Tutorial 13, Design and Analysis of Algorithms, 2019

1. We have a function $f : \{0, ..., n-1\} \to \{0, ..., m-1\}$. We know that, for $0 \leq x, y \leq n-1$, $f((x+y) \mod n) = (f(x) + f(y)) \mod m$. The only way we have for evaluating f is to use a lookup table that stores the values of f. Unfortunately, an Evil Adversary has changed the value of $\frac{1}{5}$ of the table entries when we were not looking. Describe a simple randomized algorithm that, given an input z, outputs a value that equals $f(z)$ with probability at least $\frac{1}{2}$. Your algorithm should work for every value of z, regardless of what values the Adversary changed. Your algorithm should use as few lookups and as little computation as possible.

2. In an examination of 100 objective questions, each question has four options. Only one option is correct. A student will get 1 mark for correct answer, and 0 mark for incorrect answer (there is no negative marking). Totally unprepared, a student randomly ticks the options for all the 100 questions. Find and compare the upper bounds of the probability of the student getting at least 50 marks using the *Markov's Inequality*, *Chebyshev's Inequality*, and *Chernoff Bounds.*

3. Consider a series of 1000 coin tosses with a fair coin. Suppose one wants to compute the probability of the following events.

   - The number of heads > 750.
   - The number of heads < 400.

   (a) Compute the answers obtained by applying the Markov inequality for both these cases. In case, it is not possible to apply the Markov inequality for either of the two cases, clearly explain the reason.

   (b) If one were to compute a bound using Chebyshev inequality for the above problem, explain your approach. In case it is not possible to apply Chebyshev inequality, clearly explain why.

   (c) Compute Chernoff type bound for both the cases above, clearly specify intermediate steps. Compare with the results obtained using Markov inequality.

4. Librarian Josse has been assigned to set up a new library. He has been given book racks with n shelves and $m = kn \log_e(n)$ books, where $k > 8$. Mr. Josse decided to assign a book one of the racks randomly. Show that the probability that the number of books assigned to any shelf is more than $2k \log_e(n)$ is bounded above by $\frac{1}{n^2}$.

5. A parallel computer consists of n processors and n memory modules. During a step, each processor sends a memory request to one of the memory modules. A memory module can satisfy one request in a step; modules that receive more than one requests will satisfy one request and discard the rest. Assuming that each processor chooses a memory module independently and uniformly at random, what is the expected number of processors whose requests are satisfied?

6. Consider the following Monte Carlo algorithm to find the median among a set of n numbers. "Pick an index uniformly at random and report the element at that position as the median."

   (a) What is the probability of error i.e. probability of obtaining an incorrect answer.

   (b) One scheme often used to improve the probability of obtaining a correct output from a Monte Carlo algorithm is to repeatedly run it. Explain whether or not it is possible to obtain a Las Vegas like algorithm by repeatedly running the above Monte Carlo algorithm. State any assumptions clearly.

7. Consider the following selection problem involving n numbers where n is large. Suppose it is necessary to identify two numbers such that one of them is greater than the median and the other is less than the median. Propose a randomized algorithm, which involves selecting k numbers randomly (where k is a constant). What should the minimum value of k be such that the probability of both numbers being correct i.e. the maximum greater than the median and other being less than the median is at least 0.9995 (i.e. 99.95% probability of correctness)?