SIMPLE ARRANGEMENTS AND SELECTIONS

Permutation and Combination

A **permutation** of n distinct objects is an **ordered arrangement** **(without repeatition)** of the n objects.

An ***r*-permutation** of n distinct objects is an arrangement using r of the n objects.

An ***r*-combination** of n distinct objects is an **unordered selection** **(without repeatition)**, or *subset*, of r out of the n objects.

We use $P(n, r)$ and $C(n, r)$ to denote the number of r-permutations and r-combinations, respectively, of a set of n objects.

How to derive mathematical expressions for $P(n, r)$ and $C(n, r)$

Computing $P(n, r)$

From the multiplication principle we obtain

$$P(n,2) = n(n-1), \qquad P(n,3) = n(n-1)(n-2),$$
$$P(n,n) = n(n-1)(n-2) \times \cdots \times 3 \times 2 \times 1$$

In enumerating all permutations of n objects, we have n choices for the first position in the arrangement, $n-1$ choices (the $n-1$ remaining objects) for the second position, ..., and finally one choice for the last position. Using the notation $n! = n(n-1)(n-2)\cdots \times 3 \times 2 \times 1$ ($n!$ is said "n factorial"), we have the formulas

$$P(n,n) = n!$$

and

$$P(n,r) = n(n-1)(n-2) \times \cdots \times [n-(r-1)] = \frac{n!}{(n-r)!}$$

Computing $C(n, r)$ using $P(n, r)$

Our formula for $P(n, r)$ can be used to derive a formula for $C(n, r)$. All r-permutations of n objects can be generated by first picking any r-combination of the n objects and then arranging these r objects in any order. Thus $P(n, r) = C(n, r) \times P(r, r)$, and solving for $C(n, r)$ we have

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

Q. How many ways are there to arrange the seven letters in the word SYSTEMS? In how many of these arrangements, do the three Ss appear consecutively?

How many different 8-digit binary sequences are there with six 1s and two 0s?

Permutation or Combination?

Here order is not important

A committee of k people is to be chosen from a set of seven women and four men. How many ways are there to form the committee if

(a) The committee consists of three women and two men?

(b) The committee can be any positive size but must have equal numbers of women and men?

(c) The committee has four people and one of them must be Mr. Baggins?

(d) The committee has four people and at least two are women?

(e) The committee has four people, two of each sex, and Mr. and Mrs. Baggins cannot both be on the committee?

Permutation or Combination?

In all cases, order is not required

a. A committee of k people is to be chosen from a set of seven women and four men.
How many ways are there to form the committee if **the committee consists of three women and two men?**

b. A committee of k people is to be chosen from a set of seven women and four men. How many ways are there to form the committee of any positive size with equal number of women and men.

c. A committee of k people is to be chosen from a set of seven women and four men.
How many ways are there to form the committee if the committee has four people and one of them must be Mr. Baggins?

d. How many ways are there to form the committee if the committee has four people and at least two are women?

e. A committee of k people is to be chosen from a set of seven women and four men.
How many ways are there to form the committee if the committee has four people, two of each sex, and Mr. and Mrs. Baggins cannot both be on the committee?

How many sub-cases are required.

A complementary approach.

We can consider all C(7, 2)×C(4, 2) 2-women–2-men committees and then subtract the forbidden committees that contain both Bagginses.

The forbidden committees are formed by picking one more woman and one more man to join Mr. and Mrs. Baggins—done in C(6, 1)×C(3, 1) ways.

Answer: $21\times6-6\times3 = 108$.

How to grapple with the two constraints simultaneously

How many arrangements of the seven letters in the word SYSTEMS have the E occurring somewhere before the M?

C(7, 2) * P(5, 2)
C(7, 2) * C(5, 2)
P(7, 2) * P(5, 2)
C(7, 2) + P(5, 2)
P(7, 2) + P(5, 2)
C(7, 2) + C(5, 2)

We start by picking which of the two out of the seven positions in an arrangement are where the E and M will go (Permutation or Combination)
:C(7, 2) = 21 ways (**how, why**)
(**count it by fixing the position of E and moving the position of M 6+5+4+3+2+1**).

