CS F364
Design & Analysis of Algorithms

ALGORITHMS - COMPLEXITY

-Reductions

- - Transitivity of Reduction
  - Polynomial-Time Reductions and Implications
  - Example

Problems - Reduction

- Reduction is a mechanism for capturing the relation "at least as hard as" between problems.
  - "at least as hard as" refers to "at least as hard to solve as"
    - Typically this would mean "requires at least as much time to solve as" or "requires at least as much space to solve as".
- **Definition:**
  - Let π_1 and π_2 be decision problems with input sets $\mathbf{I}(\pi_1)$ and $\mathbf{I}(\pi_2)$ respectively.
  - We say, π_1 ***reduces to*** π_2 ,
    - if there is a function $f : \mathbf{I}(\pi_1) \text{-->} \mathbf{I}(\pi_2)$ such that
    - for every $x \in \mathbf{I}(\pi_1)$,
      - $\pi_1(x) = 1$ if and only if $\pi_2(f(x)) = 1$

PROBLEMS – REDUCTION

- We are usually interested in efficient reductions:
  - i.e. the function mapping inputs to inputs should be "efficiently" computable
  - i.e. f should be computable in polynomial time.
- Thus we say π_1 ***(polynomially) reduces to*** π_2
  - if there is a polynomial-time computable function $f : \mathbf{I}(\pi_1) \dashrightarrow \mathbf{I}(\pi_2)$ such that
  - for every $x \in \mathbf{I}(\pi_1)$,
    - $\pi_1(x) = 1$ if and only if $\pi_2(f(x)) = 1$
- We use $\pi_1 \precsim \pi_2$
  - to denote that π_1 ***(polynomially) reduces to*** π_2

Problems – Reduction – Example

- Problem Definition: Hamiltonian Cycle (**HAM**)
  - Given a graph G= (V,E) is there a simple cycle including all vertices in V?
- Claim: **HAM** $\precsim$ **TSP$_d$**
  - where TSP_d is the decision version of TSP: i.e. is there a tour of length < k, for some +ve k?
- Implication: **TSP** is as hard as **HAM**.

PROBLEMS – REDUCTION – EXAMPLE

- Claim: **HAM $\precsim$ TSP$_d$**
- Reduction:
  - Given a graph G= (V,E)
  - Construct a graph G' = (V, V x V, w) such that
    - w(u,v)=1 if (u,v) $\in$ E
    - =2 otherwise
- Verify:
  - G has a Hamiltonian cycle iff

    there is a tour in G' (as in TSP) of length <= |V|.

REDUCTIONS: TRANSITIVITY

- Transitivity of Reduction:
  - If $\pi_1 \lesssim \pi_2$ and $\pi_2 \lesssim \pi_3$ then $\pi_1 \lesssim \pi_3$
- Proof: Construct a composite function from $\mathbf{I}(\pi_1)$ to $\mathbf{I}(\pi_3)$ given mappings
  - $f_{12} : \mathbf{I}(\pi_1) \dashrightarrow \mathbf{I}(\pi_2)$ and $f_{23} : \mathbf{I}(\pi_2) \dashrightarrow \mathbf{I}(\pi_3)$
- Example:
  - Definition VERTEX COVER:
    - Given a graph G = (V,E) a vertex cover for G is a subset S of V, such that for any (u1,u2) in E, u1 is in S OR u2 is in S.
  - Problem VERTEX _COVER$_d$:
    - Given a graph G = (V,E) and a positive integer k < |V| , is there a vertex cover for G of size <= k ?
  - Claim: VERTEX_COVER$_d$ $\lesssim$ HAM
  - Implication: VERTEX_COVER$_d$ $\lesssim$ TSP$_d$