DAA Comprehensive Exam Solution

1:/
Cut $\{s\}$, $\{u, v, t\}$ having capacity $2+4=6$.

Cut $\{s, u\}$, $\{v, t\}$ having capacity $4+6+4=14$.

Cut $\{s, v\}$, $\{u, t\}$ having capacity $2+2=4$.

Cut $\{s, u, v\}$, $\{t\}$ having capacity $2+4=6$.

Min Cut is $\{s, v\}$, $\{u, t\}$ having capacity 4.

(1) for each correct answer.

2:/ $A_\epsilon(I)$

1 Run the 2-approximation algorithm as given in Problem 2, Tutorial 11. Let M be the value returned by this algorithm.

2 $\mu \leftarrow \frac{\epsilon M}{n}$

3 $v_i' \leftarrow \left\lfloor \frac{v_i}{\mu} \right\rfloor \quad \forall i \in I$

4 Run the Dominating Pairs DP algorithm for Knapsack with values v_i' and return the corresponding solution.

Time complexity: Step 1 takes $O(n \log n)$ time.
Step 2 takes $O(1)$ time. Step 3 takes $O(n)$ time.
Step 1 is a 2-approximation algorithm. $\Rightarrow$

$$\frac{OPT}{M} \le 2 \iff OPT \le 2M \longrightarrow (1)$$

Let OPT' be the optimal solution of step 4.

$$OPT' = \sum_{i \in S} v_i' \le \sum_{i \in S} \frac{v_i}{\mu} \le \frac{OPT}{\mu} \longrightarrow (2)$$

From (1) and (2), we get

$$OPT' \le \frac{2M}{\mu} = \frac{2M}{\frac{\epsilon M}{n}} = \frac{2n}{\epsilon}.$$

Therefore, step 4 takes time $= n \, OPT' =$

$$\frac{2n^2}{\epsilon} = O\left(\frac{n^2}{\epsilon}\right). \Rightarrow A_\epsilon(I) \text{ takes } O\left(\frac{|I|^2}{\epsilon}\right) \text{ time.}$$

Approximation Ratio: Proof of approximation Ratio is identical to the proof given in lecture notes.

3(a): $l(t) =$

$$\sum_{i=1}^{t} i\,p(i) + (1-\alpha(t)) \sum_{i=1}^{t} (t+i)\,p(i) +$$

$$(1-\alpha(t))^2 \sum_{i=1}^{t} (2t+i)\,p(i) + \cdots$$

Success probability of $(j+1)^{st}$ loop in step i

$$+ \underbrace{(1-\alpha(t))^j}_{\text{Failure probability of first } j \text{ loops}} \sum_{i=1}^{t} \underbrace{(jt+i)}_{\text{Total Run time}} p(i) + \cdots$$

(5)

$$= \left(\sum_{i=1}^{t} i\,p(i)\right)\left(1 + (1-\alpha(t)) + (1-\alpha(t))^2 + \cdots \infty\right)$$

$$+ \; t(1-\alpha(t))\left(\sum_{i=1}^{t} p(i) + 2(1-\alpha(t)) \sum_{i=1}^{t} p(i) + \cdots\right)$$

$$= \frac{\sum_{i=1}^{t} i\,p(i)}{\alpha(t)} + t(1-\alpha(t))\,\alpha(t)\left(1 + 2(1-\alpha(t)) + 3(1-\alpha(t))^2 + \cdots\right)$$

$$= \frac{A}{\alpha(t)} + B\,t(1-\alpha(t))\,\alpha(t) \longrightarrow (1)$$

where $A = \sum_{i=1}^{t} i\,p(i) \longrightarrow (2)$

(2)