Here there is an order first E and then M but we don't use permutation because ME is not allowed. In a bigger picture, there is no ordering.

Now we fill in the five other positions in the arrangement by picking a position for the Y and the T: (Permutation or Combination)

$P(5, 2) = 5 \times 4 = 20$ ways

and then putting the three Ss in the three remaining positions.

The answer is thus $21 \times 20 = 420$.

Hw. A manufacturing plant produces ovens. At the last stage, an inspector marks the ovens A (acceptable) or U (unacceptable). How many different sequences of 15 As and Us are possible in which the third U appears as the twelfth letter in the sequence?

Ans. 440 sequences.

5.3 ARRANGEMENTS AND SELECTIONS WITH REPETITIONS

How many arrangements are there of the six letters b, a, n, a, n, a?

There are C(6, 3) × C(3, 2) × C(1, 1) = 20 × 3 × 1 = 60 arrangements.

This is equivalent to

$$P(6; 3, 2, 1) = \binom{6}{3}\binom{3}{2}\binom{1}{1} = \frac{6!}{3!3!} \times \frac{3!}{2!1!} \times \frac{1!}{1!} = \frac{6!}{3!2!1!}$$

Theorem 1

If there are n objects, with r_1 of type 1, r_2 of type 2, ..., and r_m of type m, where $r_1 + r_2 + \cdots + r_m = n$, then the number of arrangements of these n objects, denoted $P(n; r_1, r_2, \ldots, r_m)$, is

$$P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2\cdots-r_{m-1}}{r_m}$$

$$= \frac{n!}{r_1!r_2!\ldots r_m!} \qquad (*)$$

Proof. Then there are *n!* arrangements of the n distinct objects.

$$n! = P(n; r_1, r_2, \ldots, r_m) r_1! r_2! \ldots r_m!$$

or

$$P(n; r_1, r_2, \ldots, r_m) = \frac{n!}{r_1!r_2!\ldots r_m!} \quad \blacklozenge$$

How many different ways are there to select six hot dogs from three varieties of hot dog?

Can we apply Theorem 1 directly, why or why not?
It can not be applied because the number of items of type i are not specified.

Model it as arrangement-with-repetition problem

Theorem 2

The number of selections with repetition of r objects chosen from n types of objects is $C(r+n-1, r)$.

Proof

We make an "order form" for a selection just as in Example 2, with an x for each object selected. As before, the xs before the first | count the number of the first type of object, the xs between the first and second |s count the number of the second type, $\ldots$, and the xs after the $(n-1)$-st | count the number of the nth type ($n-1$ slashes are needed to separate n types). The number of sequences with r xs and $n-1$ |s is $C(r+(n-1), r)$. ◆

How many ways are there to form a sequence of 10 letters from four as, four bs, four cs, and four ds if each letter must appear at least twice?

How many ways are there to pick a collection of exactly 10 balls from a pile of red balls, blue balls, and purple balls if there must be at least five red balls? If at most five red balls?

How many arrangements are there of the letters b, a, n, a, n, a such that:

a. The b is followed (immediately) by an a
b. The pattern bnn never occurs

Ans. 30, 56

5.4 DISTRIBUTIONS

How many ways are there to assign 3 different diplomats to 2 different continents?

How many ways are there to assign 9 diplomats to 3 continents if 3 diplomats must be assigned to each continent?

How many ways are there to distribute 3 red balloons among 2 children.

Basic Models for Distributions

Distinct Objects The process of distributing *r* distinct objects into *n* different boxes is equivalent to putting the distinct objects in a row and stamping one of the *n* different box names on each object. The resulting sequence of box names is an arrangement of length *r* formed from *n* items (box names) with repetition.

