Polynomial Representations

1. **Coefficient Form**: A polynomial $f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ is represented as $(a_0, a_1, \ldots, a_{n-1})$.

Examples: $x+2$ is represented as $(1,2)$, $2x+3$ is represented as $(2,3)$, and $2x^2+7x+6$ is represented as $(6,7,2)$.

Complexity of adding two polynomials $(a_0, a_1, \ldots, a_{n-1})$ and $(b_0, b_1, \ldots, b_{n-1})$ is $\Theta(n)$ because

$$(a_0, a_1, \ldots, a_{n-1}) + (b_0, b_1, \ldots, b_{n-1}) = (a_0+b_0, a_1+b_1, \ldots, a_{n-1}+b_{n-1})$$

Example: $(1,2) + (2,3) = (3,5)$

Complexity of multiplying a polynomial $(a_0, a_1, \ldots, a_{n-1})$ with a constant c_0 is $\Theta(n)$ because

$$c_0(a_0, a_1, \ldots, a_{n-1}) = (c_0 a_0, c_0 a_1, \ldots, c_0 a_{n-1})$$

Example: $5(1,2) = (5,10)$

Complexity of multiplying two polynomials $(a_0, a_1, \ldots, a_{n-1})$ and $(b_0, b_1, \ldots, b_{n-1})$ is $\Theta(n^2)$ because

$$(a_0, a_1, \ldots, a_{n-1}) \times (b_0, b_1, \ldots, b_{n-1})$$

$= (a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, a_{n-1} b_{n-1})$ using the convolution algorithm (Iterative Multiply).

Example: $(1,2) \times (2,3) = (1\times 2, 1\times 3 + 2\times 2, 2\times 3)$

$= (2, 7, 6)$

(2) Point Value Form: A polynomial $f(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$ is represented as $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\}$ where all x_i's are different.

Examples: $x+2$ is represented as $\{(0,2), (1,3)\}$,

$2x+3$ is represented as $\{(2,7), (3,9)\}$, and

$2x^2+7x+6$ is represented as $\{(-1,1), (0,6), (1,15)\}$

complexity of adding two polynomials $\{(x_0, f_1(x_0)), (x_1, f_1(x_1)), \ldots, (x_{n-1}, f_1(x_{n-1}))\}$ and $\{(x_0, f_2(x_0)), (x_1, f_2(x_1)), \ldots, (x_{n-1}, f_2(x_{n-1}))\}$ is $\Theta(n)$ because

$$\{(x_0, f_1(x_0)), (x_1, f_1(x_1)), \ldots, (x_{n-1}, f_1(x_{n-1}))\} +$$
$$\{(x_0, f_2(x_0)), (x_1, f_2(x_1)), \ldots, (x_{n-1}, f_2(x_{n-1}))\}$$

$$= \{(x_0, f_1(x_0)+f_2(x_0)), (x_1, f_1(x_1)+f_2(x_1)), \ldots, (x_{n-1}, f_1(x_{n-1})+f_2(x_{n-1}))\}$$

complexity of multiplying a constant c_0 with a polynomial $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\}$ is $\Theta(n)$ because $c_0\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\}$

$$= \{(x_0, c_0 f(x_0)), (x_1, c_0 f(x_1)), \ldots, (x_{n-1}, c_0 f(x_{n-1}))\}$$

complexity of multiplying two polynomials $\{(x_0, f_1(x_0)), (x_1, f_1(x_1)), \ldots, (x_{2n-2}, f_1(x_{2n-2}))\}$ and $\{(x_0, f_2(x_0)), (x_1, f_2(x_1)), \ldots, (x_{2n-2}, f_2(x_{2n-2}))\}$ is $\Theta(n)$ because

$\{(x_0, f_1(x_0)), (x_1, f_1(x_1)), \ldots, (x_{2n-2}, f_1(x_{2n-2}))\} \times$
$\{(x_0, f_2(x_0)), (x_1, f_2(x_1)), \ldots, (x_{2n-2}, f_2(x_{2n-2}))\} =$

$\{(x_0, f_1(x_0) f_2(x_0)), (x_1, f_1(x_1) f_2(x_1)), \ldots,$
$(x_{n-1}, f_1(x_{2n-2}) f_2(x_{2n-2}))\}$

<u>Evaluation using Horner's Rule</u>: We can convert a polynomial $f(x)$ from coefficient form $(a_0, a_1, \ldots, a_{n-1})$ to point value form $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\}$ in $\Theta(n^2)$ time using <u>Horner's Rule</u> for polynomial evaluation:

$a_0 + a_1 x + \cdots + a_{n-2} x^{n-2} + a_{n-1} x^{n-1} =$
$a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$

<u>Example</u>: $6 + 7x + 2x^2 = 6 + x(7 + x(2))$

Horner's Rule takes $\Theta(n)$ time. Applying it n times will take $\Theta(n^2)$ time.

