CS F364: Design & Analysis of Algorithm

03

R Quick Sort
Defective Chessboard

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http://ktiwari.in/algo

Quick Sort

Which algorithm is better ?

	Best Case	Worst Case	Average Case
Algo-01	$n \log n$	$n \log n$	$n \log n$
Algo-02	$n \log n$	$n(n-1)$	$n \log n$

- If I tell you Algo-01 is merge sort and Algo-02 is quick sort then?
- Quick sort is popular because it always behaves like average case as the input size increases

Table: 1000 execution of randomized quick sort on random list

Number of times runtime exceed the average behavior	Input size (# of items) 10^2	10^3	10^4	10^5	10^6
10%	190	49	22	10	3
20%	28	17	12	3	0
50%	2	1	1	0	0
100%	0	0	0	0	0

Randomized Quick Sort

```
QuickSort(A,p,r)
  if p<r
    q = Partition(A,p,r)
    QuickSort(A,p,q-1)
    QuickSort(A,q+1,r)
```

```
Partition(A,p,r)
  x = A[r]
  i = p-1
  for j = p to r-1
    if A[j] <= x
        i = i + 1
        swap A[i] <-> A[j]
  swap A[i+1] <-> A[r]
  return i+1
```

```
rQuickSort(A,p,r)
   if(p<r)
      q = rPartition(A,p,r)
      rQuickSort(A,p,q-1)
      rQuickSort(A,q+1,r)
```

```
rPartition(A,p,r)
  i = random(p,r)
  swap A[r] <-> A[i]
  return Partition(A,p,r)
```


Analysis of Randomized Quick Sort

Estimate number of comparisons performed during execution

- Let sorted list $< S_1, S_2, S_3, ..., S_n >$ with S_i as i^{th} smallest element
- Define random variable X_{ij} as number of comparisons between S_i and S_j. X_{ij} could take a value 0 or 1
- Expected number of comparison is

$$E[\sum_{i=1}^{n}\sum_{j>i} X_{ij}] = \sum_{i=1}^{n}\sum_{j>i} E[X_{ij}]$$

- If p_{ij} be the probability of comparison between S_i and S_j. Then,

$$E[X_{ij}] = p_{ij} \times 1 + (1 - p_{ij}) \times 0 = p_{ij}$$

Randomized Quick Sort

21	15	14	18	31	6	17	11	15	8	19	16
0	1	2	3	4	5	6	7	8	9	10	11

6	8	11	14	15	15	16	17	18	19	21	31
0	1	2	3	4	5	6	7	8	9	10	11
		↑ i					↑ j				

- Pivot element can either be
  1. S_i or S_j: its probability is $\frac{2}{j-i+1}$
  2. Inside S_q from $i < q < j$, comparison not possible
  3. Outside S_r from $r < i$ or $j < r$, no effect on comparison

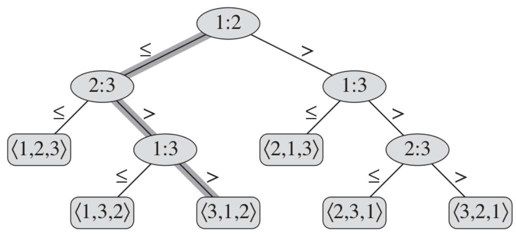
$$\sum_{i=1}^{n}\sum_{j>i} E[X_{ij}] = \sum_{i=1}^{n}\sum_{j>i} p_{ij} = \sum_{i=1}^{n}\sum_{j>i} \frac{2}{j-i+1} = 2\sum_{i=1}^{n}\sum_{k=1}^{n-i+1} \frac{1}{k}$$

$$\leq 2\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{1}{k} = 2nH_n = O(n \ln n)$$

as $H_n \sim \ln n + \Theta(1)$

Decision Tree Model of Sorting

Sort three items a_1, a_2, a_3



Is it always to be a binary tree?
What is worst case time taken by this algorithm? $O(height)$
How many leaves would be there with 4 items?

Lower Bound of Sorting

Any comparison sort needs $\Omega(n \log n)$ comparisons in the worst case.

