

**16.3-9**

Show that no compression scheme can expect to compress a file of randomly chosen 8-bit characters by even a single bit. (*Hint:* Compare the number of possible files with the number of possible encoded files.)

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**★ 16.4 Matroids and greedy methods**

In this section, we sketch a beautiful theory about greedy algorithms. This theory describes many situations in which the greedy method yields optimal solutions. It involves combinatorial structures known as “matroids.” Although this theory does not cover all cases for which a greedy method applies (for example, it does not cover the activity-selection problem of Section 16.1 or the Huffman-coding problem of Section 16.3), it does cover many cases of practical interest. Furthermore, this theory has been extended to cover many applications; see the notes at the end of this chapter for references.

**Matroids**

A **matroid** is an ordered pair  $M = (S, \mathcal{I})$  satisfying the following conditions.

1.  $S$  is a finite set.
2.  $\mathcal{I}$  is a nonempty family of subsets of  $S$ , called the **independent** subsets of  $S$ , such that if  $B \in \mathcal{I}$  and  $A \subseteq B$ , then  $A \in \mathcal{I}$ . We say that  $\mathcal{I}$  is **hereditary** if it satisfies this property. Note that the empty set  $\emptyset$  is necessarily a member of  $\mathcal{I}$ .
3. If  $A \in \mathcal{I}$ ,  $B \in \mathcal{I}$ , and  $|A| < |B|$ , then there exists some element  $x \in B - A$  such that  $A \cup \{x\} \in \mathcal{I}$ . We say that  $M$  satisfies the **exchange property**.

The word “matroid” is due to Hassler Whitney. He was studying **matric matroids**, in which the elements of  $S$  are the rows of a given matrix and a set of rows is independent if they are linearly independent in the usual sense. As Exercise 16.4-2 asks you to show, this structure defines a matroid.

As another example of matroids, consider the **graphic matroid**  $M_G = (S_G, \mathcal{I}_G)$  defined in terms of a given undirected graph  $G = (V, E)$  as follows:

- The set  $S_G$  is defined to be  $E$ , the set of edges of  $G$ .
- If  $A$  is a subset of  $E$ , then  $A \in \mathcal{I}_G$  if and only if  $A$  is acyclic. That is, a set of edges  $A$  is independent if and only if the subgraph  $G_A = (V, A)$  forms a forest.

The graphic matroid  $M_G$  is closely related to the minimum-spanning-tree problem, which Chapter 23 covers in detail.

**Theorem 16.5**

If  $G = (V, E)$  is an undirected graph, then  $M_G = (S_G, \mathcal{I}_G)$  is a matroid.

**Proof** Clearly,  $S_G = E$  is a finite set. Furthermore,  $\mathcal{I}_G$  is hereditary, since a subset of a forest is a forest. Putting it another way, removing edges from an acyclic set of edges cannot create cycles.

Thus, it remains to show that  $M_G$  satisfies the exchange property. Suppose that  $G_A = (V, A)$  and  $G_B = (V, B)$  are forests of  $G$  and that  $|B| > |A|$ . That is,  $A$  and  $B$  are acyclic sets of edges, and  $B$  contains more edges than  $A$  does.

We claim that a forest  $F = (V_F, E_F)$  contains exactly  $|V_F| - |E_F|$  trees. To see why, suppose that  $F$  consists of  $t$  trees, where the  $i$ th tree contains  $v_i$  vertices and  $e_i$  edges. Then, we have

$$\begin{aligned} |E_F| &= \sum_{i=1}^t e_i \\ &= \sum_{i=1}^t (v_i - 1) \quad (\text{by Theorem B.2}) \\ &= \sum_{i=1}^t v_i - t \\ &= |V_F| - t, \end{aligned}$$

which implies that  $t = |V_F| - |E_F|$ . Thus, forest  $G_A$  contains  $|V| - |A|$  trees, and forest  $G_B$  contains  $|V| - |B|$  trees.

Since forest  $G_B$  has fewer trees than forest  $G_A$  does, forest  $G_B$  must contain some tree  $T$  whose vertices are in two different trees in forest  $G_A$ . Moreover, since  $T$  is connected, it must contain an edge  $(u, v)$  such that vertices  $u$  and  $v$  are in different trees in forest  $G_A$ . Since the edge  $(u, v)$  connects vertices in two different trees in forest  $G_A$ , we can add the edge  $(u, v)$  to forest  $G_A$  without creating a cycle. Therefore,  $M_G$  satisfies the exchange property, completing the proof that  $M_G$  is a matroid. ■

Given a matroid  $M = (S, \mathcal{I})$ , we call an element  $x \notin A$  an **extension** of  $A \in \mathcal{I}$  if we can add  $x$  to  $A$  while preserving independence; that is,  $x$  is an extension of  $A$  if  $A \cup \{x\} \in \mathcal{I}$ . As an example, consider a graphic matroid  $M_G$ . If  $A$  is an independent set of edges, then edge  $e$  is an extension of  $A$  if and only if  $e$  is not in  $A$  and the addition of  $e$  to  $A$  does not create a cycle.

