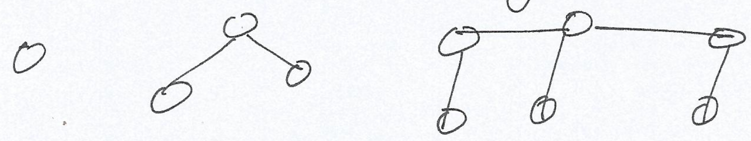
The Graphic Matroid

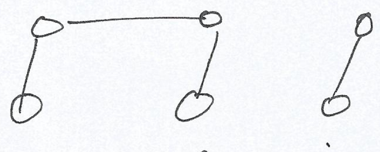
Tree : A tree is a connected, acyclic, undirected graph.

Examples :



Forest : If an undirected graph is acyclic but possibly disconnected, it is a forest.

Example :



Let E be the set of edges in a graph, and V be the set of vertices in a graph. For a tree (V, E) we have:

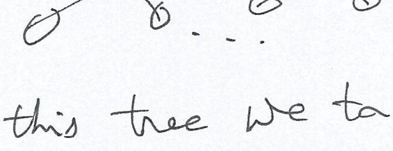
$|E| = |V| - 1 \longrightarrow (1)$

Proof by induction on $|V|$.

Basis step : $|V| = 1 \Rightarrow$ o $\quad |V| = 1, |E| = 0$ True

Induction Hypothesis : We assume (1) to be true for values of $|V| = 1, 2, \ldots, n$.

Induction step : Consider a tree having $|V| = n+1$:



From this tree we take out an edge. This will make the tree disconnected (otherwise it will have a cycle involving the edge that we have taken out, and is no longer a tree). Let the two disconnected components (trees) be (V_1, E_1) and (V_2, E_2). We have:

$|V_1| + |V_2| = |V| \longrightarrow (2) \qquad |E_1| + |E_2| = |E| - 1 \longrightarrow (3)$

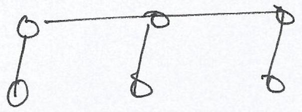
$|E_1| = |V_1| - 1 \longrightarrow (4) \qquad |E_2| = |V_2| - 1 \longrightarrow (5)$

(4) + (5) gives:

$|E_1| + |E_2| = |V_1| + |V_2| - 2$

From (2) and (3) we get:

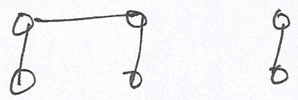
$|E| - 1 = |V| - 2 \Rightarrow |E| = |V| - 1$

Example:  $|V| = 6$, $|E| = 5 = |V| - 1$

A forest $F = (V_F, E_F)$ contains exactly $|V_F| - |E_F|$ trees.
Suppose that F consists of t trees, where the i^{th} tree contains $|V_i|$ vertices and $|E_i|$ edges. Then, we have

$$|E_F| = \sum_{i=1}^{t} |E_i| = \sum_{i=1}^{t} (|V_i| - 1) = \sum_{i=1}^{t} |V_i| - t = |V_F| - t$$

$\Rightarrow t = |V_F| - |E_F|$

Example:  $|V| = 6$, $|E| = 4$, $t = 2 = |V| - |E|$

Now we consider the Graphic Matroid $M_G = (S_G, I_G)$ defined in terms of a given undirected graph $G = (V, E)$ as follows:

(1) The set S_G is defined to be E, the set of edges of G.

(2) If A is a subset of E, then $A \in I_G$ if and only if A is acyclic. That is, a set of edges A is independent if and only if the subgraph $G_A = (V, A)$ forms a forest.

If $G = (V, E)$ is an undirected graph, then $M_G = (S_G, I_G)$ is a matroid.

(1) $S_G = E$ is a finite set.

(2) I_G is hereditary, since a subset of a forest is a forest: removing edges from an acyclic set of edges cannot create cycles.

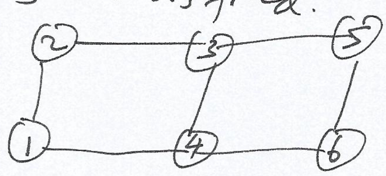
(3) Suppose that $G_A = (V, A)$ and $G_B = (V, B)$ are forests of G and that $|B| > |A|$. A and B are acyclic set of edges, and B contains more edges than A does. Forest G_A contains $|V| - |A|$ trees, and forest G_B contains $|V| - |B|$ trees.

Since forest G_B has fewer trees than forest G_A does, forest G_B must contain some tree T whose vertices are in two different trees in forest G_A. Moreover, since T is connected, it must contain an edge (u, v) such that vertices u and v are in different trees in forest G_A. Since the edge (u, v) connects vertices in two different trees in forest G_A, we can add the edge (u, v) to forest G_A without creating a cycle.

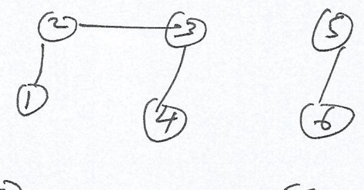
$\Rightarrow (u, v) \in B - A$ and $A \cup (u, v) \in I_G$.

Exchange property is satisfied.

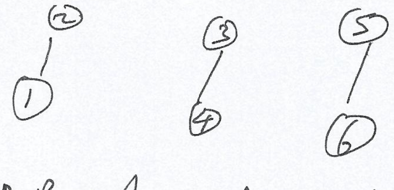
Example: let $G =$



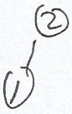
let $G_B =$



Let $G_A =$



Edge $(2,3) \in \cancel{B}\, B - A$ and it connects two different trees

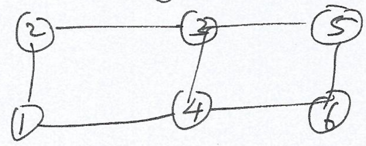


and

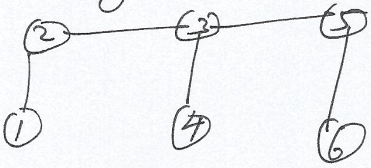
in $A \Rightarrow$ we can add $(2,3)$ to G_A to get $G_B \in I_G$ ($A \cup (2,3) = B$).