Thus there are $n \times n \times \ldots \times n$ (r ns) $= n^r$ distributions of the r distinct objects.

If r_i objects must go in box i, $1 \le i \le n$, then there are $P(r; r_1, r_2, \ldots, r_n)$ distributions.

Theorem 1

If there are n objects, with r_1 of type 1, r_2 of type 2, ..., and r_m of type m, where $r_1 + r_2 + \cdots + r_m = n$, then the number of arrangements of these n objects, denoted $P(n; r_1, r_2, \ldots, r_m)$, is

$$P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2\cdots-r_{m-1}}{r_m}$$

$$= \frac{n!}{r_1!r_2!\ldots r_m!} \qquad (*)$$

How many ways are there to assign 100 different diplomats to five different continents?

How many ways if 20 diplomats must be assigned to each continent?

How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?

Identical Objects The process of distributing r identical objects into n different boxes is equivalent to choosing an (unordered) subset of r box names with repetition from among the n choices of boxes.
Thus there are $C(r + n - 1, r) = (r + n - 1)!/ r!(n - 1)!$ distributions of the r identical objects.

Theorem 2

The number of selections with repetition of r objects chosen from n types of objects is $C(r + n - 1, r)$.

How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?

Remark.
Distributions of distinct objects are equivalent to arrangements
and
Distributions of identical objects are equivalent to selections

Ways to Arrange, Select, or Distribute r Objects from n Items or into n Boxes

	Arrangement (Ordered Outcome) or Distribution of Distinct Objects	*Combination (Unordered Outcome) or Distribution of Identical Objects*
No repetition	$P(n, r)$	$C(n, r)$
Unlimited repetition	n^r	$C(n+r-1, r)$
Restricted repetition	$P(n; r_1, r_2, \ldots r_m)$	—

How many ways are there to distribute four identical oranges and six distinct apples (each a different variety) into five distinct boxes? In what fraction of these distributions does each box get exactly two objects?

Compute the number of ways to distribute r identical balls into n distinct boxes with

- at least one ball in each box
- at least r_1 balls in the first box, at least r_2 balls in the second box, . . . , and at least r_n balls in the n^{th} box

How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$? How many solutions with $x_i \geq 1$? How many solutions with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

Remark. Equations with integer-valued variables are called *diophantine equations.* They are named after the Greek mathematician Diophantus, who studied them 2,250 years ago.

How many arrangements of the letters *a, e, i, o, u, x, x, x, x, x, x, x, x* (eight *x*s) are there if no two vowels can be consecutive?

Hw. What fraction of binary sequences of length 10 consists of a (positive) number of 1s, followed by a number of 0s, followed by a number of 1s, followed by a number of 0s? An example of such a sequence is 1110111000.

84/1024

Equivalent Forms for Selection with Repetition

1. The number of ways to select r objects with repetition from n different types of objects.
2. The number of ways to distribute r identical objects into n distinct boxes.
3. The number of nonnegative integer solutions to $x_1 + x_2 + \cdots + x_n = r$.

Ways to Arrange, Select, or Distribute *r* Objects from *n* Items or into *n* Boxes

	Arrangement (Ordered Outcome) or Distribution of Distinct Objects	*Combination (Unordered Outcome) or Distribution of Identical Objects*
No repetition	$P(n, r)$	$C(n, r)$
Unlimited repetition	n^r	$C(n+r-1, r)$
Restricted repetition	$P(n; r_1, r_2, \ldots r_m)$	—

5.5 BINOMIAL IDENTITIES

Why the numbers *C(n, r)* are called *binomial coefficients ?*

Consider the polynomial expression $(a + x)^3$.

$$(a+x)(a+x)(a+x) = aaa + aax + axa + axx + xaa + xax + xxa + xxx$$

Why there are 8 terms on RHS

There are 3 positions and each position has 2 choices (with repeation).

