Matroids: A *matroid* is an ordered pair $M = (S, I)$ satisfying the following conditions:

(1) S is a finite set.

(2) I is a nonempty family of subsets of S, called the *independent subsets* of S, such that if $B \in I$ and $A \subseteq B$, then $A \in I$. We say that I is ~~~~ *hereditary* if it satisfies this property. Note that the empty set ϕ is necessarily a member of I.

(3) If $A \in I$, $B \in I$, and $|A| < |B|$, then there exists some element $x \in B - A$ such that $A \cup \{x\} \in I$. We say that M satisfies the exchange property.

Example: (S, I_k) is a matroid, where S is any finite set and I_k is the set of all subsets of S of size at most k, where $k \le |S|$.

(1) S is a finite set.

(2) I_k is a nonempty family of subsets of S.
Suppose $B \in I_k$ and $A \subseteq B$. $\Rightarrow$ $|B| \le k$ and $|A| \le |B|$
$\Rightarrow |A| \le k$. $\quad B \subseteq S$ and $A \subseteq B \Rightarrow A \subseteq S$.
From $A \subseteq S$ and $|A| \le k$ we get $A \in I_k$

(3) Let $A \in I_k$, $B \in I_k$ and $|A| < |B|$.
$|A| < |B| \Rightarrow \exists x$ such that $x \in B$ and $x \notin A \Leftrightarrow x \in B - A$
$A \cup \{x\} \subseteq S$ because $|A \cup \{x\}| = |A| + 1 \le |B| \le k \le |S|$
and also $|A \cup \{x\}| \le k \Rightarrow A \cup \{x\} \in I_k$

Given a matroid $M = (S, I)$, we call an element $x \notin A$ an *extension* of $A \in I$ if we can add x to A while preserving independence; that is x is an extension of A if $A \cup \{x\} \in I$.

Example:

Let $S = \{a, b, c, d\}$, and

$I_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}\}$

Let $A = \{a\}$ and $x = b$. $A \cup \{x\} = \{a\} \cup \{b\} = \{a,b\} \in I_2$

$\Rightarrow$ b is an extension of $\{a\}$.

Let $A = \{a,b\}$ and $x = c$. $A \cup \{x\} = \{a,b\} \cup \{c\} = \{a,b,c\} \notin I_2$

$\Rightarrow$ c is not an extension of $\{a,b\}$.

If A is an independent subset in a matroid M, we say that A is maximal if it has no extensions. That is, A is maximal if it is not contained in any larger independent subset of M.

Example: In the above example of $M = (S, I_2)$ the maximal ~~elements~~ independent subsets of M are:

$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}$ and $\{c,d\}$.

All maximal independent subsets in a matroid have the same size.

Suppose to the contrary that A is a maximal independent subset of M and there exists another larger maximal independent subset B of M. Then, the exchange property implies that for some $x \in B - A$, we can extend A to a larger independent set $A \cup \{x\}$, contradicting the assumption that A is maximal.

Example: In the above example of (S, I_2), all maximal independent subsets have size 2.

We say that a matroid $M=(S,I)$ is <u>weighted</u> if it is associated with a weight function w that assigns a strictly positive weight $w(x)$ to each element $x \in S$. The weight function w extends to subsets of S by summation: $w(A) = \sum_{x \in A} w(x)$ for any $A \subseteq S$.

<u>Example</u>: In the above example of (S, I_2), let the weight function $w(x)$ be defined as:

$w(a) = 1$, $w(b) = 2$, $w(c) = 3$ and $w(d) = 4$.

Let $A = \{c, d\}$. Then $w(A) = w(c) + w(d) = 3+4 = 7$.

Greedy Algorithms on a Weighted Matroid

Many problems for which a greedy approach provides optimal solutions can be formulated in terms of finding a maximum-weight independent subset in a weighted matroid. That is, we are given a weighted matroid $M=(S,I)$, and we wish to find an independent set $A \in I$ such that $w(A)$ is maximized. We call such a subset that is independent and has maximum possible weight an <u>optimal</u> subset of the matroid. Because the weight $w(x)$ of any element $x \in S$ is positive an optimal subset is always a maximal independent subset — it always helps to make A as large as possible.

<u>Example</u>: In the above example of the weighted matroid $M=(S, I_2)$, the problem that we want to solve is to find the independent subset in I_2 having maximum weight. The optimal solution is $A = \{c, d\}$, with $w(A) = 7$.

