7.1 RECURRENCE RELATION MODELS

We now consider more complex recurrence relations involving two variable relations.

Let $a_{n, k}$ denote the number of ways to select a subset of k objects from a set of n distinct objects. Find a recurrence relation for $a_{n, k}$.

Find a recurrence relation for the ways to distribute n identical balls into k distinct boxes with between two and four balls in each box. Repeat the problem with balls of three colors.

The problem with all balls identical can be solved by generating functions, but recurrence relations are the only practical approach with the extra constraint of different types of balls.

Find recurrence relations for

(a) The number of n-digit ternary sequences with an even number of 0s

(b) The number of n-digit ternary sequences with an even number of 0s and an even number of 1s

(b) The number of n-digit ternary sequences with an even number of 0s and an even number of 1s

We will need simultaneous 3 recurrence relations:

a_n : the number of n-digit ternary sequences with even 0s and even 1s;
b_n : the number of n-digit ternary sequences with even 0s and odd 1s;
c_n : the number of n-digit sequences with odd 0s and even 1s.

Why not for odd 0s and odd 1s.

Observe that $3^n - a_n - b_n - c_n$ is the number of n-digit ternary sequences with odd 0s and odd 1s.

a_n is obtained either by having a
1 for the first digit followed by (n −1)-digit sequence with even 0s and odd 1s,
or a 0 followed by (n −1)-digit sequence with odd 0s and even 1s,
or a 2 followed by an (n −1)-digit sequence with even 0s and even 1s.

Thus $a_n = b_{n-1} + c_{n-1} + a_{n-1}$.

But from where we get b_{n-1} and c_{n-1}

Similar analyses yield

$b_n = a_{n-1} + (3^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) + b_{n-1} = 3^{n-1} - c_{n-1}$ and

$c_n = a_{n-1} + (3^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) + c_{n-1} = 3^{n-1} - b_{n-1}$

The initial conditions are $a_1 = b_1 = c_1 = 1$.

To recursively compute values for a_n, we must simultaneously compute b_n and c_n.