Max-Flow Min-Cut Theorem

An s-t cut is a partition (A, B) of the vertex set V, so that $s \in A$ and $t \in B$. The capacity of a cut (A, B) is simply the sum of the capacities of all edges out of A: $c(A, B) = \sum_{e \text{ out of } A} c_e$.

Let f be any s-t flow, and (A, B) any s-t cut. Then $v(f) = f^{out}(A) - f^{in}(A)$.

$v(f) = f^{out}(s)$ and $f^{in}(s) = 0 \Rightarrow v(f) = f^{out}(s) - f^{in}(s) \rightarrow (1)$

For $v \in A - s$, applying capacity condition:

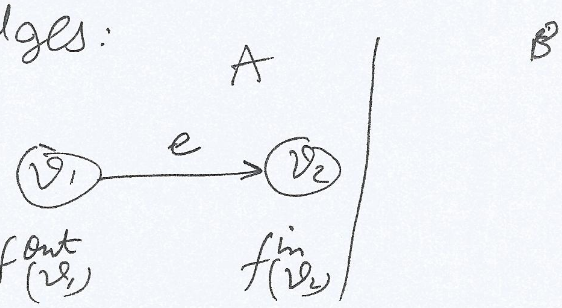
$f^{out}(v) - f^{in}(v) = 0 \rightarrow (2)$

From (1) and (2) we get

$$v(f) = \sum_{v \in A} \left(f^{out}(v) - f^{in}(v) \right) \rightarrow (3)$$

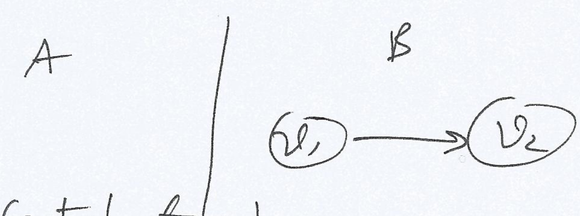
There are 4 types of edges:

1) Edges within A:



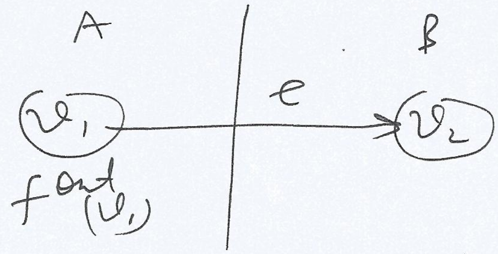
contribution in (3) $= +f(e) \quad -f(e) = 0$

2) Edges within B:



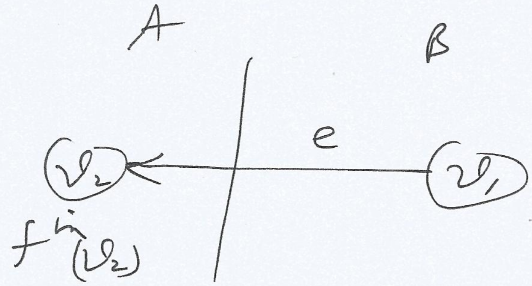
contribution in (3) $= 0$ (no contribution)

3) Edges from A to B:



Contribution in (3) $=$ $f(e)$

4) Edges from B to A:



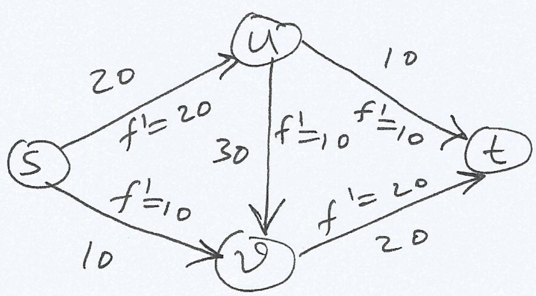
Contribution in (3) $=$ $-f(e)$

From (3) and using the above observations, we get:

$$v(f) = \sum_{v \in A} \left(f^{out}(v) - f^{in}(v) \right) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= f^{out}(A) - f^{in}(A)$$

Example:



There are 4 s-t cuts:

(1) $A=\{s\}$, $B=\{u, v, t\}$, $f^{out}(A) = 20+10=30$, $f^{in}(A)=0$

(2) $A=\{s, u\}$, $B=\{v, t\}$, $f^{out}(A) = 10+10+10=30$, $f^{in}(A)=0$

(3) $A=\{s, v\}$, $B=\{u, t\}$, $f^{out}(A) = 20+20=40$, $f^{in}(A)=10$

(4) $A=\{s, u, v\}$, $B=\{t\}$, $f^{out}(A) = 10+20=30$, $f^{in}(A)=0$

In all the above cases, we have: $f^{out}(A) - f^{in}(A) = 30 = v(f)$

Let f be any s-t flow, and (A, B) any s-t cut.
Then $v(f) \le c(A, B)$.

$$v(f) = f^{out}(A) - f^{in}(A) \le f^{out}(A) = \sum_{e \text{ out of } A} f(e)$$

$$\le \sum_{e \text{ out of } A} c_e = c(A, B)$$

Example:

