Iterative Matrix Multiplication

Multiplication of two $n \times n$ matrices $A_{n\times n} = [a_{ij}]$, $B_{n\times n} = [b_{ij}]$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \\ \boxed{a_{i1} & a_{i2} & \cdots & a_{ij} \cdots & a_{in}} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & & \\ & & & b_{ij} & \cdots & b_{in} \\ b_{i1} & b_{i2} & \cdots & & & \\ \vdots & & & & & \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix}$$

is $C_{n\times n} = [C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & & C_{1n} \\ C_{21} & C_{22} & \cdots & & C_{2n} \\ \vdots & & & & \\ C_{i1} & C_{i2} & \cdots & \boxed{C_{ij}} & \cdots C_{in} \\ \vdots & & & & \\ C_{n1} & C_{n2} & \cdots & & C_{nn} \end{bmatrix}$

Such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Example: $\begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 8 \\ 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (1\times 6 + 3\times 4) & (1\times 8 + 3\times 2) \\ (7\times 6 + 5\times 4) & (7\times 8 + 5\times 2) \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}$$

Iterative-~~Multiply(A, B)~~ Matrix (A, B)

```
1  n ← A.rows
2  let C be a new n×n matrix
3  for i = 1 to n
4      for j = 1 to n
5          c_ij ← 0
6          for k = 1 to n
7              c_ij = c_ij + a_ik . b_kj
8  return C
```

<u>Time Complexity of Iterative-Multiply</u>

Lines 1, 2, and 3 contribute: 3.

Total contribution of line 5 is: n^2

Total contribution of line 7 is: n^3

$$T(n) = n^3 + n^2 + 3 = \Theta(n^3)$$

<u>A Divide and Conquer Matrix Multiplication Algorithm</u>

Here we partition the A, B, and C matrices into four $\frac{n}{2} \times \frac{n}{2}$ matrices as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

and write the equation $C = A \times B$ as:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Now we get 8 subproblems of size $\frac{n}{2} \times \frac{n}{2}$:

$$C_{11} = \boxed{A_{11} \times B_{11}} + \boxed{A_{12} \times B_{21}}$$

$$C_{12} = \boxed{A_{11} \times B_{12}} + \boxed{A_{12} \times B_{22}}$$

$$C_{21} = \boxed{A_{21} \times B_{11}} \cancel{A} + \boxed{A_{22} \times B_{21}}$$

$$C_{22} = \boxed{A_{21} \times B_{12}} + \boxed{A_{22} \times B_{22}}$$

Example:

