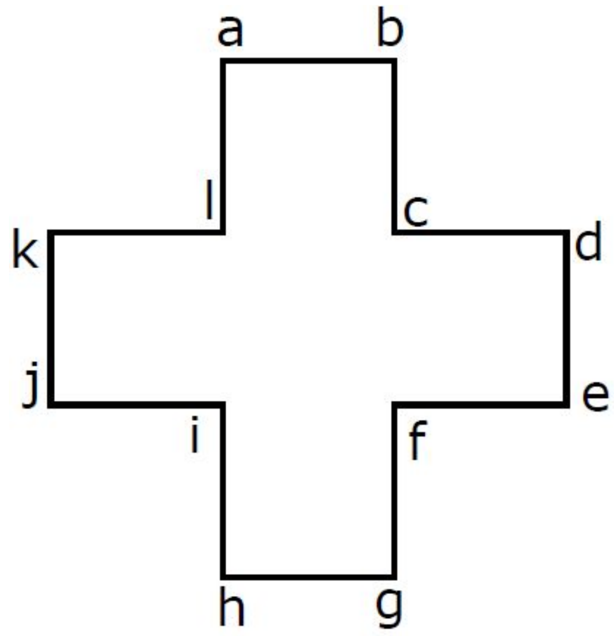
Compute the number of ways to color the vertices of a rhombus (which is not a square) with k colors.

¼ $(x_1^4 + x_2^2 + 2x_1^2x_2)$

Compute the number of ways to color the vertices of an equilateral triangle with 5 colors while using at least 2 colors.

6 symmetries are there.
For 0 degree, $5^3 – 5 = 120$.
For 120 and 240, $5 – 5 = 0$ and for 3 flips, $5^2 – 5 = 20$.
$1/6 (120 + 0 + 3.20) = 30$.

Find the number of different k-colorings of the vertices of the following figure



For 0 degree, x_1^{12}

For 90=270 degree, x_4^{3}

For 180 degree, x_2^{6}

For horizontal and vertical reflections, x_2^{6}

For diagonal reflections, $x_1^{2}x_2^{5}$

POLYA’S FORMULA

(*i*) Motion π_i	(*ii*) Colorings Left Fixed by π_i	(*iii*) Cycle Structure Representation
π_1	16—all colorings	x_1^4
π_2	2—C_1, C_{16}	x_4
π_3	2—$C_1, C_{10}, C_{11}, C_{16}$	x_2^2
π_4	2—C_1, C_{16}	x_4
π_5	2—C_1, C_6, C_8, C_{16}	x_2^2
π_6	2—$C_1, C_7, C_9 C_{16}$	x_2^2
π_7	8—C_1, C_2, C_4, C_{10} $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2 x_2$
π_8	8—C_1, C_3, C_5, C_{10} $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2 x_2$

We are now ready to address our ultimate goal of a formula for the **pattern inventory**.

The pattern inventory is a generating function that tells how many colorings of an unoriented figure there are using different possible collections of colors.

For black–white colorings of the unoriented square, the pattern inventory is

$1b^4 + 1b^3w + 2b^2w^2 + 1bw^3 + 1w^4$.

For example, the term $1b^3w$ tells us that there is one nonequivalent coloring with three black (b) corners and one white (w) corner. Our aim is to compute the coefficient of $b^{4-k}w^k$ in the pattern inventory. For computing the coefficient of $b^{4-k}w^k$, refer to the Table below.

(i) Motion π_i	(ii) Colorings Left Fixed by π_i	(iii) Cycle Structure Representation	(iv) Inventory of Colorings Left Fixed by π_i
π_1	16—all colorings	x_1^4	$(b+w)^4 = 1b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + 1w^4$
π_2	2—C_1, C_{16}	x_4	$(b^4+w^4) = 1b^4 + 1w^4$
π_3	2—$C_1, C_{10}, C_{11}, C_{16}$	x_2^2	$(b^2+w^2)^2 = 1b^4 + 2b^2w^2 + 1w^4$
π_4	2—C_1, C_{16}	x_4	$(b^4+w^4) = 1b^4 + 1w^4$
π_5	2—C_1, C_6, C_8, C_{16}	x_2^2	$(b^2+w^2)^2 = 1b^4 + 2b^2w^2 + 1w^4$
π_6	2—C_1, C_7, C_9, C_{16}	x_2^2	$(b^2+w^2)^2 = 1b^4 + 2b^2w^2 + 1w^4$
π_7	8—C_1, C_2, C_4, C_{10} $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2x_2$	$(b+w)^2(b^2+w^2) = 1b^4 + 2b^3w + 2b^2w^2 + 2bw^3 + 1w^4$
π_8	8—C_1, C_3, C_5, C_{10} $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2x_2$	$(b+w)^2(b^2+w^2) = 1b^4 + 2b^3w + 2b^2w^2 + 2bw^3 + 1w^4$

