Complexity classes for Approximation: We started with the definition of NP-optimization problems by defining the complexity class NPO. Then we studied a 2-approximation algorithm for cardinality vertex cover. We define the complexity class APX to be NPO problems having a constant factor approximation algorithm (like cardinality vertex cover). From the definitions of NPO and APX: $APX \subseteq NPO$. Under the assumption of $P \neq NP$, we will show that $APX \subsetneq NPO$. One example of a NPO problem that does not belong to APX is the Travelling Salesman Problem (TSP):

Given a complete graph with nonnegative edge costs, find a minimum cost cycle visiting every vertex exactly once.

For any polynomial time computable function $\alpha(n)$, TSP cannot be approximated within a factor of $\alpha(n)$, unless $P=NP$.

Assume, for a contradiction, that there is a factor $\alpha(n)$ polynomial time approximation algorithm, A, for the general TSP problem. We will show that A can be used for deciding the Hamiltonian cycle problem (which is NP-hard) in polynomial time, thus implying $P=NP$.

Consider the following reduction from a graph G on n vertices to an edge-weighted complete graph G' on n vertices: assign a weight of 1 to edges of G, and a weight $\alpha(n) \cdot n$ to nonedges, to obtain G'. Now, if G has a Hamiltonian cycle, then the corresponding tour in G' has cost n. On the other hand, if G has no Hamiltonian cycle, any tour in G' must use an edge of cost $\alpha(n) \cdot n$, and therefore has cost $> \alpha(n) \cdot n$. Now we can use A for deciding which of the two cases holds, and then return our answer accordingly.

We defined the complexity class PTAS as the NPO problems that may be approximated to any constant factor close to 1 ($1+\epsilon$ or $1-\epsilon$). For a fixed ϵ the running time will be polynomial in input size n. In general the complexity may not be polynomial in $(1/\epsilon)$. From the definition of PTAS: PTAS $\subseteq$ APX. Under the assumption of $P \neq NP$, we will show that PTAS $\subsetneq$ APX.

One example of a APX problem that does not belong to PTAS is the <u>Bin Packing Problem</u>:

Given n items with sizes $a_1, \dots, a_n \in (0,1]$, find a packing in unit-sized bins that minimizes the number of bins used.

The First-Fit algorithm considers items in an arbitrary order. In the i^{th} step, it has a list of partially packed bins, say $B_1, \dots, B_k$. It attempts to put the next item, a_i, in one of these bins, in this order. If a_i does not fit into any of these bins, it opens a new bin B_{k+1}, and puts a_i in it. If the algorithm uses m bins, then at least $m-1$ bins are more than half full. Therefore, $\sum_{i=1}^{n} a_i > \frac{m-1}{2}$. Since the sum of the item sizes is a lower bound on OPT, $m-1 < 2\,OPT \Rightarrow \underline{m \leq 2\,OPT}$.

For any $\epsilon > 0$, there is no approximation algorithm having a guarantee of $3/2 - \epsilon$ for the bin packing problem, assuming $P \neq NP$.

If there were such an algorithm, then we can use it to solve the NP-hard problem of deciding if there is a way to partition n nonnegative numbers $a_1, \dots, a_n$ into two sets, each adding up to $(1/2)\sum_i a_i$. Clearly, the answer to this question is "yes" $\Leftrightarrow$ the n items can be packed in 2 bins of size $(1/2)\sum a_i$. If the answer is "yes" the $3/2-\epsilon$ factor algorithm will have to give an optimal packing, and thereby solving the partition problem.

We defined the complexity class FPTAS as the NPO problems that may be approximated to any constant factor close to 1 ($1+\epsilon$ or $1-\epsilon$), where the complexity is polynomial in both the input size n as well as $(1/\epsilon)$. From the definition of FPTAS: FPTAS $\subseteq$ PTAS. Under the assumption of $P \neq NP$, we will show that FPTAS $\subsetneq$ PTAS.

An optimization problem Π is polynomially bounded if there exists a polynomial p such that, for any instance x and for any $y \in S_\Pi(x)$, $Obj_\Pi(x, y) \leq p(|x|)$.

No NP-hard polynomially bounded optimization problem belongs to the class FPTAS unless $P = NP$.

Suppose we have a FPTAS A for Π (max problem) which, for any instance x and for any rational ϵ $(0<\epsilon<1)$ runs in time bounded by $q(|x|, 1/\epsilon)$ for a suitable polynomial q. Since Π is polynomially bounded, $\exists$ polynomial p such that, for any instance x, $OPT(x) \leq p(|x|)$. If we choose $\epsilon = \frac{1}{p(|x|)}$, then $A(x, \epsilon)$ provides an optimal solution of x as follows:

$$A \text{ is FPTAS} \Rightarrow \frac{OPT(x)}{Obj_\Pi(x, A(x,\epsilon))} \leq 1 + \frac{1}{p(|x|)}$$

$$\Rightarrow Obj_\Pi(x, A(x,\epsilon)) \geq OPT(x) \cdot \frac{p(|x|)}{p(|x|)+1} = \frac{-OPT(x)}{p(|x|)+1} + OPT(x) > OPT(x) - 1$$

as Obj_Π returns an integer $\Rightarrow Obj_\Pi(x, A(x,\epsilon)) = OPT(x)$

$\Rightarrow A(x,\epsilon)$ is an optimal solution.

$\Rightarrow$ If $P \neq NP$ then FPTAS $\subsetneq$ PTAS

One example of a problem in PTAS with polynomially bounded objective function (integer) is the Maximum Independent Set restricted to planar graphs.

A planar graph can be drawn on the plane in such a way that its edges intersect only at their endpoints.