Consider a graphic matroid M_G for a connected, undirected graph G. Every maximal independent subset of M_G must be a tree with exactly $|V|-1$ edges that connects all the vertices of G. Such a tree is called a <u>Spanning Tree</u> of G.

<u>Example</u>: Let $G =$



A spanning tree of G (a maximal independent subset of M_G) is:



<u>The Minimum Spanning Tree Problem</u>:

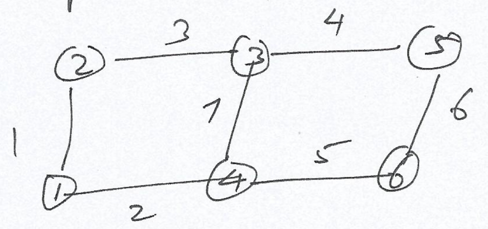
We are given a connected undirected graph $G=(V,E)$ and a length function w such that $w(e)$ is the positive length of edge e. We wish to find a subset of the edges that connects all of the vertices together and has minimum total length. To view this as a problem of finding an optimal subset of a matroid, consider the weighted matroid M_G with weight function w', where $w'(e) = w_0 - w(e)$ and w_0 is larger than the maximum length of any edge. In this weighted matroid, all weights are positive and an optimal subset is a spanning tree of minimum total length in the original graph. More specifically, each maximal independent subset A corresponds to a spanning tree with $|V|-1$ edges, and since

$$w'(A) = \sum_{e \in A} w'(e) = \sum_{e \in A} (w_0 - w(e)) = (|V|-1) w_0 - \sum_{e \in A} w(e)$$

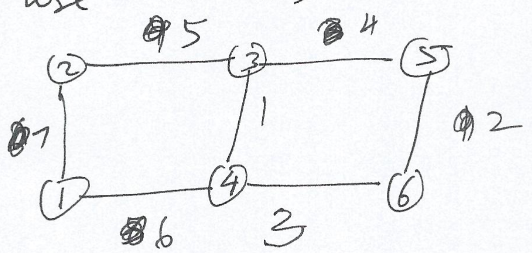
$$= (|V|-1) w_0 - w(A)$$

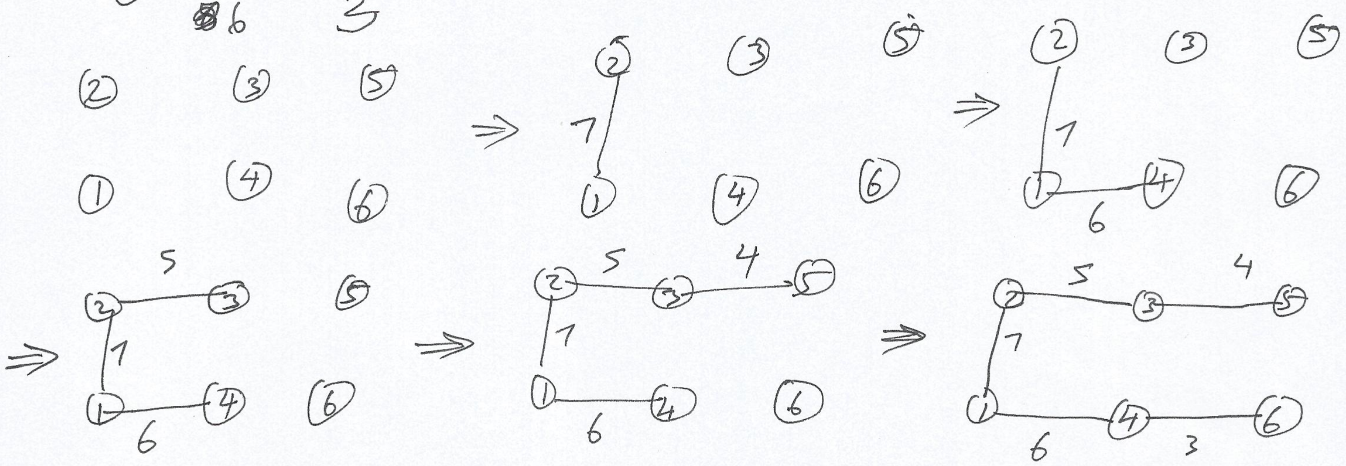
for any maximal independent subset A, an independent subset that maximizes the quantity $w'(A)$ must minimize $w(A)$. Thus, we can use the Greedy algorithm on a weighted matroid to solve the minimum spanning tree problem.

<u>Example</u>: <u>Let G be:</u>

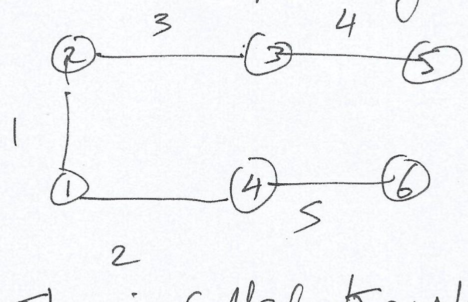


We take $w_0 = 8$ (greater than all edge weights in G) and use $w'(e) = w_0 - w(e)$ to get G':





The corresponding minimum spanning tree is:



with total weight 15.

This is called <u>Kruskal's algorithm</u> and can be implemented in time $O(|E| \log|V|)$.