CS F364
Design & Analysis of Algorithms

ALGORITHMS - COMPLEXITY

Complexity Classes

- $\mathcal{NP}$-Completeness Via Reductions
- Reduction Techniques:
  - Component Design
    - Example: Vertex Cover

PROBLEM: VERTEX-COVER

- Definition: ***Vertex Cover of a graph***
  - Give an undirected graph G= (V,E), a subset S of V, is said to be a vertex cover of G
    - if for each edge $(u, v) \in E$, either $u \in S$ or $v \in S$
- VERTEX-COVER:
  - Given an undirected graph G and a positive integer k, find whether there is a vertex cover of size at most k.
- VERTEX-COVER is $\mathbb{NP}$-complete:
  - Proof:
    1. VERTEX-COVER is in $\mathbb{NP}$
    2. VERTEX-COVER is hard $\mathbb{NP}$-hard

VERTEX-COVER IS IN $\mathbb{NP}$

- VERTEX-COVER is in $\mathbb{NP}$
  - Proof:
    - Input G=(V,E) and k
    - Certificate: a set $S \subseteq V$, such that $|S| <= k$
    - Verification Algorithm $O(|E|*\log|V|)$ time:
      - For each edge (u,v) in E, verify u or v is in S.

VERTEX-COVER IS $\mathbb{NP}$-HARD

- Proof: 3SAT ≾ VERTEX-COVER
- Reduction:
  - Given a Boolean expression B in 3CNF (i.e. with exactly 3 literals in each clause), construct a graph G and an integer k as follows:
    - For each of the variables x_i occurring in B, add to G:
      - two vertices – labeled x_i and $\sim x_i$ – and an edge $(x_i, \sim x_i)$
    - For each clause C_i = (l1 V l2 V l3) in B , add a triangle to G:
      - three new vertices, say v_{i1}, v_{i2}, and v_{i3} and edges connecting them to each other.
    - For each clause C_i = (l1 V l2 V l3) connect the variable and the triangle in G:
      - i.e. add edges to G: (l1, v_{i1}), (l2, v_{i2}), and (l3, v_{i3})
    - Let k = n+2m where n is # variables in B, and m is # clauses in B.

VERTEX-COVER IS $\mathbb{NP}$-HARD [2]

- 3SAT ≾ VERTEX-COVER
- Reduction: [contd…]
  - Given the graph G and integer k constructed from B as above we claim:
    1. If there is a vertex cover, of size at most k, for G then it must be of size exactly k
    2. Construction of G and k takes polynomial time
    3. If there is a satisfying assignment for B, then there is a vertex cover of size k for G.
    4. If there is a vertex cover of size at most k for G, then there is a satisfying assignment for B.
  - Summary claim:
    - G has a vertex cover of size at most k iff B is satisfiable
    - i.e. *there is a poly-time reduction from 3SAT to VERTEX-COVER*

REDUCTION TECHNIQUES – COMPONENT DESIGN

- The reduction used for hardness of VERTEX-COVER is an instance of a reduction technique called “Component Design”:
  - Central Idea:
    - Use constituents of the target problem to
      - design “components” that can be combined to “realize” instances of the known hard problem.

Reduction Techniques – Component Design [2]

- Component Design – Example
  - Target problem:
    - VERTEX-COVER
  - Known hard problem:
    - 3SAT
  - Components:
    - Selection of vertices, testing each edge is covered
  - Realization:
    - each variable or its negative literal would be satisfied,
    - at least one literal in each clause would be satisfied.
- Exercise:
  - Identify the application of this technique in the hardness proof for CIRCUIT-SAT.