Chapter A

Linear programming

In this appendix we give a quick overview of linear programming. In linear programming, we find a non-negative, rational vector x that minimizes a given linear objective function in x subject to linear constraints on x. More formally, given an n-vector $c \in \mathbb{Q}^n$, an m-vector $b \in \mathbb{Q}^m$, and an $m \times n$ matrix $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$, an *optimal solution* to the linear programming problem

$$
\begin{aligned}
& & \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j & & \\
(P) & & \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, & i = 1, \ldots, m, & \qquad \text{(A.1)} \\
& & & x_j \geq 0, & j = 1, \ldots, n. & \qquad \text{(A.2)}
\end{aligned}
$$

is an n-vector x that minimizes the linear *objective function* $\sum_{j=1}^{n} c_j x_j$ subject to the *constraints* (A.1) and (A.2). The vector x is called the *variable*. Any x which satisfies the constraints is said to be *feasible*, and if such an x exists, the linear program is said to be *feasible* or is a *feasible solution*. We say that we *solve* the linear program if we find an optimal solution x. If there does not exist any feasible x, the linear program is called *infeasible*. The term "linear program" is frequently abbreviated to *LP*. There are very efficient, practical algorithms to solve linear programs; LPs with tens of thousands of variables and constraints are solved routinely.

One could imagine variations and extensions of the linear program above: for example, maximizing the objective function rather than minimizing it, having equations in addition to inequalities, and allowing variables x_j to take on negative values. However, the linear program (P) above is sufficiently general that it can capture all these variations, and so is said to be in *canonical form*. To see this, observe that maximizing $\sum_{j=1}^{n} c_j x_j$ is equivalent to minimizing $-\sum_{j=1}^{n} c_j x_j$, and that an equation $\sum_{j=1}^{n} a_{ij} x_j = b_i$ can be expressed as a pair of inequalities $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ and $-\sum_{j=1}^{n} a_{ij} x_j \geq -b_i$. Finally, a variable x_j which is allowed to be negative can be expressed in terms of two nonnegative variables x_j^+ and x_j^- by substituting $x_j^+ - x_j^-$ for x_j in the objective function and the constraints.

Another variation of linear programming, called *integer linear programming* or *integer programming*, allows constraints that require variable x_j to be an integer. For instance, we can require that $x_j \in \mathbb{N}$, or that x_j be in a bounded range of integers, such as $x_j \in \{0, 1\}$. Unlike linear programming, there is currently no efficient, practical algorithm to solve general integer programs; in fact, many quite small integer programs are very difficult to solve. Integer programming is known to be NP-complete, so no efficient algorithm is likely to exist. Nevertheless, integer programming remains a useful tool because it is a compact way to model problems in combinatorial optimization, and because there are several important special cases that do have efficient algorithms.

Linear programming has a very interesting and useful concept of *duality*. To explain it, we begin with a small example. Consider the following linear program in canonical form:

$$\begin{aligned}
\text{minimize} \quad & 6x_1 + 4x_2 + 2x_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 + x_3 \geq 5, \\
& x_1 + x_2 \geq 3, \\
& x_2 + x_3 \geq 4, \\
& x_i \geq 0, \qquad \text{for } i = 1, 2, 3.
\end{aligned}$$

Observe that because all variables x_j are nonnegative, it must be the case that the objective function $6x_1 + 4x_2 + 2x_3 \geq 4x_1 + 2x_2 + x_3$. Furthermore, $4x_1 + 2x_2 + x_3 \geq 5$ by the first constraint. Thus we know that the value of the objective function of an optimal solution to this linear program (called the *optimal value* of the linear program) is at least 5. We can get an improved lower bound by considering combinations of the constraints. It is also the case that $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + 2 \cdot (x_1 + x_2) \geq 5 + 2 \cdot 3 = 11$, which is the first constraint summed together with twice the second constraint. Even better, $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + (x_1 + x_2) + (x_2 + x_3) \geq 5 + 3 + 4 = 12$, by summing all three constraints together. Thus the optimal value of the LP is at least 12.

In fact, we can set up a linear program to determine the best lower bound obtainable by various combinations of the constraints. Suppose we take y_1 times the first constraint, y_2 times the second, and y_3 times the third, where the y_i are non-negative. Then the lower bound achieved is $5y_1 + 3y_2 + 4y_3$. We need to ensure that

$$6x_1 + 4x_2 + 2x_3 \geq y_1(4x_1 + 2x_2 + x_3) + y_2(x_1 + x_2) + y_3(x_2 + x_3),$$

which we can do by ensuring that no more than 6 copies of x_1, 4 copies of x_2, and 2 copies of x_3 appear in the sum; that is, $4y_1 + y_2 \leq 6$, $2y_1 + y_2 + y_3 \leq 4$, and $y_1 + y_3 \leq 2$. We want to maximize the lower bound achieved subject to these constraints, which gives the linear program

$$\begin{aligned}
\text{maximize} \quad & 5y_1 + 3y_2 + 4y_3 \\
\text{subject to} \quad & 4y_1 + y_2 \leq 6, \\
& 2y_1 + y_2 + y_3 \leq 4, \\
& y_1 + y_3 \leq 2, \\
& y_i \geq 0, \qquad i = 1, 2, 3.
\end{aligned}$$

This maximization linear program is called the *dual* of the previous minimization linear program, which is referred to as the *primal*. It is not hard to see that any feasible solution to the dual gives an objective function value that is a lower bound on the optimal value of the primal.

