CS F364: Design & Analysis of Algorithm

09 Matroids Application Mimimum Spanning Tree

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Matroids

Theory for some situations in which the greedy yields optimal solutions

- **Matroids:** ordered pair $M = (S, I)$ satisfying the following
  1. S is a finite set
  2. **Hereditary property:** I is a nonempty family of subsets of S, called the independent subsets of S, such that $B \in I$ and $A \subseteq B$, then $A \in I$. (Question: is ϕ a member of I? Y)
  3. **Exchange property:** If $A \in I$, $B \in I$, and $|A| < |B|$ then there exists some element $x \in B - A$ such that $A \cup \{x\} \in I$.

Graphic Matroid $M_G = (S_G, I_G)$

Defined in terms of a given undirected graph $G = (V, E)$
1. $S = E$
2. If A is a subset of E, then $A \in I_G$ if and only of A is acyclic. So (V, A) is a forest

For $G = (V, E)$ a undirected graph $M_G = (S_G, I_G)$ is a matroid

Proof:
- S_G is finite
- Hereditary property: subset of a forest is a forest
- Exchange property:
  Number of trees in a forest (V, E_f) is $|V| - |E_f|$
  For $A, B \in I$ if $|A| < |B|$ then B has fewer trees
  $\rightarrow$ Consider and edge $x \in B$ that links two trees in A

Extention: x extends A
Maximal independent set: set that can not be extended

All maximal independent set have same size

Suppose to the contrary that A is a maximal independent subset of M and there exists another larger maximal independent subset B of M

- Then due to exchange property $\exists x \in B - A$ so that A could be extended
- so A is not maximal independent set
- **Contradiction.**

Weighted matroid

- $M = (S, I)$ is weighted if it is associated with weight function $w(x)$ for all $x \in S$
- $w(A)$ is defined as

$$w(A) = \sum_{x \in A} w(x)$$

- One example of w be the weight of the edge
- $w'(e) = w_0 - w(e)$

See a minimization problem as maximization one

$$\begin{aligned} w'(A) &= \sum_{e \in A} w'(e) \\ &= \sum_{e \in A} (w_0 - w(e)) \\ &= (|V| - 1)w_0 - \sum_{e \in A} w(e) \\ &= (|V| - 1)w_0 - w(A) \end{aligned}$$

Minimum Spanning Tree Problem

- Subset of the edges that connects all of the vertices together and has minimum total length
- It is like finding maximal independent set in M_G

Algorithm 1: Greedy(M, w)

```
A = φ
sort M.S in decreasing order of weight w
for x ∈ M.s take in order do
    if A ∪ {x} ∈ M.I then
        A = A ∪ {x}
return A
```

Complexity $O(n \log n + nf(n))$

Matroids exhibit the greedy-choice property

Consider $M = (S, I)$ with weight function w. Let S sorted in decreasing order. Consider x, the the first element of S such that $\{x\}$ is independent. if $\exists$ x then there exists an optimal subset A containing x

- Let B be any nonempty optimal subset with $x \notin B$
- No element of B has weight greater then $w(x)$
- Construct A by taking x and then items from B
- A and B are of same size differing on only one item $y \in B$

$$\begin{aligned} w(A) &= w(B) - w(y) + w(x) \\ &\geq w(B) \end{aligned}$$

- Contradiction. As B was optimal

Matroids exhibit the greedy-choice property

Let $M = (S, I)$ be any matroid. If x is an element of S that is an extension of some independent subset A of S, then x is also an extension of ϕ

- Since x is an extension of A, we have that $A \cup \{x\}$ is independent. Since I is hereditary, $\{x\}$ must be independent. Thus, x is an extension of ϕ.

Let $M = (S, I)$ be any matroid. If x is an element of S such that x is not an extension of ϕ, then x is not an extension of any independent subset A of S
contrapositive of above

Any elements that GREEDY passes over initially because they are not extensions of ϕ can be forgotten about, since they can never be useful.

Matroids exhibit the optimal-substructure property

Let Let x be the first element of S chosen by GREEDY for the weighted matroid $M = (S, I)$.
We can reduce the problem to $M' = (S', I')$.

- $s' = \{y \in S : \{x, y\} \in I\}$
- $I' = \{B \subseteq S - \{x\} : B \cup \{x\} \in I\}$

This is because $A' = A - \{x\}$ is an independent subset of M'.

Correctness of the greedy algorithm on matroids

If $M = (S, I)$ is a weighted matroid with weight function w, then GREEDY(M,w) returns an optimal subset

- Any elements that GREEDY passes over initially because they are not extensions of ϕ can be forgotten about, since they can never be useful.
- Once GREEDY selects the first element x, the algorithm does not err by adding x to A, since there exists an optimal subset containing x.
- Finally, the remaining problem is one of finding an optimal subset in the matroid M' that is the contraction of M by x.

Thank You!

Thank you very much for your attention! (Reference[1])

Queries ?

[1] [1] Book - *Introduction to Algorithm*, By THOMAS H. CORMEN, CHARLES E. LEISERSON, RONALD L. RIVEST, CLIFFORD STEIN