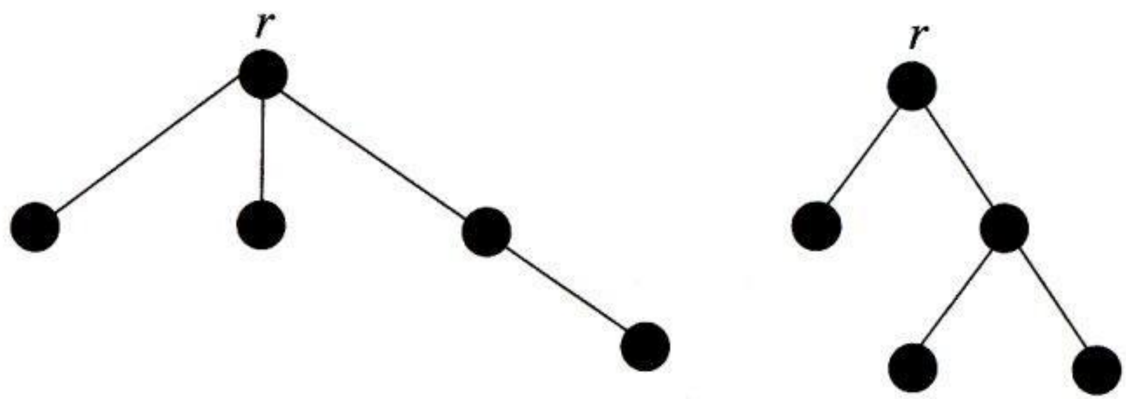
Catalon Numbers

Rooted Trees

Let T be a tree and $r \in V(T)$. A *rooted tree* is the ordered pair (T, r). The vertex r is called the *root* of T.

Example: Are the following trees isomorphic

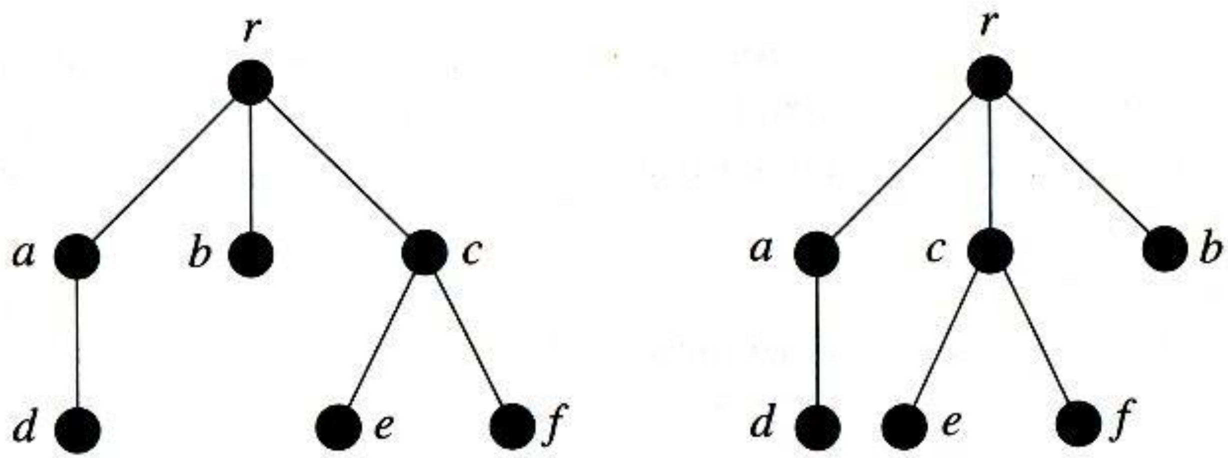


Two rooted trees, isomorphic as trees but not as rooted trees.

Ordered rooted tree

A rooted tree (T, r) in which the left/right order of every set of siblings is specified is called an *ordered rooted tree*.

Example

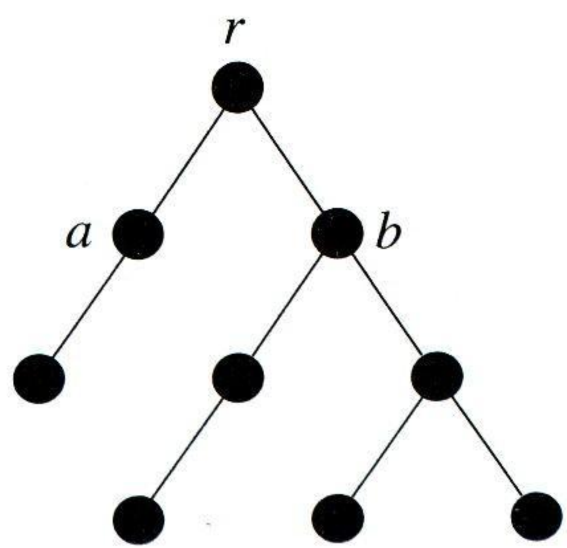


(a) (b)

Two distinct ordered rooted trees that are identical when viewed as rooted trees.