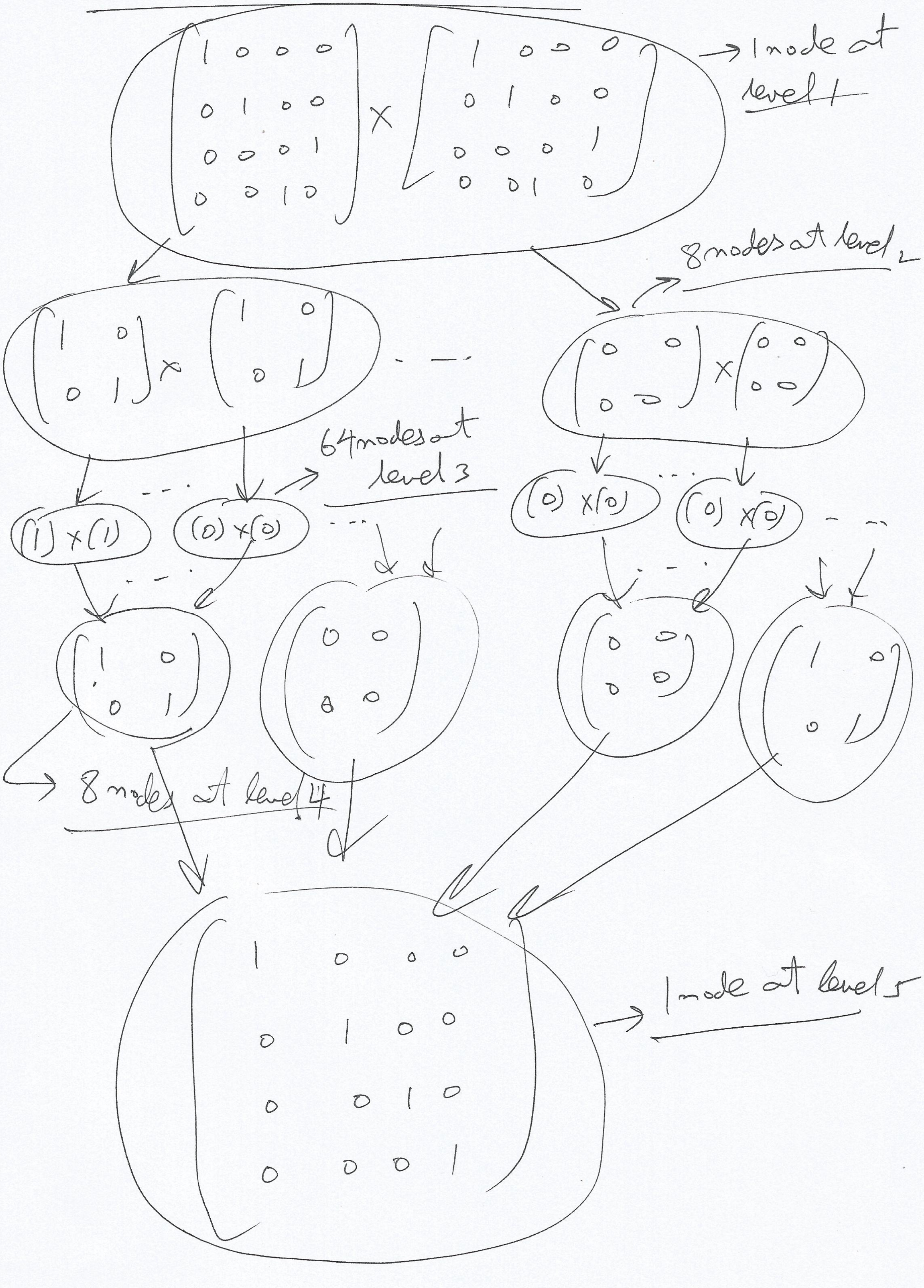
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \times \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) & \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \end{bmatrix}$$

$$= \begin{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Divide and Conquer Graph



DAC-Matrix (A, B)

$n \leftarrow A.rows$

Let C be a new $n \times n$ matrix

if $n = 1$

$c_{11} = a_{11}\, b_{11}$ } Base Case

else Partition A as $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

Partition B as $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ Divide Step

Partition C as $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

$C_{11} \Leftarrow$ DAC-Matrix (A_{11}, B_{11}) + DAC-Matrix (A_{12}, B_{21})

$C_{12} \Leftarrow$ DAC-Matrix (A_{11}, B_{12}) + DAC-Matrix (A_{12}, B_{22})

$C_{21} \Leftarrow$ DAC-Matrix (A_{21}, B_{11}) + DAC-Matrix (A_{22}, B_{21})

$C_{22} \Leftarrow$ DAC-Matrix (A_{21}, B_{12}) + DAC-Matrix (A_{22}, B_{22})

