

The professor can carry two liters of water, and he can skate m miles before running out of water. (Because North Dakota is relatively flat, the professor does not have to worry about drinking water at a greater rate on uphill sections than on flat or downhill sections.) The professor will start in Grand Forks with two full liters of water. His official North Dakota state map shows all the places along U.S. 2 at which he can refill his water and the distances between these locations.

The professor's goal is to minimize the number of water stops along his route across the state. Give an efficient method by which he can determine which water stops he should make. Prove that your strategy yields an optimal solution, and give its running time.

16.2-5

Describe an efficient algorithm that, given a set $\{x_1, x_2, \dots, x_n\}$ of points on the real line, determines the smallest set of unit-length closed intervals that contains all of the given points. Argue that your algorithm is correct.

16.2-6 ★

Show how to solve the fractional knapsack problem in $O(n)$ time.

16.2-7

Suppose you are given two sets A and B , each containing n positive integers. You can choose to reorder each set however you like. After reordering, let a_i be the i th element of set A , and let b_i be the i th element of set B . You then receive a payoff of $\prod_{i=1}^n a_i^{b_i}$. Give an algorithm that will maximize your payoff. Prove that your algorithm maximizes the payoff, and state its running time.

16.3 Huffman codes

Huffman codes compress data very effectively: savings of 20% to 90% are typical, depending on the characteristics of the data being compressed. We consider the data to be a sequence of characters. Huffman's greedy algorithm uses a table giving how often each character occurs (i.e., its frequency) to build up an optimal way of representing each character as a binary string.

Suppose we have a 100,000-character data file that we wish to store compactly. We observe that the characters in the file occur with the frequencies given by Figure 16.3. That is, only 6 different characters appear, and the character **a** occurs 45,000 times.

We have many options for how to represent such a file of information. Here, we consider the problem of designing a *binary character code* (or *code* for short)

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

Figure 16.3 A character-coding problem. A data file of 100,000 characters contains only the characters a–f, with the frequencies indicated. If we assign each character a 3-bit codeword, we can encode the file in 300,000 bits. Using the variable-length code shown, we can encode the file in only 224,000 bits.

in which each character is represented by a unique binary string, which we call a **codeword**. If we use a **fixed-length code**, we need 3 bits to represent 6 characters: a = 000, b = 001, ..., f = 101. This method requires 300,000 bits to code the entire file. Can we do better?

A **variable-length code** can do considerably better than a fixed-length code, by giving frequent characters short codewords and infrequent characters long codewords. Figure 16.3 shows such a code; here the 1-bit string 0 represents a, and the 4-bit string 1100 represents f. This code requires

$$(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1,000 = 224,000 \text{ bits}$$

to represent the file, a savings of approximately 25%. In fact, this is an optimal character code for this file, as we shall see.

Prefix codes

We consider here only codes in which no codeword is also a prefix of some other codeword. Such codes are called **prefix codes**.³ Although we won't prove it here, a prefix code can always achieve the optimal data compression among any character code, and so we suffer no loss of generality by restricting our attention to prefix codes.

Encoding is always simple for any binary character code; we just concatenate the codewords representing each character of the file. For example, with the variable-length prefix code of Figure 16.3, we code the 3-character file abc as 0·101·100 = 0101100, where “·” denotes concatenation.

Prefix codes are desirable because they simplify decoding. Since no codeword is a prefix of any other, the codeword that begins an encoded file is unambiguous. We can simply identify the initial codeword, translate it back to the original char-

³Perhaps “prefix-free codes” would be a better name, but the term “prefix codes” is standard in the literature.