<u>Interpolation using Gaussian Elimination</u>: We can convert a polynomial $f(x)$ from point value form $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\}$ to coefficient form $(a_0, a_1, \ldots, a_{n-1})$ in $\Theta(n^3)$ time using the <u>Gaussian Elimination Algorithm</u>

Example: let's take $f(x) = \{(-1,1), (0,6), (1,15)\}$
$= (a_0, a_1, a_2) = a_0 + a_1 x + a_2 x^2$. We write the three linear equations as in three unknowns

$$a_0 - a_1 + a_2 = 1 \longrightarrow (1)$$

$$a_0 \qquad = 6 \longrightarrow (2)$$

$$a_0 + a_1 + a_2 = 15 \longrightarrow (3)$$

First we eliminate a_0. From equation (2), we put the value of $a_0 = 6$ in equations (1) and (3) to get two linear equations in two unknowns:

$$6 - a_1 + a_2 = 1 \iff a_1 - a_2 = 5 \longrightarrow (4)$$

$$6 + a_1 + a_2 = 15 \iff a_1 + a_2 = 9 \longrightarrow (5)$$

Next we eliminate a_1. From equation (4), we put the value of $a_1 = a_2 + 5$ in equation (5) to get one linear equation in one unknown:

$$a_2 + 5 + a_2 = 9 \iff 2a_2 = 4 \iff a_2 = 2 \longrightarrow (6)$$

$\Rightarrow a_1 = a_2 + 5 = 7$ and $a_0 = 6$. The coefficient form is $(6, 7, 2)$.

Divide and Conquer Graph

$$\begin{aligned} a_0 - a_1 + a_2 &= 1 \\ a_0 &= 6 \\ a_0 + a_1 + a_2 &= 15 \end{aligned}$$

Divide Step

$$\downarrow$$

$$\begin{aligned} a_1 - a_2 &= 5 \\ a_1 + a_2 &= 9 \end{aligned}$$

$$\downarrow$$

$$a_2 = 2$$

Conquer Step

$$\downarrow$$

$$a_1 = 7$$

$$\downarrow$$

$$a_0 = 6$$

Complexity of Gaussian Elimination Algorithm:

Let $T(n)$ denote the complexity of Gaussian Elimination Algorithm:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \longrightarrow (1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \longrightarrow (2)$$

$$\vdots \qquad\qquad \vdots$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \longrightarrow (i)$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \longrightarrow (n)$$

First we eliminate x_n. From equation (n), we put the value of $x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \cdots - \frac{a_{(n-1)2}}{a_{nn}}x_{n-1}$

in equations $(1), (2), \ldots (n-1)$ to get $n-1$ linear equations in $n-1$ unknowns:

$$a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1(n-1)}x_{n-1} = b'_1 \longrightarrow (1)'$$

$$a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2(n-1)}x_{n-1} = b'_2 \longrightarrow (2)'$$

$$\vdots \qquad\qquad \vdots$$

$$a'_{i1}x_1 + a'_{i2}x_2 + \cdots + a'_{i(n-1)}x_{n-1} = b'_i \longrightarrow (i)'$$

$$\vdots \qquad\qquad \vdots$$

$$a'_{(n-1)1}x_1 + a'_{(n-1)2}x_2 + \cdots + a'_{(n-1)(n-1)} = b_{n-1} \longrightarrow (n-1)'$$

Complexity of Divide Step involves putting the value of x_n in equations $(1), (2), \ldots, (n-1)$ and simplifying them to get the equations $(1)', (2)', \ldots (n-1)'$.