Conquer Step

Recursive Steps

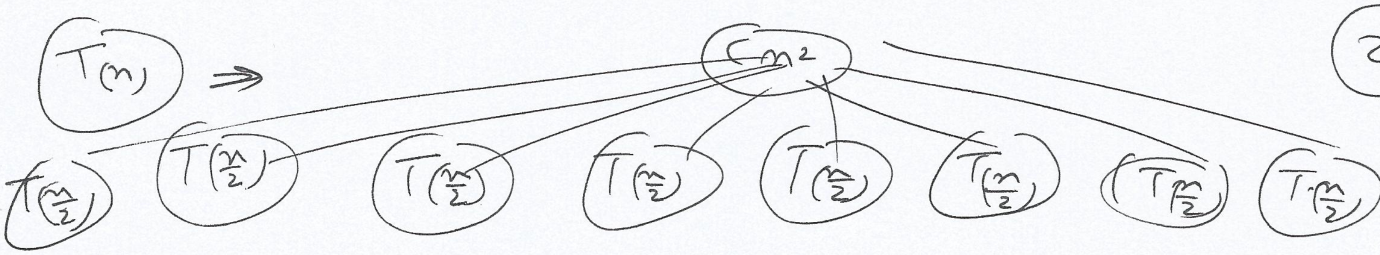
Return C

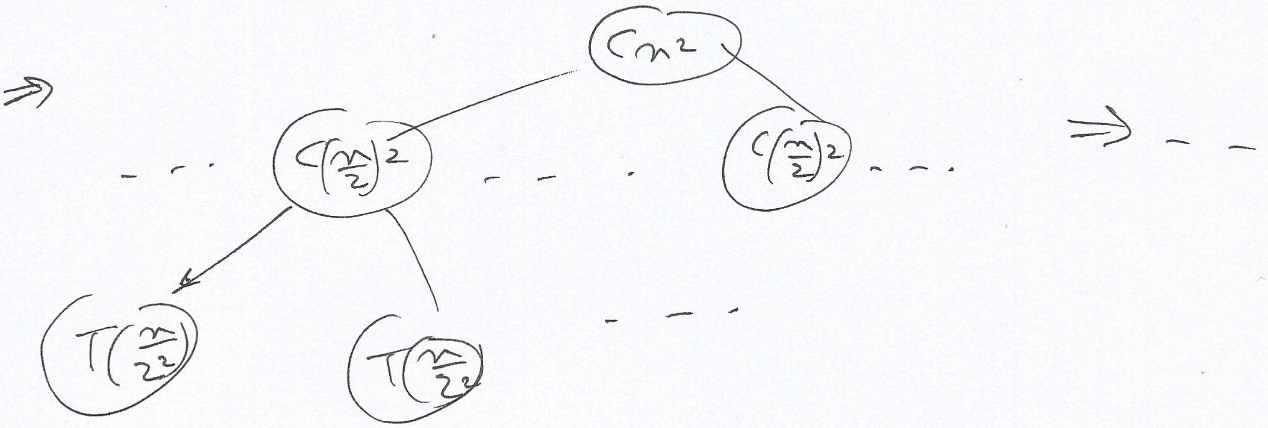
Time Complexity of DAC-Matrix

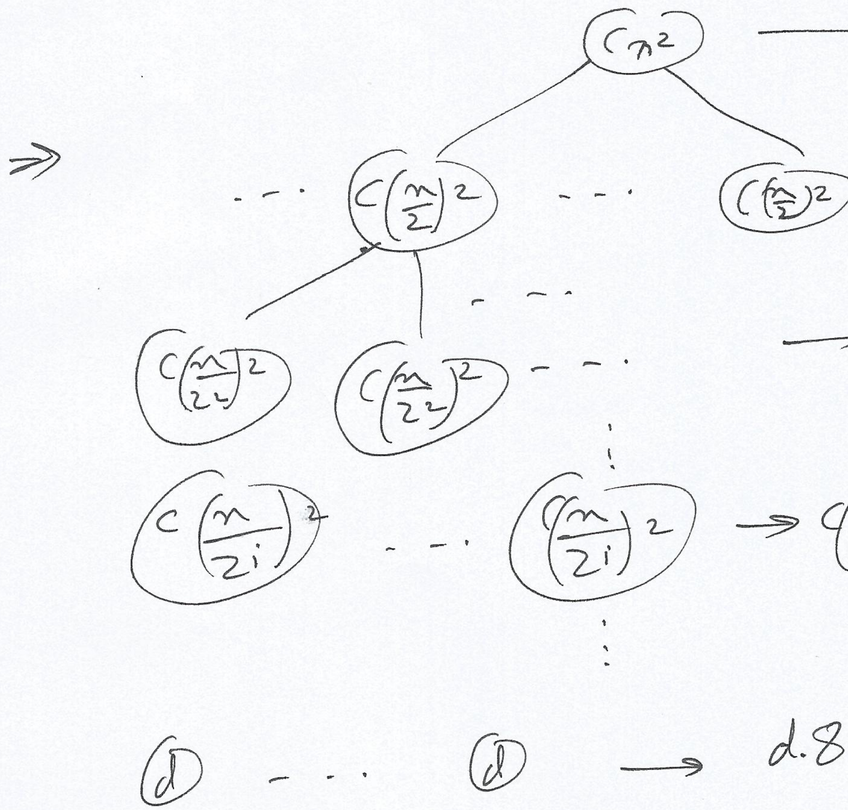
Divide and Conquer Steps take time linear in input size $= cn^2$ (input size is $n \times n = n^2$). There are 8 subproblems of size $\frac{n}{2} \times \frac{n}{2}$. $\Rightarrow$

