CS F364
Design & Analysis of Algorithms

ALGORITHM DESIGN TECHNIQUES - GREEDY

Matroids – A Theoretical Framework for Greedy Algorithms

- Introduction to Matroids:

Definition and Examples

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Matroids: Motivation

- **Matroids** provide a ***sound*** but ***incomplete*** theory of the ***greedy method*** i.e.
  - The theory of matroids can be used to derive **optimal solutions** for many problems / scenarios but
  - the theory **does not cover all cases** for which a greedy method applies.

MATROIDS: DEFINITIONS

- Definitions (Hereditary Property and Exchange Property)
  - Given a finite set S and a nonempty family I of subsets – referred to as ***independent subsets*** – of S:
    - I is said to be ***hereditary*** if it satisfies the following property:
      - if B is in I, and A is subset of B, then A is in I
    - I is said to satisfy the ***exchange*** property if the following is true:
      - if A is in I, B is in I, and |A| < |B|,
      - then there exists x in B\A such that (A U {x}) is in I

MATROIDS: DEFINITIONS [2]

- Definition:
  - A ***matroid*** is an ordered pair M = <S, I> such that
    1. S is a finite set
    2. I is a nonempty family of subsets of S, satisfying
       a. the hereditary property and
       b. the exchange property
- Example:
  - Let S be any finite set and
  - I_k be the set of all subsets of S of size at most k, where k<=|S|
    - Then $\langle S, I_k \rangle$ is a matroid.
- Example:
  - Linearly independent columns of a matrix.

GRAPHIC MATROID

- Given, an undirected graph G=(V,E), consider $M_G = <S_G,I_G>$ defined as:
  - $S_G = E$
  - If $A \subseteq S$, then A is in I_G iff A is acyclic
    - i.e. a set of edges A is independent iff the subgraph $G_A = (V, A)$ forms a forest (of trees).
- Theorem:
  - If G=(V,E) is an undirected graph, then $M_G=<S_G,I_G>$ is a matroid.
- Given G=(V,E), $\mathbf{M_G=<S_G,I_G>}$ is referred to as the ***Graphic Matroid.***

Graphic Matroid

Theorem (Graphic Matroid):

If $G=(V,E)$ is an undirected graph, then $M_G=\langle S_G, I_G\rangle$ is a matroid.

Proof:

- S_G is finite.
- I_G is hereditary: *the subset of a forest is a forest*
- I_G satisfies the exchange property:
  - Let $G_A = (V,A)$ and $G_B = (V, B)$ be forests of G and let $|B| > |A|$
  - Then G_A and G_B contain $|V|-|A|$ and $|V|-|B|$ trees resp.
  - Therefore there must be a tree T in G_B
    i. whose vertices are from two different trees in G_A and
    ii. there must be an edge (u,v) in T such that u and v are from different trees in G_A
    - Adding (u,v) to G_A preserves acyclicity:

      i.e. (G_A U {(u,v)}) is in I_G
    - i.e. I_G satisfies the exchange property