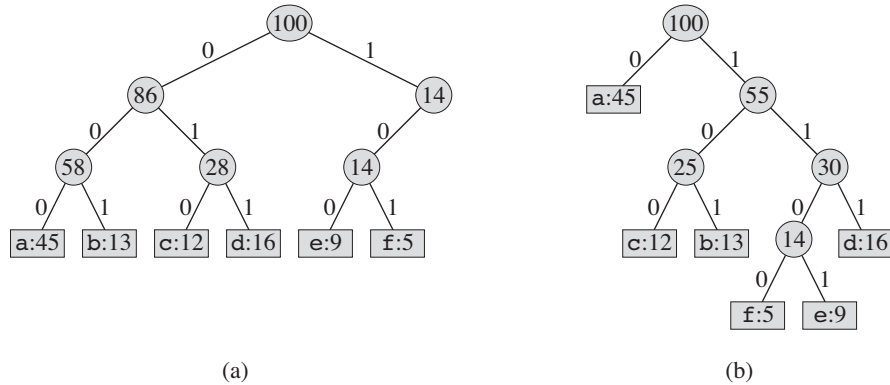


Figure 16.4 Trees corresponding to the coding schemes in Figure 16.3. Each leaf is labeled with a character and its frequency of occurrence. Each internal node is labeled with the sum of the frequencies of the leaves in its subtree. (a) The tree corresponding to the fixed-length code $a = 000, \dots, f = 101$. (b) The tree corresponding to the optimal prefix code $a = 0, b = 101, \dots, f = 1100$.

acter, and repeat the decoding process on the remainder of the encoded file. In our example, the string 001011101 parses uniquely as $0 \cdot 0 \cdot 101 \cdot 1101$, which decodes to **aabe**.

The decoding process needs a convenient representation for the prefix code so that we can easily pick off the initial codeword. A binary tree whose leaves are the given characters provides one such representation. We interpret the binary codeword for a character as the simple path from the root to that character, where 0 means “go to the left child” and 1 means “go to the right child.” Figure 16.4 shows the trees for the two codes of our example. Note that these are not binary search trees, since the leaves need not appear in sorted order and internal nodes do not contain character keys.

An optimal code for a file is always represented by a *full* binary tree, in which every nonleaf node has two children (see Exercise 16.3-2). The fixed-length code in our example is not optimal since its tree, shown in Figure 16.4(a), is not a full binary tree: it contains codewords beginning 10..., but none beginning 11.... Since we can now restrict our attention to full binary trees, we can say that if C is the alphabet from which the characters are drawn and all character frequencies are positive, then the tree for an optimal prefix code has exactly $|C|$ leaves, one for each letter of the alphabet, and exactly $|C| - 1$ internal nodes (see Exercise B.5-3).

Given a tree T corresponding to a prefix code, we can easily compute the number of bits required to encode a file. For each character c in the alphabet C , let the attribute $c.freq$ denote the frequency of c in the file and let $d_T(c)$ denote the depth

of c 's leaf in the tree. Note that $d_T(c)$ is also the length of the codeword for character c . The number of bits required to encode a file is thus

$$B(T) = \sum_{c \in C} c.freq \cdot d_T(c) , \quad (16.4)$$

which we define as the *cost* of the tree T .

Constructing a Huffman code

Huffman invented a greedy algorithm that constructs an optimal prefix code called a *Huffman code*. In line with our observations in Section 16.2, its proof of correctness relies on the greedy-choice property and optimal substructure. Rather than demonstrating that these properties hold and then developing pseudocode, we present the pseudocode first. Doing so will help clarify how the algorithm makes greedy choices.

In the pseudocode that follows, we assume that C is a set of n characters and that each character $c \in C$ is an object with an attribute $c.freq$ giving its frequency. The algorithm builds the tree T corresponding to the optimal code in a bottom-up manner. It begins with a set of $|C|$ leaves and performs a sequence of $|C| - 1$ “merging” operations to create the final tree. The algorithm uses a min-priority queue Q , keyed on the *freq* attribute, to identify the two least-frequent objects to merge together. When we merge two objects, the result is a new object whose frequency is the sum of the frequencies of the two objects that were merged.

HUFFMAN(C)

```

1   $n = |C|$ 
2   $Q = C$ 
3  for  $i = 1$  to  $n - 1$ 
4      allocate a new node  $z$ 
5       $z.left = x = \text{EXTRACT-MIN}(Q)$ 
6       $z.right = y = \text{EXTRACT-MIN}(Q)$ 
7       $z.freq = x.freq + y.freq$ 
8       $\text{INSERT}(Q, z)$ 
9  return  $\text{EXTRACT-MIN}(Q)$     // return the root of the tree
```

For our example, Huffman's algorithm proceeds as shown in Figure 16.5. Since the alphabet contains 6 letters, the initial queue size is $n = 6$, and 5 merge steps build the tree. The final tree represents the optimal prefix code. The codeword for a letter is the sequence of edge labels on the simple path from the root to the letter.