Collecting similar terms, we reduce the right-hand side of this expansion to

$$a^3 + 3a^2x + 3ax^2 + x^3 \tag{1}$$

How many of the formal products in the expansion of $(a + x)^3$ contain k xs and $(3 - k)$ as?
This question is equivalent to asking for the coefficient of $a^{3-k} x^k$ in (1).

Since formal products are just three-letter sequences of as and xs, we are simply asking for the number of all three-letter sequences with k xs and $(3 - k)$ as.

The answer is C(3, k) and so the reduced expansion for $(a + x)^3$ can be written as

$$\binom{3}{0} a^3 + \binom{3}{1} a^2 x + \binom{3}{2} a x^2 + \binom{3}{3} x^3$$

By the same argument, we see that the coefficient of $a^{n-k} x^k$ in $(a + x)^n$ will be equal to the number of n-letter sequences formed by k xs and $(n - k)$ as, that is, $C(n, k)$. If we set $a = 1$, we have the following theorem.

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

Write the Equivalent Identity

The number of ways to select a subset of k objects out of a set of n objects is equal to the number of ways to select a group of n − k of the objects to set aside (the objects not in the subset).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \qquad (2)$$

Prove the following Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (3)$$

LHS: represents C(n, k) committees of k people chosen from a set of n people

To prove (3), we need to classify the C(n, k) committees of k people into two categories, depending on whether or not the committee contains a given person P.
If P is not a part of the committee,
there are C(n − 1, k) ways to form the committee from the other n − 1 people.
On the other hand, if P is on the committee,
the problem reduces to choosing the k − 1 remaining members of the committee from the other n − 1 people. This can be done C(n − 1, k − 1) ways.

Thus C(n, k) = C(n − 1, k) + C(n − 1, k − 1).

The above proof is a useful interpretation of binomial coefficients known as **committee selection model.**

Use committee selection model to show that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \qquad (4)$$

The left-hand side of (4) counts the ways to select a group of k people chosen from a set of n people and then to select a subset of m leaders within the group of k people.
Here, we are first choosing k members and then m members from k members.

Equivalently, as counted on the right side, we could first select the subset of m leaders from the set of n people and then select the remaining k − m members of the group from the remaining n − m people.
We first selected m members which means we still need to select k – m members out of n – m members.

Show that

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \quad \blacksquare \qquad (5)$$