$$T(1) = d$$

$$T(n) = 8T\left(\frac{n}{2}\right) + cn^2 \text{ for } n > 1$$







cn^2 at level 0

$c\left(\frac{n}{2}\right)^2 8 = 2cn^2$ at level 1

$c\left(\frac{n}{2^2}\right)^2 8^2 = 2^2 cn^2$ at level 2

$c\left(\frac{n}{2^i}\right)^2 8^i = 2^i cn^2$ at level i

$d\cdot 8^{\log_2 n} = d\cdot (2^3)^{\log_2 n} = d\left(2^{\log_2 n}\right)^3$

$= dn^3$ at level $\log_2 n$

$$T(n) = cn^2 + 2cn^2 + 2^2cn^2 + \cdots + 2^{\log_2 n - 1}cn^2 + dn^3$$

$$= cn^2\left(1 + 2 + 2^2 + \cdots + 2^{\log_2 n - 1}\right) + dn^3$$

$$= cn^2\left(2^{\log_2 n} - 1\right) + dn^3 = (c+d)n^3 - cn^2 = \Theta(n^3)$$

Strassen's Divide and Conquer Matrix Multiplication Algorithm

In this algorithm, we make 7 subproblems of size $\frac{n}{2} \times \frac{n}{2}$ as follows:

Let
$$S_1 = B_{12} - B_{22}$$
$$S_2 = A_{11} + A_{12}$$
$$S_3 = A_{21} + A_{22}$$
$$S_4 = B_{21} - B_{11}$$
$$S_5 = A_{11} + A_{22}$$
$$S_6 = B_{11} + B_{22}$$
$$S_7 = A_{12} - A_{22}$$
$$S_8 = B_{21} + B_{22}$$
$$S_9 = A_{11} - A_{21}$$
$$S_{10} = B_{11} + B_{12}$$

The 7 subproblems are:

$$P_1 = A_{11} \times S_1$$
$$P_2 = S_2 \times B_{22}$$
$$P_3 = S_3 \times B_{11}$$
$$P_4 = A_{22} \times S_4$$
$$P_5 = S_5 \times S_6$$
$$P_6 = S_7 \times S_8$$
$$P_7 = S_9 \times S_{10}$$

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We get the solution as:

$C_{11} = P_5 + P_4 - P_2 + P_6$

$C_{12} = P_1 + P_2$

$C_{21} = P_3 + P_4$

$C_{22} = P_5 + P_1 - P_3 - P_7$

<u>Example</u>: For $\begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 8 \\ 4 & 2 \end{bmatrix}$, we have:

$S_1 = 8 - 2 = 6$

$S_2 = 1 + 3 = 4$

$S_3 = 7 + 5 = 12$

$S_4 = 4 - 6 = -2$

$S_5 = 1 + 5 = 6$

$S_6 = 6 + 2 = 8$

$S_7 = 3 - 5 = -2$

$S_8 = 4 + 2 = 6$

$S_9 = 1 - 7 = -6$

$S_{10} = 6 + 8 = 14$

$P_1 = 1 \times 6 = 6$

$P_2 = 4 \times 2 = 8$

$P_3 = 12 \times 6 = 72$

$P_4 = 5 \times (-2) = -10$

$P_5 = 6 \times 8 = 48$

$P_6 = (-2) \times 6 = -12$

$P_7 = (-6) \times 14 = -84$

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$C_{11} = 48 + (-10) - 8 + (-12) = \boxed{18}$