Total complexity of Divide Step $= (n-1) \times cn$

$= c\,n(n-1)$

complexity of conquer step involves computing the value of x_n, given the values of $x_1, x_2, \dots, x_{n-1}$

$= bn$

There is only one subproblem of size $n-1$.

$$\Rightarrow T(n) = T(n-1) + cn(n-1) + bn \quad \text{for } n > 1$$

$$T(1) = d$$

~~Solving the~~

$$T(n) \Rightarrow \begin{array}{c} cn^2+(b-c)n \\ \downarrow \\ T(n-1) \end{array} \Rightarrow \begin{array}{c} cn^2+(b-c)n \\ \downarrow \\ c(n-1)^2+(b-c)(n-1) \end{array}$$

$\Rightarrow \dots \Rightarrow$

$cn^2+(b-c)n \rightarrow$ level 0

$c(n-1)^2+(b-c)(n-1) \rightarrow$ level 1

$c(n-2)^2+(b-c)(n-2) \rightarrow$ level 2

$\vdots$

$c(n-i)^2+(b-c)(n-i) \rightarrow$ level i

$\vdots$

$d \longrightarrow$ level $n-1$

$$T(n) = \left(cn^2+(b-c)n\right) + \left(c(n-1)^2+(b-c)(n-1)\right) + \left(c(n-2)^2+(b-c)(n-2)\right)$$
$$+ \dots + \left(c(n-i)^2+(b-c)(n-i)\right) + \dots + \left(c \cdot 2^2 + (b-c)\cdot 2\right) + d$$
$$= \frac{cn(n+1)(2n+1)}{6} - c + \frac{(b-c)\,n(n+1)}{2} - (b-c) + d$$
$$= \Theta(n^3)$$

Discrete Fourier Transform (DFT):

Consider a polynomial $(a_0, a_1, \ldots, a_{n-1}) = f(x)$. Let $w_n = e^{\frac{i2\pi}{n}}$ be n^{th} root of unity. We consider the point value form of $f(x)$ at the points

$$\{(w_n^0, f(w_n^0)), (w_n^1, f(w_n^1)), \ldots, (w_n^{n-1}, f(w_n^{n-1}))\}$$

and define the DFT of the vector $(a_0, a_1, \ldots, a_{n-1})$ ~~as the vector~~ (corresponding to the coefficient form of $f(x)$) as the vector $(f(w_n^0), f(w_n^1), \ldots, f(w_n^{n-1}))$ (corresponding to the point value form).

Examples: $DFT(1,2) = (1+2, 1-2) = (3, -1)$

$DFT(2,3) = (2+3, 2-3) = (5, -1)$, and

$$DFT(6,7,2) = \left(6+7+2,\ 6+7\left(\frac{-1+\sqrt{3}i}{2}\right)+2\left(\frac{-1-\sqrt{3}i}{2}\right),\ 6+7\left(\frac{-1-\sqrt{3}i}{2}\right)+2\left(\frac{-1+\sqrt{3}i}{2}\right)\right)$$

$$= \left(15, \frac{3+5\sqrt{3}i}{2}, \frac{3-5\sqrt{3}i}{2}\right)$$

Complexity of computing DFT using Horner's Rule is $\Theta(n^2)$.

Inverse Discrete Fourier Transform (DFT^{-1}):

Consider a polynomial $(a_0, a_1, \cdots a_{n-1}) = f(x)$.
Let $w_n = e^{i\frac{2\pi}{n}}$ be n^{th} root of unity. We consider the point value form of $f(x)$ at the points
$\{(w_n^0, f(w_n^0)), (w_n^1, f(w_n^1)), \cdots, (w_n^{n-1}, f(w_n^{n-1}))\}$ and consider the vector $(y_0, y_1, \cdots, y_{n-1})$
$= (f(w_n^0), f(w_n^1), \cdots, f(w_n^{n-1}))$ (corresponding to the point value form). We define the $\underline{DFT^{-1}}$ of the vector $(y_0, y_1, \cdots, y_{n-1})$ as the vector $(a_0, a_1, \cdots a_{n-1})$ (corresponding to the coefficient form of $f(x)$).