Line 2 initializes the min-priority queue Q with the characters in C . The **for** loop in lines 3–8 repeatedly extracts the two nodes x and y of lowest frequency

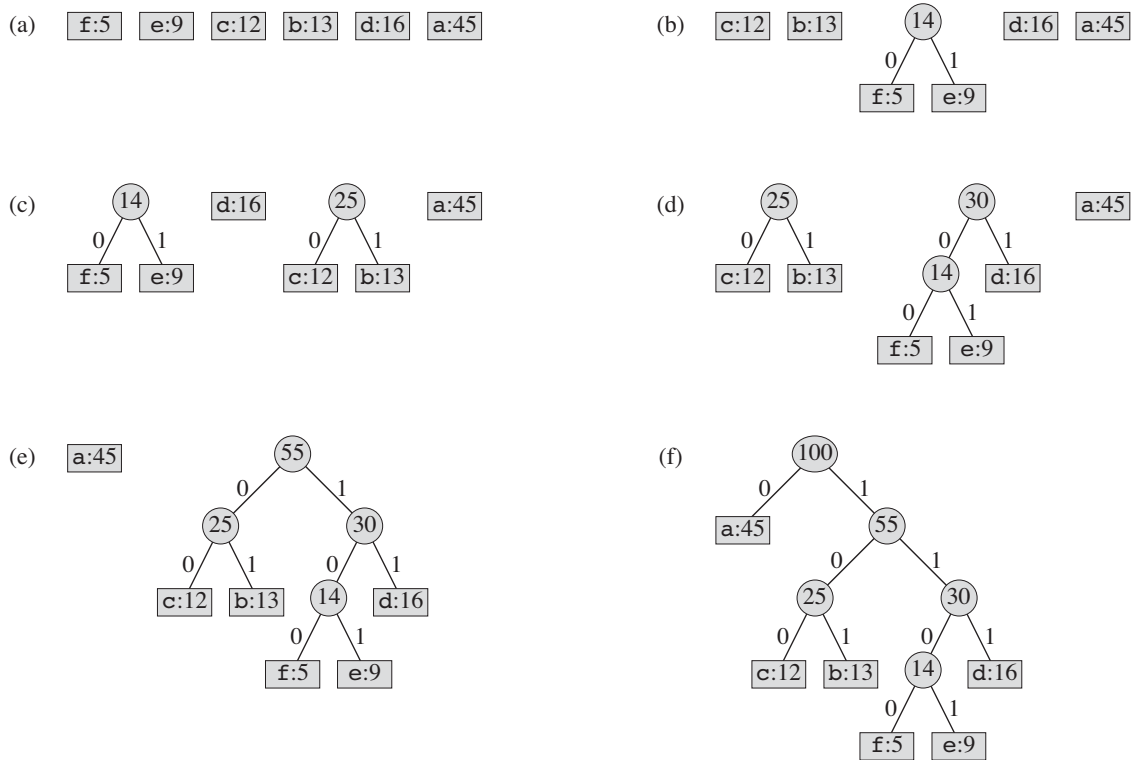


Figure 16.5 The steps of Huffman's algorithm for the frequencies given in Figure 16.3. Each part shows the contents of the queue sorted into increasing order by frequency. At each step, the two trees with lowest frequencies are merged. Leaves are shown as rectangles containing a character and its frequency. Internal nodes are shown as circles containing the sum of the frequencies of their children. An edge connecting an internal node with its children is labeled 0 if it is an edge to a left child and 1 if it is an edge to a right child. The codeword for a letter is the sequence of labels on the edges connecting the root to the leaf for that letter. (a) The initial set of $n = 6$ nodes, one for each letter. (b)–(e) Intermediate stages. (f) The final tree.

from the queue, replacing them in the queue with a new node z representing their merger. The frequency of z is computed as the sum of the frequencies of x and y in line 7. The node z has x as its left child and y as its right child. (This order is arbitrary; switching the left and right child of any node yields a different code of the same cost.) After $n - 1$ mergers, line 9 returns the one node left in the queue, which is the root of the code tree.