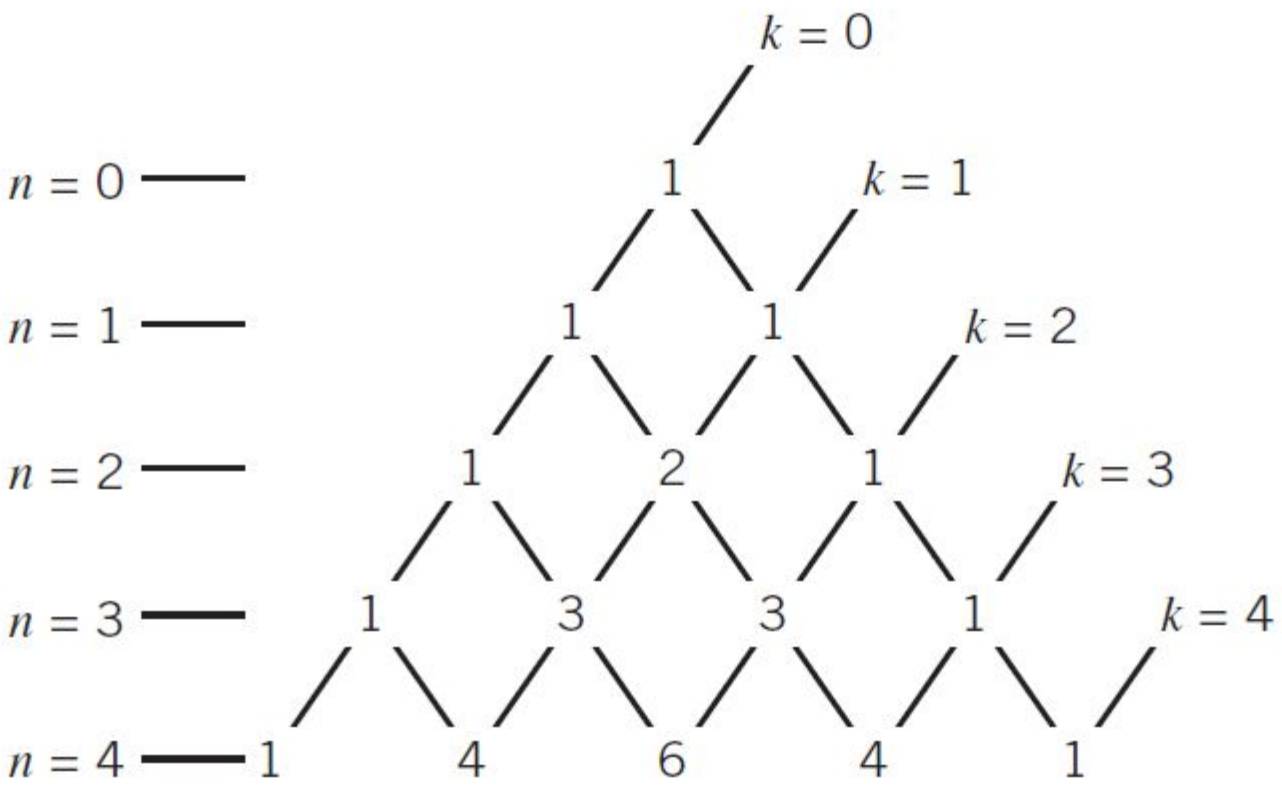
It is the special form of (4) when m = 1.

$$k\binom{n}{k} = n\binom{n-1}{k-1} \quad \text{or} \quad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Using (3) and the fact that C(n, 0) = C(n, n) = 1 for all nonnegative n, we can recursively build successive rows in the following table of binomial coefficients, called **Pascal's triangle**.
Each number in this table, except the first and last numbers in a row, is the sum of the two neighboring numbers in the preceding row.

Table of binomial coefficients: kth number in row n is $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$



Pascal’s triangle has the following combinatorial interpretation.

Consider the ways a person can traverse the blocks in the map of streets shown in Fig below. The person begins at the top of the network, at the spot marked (0, 0), and moves down the network making a choice at each intersection to go right or left.

We label each street corner in the network with a pair (*n, k*), where n is the number of blocks traversed from (0, 0) and k is the number of times the person chose the right branch at intersections.

