

Online Problems and Online Algorithms

- **Online Paging Problem:**

- Performance of Online Algorithms
- Competitive Performance

Paging Algorithm - Miss Rates

❖ Online Paging Lemma:

- ♦ For any deterministic online algorithm A there exist sequences of arbitrary length such that A misses on every request

- ♦ i.e. $f_A([\rho_0, \rho_1, \dots, \rho_n]) = n$

- ♦ Proof:

- ♦ Consider an adversary who chooses the next input ρ_j to be a page that is not one of the k pages in M_0
 - ♦ Since $|M_1| = k+1$, there always exists one such page.

❖ Implication:

- ♦ Worst case analysis is not useful in comparing these algorithms

Paging Algorithm - Competitive Analysis

❖ Definition:

- ♦ A deterministic online page replacement algorithm A is said to be C -competitive if there exists a constant b such that on every sequence of requests $\rho = \rho_0, \rho_1, \dots, \rho_n$,

- ♦ $f_A(\rho) - C * f_{OPT}(\rho) \leq b$

where the constant b must be independent of n but may depend on k .

- ♦ The competitiveness coefficient of A , denoted C_A , is the smallest C such that A is C -competitive.

❖ Online Paging Competitiveness Theorem:

- ♦ For any deterministic online algorithm A for paging, $C_A \geq k$
- ♦ Proof:
 - ♦ By GreedyPaging Lemma and Online Paging Lemma

Paging Algorithm - Competitive Analysis

❖ Claim:

- ◆ $C_{LRU} = k$

- ◆ Proof:

- ◆ Partition the input sequence into rounds $R_0, R_1 \dots R_t$
 - ◆ such that each round R_j results in exactly k misses by LRU.
- ◆ In each round R_j , all the $k+1$ pages must have been accessed.
 - ◆ Why?
- ◆ So, the ratio of misses by LRU to optimal misses is at most k i.e. for any input sequence ρ
 - ◆ $f_{LRU}(\rho) / f_{OPT}(\rho) \leq k$
 - ◆ i.e. $C_{LRU} \leq k$
- ◆ But by OPC Theorem: $C_{LRU} \geq k$.

Paging Algorithm - Competitive Analysis

❖ **Claim:** $C_{\text{FIFO}} = k$

◆ **Proof:** (similar to the proof for LRU: left as exercise)

❖ **Claim:** $C_{\text{LFU}} > k$

◆ **Proof:**

◆ **Consider a sequence ρ where**

◆ $\rho_0, \rho_1, \dots, \rho_j$ are $k-1$ distinct pages with 2 accesses each and

◆ $\rho_{j+2*i-1}, \rho_{j+2*i}$ are a pair of different pages repeated for each $i = 1, 2, \dots$

◆ and are different from $\rho_0, \rho_1, \dots, \rho_j$

◆ $f_{\text{LFU}}(\rho) = n - 2*k + 1$ and $f_{\text{OPT}}(\rho) = 1$

◆ Therefore the ratio $f_{\text{LFU}}(\rho) / f_{\text{OPT}}(\rho)$ is $O(n)$

◆ i.e. not bounded