Although the algorithm would produce the same result if we were to excise the variables x and y —assigning directly to $z.left$ and $z.right$ in lines 5 and 6, and changing line 7 to $z.freq = z.left.freq + z.right.freq$ —we shall use the node

names x and y in the proof of correctness. Therefore, we find it convenient to leave them in.

To analyze the running time of Huffman's algorithm, we assume that Q is implemented as a binary min-heap (see Chapter 6). For a set C of n characters, we can initialize Q in line 2 in $O(n)$ time using the BUILD-MIN-HEAP procedure discussed in Section 6.3. The **for** loop in lines 3–8 executes exactly $n - 1$ times, and since each heap operation requires time $O(\lg n)$, the loop contributes $O(n \lg n)$ to the running time. Thus, the total running time of HUFFMAN on a set of n characters is $O(n \lg n)$. We can reduce the running time to $O(n \lg \lg n)$ by replacing the binary min-heap with a van Emde Boas tree (see Chapter 20).

Correctness of Huffman's algorithm

To prove that the greedy algorithm HUFFMAN is correct, we show that the problem of determining an optimal prefix code exhibits the greedy-choice and optimal-substructure properties. The next lemma shows that the greedy-choice property holds.

Lemma 16.2

Let C be an alphabet in which each character $c \in C$ has frequency $c.freq$. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

Proof The idea of the proof is to take the tree T representing an arbitrary optimal prefix code and modify it to make a tree representing another optimal prefix code such that the characters x and y appear as sibling leaves of maximum depth in the new tree. If we can construct such a tree, then the codewords for x and y will have the same length and differ only in the last bit.

Let a and b be two characters that are sibling leaves of maximum depth in T . Without loss of generality, we assume that $a.freq \leq b.freq$ and $x.freq \leq y.freq$. Since $x.freq$ and $y.freq$ are the two lowest leaf frequencies, in order, and $a.freq$ and $b.freq$ are two arbitrary frequencies, in order, we have $x.freq \leq a.freq$ and $y.freq \leq b.freq$.

In the remainder of the proof, it is possible that we could have $x.freq = a.freq$ or $y.freq = b.freq$. However, if we had $x.freq = b.freq$, then we would also have $a.freq = b.freq = x.freq = y.freq$ (see Exercise 16.3-1), and the lemma would be trivially true. Thus, we will assume that $x.freq \neq b.freq$, which means that $x \neq b$.

As Figure 16.6 shows, we exchange the positions in T of a and x to produce a tree T' , and then we exchange the positions in T' of b and y to produce a tree T''

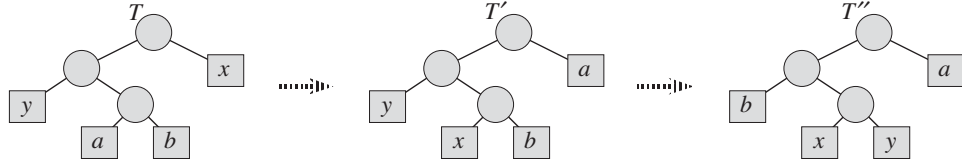


Figure 16.6 An illustration of the key step in the proof of Lemma 16.2. In the optimal tree T , leaves a and b are two siblings of maximum depth. Leaves x and y are the two characters with the lowest frequencies; they appear in arbitrary positions in T . Assuming that $x \neq b$, swapping leaves a and x produces tree T' , and then swapping leaves b and y produces tree T'' . Since each swap does not increase the cost, the resulting tree T'' is also an optimal tree.