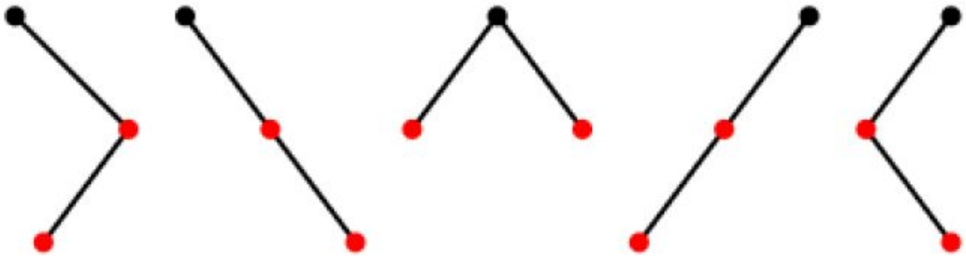
Binary Trees

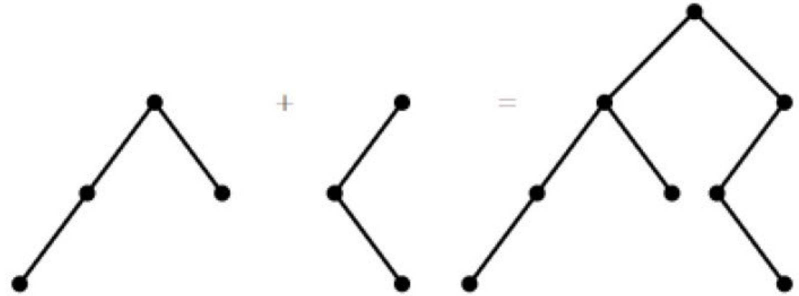
A *binary tree* is an ordered rooted tree in which each vertex has at most two children. Each child of a vertex is called either the *left* child or the *right* child. A subtree rooted at the left (right) child of a vertex u is known as u's *left* (*right*) *subtree*. By convention, the single vertex tree is considered a "trivial" binary tree.



How many different rooted binary trees are there with n vertices?

Let us denote this number by $\mathbf{C_n}$; these are the **Catalan numbers.**
For convenience, we allow a rooted binary tree to be empty, and let $C_0 = 1$. Then it is easy to see that $C_1 = 1$ and $C_2 = 2$.
Compute C_3.

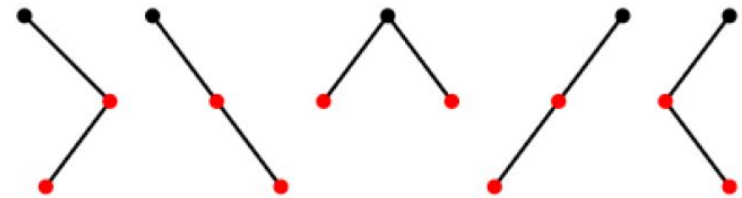




Any rooted binary tree on at least one vertex can be viewed as two binary trees joined into a new tree by introducing a new root vertex and making the children of this root the two roots of the original trees.

since we know that $C_0 = C_1 = 1$ and $C_2 = 2$,

$$C_3 = C_0C_2 + C_1C_1 + C_2C_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5,$$



Now we can write

$$C_n = \sum_{i=0}^{n-1} C_iC_{n-i-1}.$$

Now we use a generating function to find a formula for C_n. Let $f = \sum_{i=0}^{\infty} C_ix^i$. Now consider f^2: the coefficient of the term x^n in the expansion of f^2 is $\sum_{i=0}^{n} C_iC_{n-i}$, corresponding to all possible ways to multiply terms of f to get an x^n term:

$$C_0 \cdot C_nx^n + C_1x \cdot C_{n-1}x^{n-1} + C_2x^2 \cdot C_{n-2}x^{n-2} + \cdots + C_nx^n \cdot C_0.$$

Now we recognize this as precisely the sum that gives C_{n+1}, so $f^2 = \sum_{n=0}^{\infty} C_{n+1}x^n$. If we multiply this by x and add 1 (which is C_0) we get exactly f again, that is, $xf^2 + 1 = f$ or $xf^2 - f + 1 = 0$; here 0 is the zero function, that is, $xf^2 - f + 1$ is 0 for all x. Using the quadratic formula,

$$f = \frac{1 \pm \sqrt{1-4x}}{2x},$$

as long as $x \neq 0$. It is not hard to see that as x approaches 0,

$$\frac{1 + \sqrt{1-4x}}{2x}$$

goes to infinity while

$$\frac{1 - \sqrt{1-4x}}{2x}$$

goes to 1. Since we know $f(0) = C_0 = 1$, this is the f we want.

