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Agenda

RANDOMIZED ALGORITHMS – INTRODUCTION

- MOTIVATING EXAMPLE: QUICK SORT
- ALGORITHM AND ANALYSIS

Algorithm QS

- Input: A set S of numbers
- Output: Elements of S sorted in increasing order
- Steps
  1. Choose an element **p** from **S**
  2. Partition **S** into $\mathbf{S}_<$ and $\mathbf{S}_>$ s.t.
     1. all elements less than **p** are in $\mathbf{S}_<$ and all elements greater than **p** are in $\mathbf{S}_>$
  3. Sort $\mathbf{S}_<$
  4. Sort $\mathbf{S}_>$

Algorithm QS - Analysis

- The pivot chosen in Step 1 decides how
  - Step 2 partitions S into sublists
    - i.e. sizes of the sublists are dependent on the input "order" if the pivot is chosen deterministically!
  - What is the best case behavior?
    - Can you ensure that happens in every (recursive) step?
  - What is the worst case behavior?
- Compare QS with the process of "constructing a binary search tree"
  - Compare the *height of the tree* with the expected *depth of recursion in QS*

Algorithm RandQS

- Input: A set S of numbers
- Output: Elements of S sorted in increasing order
- Steps
  1. Choose an element p uniformly at random from S
  2. Partition **S** into $\mathbf{S}_<$ and $\mathbf{S}_>$ s.t.
     1. all elements less than p are in $\mathbf{S}_<$ and all elements greater than **p** are in $\mathbf{S}_>$
  3. Sort $\mathbf{S}_<$
  4. Sort $\mathbf{S}_>$

Algorithm RandQS

- What does it achieve (apart from sorting)?
  - Step 1 ensures Step 2 partitions S into sublists uniformly randomly
    - i.e. sizes of the sublists are not dependent on the input “order”
      - This is the best you can hope for! (Why?)
- Compare RandQS with the process of “constructing a binary search tree”
  - Compare the *expected height of the tree* with the expected *depth of recursion in RandQS*

Algorithm RandQS - Analysis

- What is the expected number of comparisons?
  - Observations:
    - All comparisons happen in Step 2 (partitioning)
    - No pair of elements is compared more than once.
  - Notation:
    - Let $S_{(i)}$ denote the element of rank i in S
    - Define random variable X_{ij} to denote the number of comparisons between $S_{(i)}$ and $S_{(j)}$
      - i.e. X_{ij} is 1 if $S_{(i)}$ and $S_{(j)}$ are compared; 0 otherwise
  - So, the total number of comparisons is
    - $\Sigma_{i=1}^{n} \Sigma_{j>i}^{n} X_{ij}$

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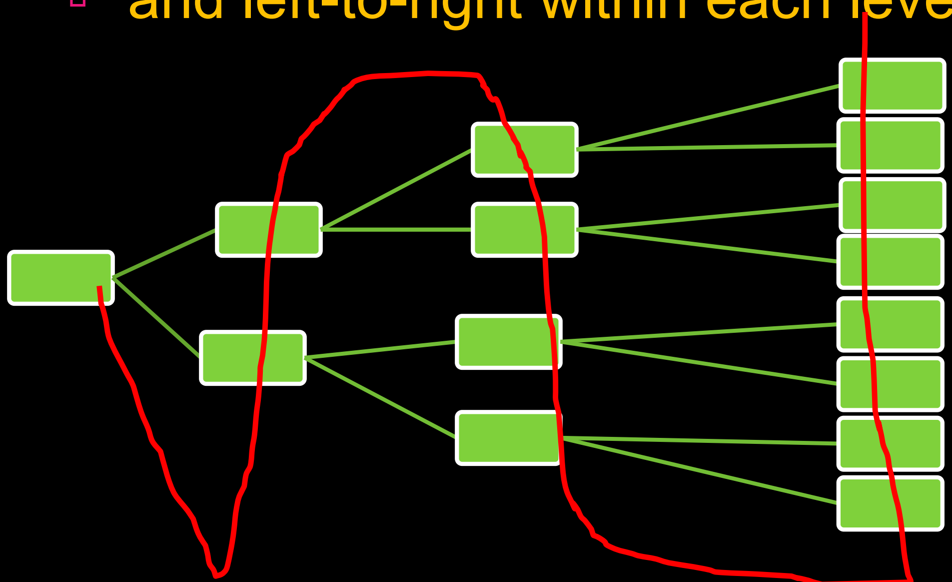
Algorithm RandQS - Analysis

- Thus the expected number of comparisons is
    - $E[\Sigma_{i=1}^{n} \Sigma_{j>i}^{n} X_{ij}]$
  - By linearity of expectations
    - $E[\Sigma_{i=1}^{n} \Sigma_{j>i}^{n} X_{ij}] = \Sigma_{i=1}^{n} \Sigma_{j>i}^{n} E[X_{ij}]$
  - By definition of expected values
    - $E[X_{ij}] = p_{ij} * 1 + (1 - p_{ij}) * 0 = p_{ij}$
      - where p_{ij} is the probability that $S_{(i)}$ and $S_{(j)}$ are compared in an execution.
- Thus the expected number of comparisons is
    - $\Sigma_{i=1}^{n} \Sigma_{j>i}^{n} p_{ij}$

Algorithm RandQS - Analysis

- $S_{(i)}$ and $S_{(j)}$ are compared if and only if:
  - one of them is an ancestor of the other in the BST
- Let π be the permutation obtained by visiting the nodes of T
  - in increasing order of level numbers
  - and left-to-right within each level



Claim:
One of $S_{(i)}$ and $S_{(j)}$ is an ancestor of the other if and only if:

one of them occurs earlier in π than any element $S_{(m)}$ such that $i < m < j$

Algorithm RandQS - Analysis

Claim:

One of $S_{(i)}$ and $S_{(j)}$ is an ancestor of the other only if:

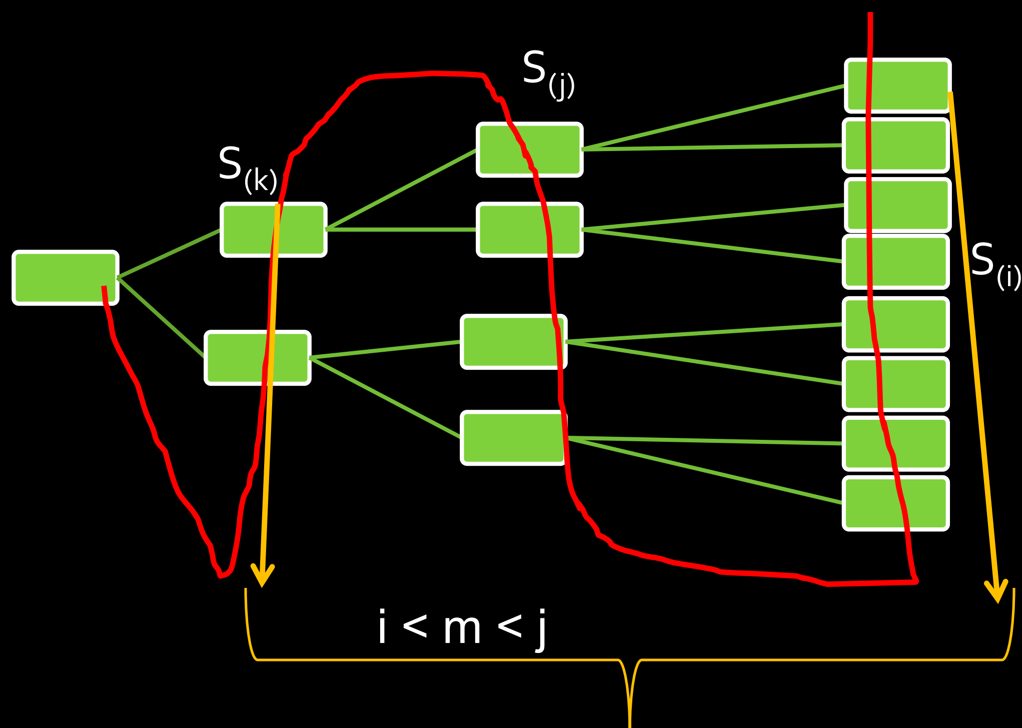
one of them occurs earlier in π than any element $S_{(m)}$ such that $i < m < j$

Proof:

Let $S_{(k)}$ be the earliest in π among elements of rank between i and j.

- Assume k<>i and k<>j
- Then $S_{(i)}$ and $S_{(j)}$ will belong to the left and right sub-trees - respectively - of $S_{(k)}$

  i.e neither is an ancestor of the other.



Algorithm RandQS - Analysis

Claim:

One of $S_{(i)}$ and $S_{(j)}$ is an ancestor of the other if :

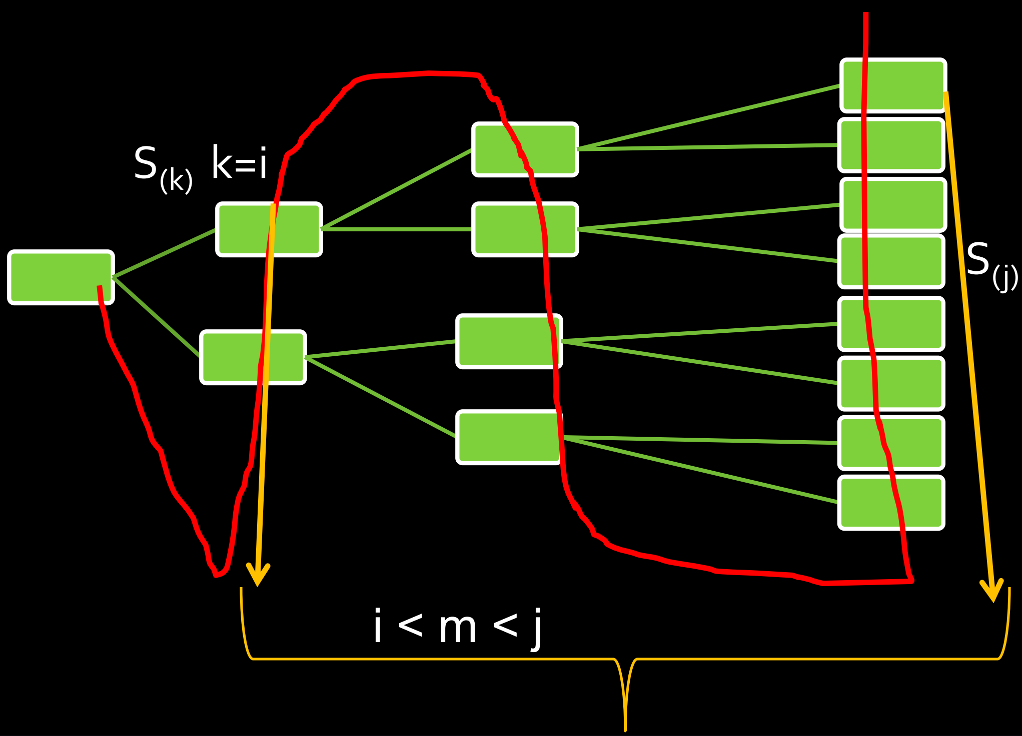
one of them occurs earlier in π than any element $S_{(m)}$ such that $i < m < j$

Proof:

Let $S_{(k)}$ be the earliest in π among elements of rank between i and j.

Assume k=i or k=j

Then this (say k=i) must be the ancestor of the other (j).



Algorithm RandQS - Analysis

- All the elements $S_{(i)}, S_{(i+1)}, \ldots S_{(j)}$ are equally likely to appear the earliest in π
  - Probability that one of the two elements (i.e. i or j) occurs the earliest in π is

$$2 / (j-i+1)$$

- Thus, $p_{ij} = 2 / (j-i+1)$
- Then
  - $\Sigma_{i=1}^{n} \Sigma_{j>i}^{n} p_{ij} = \Sigma_{i=1}^{n} \Sigma_{j>i}^{n} (2 / (j-i+1))$
  - $= \Sigma_{i=1}^{n} \Sigma_{k>1}^{n-i+1} (2/k)$ // let k=j-i+1
  - $<= 2 * \Sigma_{i=1}^{n} \Sigma_{k=1}^{n} (1/k)$
  - $= 2nH_n = O(n*\log n)$
    - where H_n is the n-th Harmonic number i.e. $\Sigma_{i=1}^{n} 1/k$
    - and H_n is $O(\log n)$