Tutorial 12, Design and Analysis of Algorithms, 2019

1. Let m be a positive integer, and let $\alpha(m)$ be the probability that a number chosen at random from $\{1, ..., m\}$ is divisible by either 4, 5, or 6. Write down an exact formula for $\alpha(m)$, and also show that $\alpha(m) = 7/15 + O(1/m)$.

2. Suppose there is a biased coin which outputs a head with probability p where $0 < p < 1$. The following scheme is suggested to use the biased coin to get a random number generator that outputs $1, 2$ or 3.
   Suggested Scheme: Toss the coin exactly 3 times. Output i only if Heads was obtained only on the i'th toss. Otherwise output 4 and repeat the experiment i.e. toss the coin exactly 3 times again.

   (a) Is/are there any value(s) of p such that the above scheme outputs $1, 2$ or 3 with equal probabilities i.e. $P[1] = P[2] = P[3]$?

   (b) Consider a series of exactly 3 coin tosses as a trial. The trial is said to be successful if the outcome is $1, 2$ or 3 as per above scheme and the experiment can be stopped. If $p = \frac{1}{4}$, what is the average number of trials required for a successful outcome?

3. A fair coin is flipped n times. Let X_{ij}, with $1 \leq i < j \leq n$, be 1 if the ith and jth flip landed on the same side; let $X_{ij} = 0$ otherwise. Prove that X_{ij} are pairwise independent, but not $3-$wise independent.
   Random variables $X_1, X_2, ..., X_n$ are pairwise independent if $\forall i \neq j$ and $\forall a, b$

   $$Pr((X_i = a) \wedge (X_j = b)) = Pr(X_i = a)Pr(X_j = b)$$

   Random variables $X_1, X_2, ..., X_n$ are $3-$wise independent if $\forall i \neq j \neq k \neq i$ and $\forall a, b, c$

   $$Pr((X_i = a) \wedge (X_j = b) \wedge (X_k = c)) = Pr(X_i = a)Pr(X_j = b)Pr(X_k = c)$$

4. Consider the following algorithm $A()$:

   $A()$
   $j \leftarrow 3$
   do
       $j \leftarrow j + 1$
       $n \leftarrow$ RANDOM$(0, j - 1)$
   while $n > 1$
   output j

   Here RANDOM(m, n) returns a random integer between m and n (both inclusive) with uniform probability.

   (a) Find the probability of halting of $A()$.

   (b) Find the expected number of iterations of the while loop in $A()$.

5. An array $A[1..n]$ contains n distinct numbers that are randomly ordered, with each permutation of the n numbers being equally likely. What is the expectation of the index of the maximum element in the array? What is the expectation of the index of the minimum element in the array?

6. A permutation on the numbers $[1..n]$ can be represented as a function $\pi : [1..n] \rightarrow [1..n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [1..n] \rightarrow [1..n]$ is a value for which $\pi(x) = x$. Find the expected number of fixed points of a permutation chosen uniformly at random from all permutations.

7. Consider a very simple online auction system that works as follows. There are n bidding agents; agent i has a bid b_i, which is a positive natural number. We will assume that all bids b_i are distinct from one another. The bidding agents appear in an order chosen uniformly at random, each proposes its bid b_i in turn, and at all times the system maintains a variable b^* equal to the highest bid seen so far. (Initially b^* is set to 0.) What is the expected number of times that b^* is updated when this process is executed, as a function of the parameters in the problem? *Example.* Suppose $b_1 = 20, b_2 = 25$, and $b_3 = 10$, and the bidders arrive in the order $1, 3, 2$. Then b^* is updated for 1 and 2, but not for 3.