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

$$
\begin{aligned}
(1-4z)^{1/2} \quad = \quad & 1-\frac{\left(\frac{1}{2}\right)}{1}4z+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2\cdot 1}(4z)^2-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2\cdot 1}(4z)^3+ \\
& \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4\cdot 3\cdot 2\cdot 1}(4z)^4-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{5\cdot 4\cdot 3\cdot 2\cdot 1}(4z)^5+\cdots
\end{aligned}
$$

We can get rid of many powers of 2 and combine things to obtain:

$$
(1-4z)^{1/2}=1-\frac{1}{1!}2z-\frac{1}{2!}4z^2-\frac{3\cdot 1}{3!}8z^3-\frac{5\cdot 3\cdot 1}{4!}16z^4-\frac{7\cdot 5\cdot 3\cdot 1}{5!}32z^5-\cdots \tag{9}
$$

From Equations 9 and 8:

$$
f(z)=1+\frac{1}{2!}2z+\frac{3\cdot 1}{3!}4z^2+\frac{5\cdot 3\cdot 1}{4!}8z^3+\frac{7\cdot 5\cdot 3\cdot 1}{5!}16z^4+\cdots \tag{10}
$$

The terms that look like $7\cdot 5\cdot 3\cdot 1$ are a bit troublesome. They are like factorials, except they are missing the even numbers. But notice that $2^2\cdot 2!=4\cdot 2$, that $2^3\cdot 3!=6\cdot 4\cdot 2$, that $2^4\cdot 4!=8\cdot 6\cdot 4\cdot 2$, et cetera. Thus $(7\cdot 5\cdot 3\cdot 1)\cdot 2^4 4!=8!$. If we apply this idea to Equation 10 we can obtain:

$$
f(z)=1+\frac{1}{2}\left(\frac{2!}{1!1!}\right)z+\frac{1}{3}\left(\frac{4!}{2!2!}\right)z^2+\frac{1}{4}\left(\frac{6!}{3!3!}\right)z^3+\frac{1}{5}\left(\frac{8!}{4!4!}\right)z^4+\cdots=\sum_{i=0}^{\infty}\frac{1}{i+1}\binom{2i}{i}z^i.
$$

From this we can conclude that the i^{th} Catalan number is given by the formula

$$C_i = \frac{1}{i+1}\binom{2i}{i}.$$

1.4 Polygon Triangulation

If you count the number of ways to triangulate a regular polygon with $n+2$ sides, you also obtain the Catalan numbers. Figure 2 illustrates the triangulations for polygons having 3, 4, 5 and 6 sides.

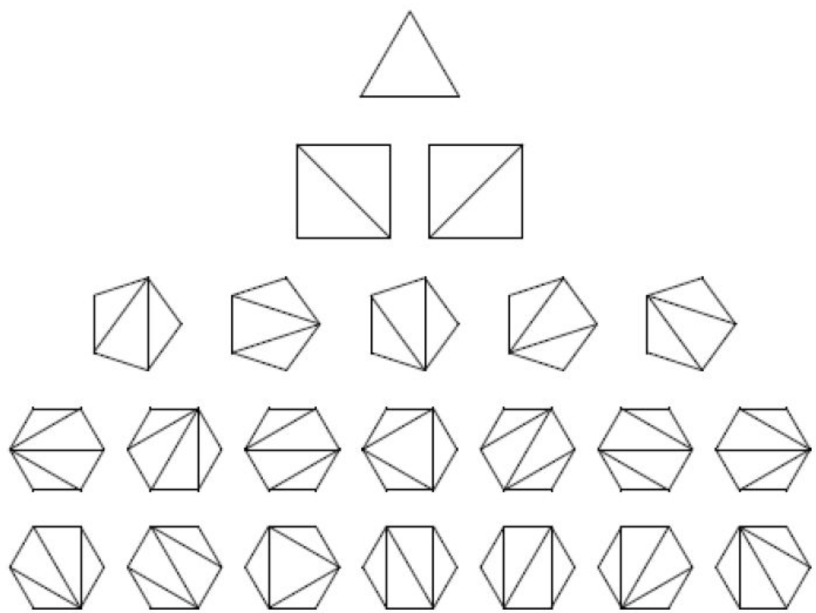


Figure 2: Polygon Triangulations

Given a “triangular” region composed of n blocks on a side, in how many different ways can the region be tiled with exactly 4 rectangles?