We can create a dual for any linear program; the dual of the canonical form LP (P) above is

$$
(D) \qquad \begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{m} b_i y_i \\
\text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \qquad \text{for } j = 1, \ldots, n, \qquad (A.3)\\
& y_i \geq 0, \qquad \text{for } i = 1, \ldots, m. \qquad (A.4)
\end{aligned}
$$

As in our small example, we introduce a variable y_i for each linear constraint in the primal, and try to maximize the lower bound achieved by summing y_i times the ith constraint, subject to the constraint that the variable x_j not appear more than c_j times in the sum.

We now formalize our argument above that the value of the dual of the canonical form LP is a lower bound on the value of the primal. This fact is called *weak duality.*

Theorem A.1 (Weak duality): *If x is a feasible solution to the LP (P), and y a feasible solution to the LP (D), then $\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$.*

Proof.

$$
\begin{aligned}
\sum_{j=1}^{n} c_j x_j & \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j & \text{(A.5)} \\
& = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i & \\
& \geq \sum_{i=1}^{m} b_i y_i, & \text{(A.6)}
\end{aligned}
$$

where the first inequality follows by the feasibility of y (via dual inequalities (A.3)) and $x_j \geq 0$, the next equality by an interchange of summations, and the last inequality by the feasibility of x (via primal inequalities (A.1)) and $y_i \geq 0$. □

A very surprising, interesting, and useful fact is that when both primal and dual LPs are feasible, their optimal values are exactly the same! This is sometimes called *strong duality.*

Theorem A.2 (Strong duality): *If the LPs (P) and (D) are feasible, then for any optimal solution x^* to (P) and any optimal solution y^* to (D), $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$.*

As an example of this, for the small, three-variable LP and its dual we saw earlier, the optimal value is 14, achieved by setting $x_1^* = 0$, $x_2^* = 3$, and $x_3^* = 1$ in the primal, and $y_1^* = 0$, $y_2^* = 2$, and $y_3^* = 2$ in the dual. A proof of Theorem A.2 is beyond the scope of this appendix, but one can be found in the textbooks on linear programming referenced in the notes at the end of Chapter 1.

An easy but useful corollary of strong duality is a set of implications called the *complementary slackness conditions.* Let $\bar{x}$ and $\bar{y}$ be feasible solutions to (P) and (D), respectively. We say that $\bar{x}$ and $\bar{y}$ obey the complementary slackness conditions if $\sum_{i=1}^{m} a_{ij} \bar{y}_i = c_j$ for each j such that $\bar{x}_j > 0$ and if $\sum_{j=1}^{n} a_{ij} \bar{x}_j = b_i$ for each i such that $\bar{y}_i > 0$. In other words, whenever $\bar{x}_j > 0$ the dual constraint that corresponds to the variable x_j is met with equality, and whenever $\bar{y}_i > 0$ the primal constraint that corresponds to the variable y_i is met with equality.

Corollary A.3 (Complementary slackness): *Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the LPs (P) and (D), respectively. Then $\bar{x}$ and $\bar{y}$ obey the complementary slackness conditions if and only if they are optimal solutions to their respective LPs.*

Proof. If $\bar{x}$ and $\bar{y}$ are optimal solutions, then by strong duality the two inequalities (A.5) and (A.6) must hold with equality, which implies that the complementary slackness conditions are obeyed. Similarly, if the complementary slackness conditions are obeyed, then (A.5) and (A.6) must hold with equality, and it must be the case that $\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i$. By weak duality, $\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$ for any feasible x and y so therefore $\bar{x}$ and $\bar{y}$ must both be optimal. □

So far we have only discussed the case in which the LPs (P) and (D) are feasible, but of course it is possible that one or both of them are infeasible. The following theorem tells us that if the primal is infeasible and the dual is feasible, the dual must be *unbounded*: that is, given a feasible y with objective function value z, then for any $z' > z$ there exists a feasible y' of value z'. Similarly, if the dual is infeasible and the primal is feasible, then the primal is unbounded: given feasible x with objective function value z, then for any $z' < z$ there exists a feasible x' with value z'. If an LP is not unbounded, we say it is *bounded*.

Theorem A.4: *For primal and dual LPs (P) and (D), one of the following four statements must hold: (i) both (P) and (D) are feasible; (ii) (P) is infeasible and (D) is unbounded; (iii) (P) is unbounded and (D) is infeasible; or (iv) both (P) and (D) are infeasible.*

Sometimes in the design of approximation algorithms it is helpful to take advantage of the fact that if an LP is feasible, there exist feasible solutions of a particular form, called *basic feasible solutions*. Furthermore, if an optimal solution exists, then there exists an *basic optimal solution*; that is, an optimal solution that is also a basic feasible solution. Most linear programming algorithms will return a basic optimal solution. Consider the canonical primal LP: there are $n + m$ constraints and n variables. A basic solution is obtained by selecting n of the constraints, treating them as equalities, and solving the resulting $n \times n$ linear system (assuming the system is consistent and the n constraints are linearly independent). The solution might not be feasible since we ignored some of the constraints. The oldest and most frequently used linear programming algorithm, called the *simplex method*, works by moving from basic solution to basic solution, at each step swapping a constraint outside of the linear system for another in the linear system in a particular manner, eventually reaching a basic feasible solution, then finally a basic optimal solution.