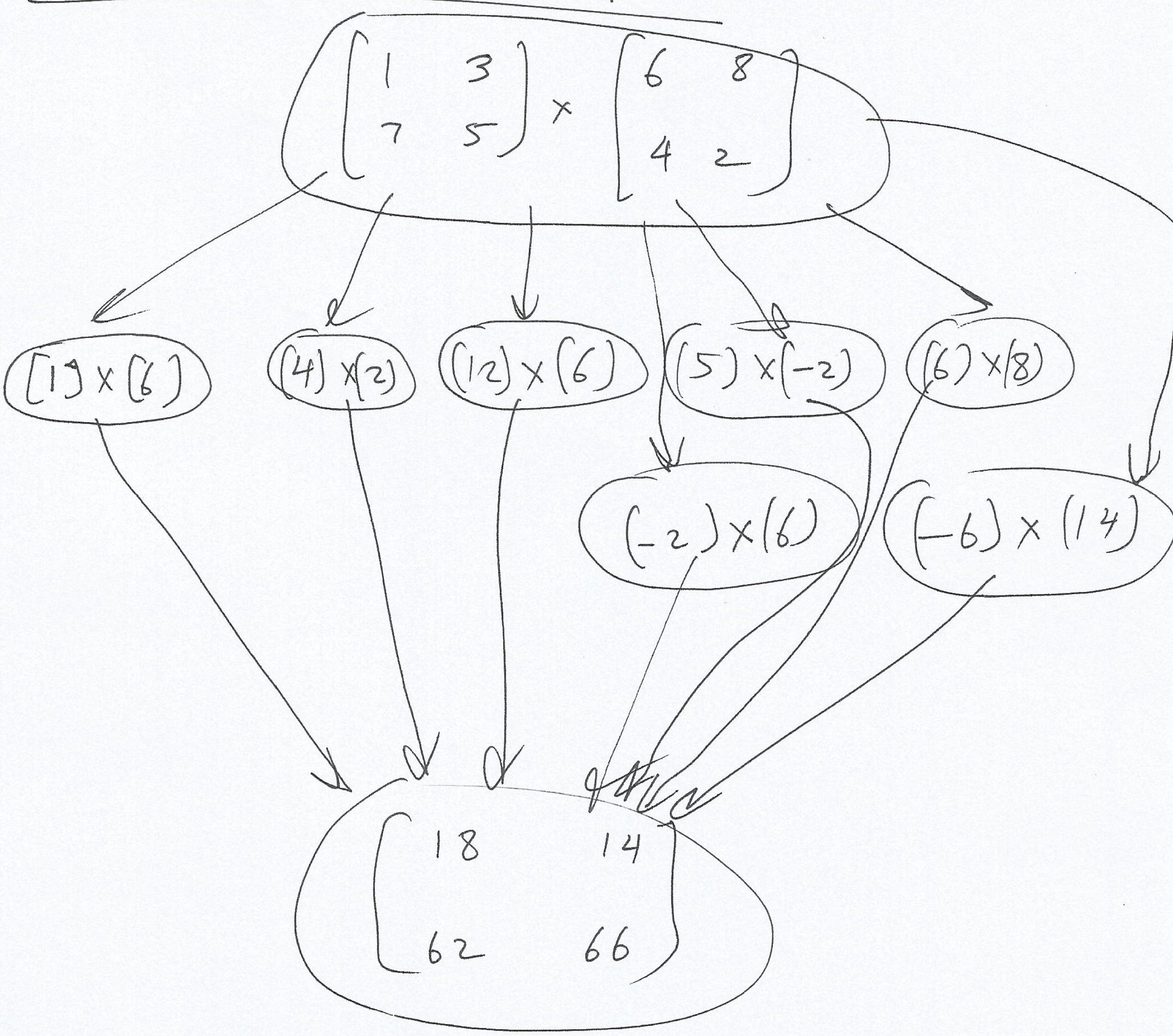
$C_{12} = 6 + 8 = \boxed{14}$

$C_{21} = 72 + (-10) = \boxed{62}$

$C_{22} = 48 + 6 - 72 - (-84) = \boxed{66}$

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \times \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 18 & 14 \\ 62 & 66 \end{pmatrix}$$

Divide and conquer Graph



Strassen-Matrix(A, B)

$n \leftarrow A.rows$

Let C be a new $n \times n$ matrix

$$\left.\begin{array}{l} \text{if } n = 1 \\ \qquad c_{11} = a_{11} b_{11} \end{array}\right\} \text{Base Case}$$

$$\left.\begin{array}{l}
\text{else Partition } A \text{ as } \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\
\qquad \text{Partition } B \text{ as } \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\
\qquad \text{Partition } C \text{ as } \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \\
\qquad S_1 \leftarrow B_{12} - B_{22} \\
\qquad S_2 \leftarrow A_{11} + A_{12} \\
\qquad S_3 \leftarrow A_{21} + A_{22} \\
\qquad S_4 \leftarrow B_{21} - B_{11} \\
\qquad S_5 \leftarrow A_{11} + A_{22} \\
\qquad S_6 \leftarrow B_{11} + B_{22} \\
\qquad S_7 \leftarrow A_{12} - A_{22} \\
\qquad S_8 \leftarrow B_{21} + B_{22} \\
\qquad S_9 \leftarrow A_{11} - A_{21} \\
\qquad S_{10} \leftarrow B_{11} + B_{12}
\end{array}\right\} \text{Divide steps}$$

