<u>Atlantic City Algorithms</u> correspond to the complexity class BPP, and they give two-sided error.

<u>Monte-Carlo Algorithms</u> correspond to the complexity classes RP, and co-RP, and they give one-sided error.

<u>Las Vegas Algorithms</u> correspond to the complexity class ZPP, and they do not give any error.

<u>$ZPP = RP \cap coRP$</u>

① <u>$ZPP \subseteq RP \cap coRP$</u> : Let $L \in ZPP$

$\Rightarrow$ $\exists$ PTM $M$ such that $x \in L \Rightarrow M(x) = 1$ and $x \notin L \Rightarrow M(x) = 0$ and expected running time of $M$ is polynomial: $E[T(n)] = p(n)$.

<u>$L \in RP$</u> : construct a PTM $M_1$ that simulates the moves of $M$ for $3p(n)$ steps. If during this time $M$ outputs 0 or 1, then $M_1$ outputs 0 or 1. If during this time $M$ does not give any output, then $M_1$ outputs 0.

<u>$L \in coRP$</u> : Same as $M_1$ except that it outputs 1 when $M$ does not give any output.

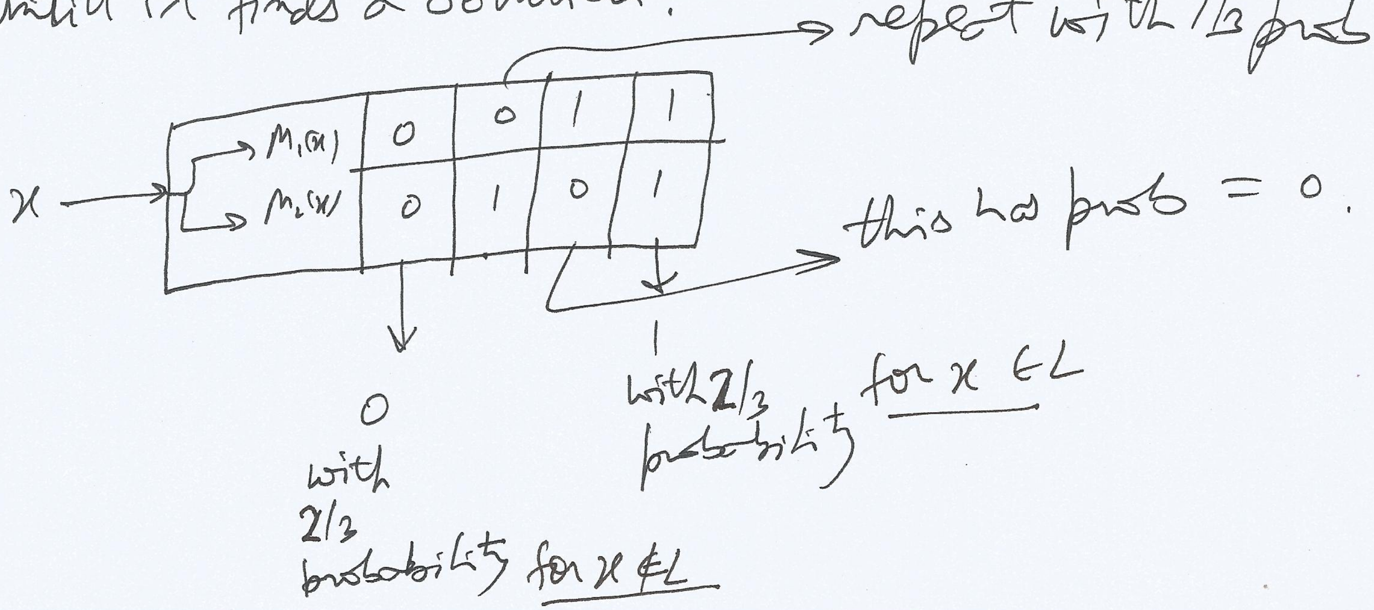
For error probability, we apply Markov's inequality: $\Pr[\text{error}] \le \Pr[T(n) \ge 3p(n)] < \frac{1}{3}$.

(2) RP $\cap$ coRP $\subseteq$ ZPP: Let $L \in RP \cap coRP$

$\Rightarrow \exists$ PTM $M_1$ and $M_2$ both running in polynomial time (let the larger time complexity be $p(n)$)

such that
$$x \in L \Rightarrow \Pr[M_1(x) = 0] \le 1/3$$
$$x \notin L \Rightarrow \Pr[M_1(x) = 1] = 0$$
$$x \in L \Rightarrow \Pr[M_2(x) = 0] = 0$$
$$x \notin L \Rightarrow \Pr[M_2(x) = 1] \le 1/3$$

Construct a PTM M that simultaneously runs $M_1(x)$ and $M_2(x)$. If they both output 0, then M outputs 0. If they both output 1, then M outputs 1. If $M_1$ outputs 0, and $M_2$ outputs 1, then M repeats running $M_1(x)$ and $M_2(x)$, untill it finds a solution.



