CS F364
Design & Analysis of Algorithms

Analysis – Problems – Reductions

Analysis of Problems

- Reduction
- Karp Reduction, 1-1 Reduction
- Lower Bounds and Reduction: Example
- Turing Reduction

RECALL: REDUCTION

- We reduced the problem of factoring a number N into the problem of computing $\varphi(N)$:
  - i.e. we argued that the former can be solved using an algorithm for the latter as a black box.

PROBLEMS - REDUCTION

- Reduction can be used as a mechanism for capturing the relation "at least as difficult as" between problems.
  - "at least as difficult as" refers to "at least as difficult to solve as"
    - Typically this would mean

      "**requires at least as much time to solve as**" or

      "**requires at least as much space to solve as**"

    (*depending on the context*).

Problems - Reduction

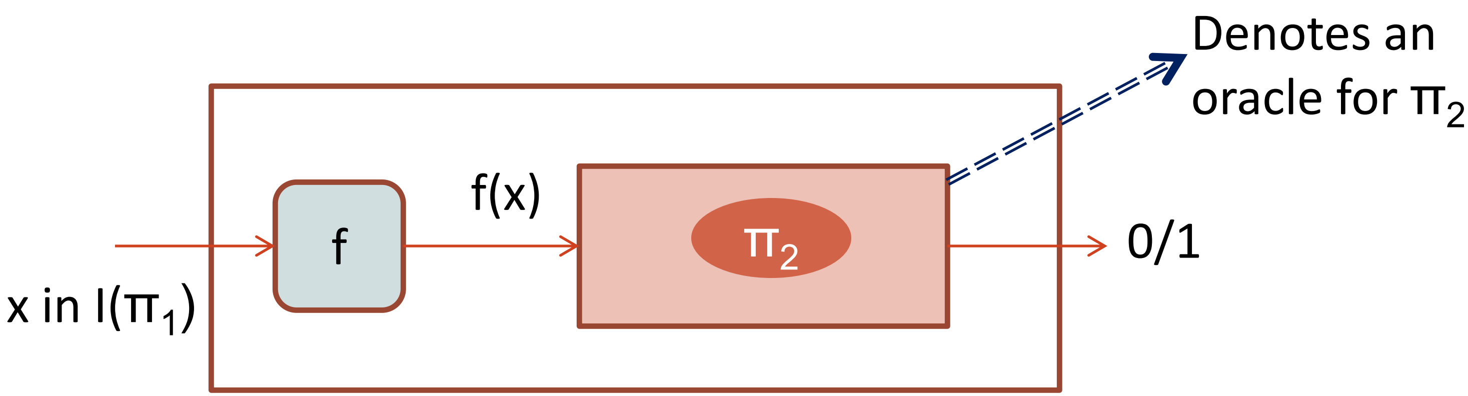
- Definition:
  - Let π_1 and π_2 be decision problems with input sets $\mathbf{I}(\pi_1)$ and $\mathbf{I}(\pi_2)$ respectively.
  - We say, π_1 ***reduces to*** π_2 ,
    - if there is a function f : $\mathbf{I}(\pi_1)$ --> $\mathbf{I}(\pi_2)$ such that
    - for every $x \in \mathbf{I}(\pi_1)$,
      - $\pi_1(x) = 1$ if and only if $\pi_2(f(x)) = 1$
- Questions:

1. What does it mean "algorithmically"?
2. What is the cost of reduction?

   i.e. what is the cost of (computing) the function f?

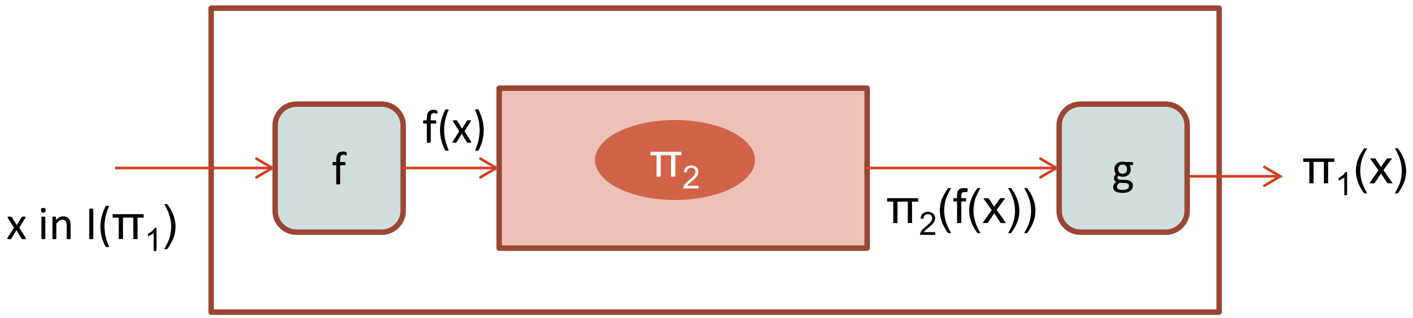
Karp Reduction

- Definition:
  - Let π_1 and π_2 be decision problems with input sets $\mathbf{I}(\pi_1)$ and $\mathbf{I}(\pi_2)$ respectively.
  - We say, π_1 ***reduces to*** π_2 (denoted $\pi_1 \precsim \pi_2$)
    - if there is a function $f : \mathbf{I}(\pi_1) \dashrightarrow \mathbf{I}(\pi_2)$ such that
    - for every $x \in \mathbf{I}(\pi_1)$,
      - $\pi_1(x) = 1$ if and only if $\pi_2(f(x)) = 1$



