CS F364 Design & Analysis of Algorithms

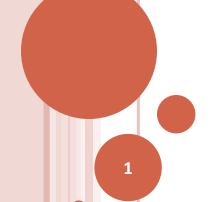
ALGORITHM DESIGN TECHNIQUES - GREEDY

Matroids – A Theoretical Framework for Greedy Algorithms

Introduction to Matroids

- Introduction to Matroids:

Definition and Examples



MATROIDS: MOTIVATION

- Matroids provide a sound but incomplete theory of the greedy method i.e.
 - The theory of matroids can be used to derive optimal solutions for many problems / scenarios but
 - the theory does not cover all cases for which a greedy method applies.

MATROIDS: DEFINITIONS

- Definitions (Hereditary Property and Exchange Property)
 - Given a finite set S and a nonempty family I of subsets
 - referred to as *independent subsets* of S:
 - ol is said to be *hereditary* if it satisfies the following property:
 - oif B is in I, and A is subset of B, then A is in I
 - ol is said to satisfy the *exchange* property if the following is true:
 - oif A is in I, B is in I, and |A| < |B|,
 - othen there exists x in B\A such that (A U {x}) is in I

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Matroids: Definitions

Definition:

- A *matroid* is an ordered pair M = <S, I> such that
 - S is a finite set
 - 2. I is a nonempty family of subsets of S, satisfying
 - the hereditary property and
 - b. the exchange property

• Example:

- Let S be any finite set and
- I_k be the set of all subsets of S of size at most k, where k<=|S|
 - Then $\langle S, I_k \rangle$ is a matroid.

• Example:

Linearly independent columns of a matrix.

GRAPHIC MATROID

- Given, an undirected graph G=(V,E), consider $M_G = \langle S_G, I_G \rangle$ defined as:
 - $S_G = E$
 - If A ⊆ S, then A is in I_G iff A is acyclic
 i.e. a set of edges A is independent iff the subgraph
 G_Δ = (V, A) forms a forest (of trees).

o Theorem:

- If G=(V,E) is an undirected graph, then $M_G=<S_G,I_G>$ is a matroid.
- Given G=(V,E), $M_G=<S_G,I_G>$ is referred to as the *Graphic Matroid*.

GRAPHIC MATROID

Theorem (Graphic Matroid):

If G=(V,E) is an undirected graph, then $M_G=<S_G,I_G>$ is a matroid.

Proof:

- S_G is finite.
- I_G is hereditary: the subset of a forest is a forest
- I_G satisfies the exchange property:
 - Let $G_A = (V,A)$ and $G_B = (V,B)$ be forests of G and let |B| > |A|
 - Then G_A and G_B contain |V|-|A| and |V|-|B| trees resp.
 - Therefore there must be a tree T in G_B
 - whose vertices are from two different trees in G_A and
 - there must be an edge (u,v) in T such that u and v are from different trees in G_A
 - Adding (u,v) to G_A preserves acyclicity:
 - i.e. $(G_A \cup \{(u,v)\})$ is in I_G
 - i.e. I_G satisfies the exchange property