<u>Examples</u>: $DFT^{-1}(3,-1) = (1,2)$,
$DFT^{-1}(5,-1) = (2,3)$, and
$DFT^{-1}\left(15, \frac{3+5\sqrt{3}i}{2}, \frac{3-5\sqrt{3}i}{2}\right) = (6,7,2)$

We write the DFT equation as:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_n & w_n^2 & \cdots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & \cdots & w_n^{2(n-1)} \\ 1 & w_n^3 & w_n^6 & \cdots & w_n^{3(n-1)} \\ \vdots & & \vdots & & \\ 1 & w_n^{n-1} & w_n^{2(n-1)} & \cdots & w_n^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = A \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

We multiply both sides by the Matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_n^{-1} & w_n^{-2} & \cdots & w_n^{-(n-1)} \\ 1 & w_n^{-2} & w_n^{-4} & \cdots & w_n^{-2(n-1)} \\ 1 & w_n^{-3} & w_n^{-6} & \cdots & w_n^{-3(n-1)} \\ 1 & w_n^{-(n-1)} & w_n^{-2(n-1)} & \cdots & w_n^{-(n-1)(n-1)} \end{bmatrix} = B$$

$$B\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1}\end{bmatrix} = BA\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1}\end{bmatrix} = C\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1}\end{bmatrix}$$

$(i,j)^{th}$ entry of BA is given by

$$C_{ij} = 1 \times 1 + w_n^{-i} \times w_n^{j} + w_n^{-2i} \times w_n^{2j} + \cdots + w_n^{-i(n-1)} \times w_n^{j(n-1)}$$

$$= 1 + w_n^{(j-i)} + w_n^{2(j-i)} + \cdots + w_n^{(n-1)(j-i)}$$

for $i=j$ we get:

$$C_{ii} = 1 + 1 + \cdots + 1 = n$$

for $i \neq j$ we get:

$$C_{ij} = \frac{w_n^{(j-i)n} - 1}{w_n^{(j-i)} - 1} = \frac{\left(w_n^{n}\right)^{(j-i)} - 1}{w_n^{(j-i)} - 1} = \frac{1-1}{w_n^{j-i} - 1} = 0$$

$$\Rightarrow C = nI = n\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & 1\end{bmatrix}$$

$$\Rightarrow B\begin{bmatrix} y_0 \\ \vdots \\ y_{n-1}\end{bmatrix} = nI\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1}\end{bmatrix} = n\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1}\end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1}\end{pmatrix} = \frac{1}{n} B \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1}\end{bmatrix} = DFT^{-1}(y_0, y_1, \cdots y_{n-1})$$

DFT and DFT^{-1} can also be computed in time $O(n^2)$ using Horner's Rule or Matrix Multiplication

Examples: For $n=2$, we have $w_2^{-1} = e^{-\frac{2\pi i}{2}} = e^{-i\pi} = -1$

$$DFT^{-1}(3,-1) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & w_2^{-1} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{pmatrix} \frac{2}{2} \\ \frac{4}{2} \end{pmatrix} = (1,2)$$

$$DFT^{-1}(5,-1) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & w_2^{-1} \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{pmatrix} \frac{4}{2} \\ \frac{6}{3} \end{pmatrix} = (2,3)$$

For $n=3$, we have $w_3^{-1} = e^{-\frac{2\pi i}{3}} = \frac{-1-\sqrt{3}i}{2}$, $w_3^{-2} = \frac{-1+\sqrt{3}i}{2}$

$$DFT^{-1}\left(15, \frac{3+5\sqrt{3}i}{2}, \frac{3-5\sqrt{3}i}{2}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & w_3^{-1} & w_3^{-2} \\ 1 & w_3^{-2} & w_3^{-4} \end{bmatrix}\begin{bmatrix} 15 \\ \frac{3+5\sqrt{3}i}{2} \\ \frac{3-5\sqrt{3}i}{2} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{-1-\sqrt{3}i}{2} & \frac{-1+\sqrt{3}i}{2} \\ 1 & \frac{-1+\sqrt{3}i}{2} & \frac{-1-\sqrt{3}i}{2} \end{bmatrix}\begin{bmatrix} 15 \\ \frac{3+5\sqrt{3}i}{2} \\ \frac{3-5\sqrt{3}i}{2} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 15 + \frac{3+5\sqrt{3}i}{2} + \frac{3-5\sqrt{3}i}{2} \\ 15 + \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{3+5\sqrt{3}i}{2}\right) + \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{3-5\sqrt{3}i}{2}\right) \\ 15 + \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{3+5\sqrt{3}i}{2}\right) + \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{3-5\sqrt{3}i}{2}\right) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 18 \\ 15 + \frac{-3+15}{2} \\ 15 + \frac{-3-15}{2} \end{bmatrix} = \begin{bmatrix} \frac{18}{3} \\ \frac{21}{3} \\ \frac{6}{3} \end{bmatrix} = \begin{pmatrix} 6 \\ 7 \\ 2 \end{pmatrix} = (6,7,2)$$