and $B = 1 + 2(1-\alpha(t)) + 3(1-\alpha(t))^2 + \cdots \longrightarrow (3)$

We have

$$\sum_{t'<t} v(t') = v(1) + v(2) + \cdots + v(t-1)$$

$$= p(1) + (p(1) + p(2)) + \cdots + (p(1) + p(2) + \cdots + p(t-1))$$

$$= (t-1)\, p(1) + (t-2)\, p(2) + \cdots + p(t-1)$$

$$= t(p(1) + p(2) + \cdots + p(t-1))$$

$$- (1 \cdot p(1) + 2 \cdot p(2) + \cdots + (t-1)\, p(t-1))$$

$$= t(p(1) + p(2) + \cdots + p(t-1) + p(t))$$

$$- (1 \cdot p(1) + 2 \cdot p(2) + \cdots + (t-1) \cdot p(t-1) + t \cdot p(t))$$

$$= t\, v(t) - \sum_{i=1}^{t} i\, p(i)$$

$$\Rightarrow \quad A = \sum_{i=1}^{t} i\, p(i) = t\, v(t) - \sum_{t'<t} v(t') \rightarrow \quad (4)$$

(3)

(5)

$$B = 1 + 2(1-a(t)) + 3(1-a(t))^2 + \cdots \rightarrow (3)$$

$$(1-a(t))B = (1-a(t)) + 2(1-a(t))^2 + \cdots \rightarrow (5)$$

(3) − (5) gives:

$$a(t)\,B = 1 + (1-a(t)) + (1-a(t))^2 + \cdots$$

$$= \frac{1}{a(t)}$$

$$\Rightarrow B = \frac{1}{a(t)^2} \rightarrow (6) \quad (3)$$

Putting the values of A and B from equations (4) and (6) in equation (1), we get

$$l(t) = \frac{t\,a(t) - \sum_{t'<t} a(t')}{a(t)} + \frac{t(1-a(t))\,a(t)}{a(t)^2}$$

$$= \frac{t - \sum_{t'<t} a(t')}{a(t)} \rightarrow (7) \quad (2)$$

3(b): $v(t) = p(1) + p(2) + \cdots + p(t)$

$$= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^t} = \frac{\frac{1}{2} - \frac{1}{2^{t+1}}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^t} \rightarrow (8)$$

(1)

$$\sum_{t' < t} v(t') = v(1) + v(2) + \cdots + v(t-1)$$

$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2^2}\right) + \cdots + \left(1 - \frac{1}{2^{t-1}}\right)$$

$$= (t-1) - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{t-1}}\right)$$

$$= (t-1) - \frac{\frac{1}{2} - \frac{1}{2^t}}{1 - \frac{1}{2}}$$

$$= (t-1) - \left(1 - \frac{1}{2^{t-1}}\right) = t - 2 + \frac{1}{2^{t-1}} \rightarrow (9)$$

(1)

Putting the values of $v(t)$ and $\sum_{t'<t} v(t')$ from equations (8) and (9) in equation (7), we get

$$l(t) = \frac{t - \left(t - 2 + \frac{1}{2^{t-1}}\right)}{1 - \frac{1}{2^t}} = \frac{2\left(1 - \frac{1}{2^t}\right)}{\left(1 - \frac{1}{2^t}\right)}$$ (1)

$$= 2 \Rightarrow l(t^*) = 2 \text{ for any } t^* \in N.$$

(1) (1)

4: Let $x = 2019\,2019\,2019\,2019\,2019$.

We have to compute $x^{x^{x}} \pmod{1000}$

To simplify our computations, we notice that

$$x \equiv 19 \pmod{1000}$$

$$\Rightarrow x^{x^{x}} \equiv 19^{x^{x}} \pmod{1000} \longrightarrow (1) \quad (1)$$

Now, we compute $\phi(1000) = 1000 \times (1 - \frac{1}{2}) \times (1 - \frac{1}{5})$

$$= 1000 \times \frac{1}{2} \times \frac{4}{5} = 400. \longrightarrow (2) \quad (1)$$

