f(p) = min\{c_f(u,v) : (u,v) \text{ is on path } p\}$
- **Define**

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on path } p \\ 0 & \text{otherwise} \end{cases}$$

- f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$
- If we augment f with f_p we get another flow in G, whose value is closer to the maximum.

Cuts of flow network

- A **cut** (S, T) of a flow network $G = (V, E)$ is a partition of V in S and $T = V - S$ such that $s \in S$ and $t \in T$
- **Net flow** across the cut is

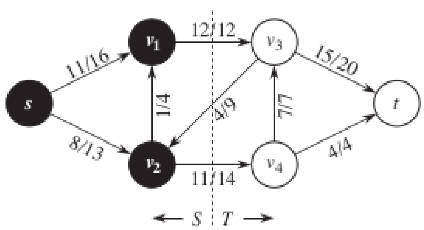
$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

- **Capacity** of the cut is

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

- **Minimum cut** of a network is a cut whose capacity is minimum over all cuts of the network

Find flow and capacity across the cut in following n/w



- flow 19, capacity 26 ?

Net flow is $|f|$

Let f be a flow in a network G with source s and sink t and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$

- $\forall u \in V - \{s,t\}$ we know $\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) = 0$

$$\begin{aligned} |f| &= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} (\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u)) \\ &= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \sum_{v \in V} f(u,v) - \sum_{u \in S-\{s\}} \sum_{v \in V} f(v,u) \\ &= \sum_{v \in V} (f(s,v) + \sum_{u \in S-\{s\}} f(u,v)) - \sum_{v \in V} (f(v,s) + \sum_{u \in S-\{s\}} f(v,u)) \\ &= \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u) \\ &= \sum_{v \in S} \sum_{u \in S} f(u,v) + \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\ &= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) + (\sum_{v \in S} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u)) \\ &= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\ &= f(S,T) \end{aligned}$$

Flow is upper bounded

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \\
&\le \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)
\end{aligned}
$$

Max-Flow min-cut theorem

If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then following conditions are equivalent

1. f is maximum flow in G
2. The residual network G_f contains no augmenting paths
3. $|f| = c(S, T)$ for some cut (S, T) of G

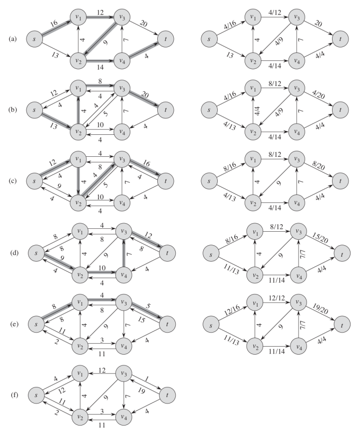
Proof:

- $(1) \rightarrow (2)$ if residual network G_f contains augmenting paths p, then augment p in G to get more flow.
- $(2) \rightarrow (3)$ let $S = \{v \in V :$ there is a path from s to v in $G_f\}$, also $T - V - S$. Consider $u \in S$ and $v \in T$; if $(u, v) \in E$ then $f(u, v) = c(u, v)$ and if $(v, u) \in E$ then $f(v, u) = 0$

$$f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) = \sum_{u\in S}\sum_{v\in T} c(u,v) = C(S,T)$$

- $(3) \rightarrow (1)$ since flow is upper bounded by $c(S, T)$ shown earlier

Example



Thank You!

Thank you very much for your attention! (Reference[1])

Queries ?

[1] [1] Book - *Algorithm*, Kormen