Fast Fourier Transform (FFT)

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$

We separate $f(x)$ into terms having odd and even powers of x as follows (assuming n to be a power of 2):

$$f(x) = (a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{n-2} x^{n-2}) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots + a_{n-1} x^{n-1})$$

$$= f_e(x^2) + x f_o(x^2)$$

where $f_e(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{(\frac{n}{2}-1)}$

and $f_o(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{(\frac{n}{2}-1)}$

We have $W_n^2 = e^{\frac{2 \cdot 2\pi i}{n}} = e^{2\pi i/(n/2)} = W_{n/2}$

and $W_n^{n/2} = e^{\frac{2\pi i}{n} \cdot \frac{n}{2}} = e^{i\pi} = -1$

$$f(W_n^i) = f_e((W_n^i)^2) + W_n^i f_o((W_n^i)^2)$$

$$= f_e((W_n^2)^i) + W_n^i f_o((W_n^2)^i)$$

$$= f_e(W_{n/2}^i) + W_n^i f_o(W_{n/2}^i)$$

As i varies from 0 to $n-1$, we observe that the values of $W_{n/2}^i$ repeats at a distance of $n/2$:

$$W_{n/2}^{i+n/2} = W_{n/2}^{n/2} \cdot W_{n/2}^i = W_{n/2}^i \longrightarrow (1)$$

and $W_n^{i+\frac{n}{2}} = W_n^{n/2} \cdot W_n^i = -W_n^i \longrightarrow (2)$

We have divided the problem of finding DFT of $f(x)$ of size n into two subproblems of finding DFT of $f_e(x)$ and $f_o(x)$ of size $\frac{n}{2}$. (Divide Step)

DFT($f(x)$) is a vector of size n, while DFT($f_e(x)$) and DFT($f_o(x)$) are vectors of size $n/2$. Given the vectors DFT($f_e(x)$) and DFT($f_o(x)$), we recover the vector DFT($f(x)$) as follows (Conquer Step): using equations ① and ②, we write (for $0 \le i \le \frac{n}{2}-1$)

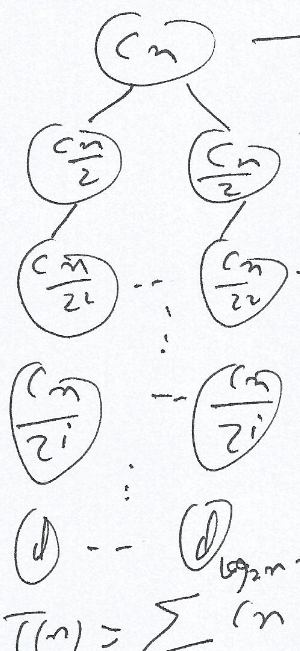
$$f(w_n^i) = f_e(w_{n/2}^i) + w_n^i f_o(w_{n/2}^i)$$

$$f(w_n^{i+\frac{n}{2}}) = f_e(w_{n/2}^i) - w_n^i f_o(w_{n/2}^i)$$

<u>Complexity of FFT</u>: Both Divide Step and Conquer Steps take time linear in input $= cn$. There are two subproblems of size $\frac{n}{2}$. We get the following recurrence:

$$T(1) = d$$

$$T(n) = 2T(n/2) + cn \quad \text{for } n > 1$$



$\rightarrow$ cn at level 0

$\rightarrow 2\left(\frac{cn}{2}\right) = cn$ at level 1

$\rightarrow 2^2\left(\frac{cn}{2^2}\right) = cn$ at level 2

$\vdots$

$\rightarrow 2^i\left(\frac{cn}{2^i}\right) = cn$ at level i

$\vdots$

$d \cdot 2^{\log_2 n} = dn$ at level $\log_2 n$

$$T(n) = \sum_{i=0}^{\log_2 n - 1} cn + dn = cn\log_2 n + dn = \boxed{\Theta(n \log_2 n)}$$

Recursive-FFT(a)