If  $A$  is an independent subset in a matroid  $M$ , we say that  $A$  is **maximal** if it has no extensions. That is,  $A$  is maximal if it is not contained in any larger independent subset of  $M$ . The following property is often useful.

**Theorem 16.6**

All maximal independent subsets in a matroid have the same size.

**Proof** Suppose to the contrary that  $A$  is a maximal independent subset of  $M$  and there exists another larger maximal independent subset  $B$  of  $M$ . Then, the exchange property implies that for some  $x \in B - A$ , we can extend  $A$  to a larger independent set  $A \cup \{x\}$ , contradicting the assumption that  $A$  is maximal. ■

As an illustration of this theorem, consider a graphic matroid  $M_G$  for a connected, undirected graph  $G$ . Every maximal independent subset of  $M_G$  must be a free tree with exactly  $|V| - 1$  edges that connects all the vertices of  $G$ . Such a tree is called a **spanning tree** of  $G$ .

We say that a matroid  $M = (S, \mathcal{I})$  is **weighted** if it is associated with a weight function  $w$  that assigns a strictly positive weight  $w(x)$  to each element  $x \in S$ . The weight function  $w$  extends to subsets of  $S$  by summation:

$$w(A) = \sum_{x \in A} w(x)$$

for any  $A \subseteq S$ . For example, if we let  $w(e)$  denote the weight of an edge  $e$  in a graphic matroid  $M_G$ , then  $w(A)$  is the total weight of the edges in edge set  $A$ .

**Greedy algorithms on a weighted matroid**

Many problems for which a greedy approach provides optimal solutions can be formulated in terms of finding a maximum-weight independent subset in a weighted matroid. That is, we are given a weighted matroid  $M = (S, \mathcal{I})$ , and we wish to find an independent set  $A \in \mathcal{I}$  such that  $w(A)$  is maximized. We call such a subset that is independent and has maximum possible weight an **optimal** subset of the matroid. Because the weight  $w(x)$  of any element  $x \in S$  is positive, an optimal subset is always a maximal independent subset—it always helps to make  $A$  as large as possible.

For example, in the **minimum-spanning-tree problem**, we are given a connected undirected graph  $G = (V, E)$  and a length function  $w$  such that  $w(e)$  is the (positive) length of edge  $e$ . (We use the term “length” here to refer to the original edge weights for the graph, reserving the term “weight” to refer to the weights in the associated matroid.) We wish to find a subset of the edges that connects all of the vertices together and has minimum total length. To view this as a problem of finding an optimal subset of a matroid, consider the weighted matroid  $M_G$  with weight function  $w'$ , where  $w'(e) = w_0 - w(e)$  and  $w_0$  is larger than the maximum length of any edge. In this weighted matroid, all weights are positive and an optimal subset is a spanning tree of minimum total length in the original graph. More specifically, each maximal independent subset  $A$  corresponds to a spanning tree

with  $|V| - 1$  edges, and since

$$\begin{aligned}
 w'(A) &= \sum_{e \in A} w'(e) \\
 &= \sum_{e \in A} (w_0 - w(e)) \\
 &= (|V| - 1)w_0 - \sum_{e \in A} w(e) \\
 &= (|V| - 1)w_0 - w(A)
 \end{aligned}$$

for any maximal independent subset  $A$ , an independent subset that maximizes the quantity  $w'(A)$  must minimize  $w(A)$ . Thus, any algorithm that can find an optimal subset  $A$  in an arbitrary matroid can solve the minimum-spanning-tree problem.

Chapter 23 gives algorithms for the minimum-spanning-tree problem, but here we give a greedy algorithm that works for any weighted matroid. The algorithm takes as input a weighted matroid  $M = (S, \mathcal{I})$  with an associated positive weight function  $w$ , and it returns an optimal subset  $A$ . In our pseudocode, we denote the components of  $M$  by  $M.S$  and  $M.\mathcal{I}$  and the weight function by  $w$ . The algorithm is greedy because it considers in turn each element  $x \in S$ , in order of monotonically decreasing weight, and immediately adds it to the set  $A$  being accumulated if  $A \cup \{x\}$  is independent.