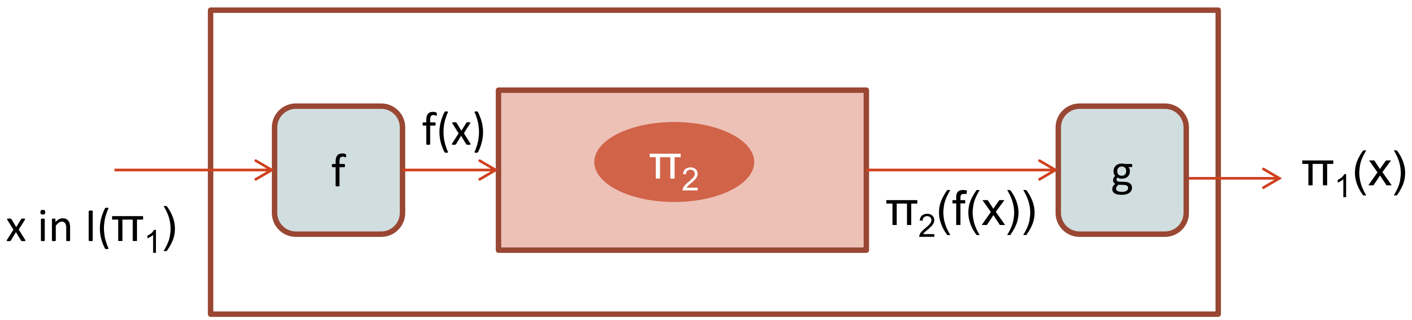
1-1 Reduction

- Karp reduction is a special case of 1-1 reduction:
  - Let π_1 and π_2 be problems with input sets $\mathbf{I}(\pi_1)$ and $\mathbf{I}(\pi_2)$ respectively.
  - We say, π_1 ***reduces to*** π_2 (denoted $\pi_1 \precsim \pi_2$)
    - if one can obtain an algorithm for π_1 given an algorithm for π_2
    - using mapping functions **f** and **g** on inputs to π_1 and outputs of π_2 respectively:



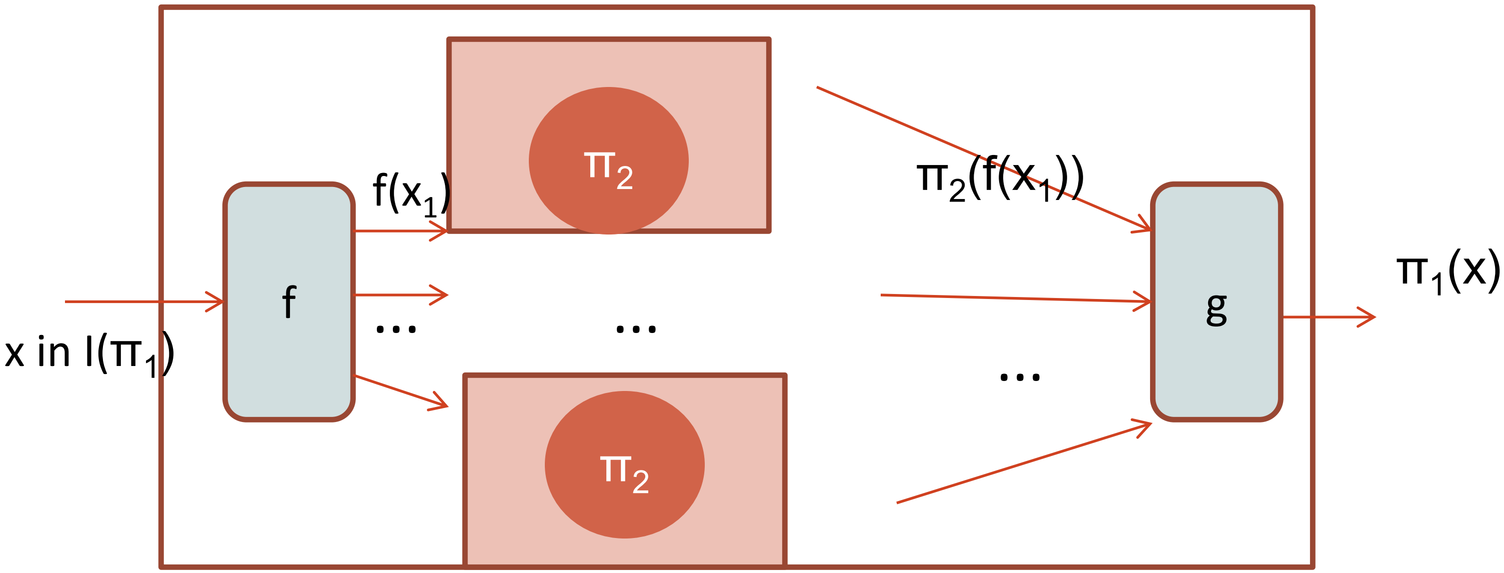
Problems – Reduction

- We are usually interested in efficient reductions:
  - i.e. the function mapping inputs to inputs should be "efficiently" computable
  - e.g. **f** should be computable in polynomial time.
- We use $\pi_1 \precsim_{t(n)} \pi_2$
  - to denote that π_1 ***reduces to*** π_2 in time t(n)
    - t(n) is the total cost of functions f and g.



TURING REDUCTIONS

- We use $\pi_1 \precsim_{T,t(n)} \pi_2$
  - to denote that π_1 ***Turing-reduces to*** π_2 in time t(n)
    - t(n) is the total cost of functions f and g and the cost of calls to π_2



Implications: ???