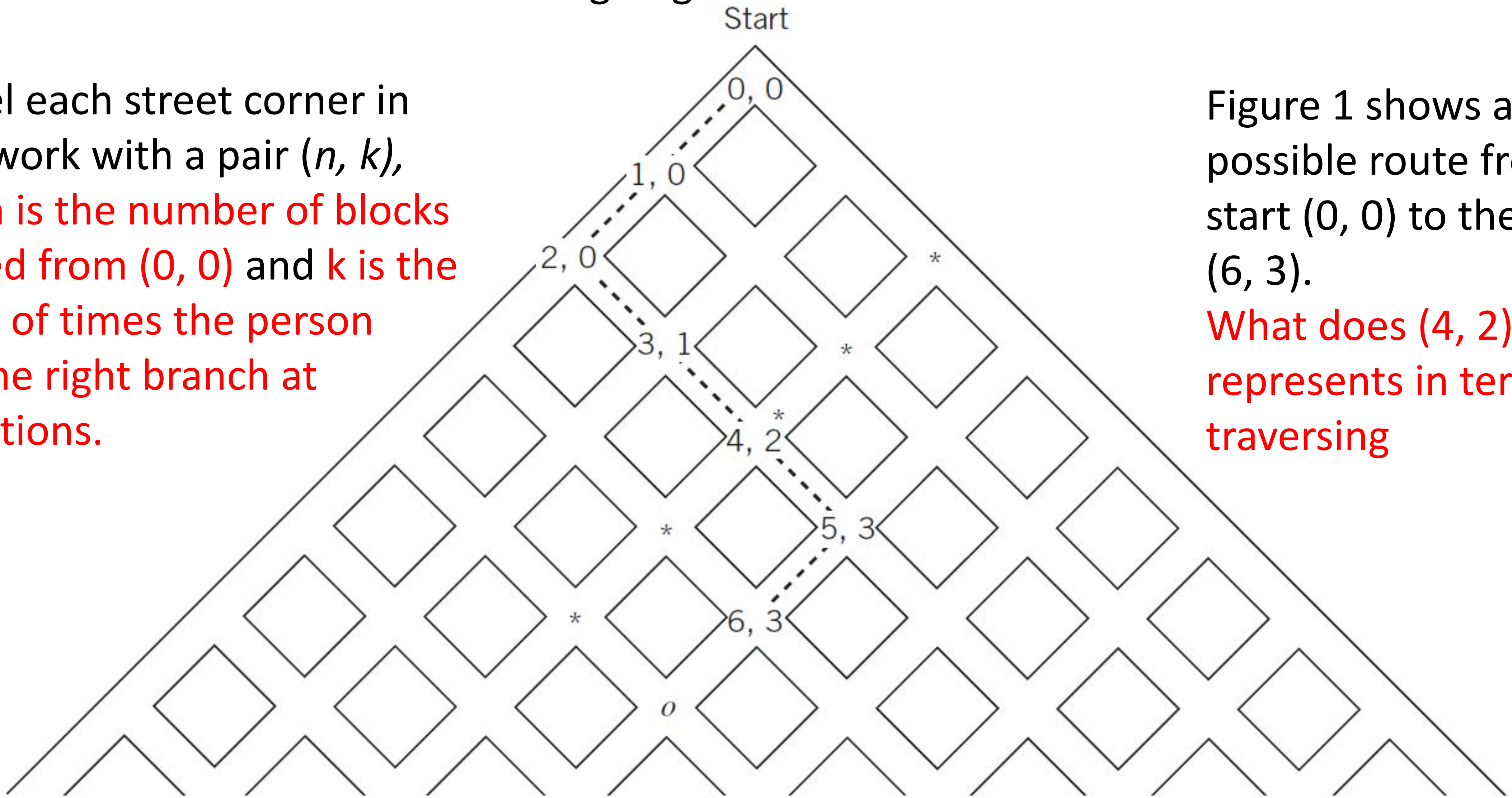


Figure 1 shows a possible route from the start (0, 0) to the corner (6, 3).

What does (4, 2) represents in terms of traversing

To go to corner (6, 3) following the route shown in Figure 1, we have the sequence of turns LLRRRL.

Any route to corner (n, k) can be written as a list of the branches (left or right) chosen at the successive corners on the path from (0, 0) to (n, k). Such a list is a sequence of k Rs (right branches) and (n − k) Ls (left branches).

What is the number of possible routes from the start (0, 0) to corner (n, k).
Let s(n, k) be the number of possible routes from (0, 0) to (n, k).
This is the number of sequences of k Rs and (n − k) Ls,
and hence s(n, k) = C(n, k).

Using “block-walking” model for binomial coefficients prove identity (3).

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n \tag{6}$$

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r} \tag{7}$$

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1} \tag{8}$$

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n} \tag{9}$$

$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r} \tag{10}$$

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k}=\binom{m+n}{m+r} \tag{11}$$

$$\sum_{k=n-s}^{m-r}\binom{m-k}{r}\binom{n+k}{s}=\binom{m+n+1}{r+s+1} \tag{12}$$

Here $C(n, r) = 0$ if $0 \leq n < r$.