1 $n = a.length$ // n is a power of 2

2 if $n = 1$

3   return a

4 $w_n \leftarrow e^{2\pi i/n}$

5 $w \leftarrow 1$

6 $a^{(0)} \leftarrow (a_0, a_2, \ldots, a_{n-2})$

7 $a^{(1)} \leftarrow (a_1, a_3, \ldots, a_{n-1})$

8 $y^{(0)} \leftarrow$ Recursive-FFT$(a^{(0)})$

9 $y^{(1)} \leftarrow$ Recursive-FFT$(a^{(1)})$

10 for $k = 0$ to $\frac{n}{2} - 1$

11     $y_k \leftarrow y_k^{(0)} + w y_k^{(1)}$

12     $y_{k+(\frac{n}{2})} \leftarrow y_k^{(0)} - w y_k^{(1)}$

13     $w \leftarrow w w_n$

14 return y

For computing DFT^{-1} using FFT, we replace $w_n = e^{\frac{2\pi i}{n}}$ with $e^{-\frac{2\pi i}{n}}$ in line 4 in the above algorithm and return $\frac{y}{n}$ in line 14 (only in the original problem, we return $\frac{y}{n}$, in subproblems, we return y). Recursive-FFT$^{-1}(a)$ is same as Recursive-FFT(a) except for line 4: $w_n \leftarrow e^{-2\pi i n}$

$FFT^{-1}(a)$ will return

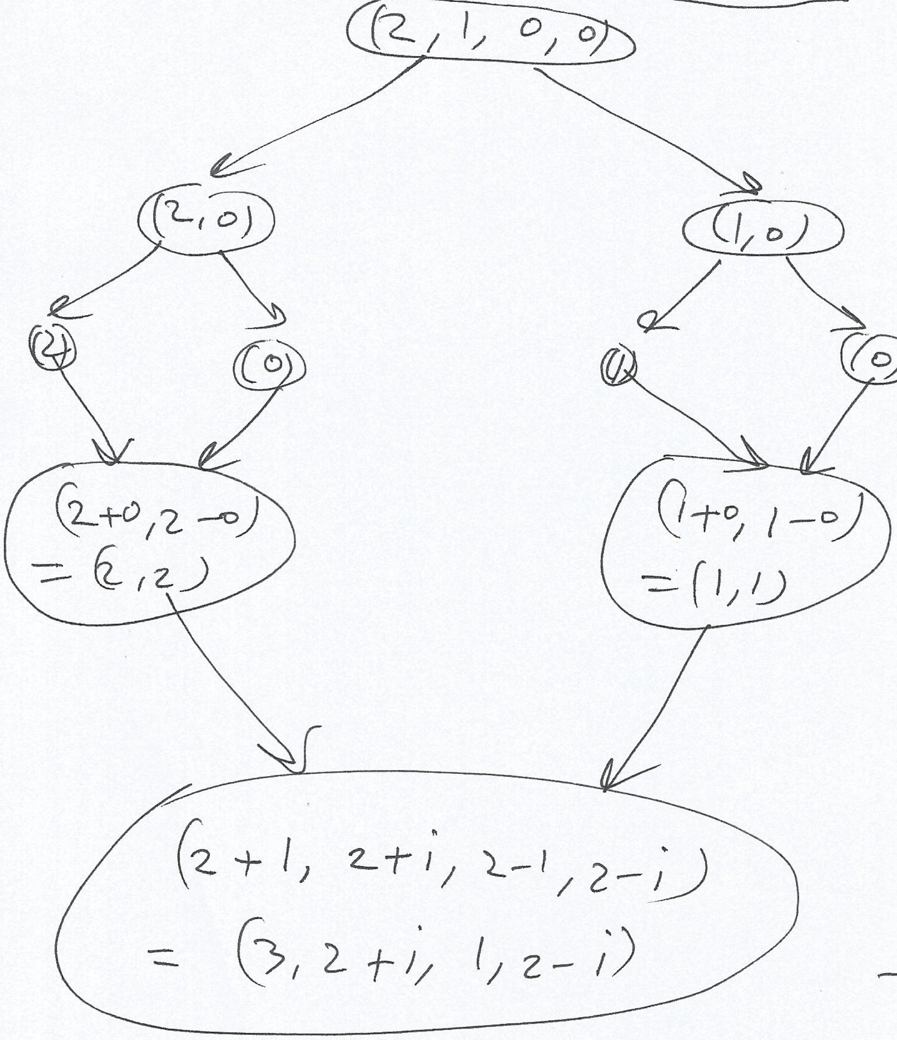
$\left(\frac{1}{n}\right)$ Recursive-FFT$^{-1}(a)$

