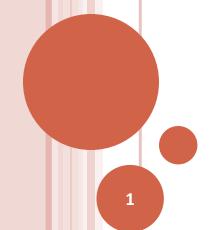
# CS F364 Design & Analysis of Algorithms

## **ALGORITHM DESIGN - APPROXIMATION**

#### **Approximation Algorithms**

- Non-approximability of TSP
- Relative Approximation Algorithm for Metric TSP



## RELATIVE APPROXIMATION

- On Relative Approximation:
  - Is it possible to find r-approximation algorithms for all NP-complete problems?
    - Note that **r** must be a constant.
  - For a given NP-complete problem, what is the best approximation ration obtainable?

# Non-Approximability of TSP

#### • Theorem:

- There exists no polynomial time r-approximation algorithm for TSP for any constant integer r > 1, unless  $P = \mathbb{N}P$
- Proof (By contradiction):
  - Reduce HAM (i.e the Hamiltonian Cycle problem) to TSP such that
    - o If there is a polynomial time r-approximation algorithm A for TSP there exists a polynomial time algorithm for HAM.
    - o See next slides for the reduction and Lemma 3.
  - But HAM is an NP-complete problem
    - i.e. there is no polynomial time algorithm for HAM unless P = NP.

# Non-Approximability of TSP

- [2]
- Reduction from HAM to (approximation of) TSP:
  - Given an instance of HAM, say a graph G = (V,E)
    - oConstruct a new graph H = (V, V x V, w) such that
      - w(e) = 1 if e is in E
      - w(e) = 1 + r\*|V| otherwise, where r is a +ve integer > 1
- o Lemma 1:
  - G has a Hamiltonian cycle iff H has a tour of weight |V|
  - Proof: (*Trivial*)
- o Lemma 2:
  - Any sub-optimal tour T for H will have at least one edge with weight 1+r\* | V | i.e. m(H, T) > r \* m\*(H)

[3]

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#### o Lemma 3:

• If there is a polynomial time r-approximation algorithm A for TSP then there exists a polynomial time algorithm for HAM.

#### • Proof:

- Assume A is a polynomial time r-approximation algorithm for TSP: then m(H, A(H,r)) <= r \* |V| if m\*(H)=|V|</li>
- 2. But by the reduction (see previous slide) and Lemma 2, the only solution A can return is the optimal solution!
- 3. Now define A'(G):
  - o construct H from G as defined in the reduction;
  - o Let T=A(H, r); if  $(m(H,T) \le r * |V|)$  return 1; else return 0;
- A' solves HAM in polynomial time.
  - o By Lemma 1 (prev. slide), (2) above, and (1) above.

## PROBLEM - METRIC TSP

- Openition: Triangle Inequality on Graphs:
  - A weighted graph G = (V,E,w) is said to satisfy if the triangle inequality if for any three vertices v1, v2, and v3 in V

$$ow((v1,v2)) + w((v2,v3)) >= w((v1,v3))$$

- Problem Definition: Metric TSP:
  - Given a weighted graph G that satisfies the triangle inequality, find a minimum weight tour for G.
- O Metric TSP is №-complete:
  - TSP is NP-complete even in the special case where all weights are the same.

## RELATIVE APPROXIMATION - METRIC TSP

- Algorithm A2\_MTSP(G)
  - 1. Construct a minimum spanning tree M of G
  - Construct an Euler Walk T of M
    - i.e. a walk where each edge is visited exactly once in each direction.
  - 3. Construct a short-circuited tour S from T such that if (v1, v2) and (v2, v3) are in T where v2 has already been visited replace these two edges by (v1, v3).
  - 4. return S
- o Claim 1:
  - A2\_MTSP(G) is a TSP tour of G
  - Proof:
    - o Step 2 ensures that all vertices are visited.
    - Step 3 ensures that each vertex is visited exactly once.

#### RELATIVE APPROXIMATION — METRIC TSP

- Algorithm A2\_MTSP(G)
  - 1. Construct a minimum spanning tree M of G
  - Construct an Euler Walk T of M
    - i.e. a walk where each edge is visited exactly once in each direction.
  - 3. Construct a short-circuited tour S from T such that if (v1, v2) and (v2, v3) are in T where v2 has already been visited replace these two edges by (v1, v3).
  - 4. return S after adding an edge from the last vertex to the first vertex.

#### o Claim 2:

- A2\_MTSP(G) runs in polynomial time
- Proof:
  - o Each of the steps 1, 2, and 3 run in polynomial time

# RELATIVE APPROXIMATION – EXAMPLE – METRIC TSP [3]

#### o Claim 3:

- The cost of a minimum spanning tree M is a lower bound on the cost of a TSP tour M for a given graph.
- Proof: The (optimal) tour T' with one edge removed is a spanning tree. Therefore  $w(T') \ge w(M)$

#### o Claim 3:

- m(G,A2\_MTSP(G)) <= 2 \* m\*(G)</li>
- Proof:
  - 1. For any H, let  $w(H) = \sum_{e \text{ in } H} w(e)$
  - 2. From step 2 of A2\_MTSP(G) w(T) = 2 \* w(M)
  - From step 3 of A2\_MTSP(G)  $w(S) \le w(T)$
  - o From steps 1 to 3 of the proof and Claim 3:
    - $\circ$  w(S) <= 2 \* w(T') QED

## RELATIVE APPROXIMATION — EXAMPLE — METRIC TSP [4]

#### • Theorem:

- A2\_MTSP(G) is a polynomial time 2-approximation algorithm for Metric TSP.
- Proof:
  - o By claims 1,2, and 3 (see previous slides).

### o Improvement:

- The approximation factor can be improved:
  - o observe that we are doubling all the edges in the spanning tree to ensure that all vertices are of even degree (so that an Euler Tour can be constructed).
  - o but one can double only those edges incident on vertices with odd degree i.e. cost of the Euler tour will be lower than twice the cost of the spanning tree:
- Derive the bound for this modified algorithm!