① $A = \{s\}$, $B = \{u, v, t\}$, $c(A, B) = 20 + 10 = 30$

② $A = \{s, u\}$, $B = \{v, t\}$, $c(A, B) = 10 + 30 + 10 = 50$

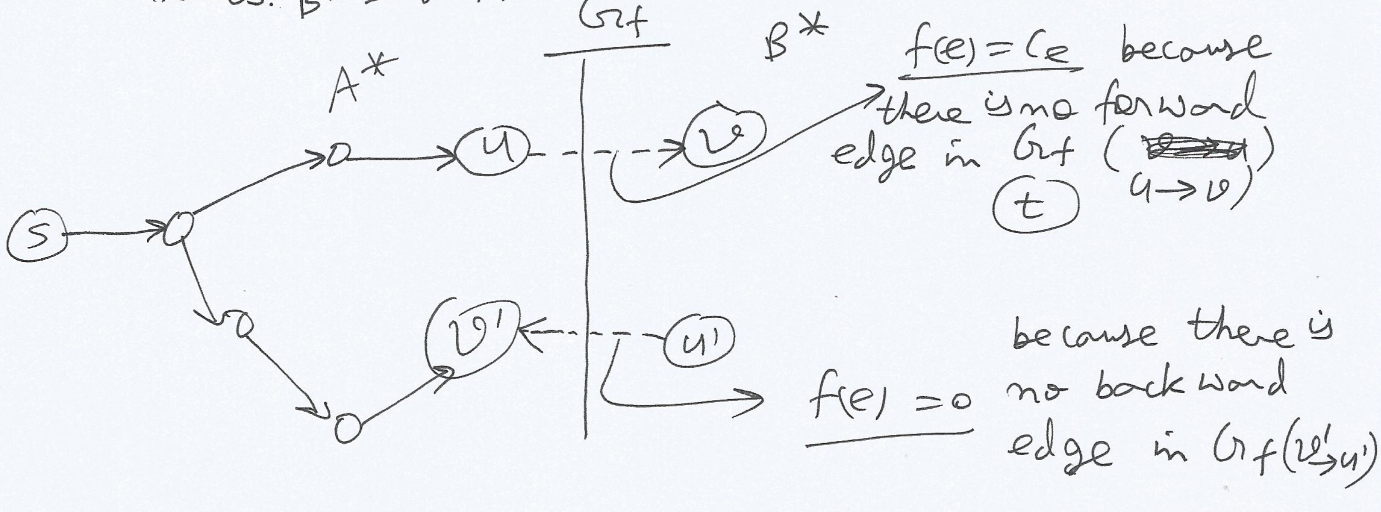
③ $A = \{s, v\}$, $B = \{u, t\}$, $c(A, B) = 20 + 20 = 40$

④ $A = \{s, u, v\}$, $B = \{t\}$, $c(A, B) = 10 + 20 = 30$

In all the above cases, we have $v(f) = 30 \le c(A, B)$

If f is an s-t flow such that there is no s-t path in the residual graph G_f, then there is an s-t cut (A^*, B^*) in G for which $v(f) = c(A^*, B^*)$. Consequently, f has the maximum value of any flow in G, and (A^*, B^*) has the minimum capacity of any s-t cut in G.

Let A^* denote the set of all nodes v in G for which there is an s-v path in G_f. Let B^* denote the set of all other nodes: $B^* = V - A^*$



$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$

$$= \sum_{e \text{ out of } A^*} f(e) - \sum_{e \text{ into } A^*} f(e)$$

$$= \sum_{e \text{ out of } A^*} c_e - 0 = c(A^*, B^*)$$

$\Rightarrow$ f is the max flow and (A^*, B^*) is the min cut.

If $v(f') > v(f)$ then we will have

$v(f') > v(f) = c(A^*, B^*)$ which is a contradiction.

(A^*, B^*) is a cut in the Ford-Fulkerson algorithm, because in G_f there is no s-t path $\Rightarrow$ $s \in A^*$ and $t \in B^*$. (A^*, B^*) is a partition of V.

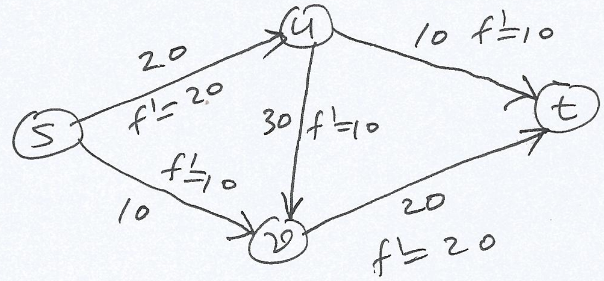
$\Rightarrow$ The flow f returned by the Ford-Fulkerson Algorithm is a maximum flow.

Given a flow f of maximum value, we can compute an s-t cut of minimum capacity in $O(|E|)$ time.

<u>Max-Flow Min-Cut Theorem</u>: In every flow network, the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.

If all capacities in the flow network are integers, then there is a maximum flow f for which every flow value $f(e)$ is an integer.

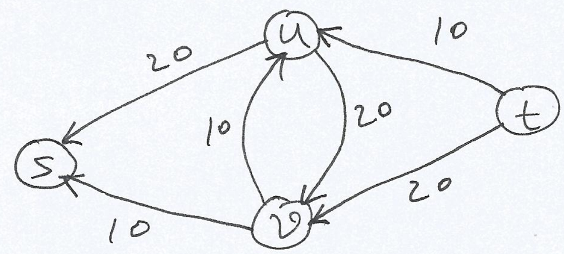
Example1 : After running the Ford-Fulkerson Algorithm, we get the following flow:



f'

$v(f') = 20+10 = 30$

and the residual graph $G_{f'}$



$A^* = \{s\}, \quad B^* = \{u, v, t\}$

(A^*, B^*) is the min cut with $c(A^*, B^*) = 20+10 = 30 = v(f')$

Example2 : Find the max-flow and min-cut using the Ford-Fulkerson Algorithm for the following flow graph: