DAA Tutorial 3 Solution

1: **Algorithm:** Sort both the sets A and B in increasing order. You can also sort both the sets A and B in decreasing order. (4)

complexity of the above algorithm is $O(n \log n)$. (2)

Proof of Correctness: **Case 1:** $n=1$ (2)

Payoff $= a_1^{b_1}$ is the only choice, hence it is maximum.

Case 2: $n=2$:

Let $a_1 \le a_2$ and $b_1 \le b_2$.

We can have only two possibilities for payoff:

According to our algorithm:

$$P_1 = a_1^{b_1} + a_2^{b_2} \longrightarrow (1)$$

Another possibility corresponding to $\begin{pmatrix} a_1 & a_2 \\ b_2 & b_1 \end{pmatrix}$:

$$P_2 = a_1^{b_2} + a_2^{b_1} \longrightarrow (2)$$

From (1) and (2), we get:

$$P_1 - P_2 = \left(a_1^{b_1} + a_2^{b_2}\right) - \left(a_1^{b_2} + a_2^{b_1}\right) = \left(a_1^{b_1} - a_1^{b_2}\right) + \left(a_2^{b_2} - a_2^{b_1}\right)$$

$$= a_2^{b_1}\left(a_2^{b_2-b_1} - 1\right) - a_1^{b_1}\left(a_1^{b_2-b_1} - 1\right) \longrightarrow (3)$$

Now $a_2 \geq a_1 \Rightarrow a_2^{b_1} \geq a_1^{b_1} \longrightarrow$ (4)

and also $a_2^{b_2-b_1} \geq a_1^{b_2-b_1} \longrightarrow$ (5)

(4) $\times$ ((5)-1) gives:

$$a_2^{b_1}(a_2^{b_2-b_1}-1) \geq a_1^{b_1}(a_1^{b_2-b_1}-1) \longrightarrow (6)$$

From (3) and (6) we get:

$P_1 - P_2 \geq 0 \Rightarrow P_1 \geq P_2 \Rightarrow$ The algorithm is correct for $n=2$.

(6)

Case 3: $n>2$:

Given any ordering of A and B, we will show that the payoff is upper bounded by our algorithm. Consider any "out of order" sequence

$$\begin{pmatrix} a_1' & a_2' & \cdots & a_i' & a_{i+1}' & \cdots & a_n' \\ b_1' & b_2' & \cdots & b_i' & b_{i+1}' & \cdots & b_n' \end{pmatrix}$$

We apply "Bubble Sort" on this sequence to finally get the numbers in sorted order. Now any step in Bubble Sort applies transformation to adjacent "out of order" numbers. For example, suppose in a particular step, the transformation is for

$$\cdots \begin{pmatrix} a_i' & a_{i+1}' \\ b_i' & b_{i+1}' \end{pmatrix} \cdots$$ because they are "out of order".

Now we apply the result of $n=2$ for this step, and we find that any step of Bubble Sort will increase the payoff (or at least it will not decrease the payoff). The payoff in the final step of Bubble Sort is identical to the payoff of our algorithm.

$\Rightarrow$ any sequence of A and B will have payoff less than or equal to the payoff when both A and B are sorted in the same order.

$\Rightarrow$ Our algorithm gives the maximum payoff for $n > 2$. (6)

5: Everything is identical to (7) except for case 2:

$$P_1 = a_1^{b_1} a_2^{b_2} \longrightarrow (1)$$

$$P_2 = a_1^{b_2} a_2^{b_1} \longrightarrow (2)$$

From (1) and (2) we get:

$$\frac{P_1}{P_2} = \frac{a_1^{b_1} a_2^{b_2}}{a_1^{b_2} a_2^{b_1}} = \left(\frac{a_2}{a_1}\right)^{b_2 - b_1} \geq 1$$ because

$\frac{a_2}{a_1} \geq 1$ and $b_2 - b_1 \geq 0 \Rightarrow P_1 \geq P_2$.