in which x and y are sibling leaves of maximum depth. (Note that if $x = b$ but $y \neq a$, then tree T'' does not have x and y as sibling leaves of maximum depth. Because we assume that $x \neq b$, this situation cannot occur.) By equation (16.4), the difference in cost between T and T' is

$$\begin{aligned}
 B(T) - B(T') &= \sum_{c \in C} c.\text{freq} \cdot d_T(c) - \sum_{c \in C} c.\text{freq} \cdot d_{T'}(c) \\
 &= x.\text{freq} \cdot d_T(x) + a.\text{freq} \cdot d_T(a) - x.\text{freq} \cdot d_{T'}(x) - a.\text{freq} \cdot d_{T'}(a) \\
 &= x.\text{freq} \cdot d_T(x) + a.\text{freq} \cdot d_T(a) - x.\text{freq} \cdot d_T(a) - a.\text{freq} \cdot d_T(x) \\
 &= (a.\text{freq} - x.\text{freq})(d_T(a) - d_T(x)) \\
 &\geq 0,
 \end{aligned}$$

because both $a.\text{freq} - x.\text{freq}$ and $d_T(a) - d_T(x)$ are nonnegative. More specifically, $a.\text{freq} - x.\text{freq}$ is nonnegative because x is a minimum-frequency leaf, and $d_T(a) - d_T(x)$ is nonnegative because a is a leaf of maximum depth in T . Similarly, exchanging y and b does not increase the cost, and so $B(T') - B(T'')$ is nonnegative. Therefore, $B(T'') \leq B(T)$, and since T is optimal, we have $B(T) \leq B(T'')$, which implies $B(T'') = B(T)$. Thus, T'' is an optimal tree in which x and y appear as sibling leaves of maximum depth, from which the lemma follows. ■

Lemma 16.2 implies that the process of building up an optimal tree by mergers can, without loss of generality, begin with the greedy choice of merging together those two characters of lowest frequency. Why is this a greedy choice? We can view the cost of a single merger as being the sum of the frequencies of the two items being merged. Exercise 16.3-4 shows that the total cost of the tree constructed equals the sum of the costs of its mergers. Of all possible mergers at each step, HUFFMAN chooses the one that incurs the least cost.

The next lemma shows that the problem of constructing optimal prefix codes has the optimal-substructure property.

Lemma 16.3

Let C be a given alphabet with frequency $c.\text{freq}$ defined for each character $c \in C$. Let x and y be two characters in C with minimum frequency. Let C' be the alphabet C with the characters x and y removed and a new character z added, so that $C' = C - \{x, y\} \cup \{z\}$. Define f for C' as for C , except that $z.\text{freq} = x.\text{freq} + y.\text{freq}$. Let T' be any tree representing an optimal prefix code for the alphabet C' . Then the tree T , obtained from T' by replacing the leaf node for z with an internal node having x and y as children, represents an optimal prefix code for the alphabet C .

Proof We first show how to express the cost $B(T)$ of tree T in terms of the cost $B(T')$ of tree T' , by considering the component costs in equation (16.4). For each character $c \in C - \{x, y\}$, we have that $d_T(c) = d_{T'}(c)$, and hence $c.\text{freq} \cdot d_T(c) = c.\text{freq} \cdot d_{T'}(c)$. Since $d_T(x) = d_T(y) = d_{T'}(z) + 1$, we have

$$\begin{aligned} x.\text{freq} \cdot d_T(x) + y.\text{freq} \cdot d_T(y) &= (x.\text{freq} + y.\text{freq})(d_{T'}(z) + 1) \\ &= z.\text{freq} \cdot d_{T'}(z) + (x.\text{freq} + y.\text{freq}), \end{aligned}$$

from which we conclude that

$$B(T) = B(T') + x.\text{freq} + y.\text{freq}$$

or, equivalently,

$$B(T') = B(T) - x.\text{freq} - y.\text{freq}.$$

We now prove the lemma by contradiction. Suppose that T does not represent an optimal prefix code for C . Then there exists an optimal tree T'' such that $B(T'') < B(T)$. Without loss of generality (by Lemma 16.2), T'' has x and y as siblings. Let T''' be the tree T'' with the common parent of x and y replaced by a leaf z with frequency $z.\text{freq} = x.\text{freq} + y.\text{freq}$. Then

$$\begin{aligned} B(T''') &= B(T'') - x.\text{freq} - y.\text{freq} \\ &< B(T) - x.\text{freq} - y.\text{freq} \\ &= B(T'), \end{aligned}$$

yielding a contradiction to the assumption that T' represents an optimal prefix code for C' . Thus, T must represent an optimal prefix code for the alphabet C . ■

Theorem 16.4

Procedure HUFFMAN produces an optimal prefix code.