GREEDY( $M, w$ )

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1   $A = \emptyset$ 
2  sort  $M.S$  into monotonically decreasing order by weight  $w$ 
3  for each  $x \in M.S$ , taken in monotonically decreasing order by weight  $w(x)$ 
4      if  $A \cup \{x\} \in M.\mathcal{I}$ 
5           $A = A \cup \{x\}$ 
6  return  $A$ 

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Line 4 checks whether adding each element  $x$  to  $A$  would maintain  $A$  as an independent set. If  $A$  would remain independent, then line 5 adds  $x$  to  $A$ . Otherwise,  $x$  is discarded. Since the empty set is independent, and since each iteration of the **for** loop maintains  $A$ 's independence, the subset  $A$  is always independent, by induction. Therefore, GREEDY always returns an independent subset  $A$ . We shall see in a moment that  $A$  is a subset of maximum possible weight, so that  $A$  is an optimal subset.

The running time of GREEDY is easy to analyze. Let  $n$  denote  $|S|$ . The sorting phase of GREEDY takes time  $O(n \lg n)$ . Line 4 executes exactly  $n$  times, once for each element of  $S$ . Each execution of line 4 requires a check on whether or not the set  $A \cup \{x\}$  is independent. If each such check takes time  $O(f(n))$ , the entire algorithm runs in time  $O(n \lg n + nf(n))$ .

We now prove that GREEDY returns an optimal subset.

**Lemma 16.7 (Matroids exhibit the greedy-choice property)**

Suppose that  $M = (S, \mathcal{I})$  is a weighted matroid with weight function  $w$  and that  $S$  is sorted into monotonically decreasing order by weight. Let  $x$  be the first element of  $S$  such that  $\{x\}$  is independent, if any such  $x$  exists. If  $x$  exists, then there exists an optimal subset  $A$  of  $S$  that contains  $x$ .

**Proof** If no such  $x$  exists, then the only independent subset is the empty set and the lemma is vacuously true. Otherwise, let  $B$  be any nonempty optimal subset. Assume that  $x \notin B$ ; otherwise, letting  $A = B$  gives an optimal subset of  $S$  that contains  $x$ .

No element of  $B$  has weight greater than  $w(x)$ . To see why, observe that  $y \in B$  implies that  $\{y\}$  is independent, since  $B \in \mathcal{I}$  and  $\mathcal{I}$  is hereditary. Our choice of  $x$  therefore ensures that  $w(x) \geq w(y)$  for any  $y \in B$ .

Construct the set  $A$  as follows. Begin with  $A = \{x\}$ . By the choice of  $x$ , set  $A$  is independent. Using the exchange property, repeatedly find a new element of  $B$  that we can add to  $A$  until  $|A| = |B|$ , while preserving the independence of  $A$ . At that point,  $A$  and  $B$  are the same except that  $A$  has  $x$  and  $B$  has some other element  $y$ . That is,  $A = B - \{y\} \cup \{x\}$  for some  $y \in B$ , and so

$$\begin{aligned} w(A) &= w(B) - w(y) + w(x) \\ &\geq w(B). \end{aligned}$$

Because set  $B$  is optimal, set  $A$ , which contains  $x$ , must also be optimal. ■

We next show that if an element is not an option initially, then it cannot be an option later.

**Lemma 16.8**

Let  $M = (S, \mathcal{I})$  be any matroid. If  $x$  is an element of  $S$  that is an extension of some independent subset  $A$  of  $S$ , then  $x$  is also an extension of  $\emptyset$ .

**Proof** Since  $x$  is an extension of  $A$ , we have that  $A \cup \{x\}$  is independent. Since  $\mathcal{I}$  is hereditary,  $\{x\}$  must be independent. Thus,  $x$  is an extension of  $\emptyset$ . ■

**Corollary 16.9**

Let  $M = (S, \mathcal{I})$  be any matroid. If  $x$  is an element of  $S$  such that  $x$  is not an extension of  $\emptyset$ , then  $x$  is not an extension of any independent subset  $A$  of  $S$ .

**Proof** This corollary is simply the contrapositive of Lemma 16.8. ■

Corollary 16.9 says that any element that cannot be used immediately can never be used. Therefore, GREEDY cannot make an error by passing over any initial elements in  $S$  that are not an extension of  $\emptyset$ , since they can never be used.

