DAA Tutorial 1 Solution

1(a): $F_n = F_{n-1} + F_{n-2} \geq F_{n-2} + F_{n-2} = 2F_{n-2}$

$\geq 2^2 F_{n-2\cdot 2} \geq \cdots 2^{\lfloor \frac{n}{2} \rfloor} F_{n-2\cdot\lfloor \frac{n}{2} \rfloor}$

$n - 2\lfloor \frac{n}{2} \rfloor = 0$ if n is even, $F_0 = 1$

$n - 2\lfloor \frac{n}{2} \rfloor = 1$ if n is odd, $F_1 = 1$

$\Rightarrow F_n \geq 2^{\lfloor \frac{n}{2} \rfloor}$

if n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2} \Rightarrow F_n \geq 2^{\frac{n}{2}} \longrightarrow (1)$

if n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2} - \frac{1}{2}$

$\Rightarrow F_n \geq 2^{\frac{n}{2} - \frac{1}{2}} = \frac{2^{\frac{n}{2}}}{\sqrt{2}} \longrightarrow (2)$

From (1) and (2), we get:

$F_n \geq \frac{2^{n/2}}{\sqrt{2}} \Rightarrow F_n = \Omega\left(2^{\frac{n}{2}}\right)$

(for all $n \geq 0$)

[5]

1(b): $F_n = F_{n-1} + F_{n-2} \leq F_{n-1} + F_{n-1} = 2F_{n-1}$

$\leq 2^2 F_{n-2} \cdots \leq 2^n F_0 = 2^n$

$\Rightarrow F_n = O(2^n)$ [5]

1(c): To prove that $F_n = \Theta\left(\left(\frac{\sqrt{5}+1}{2}\right)^n\right)$,
we will prove that $F_n < 2\left(\frac{\sqrt{5}+1}{2}\right)^n$ and
$F_n > \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^n$ for all $n \geq 0$. We will prove by induction.

(1) To prove: $F_n < 2\left(\frac{\sqrt{5}+1}{2}\right)^n$

Basis: $n=0$: $F_0 = 1 < 2\left(\frac{\sqrt{5}+1}{2}\right)^0 = 2$ is true.

Induction Hypothesis: $F_i < 2\left(\frac{\sqrt{5}+1}{2}\right)^i$ for all $i < n$.

Induction step: $F_n = F_{n-1} + F_{n-2}$

$< 2\left(\frac{\sqrt{5}+1}{2}\right)^{n-1} + 2\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}$

$= 2\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}\left[\frac{\sqrt{5}+1}{2} + 1\right] = 2\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}\left(\frac{\sqrt{5}+1}{2}\right)^2$

$= 2\left(\frac{\sqrt{5}+1}{2}\right)^n$ [5]

(2) <u>To prove:</u> $F_n > \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^n$

<u>Basis</u>: <u>$n=0$</u>: $F_0 = 1 > \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^0 = \frac{1}{2}$ is true.

<u>Induction Hypothesis</u>: $F_i > \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^i$ for all $i<n$.

<u>Induction Step</u>: $F_n = F_{n-1} + F_{n-2}$

$$> \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^{n-1} + \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}\left[\frac{\sqrt{5}+1}{2}+1\right] = \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}\left(\frac{\sqrt{5}+1}{2}\right)^2$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^n$$ [5]

<u>3(a)</u>: $T(n) = T(n^{\frac{1}{2}}) + c = T(n^{\frac{1}{2^2}}) + 2c = \cdots$

$$= T(n^{\frac{1}{2^i}}) + ic$$

We choose i so that $n^{\frac{1}{2^i}} \le 4$

$$\Longleftrightarrow \frac{1}{2^i}\log_2 n \le 2 \Leftrightarrow \log_2 n \le 2^{i+1} \Leftrightarrow i \ge \log_2\log_2 n - 1$$

We take $i = \log_2\log_2 n - 1$ so that $T(4)=1$

$$\Rightarrow T(n) = T(4) + (\log_2\log_2 n - 1)c$$

$$= (1-c) + c\log_2\log_2 n = \Theta(\log_2\log_2 n)$$

$$= \Theta(\log\log n)$$ [10]

3(b): $T(n) = 2T\left(n^{\frac{1}{2}}\right) + \log n$

$= 2\left[2T\left(n^{\frac{1}{2^2}}\right) + \log\left(n^{\frac{1}{2}}\right)\right] + \log n = 2^2 T\left(n^{\frac{1}{2^2}}\right) + 2\log n$

$= 2^2\left[2T\left(n^{\frac{1}{2^3}}\right) + \log\left(n^{\frac{1}{2^2}}\right)\right] + 2\log n = 2^3 T\left(n^{\frac{1}{2^3}}\right) + 3\log n$

$= \cdots = 2^i T\left(n^{\frac{1}{2^i}}\right) + i\log n$

We choose i so that $n^{\frac{1}{2^i}} \leq 4 \iff \frac{1}{2^i}\log_2 n \leq 2$

$\iff \log_2 n \leq 2^{i+1} \iff i \geq \log_2\log_2 n - 1$

We take $i = \log_2\log_2 n - 1$ so that $T(4) = 1$

$\Rightarrow T(n) = 2^{\log_2\log_2 n - 1}\, T(4) + (\log_2\log_2 n - 1)\log n$

$= \frac{1}{2}\log_2 n - \log n + \log n \log_2\log_2 n$

$= \Theta(\log_2 n \log_2\log_2 n) = \Theta(\log n \log\log n)$

[10]