In the first row of column (iv), we write a polynomial whose coefficients give the numbers of 2-colorings in each T_k left fixed by π_1, then in the second row of the table we write a polynomial for the numbers of 2-colorings in each T_k left fixed by π_2, then by π_3, and so forth. Then we total up the b^4 term in each row (the number of 2-colorings with four blacks) and divide by 8 to get the coefficient of b^4 in the pattern inventory, total up the b^3w term in each row and divide by 8 to get the coefficient of b^3w, and so forth.

Since the action of π_1 leaves all C_s fixed, the first row's coefficients are 1, 4, 6, 4, 1. We write $b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4$; this is an *inventory of fixed colorings*. For π_1, the inventory of fixed colorings is an inventory of all colorings. Observe that this inventory is simply $(b+w)^4 = (b+w)(b+w)(b+w)(b+w)$, **why**

Since each corner can have any color, black or white

For π_2, the inventory is $b^4 + w^4$.

If all corners have same color, then only coloring is fixed by π_2

The motion π_3 has two 2-cycles (ac) and (bd). Each 2-cycle uses two blacks or two whites in a fixed coloring. Hence the inventory of a cycle of size two is $b^2 + w^2$. The possibilities with two such cycles have the inventory $(b^2 + w^2)(b^2 + w^2)$.

The inventory of fixed colorings for π_i will be a product of factors $(b^j + w^j)$, one factor for each j-cycle of the π_i. So we need to know the number of cycles in π_i of each size. But this is exactly the information encoded in the cycle structure representation. Indeed, setting $x_j = (b^j + w^j)$ in the representation yields precisely the inventory of fixed colorings for π_i. By this method we compute the rest of the inventories of fixed colorings.

If three colors, black, white, and red, were permitted, each cycle of size j would have an inventory of $(b^j + w^j + r^j)$ in a fixed coloring. So we would set $x_j = (b^j + w^j + r^j)$ in P_G. The preceding argument applies for any number of colors and any figure. In greater generality we have the following theorem.

Theorem **(Polya's Enumeration Formula)**

Let S be a set of elements and G be a group of permutations of S that acts to induce an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using two colors is given by the generating function $P_G((b+w), (b^2+w^2), (b^3+w^3), \ldots, (b^k+w^k))$. The inventory using colors $c_1, c_2, \ldots, c_m$ is

$$P_G\left(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, \ldots, \sum_{j=1}^{m} c_j^k\right)$$

Determine the pattern inventory for 3-bead necklaces distinct under rotations using black and white beads.

From Example 1 of Section 9.3, we know $P_G = \frac{1}{3}(x_1^3 + 2x_3)$. Substituting $x_j = (b^j + w^j)$, we get

$$\begin{aligned}\frac{1}{3}[(b+w)^3 + 2(b^3+w^3)] &= \frac{1}{3}[(b^3+3b^2w+3bw^2+w^3)+(2b^3+2w^3)]\\ &= \frac{1}{3}(3b^3+3b^2w+3bw^2+3w^3)\\ &= b^3+b^2w+bw^2+w^3\end{aligned}$$

Find the number of 7-bead necklaces distinct under rotations using three black and four white beads.

We need to determine the coefficient of b^3w^4 in the pattern inventory. Each rotation, except the 0° rotation, is a cyclic permutation

so $P_G = \frac{1}{7}(x_1^7 + 6x_7)$. The pattern inventory is $\frac{1}{7}[(b+w)^7 + 6(b^7+w^7)]$.

Since the factor $6(b^7 + w^7)$ in the pattern inventory contributes nothing to the b^3w^4 term, we can neglect it. Thus the number of 3-black, 4-white necklaces is simply

$$\frac{1}{7}[(\text{coefficient of } b^3w^4 \text{ in } (b+w)^7] = \frac{1}{7}\binom{7}{3} \quad \blacksquare$$

Compute the number of 6-bead distinct necklaces using 3 black, 2 white and 1 grey beads (consider both rotation and reflection).

$1/12\ (x_1^{\ 6} + 4x_2^{\ 3} + 2x_3^{\ 2} + 2x_6 + 3x_1^{\ 2}x_2^{\ 2})$

Replace x_i by $b^i + w^i + g^i$

Coefficient of b^3w^2g is $1/12(\ C(6;\ 3,2,1) + 3.2.2) = 6$.