We have $(19, 1000) = 1$. Therefore, by Euler's Theorem, we have

$$19^{400} \equiv 1 \pmod{1000} \longrightarrow (3) \quad (1)$$

Again, to simplify our computations, we notice that

$$x \equiv 19 \pmod{400}$$

$$\Rightarrow x^{x} \equiv 19^{x} \pmod{400} \longrightarrow (4) \quad (1)$$

Now, we compute $\phi(400) = 400 \times (1 - \frac{1}{2}) \times (1 - \frac{1}{5})$

$$= 400 \times \frac{1}{2} \times \frac{4}{5} = 160 \longrightarrow (5) \quad (1)$$

We have $(19, 400) = 1$. Therefore, by Euler's Theorem, we have

$$19^{160} \equiv 1 \pmod{400} \rightarrow (6) \qquad (1)$$

Once again, to simplify our computations, we notice that

$$x = 92019 \equiv 19 \pmod{160}$$

Therefore, by Euler's Theorem, we have

$$19^{x} \equiv 19^{19} \pmod{400} \rightarrow (7) \qquad (1)$$

We will use the repeated squaring algorithm to compute $19^{19} \pmod{400}$ as follows:

$$19^{19} = 19^{(10011)_2}$$

$$19^{(1)_2} \equiv 19 \pmod{400}$$

$$19^{(10)_2} \equiv 19^2 \equiv 361 \pmod{400}$$

$$19^{(100)_2} \equiv 361^2 \equiv 321 \pmod{400}$$

$$19^{(1000)_2} \equiv 321^2 \equiv 241 \pmod{400}$$

$$19^{(1001)_2} \equiv 19 \times 241 \equiv 179 \pmod{400}$$

$$19^{(10010)_2} \equiv 179^2 \equiv 41 \pmod{400}$$

$$19^{(10011)_2} \equiv 41 \times 19 \equiv 379 \pmod{400} \quad (5)$$

$$\Rightarrow 19^{19} \equiv 379 \pmod{400} \longrightarrow (8) \quad (1)$$

From (7) and (8), we get

$$19^{x} \equiv 379 \pmod{400} \longrightarrow (9)$$

From (4) and (9), we get

$$x^{x} \equiv 379 \pmod{400} \longrightarrow (10)$$

From (1) and (10), we get

$$x^{x^{x}} \equiv 19^{379} \pmod{1000}$$

We will use the repeated squaring algorithm to compute $19^{379} \pmod{1000}$ as follows:

$$19^{379} = 19^{(101111011)_2}$$

$$19^{(1)_2} \equiv 19 \pmod{1000}$$

$$19^{(10)_2} \equiv 19^2 \equiv 361 \pmod{1000}$$

$$19^{(100)_2} \equiv 361^2 \equiv 321 \pmod{1000}$$

$$19^{(101)_2} \equiv 19 \times 321 \equiv 099 \pmod{1000}$$

$19^{(1010)_2} \equiv 99^2 \equiv 801 \pmod{1000}$

$19^{(1011)_2} \equiv 19 \times 801 \equiv 219 \pmod{1000}$

$19^{(10110)_2} \equiv 219^2 \equiv 961 \pmod{1000}$

$19^{(10111)_2} \equiv 19 \times 961 \equiv 259 \pmod{1000}$

$19^{(101110)_2} \equiv 259^2 \equiv 81 \pmod{1000}$

$19^{(101111)_2} \equiv 19 \times 81 \equiv 539 \pmod{1000}$

$19^{(1011110)_2} \equiv 539^2 \equiv 521 \pmod{1000}$

$19^{(10111100)_2} \equiv 521^2 \equiv 441 \pmod{1000}$

$19^{(10111101)_2} \equiv 19 \times 441 \equiv 379 \pmod{1000}$

$19^{(101111010)_2} \equiv 379^2 \equiv 641 \pmod{1000}$

$19^{(101111011)_2} \equiv 19 \times 641 \equiv 179 \pmod{1000}$ (6)

Therefore, the last 3 digits of x^{x^x} are 179. (1)

(0) for incorrect method (even if the answer is correct)

(5) for using inefficient algorithm.