$P_1 \leftarrow$ Strassen-Matrix(A_{11}, S_1)
$P_2 \leftarrow$ Strassen-Matrix(S_2, B_{22})
$P_3 \leftarrow$ Strassen-Matrix(S_3, B_{11})
$P_4 \leftarrow$ Strassen-Matrix(A_{22}, S_4)
$P_5 \leftarrow$ Strassen-Matrix(S_5, S_6)
$P_6 \leftarrow$ Strassen-Matrix(S_7, S_8)
$P_7 \leftarrow$ Strassen-Matrix(S_9, S_{10})

Recursion steps

$C_{11} \leftarrow P_5 + P_4 - P_2 + P_6$
$C_{12} \leftarrow P_1 + P_2$
$C_{21} \leftarrow P_3 + P_4$
$C_{22} \leftarrow P_5 + P_1 - P_3 - P_7$

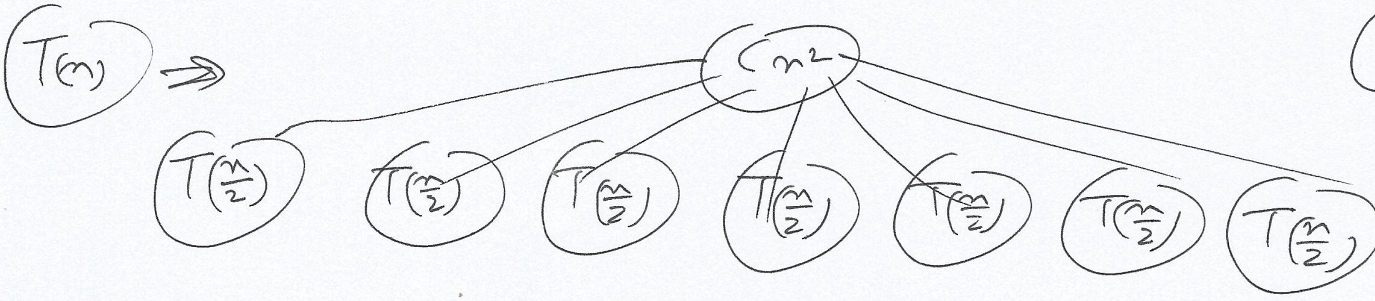
Conquer steps

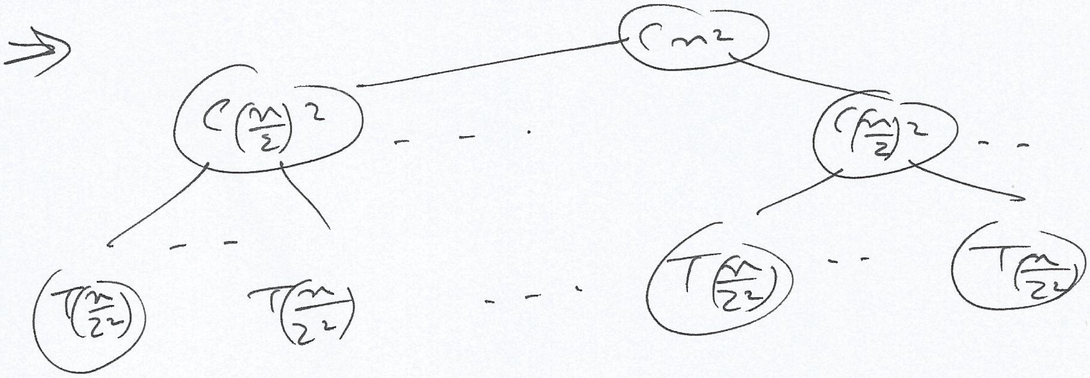
Return C

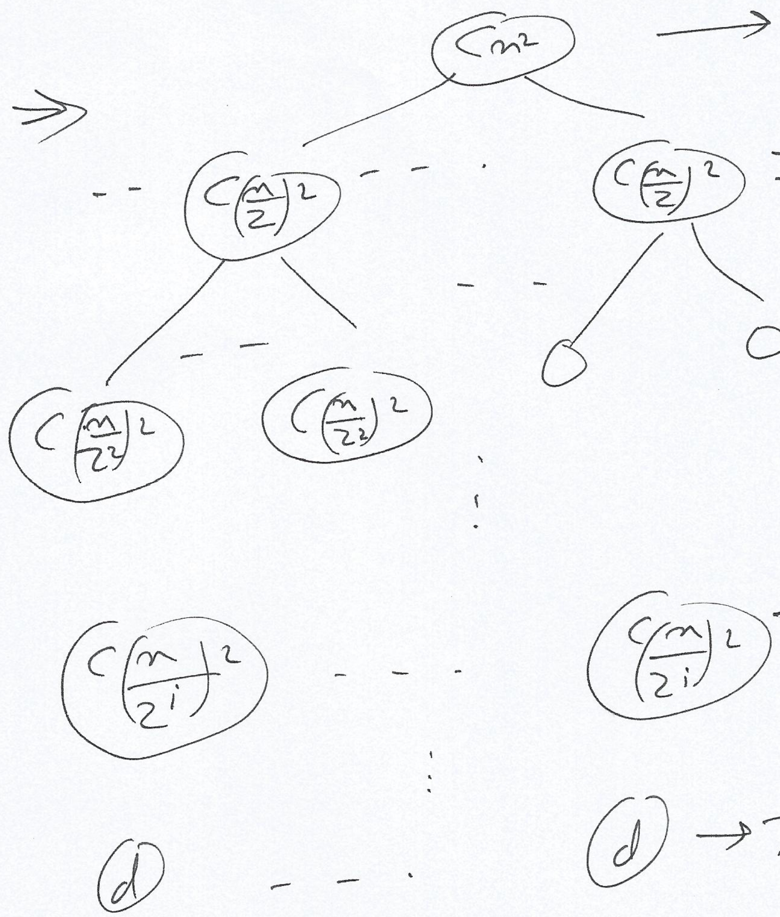
Time Complexity of Strassen-Matrix

Divide and Conquer steps take time linear in input (input size is $n \times n = n^2$): cn^2. There are 7 subproblems of size $\frac{n}{2} \times \frac{n}{2}$ $\Rightarrow$ $T(1) = d$

$$T(n) = 7\,T\left(\tfrac{n}{2}\right) + cn^2 \quad \text{for } n > 1$$







$\longrightarrow cn^2$ at level 0

$\longrightarrow 7c\left(\frac{n}{2}\right)^2 = \left(\frac{7}{4}\right)cn^2$ at level 1

$\longrightarrow 7^2 c\left(\frac{n}{2^2}\right)^2 = \left(\frac{7}{4}\right)^2 cn^2$ at level 2

