6.5 A SUMMATION METHOD

In this section we show how to construct an ordinary generating function h(x) whose coefficient of x^r is some specified function p(r) of r, such as r^2 or C(r, 3).

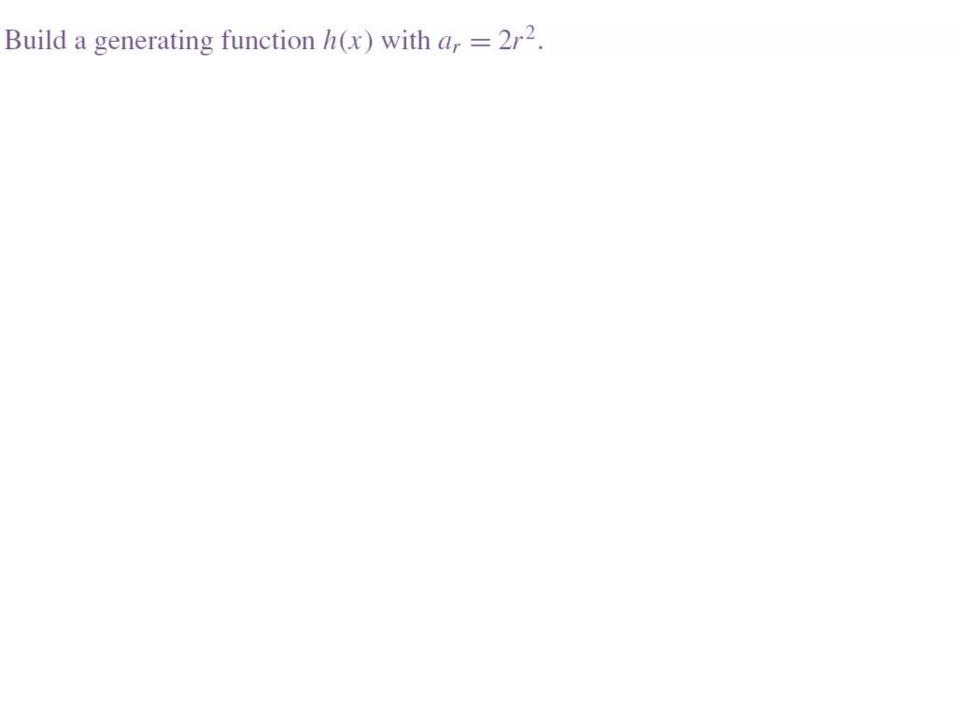
How to construct generating function $g^*(x)$ with $a_r^* = ra_r$ from the following generating function:

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_rx^r + \dots$$

The next question is from where to start, i.e., what could be initial g(x).

The natural answer is: When in doubt start with the unit coefficients $a_r = 1$ of the generating function

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^r + \dots$$



Build a generating function h(x) with $a_r = (r+1)r(r-1)$.

We could multiply (r+1)r(r-1) out getting $a_r = r^3 - r$, obtain generating functions for r^3 and r as an Example 1, and then subtract one generating function from the other. It is easier, however, to start with $3!(1-x)^{-4}$, whose coefficient a_r equals (r+4-1) (r+3)! (r+3)!

$$a_r = 3! {r+4-1 \choose r} = 3! \frac{(r+3)!}{r!3!} = \frac{(r+3)!}{r!} = (r+3)(r+2)(r+1)$$

Theorem

If h(x) is a generating function where a_r is the coefficient of x^r , then $h^*(x) = h(x)/(1-x)$ is a generating function of the sums of the a_r s. That is,

$$h^*(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots + \left(\sum_{i=0}^r a_i\right)x^r + \dots$$

Using generating function

Evaluate the sum $2 \times 1^2 + 2 \times 2^2 + 2 \times 3^2 + \dots + 2n^2$.

Using generating function

Evaluate the sum $3 \times 2 \times 1 + 4 \times 3 \times 2 + \cdots + (n+1)n(n-1)$.