**Lemma 16.10 (Matroids exhibit the optimal-substructure property)**

Let  $x$  be the first element of  $S$  chosen by GREEDY for the weighted matroid  $M = (S, \mathcal{I})$ . The remaining problem of finding a maximum-weight independent subset containing  $x$  reduces to finding a maximum-weight independent subset of the weighted matroid  $M' = (S', \mathcal{I}')$ , where

$$\begin{aligned} S' &= \{y \in S : \{x, y\} \in \mathcal{I}\}, \\ \mathcal{I}' &= \{B \subseteq S - \{x\} : B \cup \{x\} \in \mathcal{I}\}, \end{aligned}$$

and the weight function for  $M'$  is the weight function for  $M$ , restricted to  $S'$ . (We call  $M'$  the **contraction** of  $M$  by the element  $x$ .)

**Proof** If  $A$  is any maximum-weight independent subset of  $M$  containing  $x$ , then  $A' = A - \{x\}$  is an independent subset of  $M'$ . Conversely, any independent subset  $A'$  of  $M'$  yields an independent subset  $A = A' \cup \{x\}$  of  $M$ . Since we have in both cases that  $w(A) = w(A') + w(x)$ , a maximum-weight solution in  $M$  containing  $x$  yields a maximum-weight solution in  $M'$ , and vice versa. ■

**Theorem 16.11 (Correctness of the greedy algorithm on matroids)**

If  $M = (S, \mathcal{I})$  is a weighted matroid with weight function  $w$ , then GREEDY( $M, w$ ) returns an optimal subset.

**Proof** By Corollary 16.9, any elements that GREEDY passes over initially because they are not extensions of  $\emptyset$  can be forgotten about, since they can never be useful. Once GREEDY selects the first element  $x$ , Lemma 16.7 implies that the algorithm does not err by adding  $x$  to  $A$ , since there exists an optimal subset containing  $x$ . Finally, Lemma 16.10 implies that the remaining problem is one of finding an optimal subset in the matroid  $M'$  that is the contraction of  $M$  by  $x$ . After the procedure GREEDY sets  $A$  to  $\{x\}$ , we can interpret all of its remaining steps as acting in the matroid  $M' = (S', \mathcal{I}')$ , because  $B$  is independent in  $M'$  if and only if  $B \cup \{x\}$  is independent in  $M$ , for all sets  $B \in \mathcal{I}'$ . Thus, the subsequent operation of GREEDY will find a maximum-weight independent subset for  $M'$ , and the overall operation of GREEDY will find a maximum-weight independent subset for  $M$ . ■

**Exercises****16.4-1**

Show that  $(S, \mathcal{I}_k)$  is a matroid, where  $S$  is any finite set and  $\mathcal{I}_k$  is the set of all subsets of  $S$  of size at most  $k$ , where  $k \leq |S|$ .

**16.4-2 ★**

Given an  $m \times n$  matrix  $T$  over some field (such as the reals), show that  $(S, \mathcal{I})$  is a matroid, where  $S$  is the set of columns of  $T$  and  $A \in \mathcal{I}$  if and only if the columns in  $A$  are linearly independent.

**16.4-3 ★**

Show that if  $(S, \mathcal{I})$  is a matroid, then  $(S, \mathcal{I}')$  is a matroid, where

$$\mathcal{I}' = \{A' : S - A' \text{ contains some maximal } A \in \mathcal{I}\}.$$

That is, the maximal independent sets of  $(S, \mathcal{I}')$  are just the complements of the maximal independent sets of  $(S, \mathcal{I})$ .

**16.4-4 ★**

Let  $S$  be a finite set and let  $S_1, S_2, \dots, S_k$  be a partition of  $S$  into nonempty disjoint subsets. Define the structure  $(S, \mathcal{I})$  by the condition that  $\mathcal{I} = \{A : |A \cap S_i| \leq 1 \text{ for } i = 1, 2, \dots, k\}$ . Show that  $(S, \mathcal{I})$  is a matroid. That is, the set of all sets  $A$  that contain at most one member of each subset in the partition determines the independent sets of a matroid.

**16.4-5**

Show how to transform the weight function of a weighted matroid problem, where the desired optimal solution is a *minimum-weight* maximal independent subset, to make it a standard weighted-matroid problem. Argue carefully that your transformation is correct.

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**★ 16.5 A task-scheduling problem as a matroid**

An interesting problem that we can solve using matroids is the problem of optimally scheduling unit-time tasks on a single processor, where each task has a deadline, along with a penalty paid if the task misses its deadline. The problem looks complicated, but we can solve it in a surprisingly simple manner by casting it as a matroid and using a greedy algorithm.

A *unit-time task* is a job, such as a program to be run on a computer, that requires exactly one unit of time to complete. Given a finite set  $S$  of unit-time tasks, a