Proof Immediate from Lemmas 16.2 and 16.3. ■

Exercises**16.3-1**

Explain why, in the proof of Lemma 16.2, if $x.freq = b.freq$, then we must have $a.freq = b.freq = x.freq = y.freq$.

16.3-2

Prove that a binary tree that is not full cannot correspond to an optimal prefix code.

16.3-3

What is an optimal Huffman code for the following set of frequencies, based on the first 8 Fibonacci numbers?

a:1 b:1 c:2 d:3 e:5 f:8 g:13 h:21

Can you generalize your answer to find the optimal code when the frequencies are the first n Fibonacci numbers?

16.3-4

Prove that we can also express the total cost of a tree for a code as the sum, over all internal nodes, of the combined frequencies of the two children of the node.

16.3-5

Prove that if we order the characters in an alphabet so that their frequencies are monotonically decreasing, then there exists an optimal code whose codeword lengths are monotonically increasing.

16.3-6

Suppose we have an optimal prefix code on a set $C = \{0, 1, \dots, n-1\}$ of characters and we wish to transmit this code using as few bits as possible. Show how to represent any optimal prefix code on C using only $2n - 1 + n \lceil \lg n \rceil$ bits. (*Hint:* Use $2n - 1$ bits to specify the structure of the tree, as discovered by a walk of the tree.)

16.3-7

Generalize Huffman's algorithm to ternary codewords (i.e., codewords using the symbols 0, 1, and 2), and prove that it yields optimal ternary codes.

16.3-8

Suppose that a data file contains a sequence of 8-bit characters such that all 256 characters are about equally common: the maximum character frequency is less than twice the minimum character frequency. Prove that Huffman coding in this case is no more efficient than using an ordinary 8-bit fixed-length code.

16.3-9

Show that no compression scheme can expect to compress a file of randomly chosen 8-bit characters by even a single bit. (*Hint:* Compare the number of possible files with the number of possible encoded files.)

★ 16.4 Matroids and greedy methods

In this section, we sketch a beautiful theory about greedy algorithms. This theory describes many situations in which the greedy method yields optimal solutions. It involves combinatorial structures known as “matroids.” Although this theory does not cover all cases for which a greedy method applies (for example, it does not cover the activity-selection problem of Section 16.1 or the Huffman-coding problem of Section 16.3), it does cover many cases of practical interest. Furthermore, this theory has been extended to cover many applications; see the notes at the end of this chapter for references.

Matroids

A **matroid** is an ordered pair $M = (S, \mathcal{I})$ satisfying the following conditions.

1. S is a finite set.
2. \mathcal{I} is a nonempty family of subsets of S , called the **independent** subsets of S , such that if $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$. We say that \mathcal{I} is **hereditary** if it satisfies this property. Note that the empty set \emptyset is necessarily a member of \mathcal{I} .
3. If $A \in \mathcal{I}$, $B \in \mathcal{I}$, and $|A| < |B|$, then there exists some element $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$. We say that M satisfies the **exchange property**.

The word “matroid” is due to Hassler Whitney. He was studying **matric matroids**, in which the elements of S are the rows of a given matrix and a set of rows is independent if they are linearly independent in the usual sense. As Exercise 16.4-2 asks you to show, this structure defines a matroid.

As another example of matroids, consider the **graphic matroid** $M_G = (S_G, \mathcal{I}_G)$ defined in terms of a given undirected graph $G = (V, E)$ as follows:

- The set S_G is defined to be E , the set of edges of G .
- If A is a subset of E , then $A \in \mathcal{I}_G$ if and only if A is acyclic. That is, a set of edges A is independent if and only if the subgraph $G_A = (V, A)$ forms a forest.

The graphic matroid M_G is closely related to the minimum-spanning-tree problem, which Chapter 23 covers in detail.