$\longrightarrow M_2(x)=1,\ \Pr[M_1(x)=1] \geq \frac{2}{3},\ \Pr[M_1(x)=0] \leq \frac{1}{3}$

$x \in L \Rightarrow E(T(n)) = \left(\frac{2}{3}\right)p(n) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(2p(n)) + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)(3p(n)) + \dots$

$x \notin L \Rightarrow E(T(n)) = \left(\frac{2}{3}\right)p(n) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(2p(n)) + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)(3p(n)) + \dots$

$\longrightarrow M_1(x)=0,\ \Pr[M_2(x)=0] \geq \frac{2}{3},\ \Pr[M_2(x)=1] \leq 1/3$

$$\Rightarrow E(T(n)) = \left(\frac{2}{3}\right)p(n)\left[1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \dots\right]$$

$$\Rightarrow 3E(T(n)) = \left(\frac{2}{3}\right)p(n)\left[3 + 2 + \frac{3}{3} + \frac{4}{3^2} + \dots\right]$$

$$\Rightarrow 2E(T(n)) = \frac{2}{3}p(n)\left[3 + 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\right]$$

$$\Rightarrow 2E(T(n)) = \frac{2}{3}p(n)\left[3 + \frac{1}{1-\frac{1}{3}}\right]$$

$$\Rightarrow E(T(n)) = p(n)\left[1 + \frac{1}{2}\right] = \frac{3}{2}p(n)$$

## Error Reduction of Monte-Carlo Algorithms:

Let $M$ be a Monte-Carlo algorithm with error probability bounded above by $p$ (corresponding to the complexity class $RP$) accepting $L$:

$x \in L \Rightarrow \Pr[M(x)=0] \leq p$

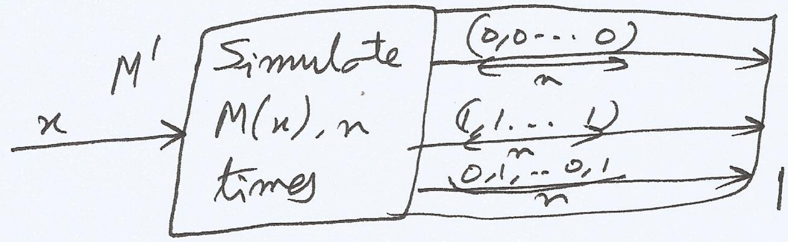
$x \notin L \Rightarrow \Pr[M(x)=1] = 0$

We have to reduce error probability to $\epsilon$:

$x \in L \Rightarrow \Pr[M'(x)=0] \leq \epsilon$

$x \notin L \Rightarrow \Pr[M'(x)=1] = 0$

Consider a PTM $M'$ which simulates $M$, $n$ times




0 : $Pr(\text{error}) = p^n$
1 : $Pr(\text{error}) = 0$
1 : $Pr(\text{error}) = 0$

$Pr(\text{error}) = p^n \leq \epsilon \Rightarrow n \log p \leq \log \epsilon$

$$\Rightarrow \boxed{n \geq \frac{\log \epsilon}{\log p}}$$

Error Reduction of Atlantic City Algorithms :

Let $M$ be an Atlantic City algorithm accepting $L$ with error probability bounded above by $p$ (corresponding to the complexity class BPP):

$x \in L \Rightarrow Pr[M(x) = 0] \leq p < \frac{1}{2}$

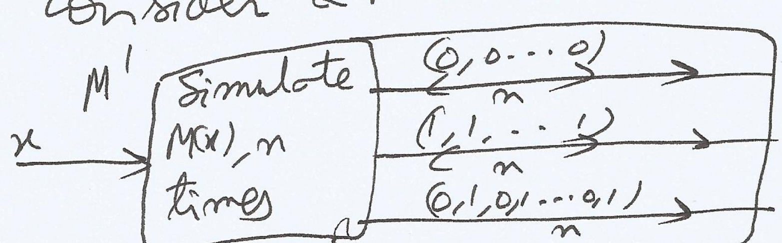
$x \notin L \Rightarrow Pr[M(x) = 1] \leq p < \frac{1}{2}$

We have to reduce error probability to $\epsilon$:

$x \in L \Rightarrow Pr[M'(x) = 0] \leq \epsilon$

$x \notin L \Rightarrow Pr[M'(x) = 1] \leq \epsilon$

Consider a PTM $M'$ which simulates $M$, $n$ times:




0 : $Pr(\text{error}) = p^n$
1 : $Pr(\text{error}) = p^n$
Majority: $Pr(\text{error}) = Pr[X < \frac{n}{2}]$

where $X$ is a Random Variable defined by: $X = \sum_{i=1}^{n} X_i$

$X_i$ ($1 \le i \le n$) are independent Random Variables corresponding to $i$th trial such that $X_i = 1$ if $i$th simulation of $M(x)$ gives correct answer.

$Pr[X_i = 1] \ge 1-p$, $Pr[X_i = 0] \le p$

$E(X_i) \ge 1-p$

$$\mu = \sum_{i=1}^{n} E(X_i) \ge n(1-p) > \frac{n}{2}$$

We apply Chernoff's Bound:

$$Pr[\text{error}] = Pr[X < \tfrac{n}{2}] = Pr\left[X < \left(\frac{1}{2(1-p)}\right)(n(1-p))\right]$$

$$< \left( \frac{e^{-\frac{1}{2(1-p)}}}{\left(1 - \frac{1}{2(1-p)}\right)^{1-\frac{1}{2(1-p)}}} \right)^{n(1-p)} \le \epsilon$$

Taking $\log_e$ on both sides:

$$n\left[-\frac{1}{2} - \left(\frac{1}{2} - p\right)\log\left(\frac{\frac{1}{2} - p}{1-p}\right)\right] \le \log(\epsilon)$$

$$\Rightarrow \boxed{n \ge \frac{\log_e(\epsilon)}{\frac{1}{2} + \left(\frac{1}{2} - p\right)\log_e\left(\frac{\frac{1}{2} - p}{1-p}\right)}}$$