Polynomial Multiplication using FFT

We will multiply $f_1(x) = x+2$, and $f_2(x) = 2x+3$ using the FFT algorithm. $f_1(x)\, f_2(x)$ will have 3 terms. We pad all polynomials with 0 to the nearest power of 2 terms. (4 terms in this example). We write $f_1(x)$ and $f_2(x)$ in coefficient form as follows:

$f_1(x) = (2, 1, 0, 0)$ and $f_2(x) = (3, 2, 0, 0)$

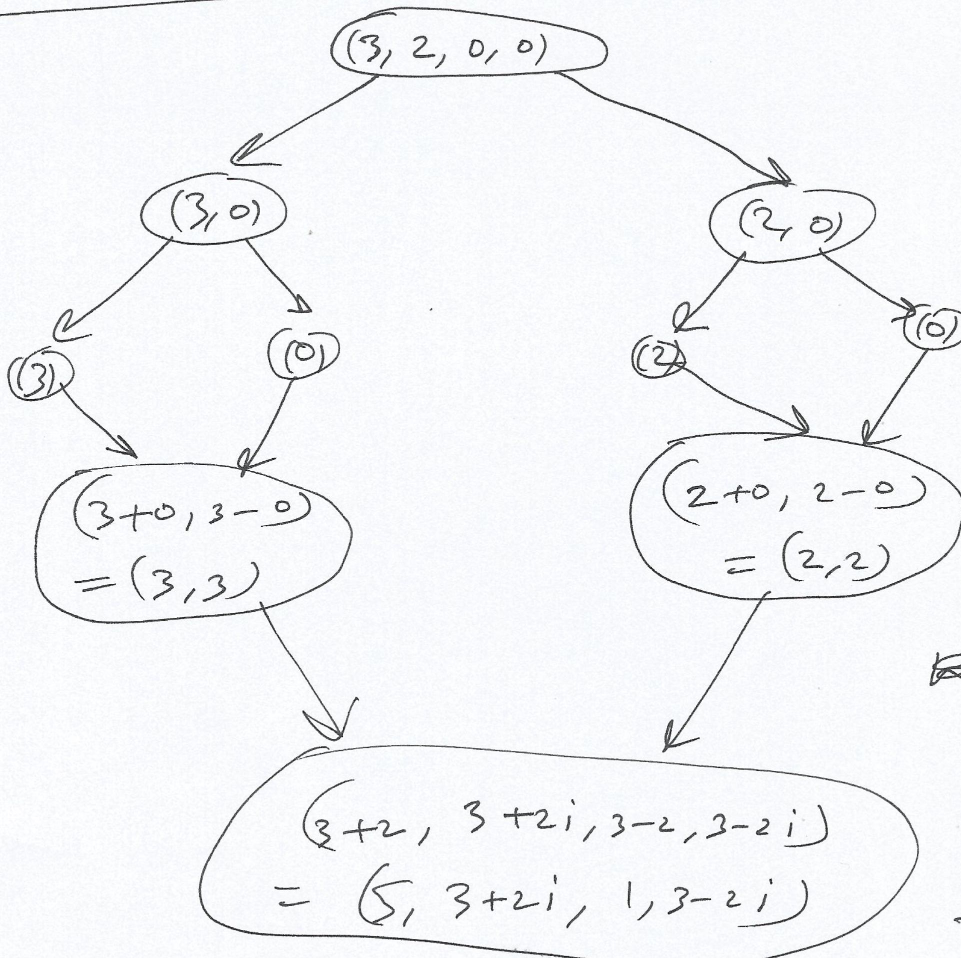
Divide and Conquer Graph for $f_1(x)$