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Greedy (M, w)
A ← ϕ
sort M.S into monotonically decreasing order by weight w
for each x ∈ M.S, taken in monotonically decreasing order
      by weight w(x)
      if A ∪ {x} ∈ M.I
            A ← A ∪ {x}
return A
```

Example: If we run the Greedy(M, w) on our example of the weight matroid $M = (S, I_2)$:

Initially: $A = \phi$

S sorted in decreasing order by weight w: $\{d, c, b, a\}$

First iteration of for loop: we take d and $\{d\} \in I_2$

$\Rightarrow A = \{d\}$

Second iteration of for loop: we take c and $\{c, d\} \in I_2$

$\Rightarrow A = \{c, d\}$

Third iteration of for loop: we take b and $\{c, d, b\} \notin I_2$.

$\Rightarrow$ we discard b.

Fourth iteration of for loop: we take a and $\{c, d, a\} \notin I_2$.

$\Rightarrow$ we discard a.

We get the optimal solution $A = \{c, d\}$ with $w(A) = 7$.

Complexity of Greedy (M, w) is $O(n \log n + n f(n))$ where $f(n)$ is the time required for checking the membership in $M.I$: $A \cup \{x\} \in M.I$?

Greedy-Choice Property of Matroids:

Suppose that $M = (S, I)$ is a weighted matroid with weight function w and that S is sorted into monotonically decreasing order by weight. Let x be the first element of S such that $\{x\}$ is independent, if any such x exists. If x exists, then there exists an optimal subset A of S that contains x.

Let B be any nonempty optimal subset with $x \notin B$. No element of B has weight greater than $w(x)$.

Example: in (S, I_2) $\{a,b\}$, $\{a,c\}$, $\{b,c\}$ all have elements with $w(x) < 4$.

This is because $y \in B \Rightarrow \{y\}$ is independent, since $B \in I$ and I is hereditary ($\{y\} \subseteq B \Rightarrow \{y\} \in I$).

$\{y\} \in I$. But our algo has chosen x instead of y

$\Rightarrow w(x) \geq w(y)$ for any $y \in B$.

We construct a new set A starting with $A = \{x\} \in I$ by using the exchange property repeatedly as follows: Find a new element of B that we can add to A until $|A| = |B|$, while preserving the independence of A:

$$|A| < |B| \Rightarrow \exists\, y \in B \text{ such that } A \cup \{y\} \in I$$

At the end we get $|A| = |B|$ with $A = (B - \{y\}) \cup \{x\}$ for some $y \in B$. (A and B are same except that A has x and B has y.

$\Rightarrow w(A) = w(B) - w(y) + w(x) = w(B) + (w(x) - w(y)) \geq w(B)$

because $w(x) \geq w(y) \Rightarrow A$ is optimal containing x and B may not be optimal.

<u>Example</u> : If we take $B = \{a, b\}$ in (S, I_2).

We start with $A = \{d\}$. We can add a in A:

$A = \{d\} \cup \{a\} = \{a, d\}$. Now $|A| = |B| = 2$.

$W(B) = 1+2 = 3$ and $W(A) = 1+4 = 5 \Rightarrow W(A) > W(B)$

$\Rightarrow B$ is not optimal solution.

Let $M = (S, I)$ be any matroid. If x is an element of S that is an extension of some independent subset A of S, then x is also an extension of ϕ.

x is an extension of $A \Rightarrow A \cup \{x\} \in I$. Since I is hereditary, $\{x\} \subseteq A \cup \{x\} \in I \Rightarrow \{x\} \in I \Rightarrow$ x is an extension of ϕ.

Let $M = (S, I)$ be any matroid. If x is an element of S such that x is not an extension of ϕ, then x is not an extension of any independent subset A of S.

x is not an extension of $\phi \Rightarrow \{x\} \notin I \Rightarrow$ If $A \cup \{x\} \in I$ for some $A \in I$, then $\{x\} \in I$ by hereditary property $\Rightarrow$ a contradiction.

<u>Optimal-Substructure Property of Matroids</u>:

Let x be the first element of S chosen by Greedy for the weighted matroid $M = (S, I)$. The remaining problem of finding a maximum-weight independent subset containing x reduces to finding a maximum-weight independent subset of the weighted matroid $M' = (S', I')$, where

$S' = \{ y \in S \mid \{x, y\} \in I \}$,

$I' = \{ B \subseteq S - \{x\} \mid B \cup \{x\} \in I \}$,

and the weight function for M' is the weight function for M, restricted to S'. (M' is the contraction of M by the element x).

Example: $x = d$ is the first element of S chosen by Greedy$((S, I_2), w)$. M' (contraction of M by $x = d$) is:

$S = \{a, b, c, d\}$

$I_2 = \{\phi, \{a\}, \{b\}, \{c\}, \underline{\{d\}}, \{a,b\}, \{a,c\}, \underline{\{a,d\}}, \{b,c\}, \underline{\{b,d\}}, \underline{\{c,d\}}\}$

From I_2, we identify all independent subsets having d:

$\{d\}, \{a,d\}, \{b,d\}, \{c,d\}$

and in S', we choose the elements with d:

$S' = \{a, b, c\}$

I_2' is the set of subsets of S' ($B \subseteq S'$) such that $B \cup \{d\} \in I_2$.

$2^{I_2} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

We cannot have $|B| \geq 2$ because then $|B \cup \{d\}| > 2 \notin I_2$

$\Rightarrow |B| \leq 1$.

we get $I_2' = \{\phi, \{a\}, \{b\}, \{c\}\}$ because all $\{d\}, \{a,d\}, \{b,d\}$, and $\{c,d\} \in I_2$

The weight function is the same: $w(a) = 1, w(b) = 2, w(c) = 3$.

If A is any maximum weight independent subset of M containing x, then $A' = A - \{x\}$ is an independent subset of M'. Conversely, any independent subset A' of M' yields an independent subset $A = A' \cup \{x\}$ of M. Since we have in both cases $w(A) = w(A') + w(x)$, a maximum-weight solution in M containing x yields a maximum-weight solution in M' and vice versa.