- There are $n!$ permutations of n items. Each should be at leaf
- Binary tree of height h has at most 2^h leaves

$$n! \le 2^h$$
$$h \ge \log(n!)$$

- Stirling's approximation of $n!$ is $(n/e)^n$

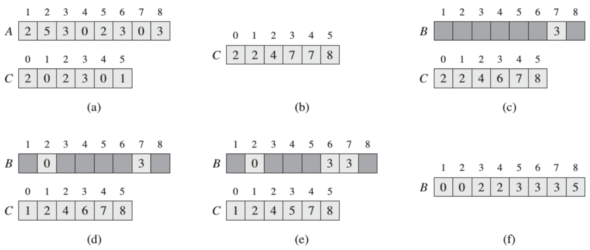
$$h \ge n \log(n) - n \log(e)$$
$$h = \Omega(n \log(n))$$

Wish to sort in linear time $O(n)$? use Counting Sort

```
CountingSort(A,B,k)
  Let C[0..k] be new array of zeros

  for j=1 to A.length
    C[A[j]] = C[A[j]] + 1

  for i = 1 to k
    C[i] = C[i] + C[i-1]

  for j = A.length down to 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] -1
```

Apply to Sort: 2, 5, 3, 0, 2, 3, 0, 3

Counting Sort in action



Radix Sort

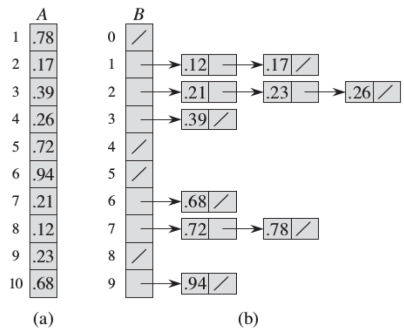
Use a stable sorting algorithm to sort array A on digit (1 to d)

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

Bucket Sort

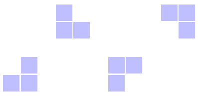
```
bucketSort(A)

  n = A.length
  Let B[0..n-1] be new array

  for i = 0 to n-1
    make B[i] as empty list

  for i = 1 to n
    inset A[i] into list B[[nA[i]]]

  for i = 0 to n-1
    Sort list B[i] with insertion sort

  concatinate B[0], B[1], ..., B[n-1] in order
```


Bucket Sort



Defective Chessboard

- Consider a chessboard of size $2^k \times 2^k$ where one cell is defective. Your task to cover it using a triomino.



Obviously

- Triomino cannot cover the defected one
- Triomino should not overlap
- Triomino must cover all other squares

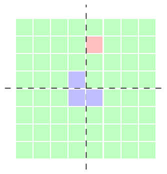
Note: $4^k - 1$ is divisible by 3

Defective Chessboard

Divide and conquer

1. **Divide:** in smaller size instances.
2. **Recursive step:** use same framework till it is trivially solvable
3. **Conquer:** combine solutions of smaller instances to get overall solution

- $2^k \times 2^k$ size board is divided in four $2^{k-1} \times 2^{k-1}$ size board
  - Let $T(n)$ be time to tile $2^k \times 2^k$ board



$$\begin{aligned} T(n) &= t_d + 4T(n/2) + t_c = 4T(n/2) + c \\ &= c + 4c + \ldots + 4^{n-2}c + 4^{n-1}d \\ &= c\frac{4^{n-1} - 1}{4 - 1} + 4^{n-1}d \\ &= \left(\frac{c}{12} + \frac{d}{4}\right) \times 4^n - \frac{c}{3} = \Theta(4^n) \end{aligned}$$

Thank You!

Thank you very much for your attention! (Refere

Queries ?

[1] Book - *Introduction to Algorithm*, By THOMAS H. CORMEN, CHARLES E. LEISERSON, RONAL
CLIFFORD STEIN

Name	Time Complexity (Best)	Time Complexity (Average)	Time Complexity (Worst)	Space Complexity	Stability
Bubble Sort	Ω(n)	Θ(n²)	O(n²)	O(1)	Stable
Selection Sort	Ω(n²)	Θ(n²)	O(n²)	O(1)	Unstable
Insertion Sort	Ω(n)	Θ(n²)	O(n²)	O(1)	Stable
Merge Sort	Ω(n log(n))	Θ(n log(n))	O(n log(n))	O(n)	Stable
Quick Sort	Ω(n log(n))	Θ(n log(n))	O(n²)	O(log(n))	Unstable
Heap Sort	Ω(n log(n))	Θ(n log(n))	O(n log(n))	O(1)	Unstable
Counting Sort	Ω(n+k)	Θ(n+k)	O(n+k)	O(k)	Stable
Radix Sort	Ω(nk)	Θ(nk)	O(nk)	O(n+k)	Stable