$w_2 = -1$

$w_4 = i$

Divide and Conquer Graph for $f_2(x)$



$W_2 = -1$

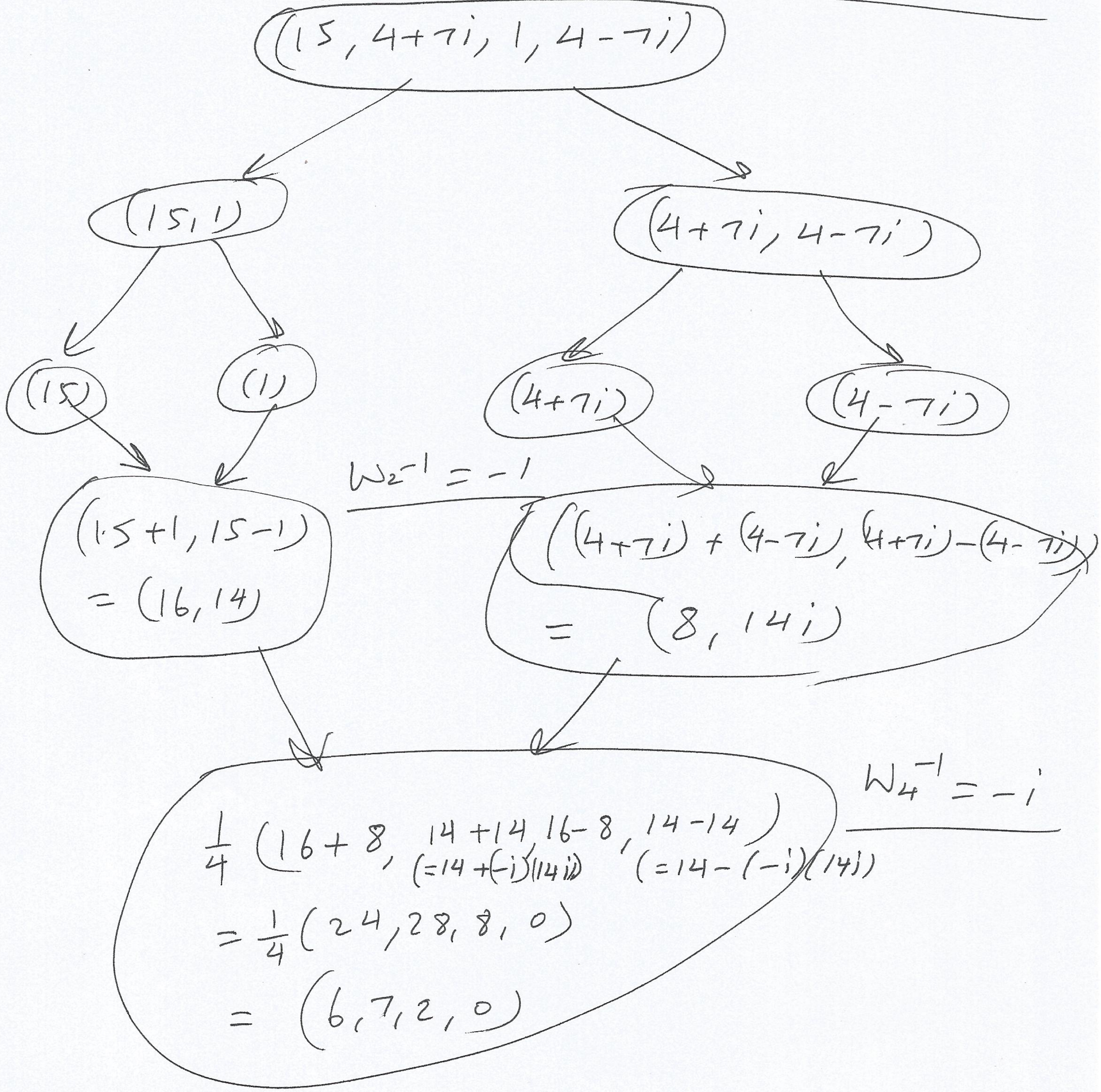
$W_4 = i$

$$DFT(f_1(x)) \cdot DFT(f_2(x)) = (3 \times 5,\ (2+i) \times (3+2i),\ 1 \times 1,\ (2-i) \times (3-2i))$$

$$= (15,\ 4+7i,\ 1,\ 4-7i)$$

Now we compute $DFT^{-1}(15, 4+7i, 1, 4-7i)$.

Divide and Conquer Graph for $DFT^{-1}(f_1(x)\, f_2(x))$



We get $f_1(x)\, f_2(x) = (6, 7, 2, 0) = 6 + 7x + 2x^2$

$= (x+2)(2x+3)$