$\vdots$

$\longrightarrow 7^i c\left(\frac{n}{2^i}\right)^2 = \left(\frac{7}{4}\right)^i cn^2$ at level i

$$d \to 7^{\log_2 n}\cdot d = \left(2^{\log_2 7}\right)^{\log_2 n}\cdot d = \left(2^{\log_2 n}\right)^{\log_2 7}\cdot d$$

$= dn^{\log_2 7}$ at level $2^{\log_2 n}$

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$$T(n) = cn^2 + \left(\frac{7}{4}\right)cn^2 + \left(\frac{7}{4}\right)^2 cn^2 + \cdots + \left(\frac{7}{4}\right)^{\log_2 n - 1} cn^2 + \quad d \cdot n^{\log_2 7}$$

$$= cn^2\left(1 + \left(\frac{7}{4}\right) + \left(\frac{7}{4}\right)^2 + \cdots + \left(\frac{7}{4}\right)^{\log_2 n - 1}\right) + d \cdot n^{\log_2 7}$$

$$= cn^2\left(\frac{\left(\frac{7}{4}\right)^{\log_2 n} - 1}{\frac{7}{4} - 1}\right) + d \cdot n^{\log_2 7}$$

$$= \left(\frac{7^{\log_2 n}}{4^{\log_2 n}} - 1\right)\left(\frac{4}{3} cn^2\right) + d \cdot n^{\log_2 7}$$

$$= \left(\frac{n^{\log_2 7}}{n^2} - 1\right)\left(\frac{4}{3} cn^2\right) + d \cdot n^{\log_2 7}$$

$$= \left(\frac{4c}{3} + d\right) n^{\log_2 7} - \left(\frac{4c}{3}\right) n^2 = \Theta\left(n^{\log_2 7}\right)$$