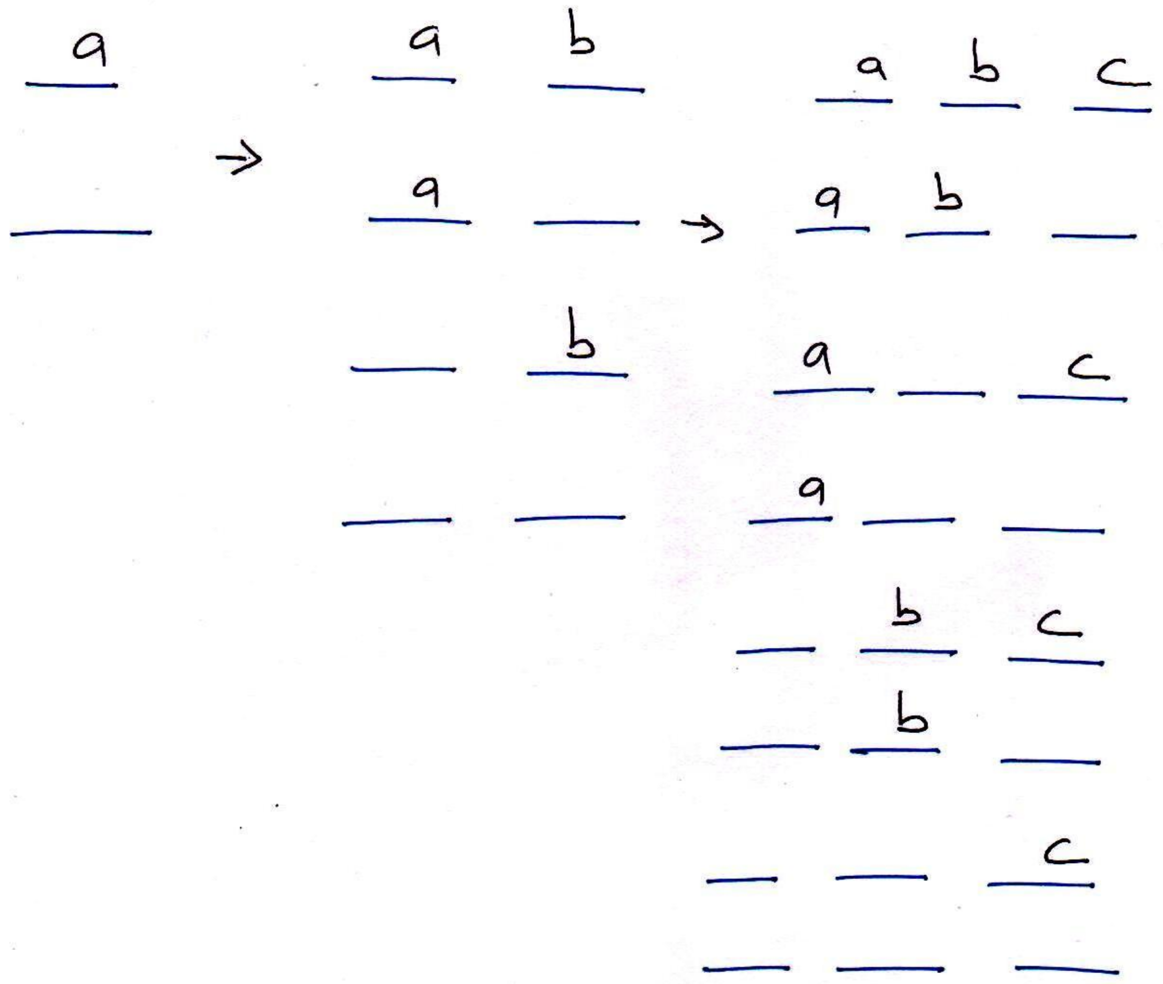
These identities can be explained by the "committee" type of combinatorial argument or by "block-walking arguments."

Using committee argument prove

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \tag{6}$$

1^{st} 2^{nd} 3^{rd} person

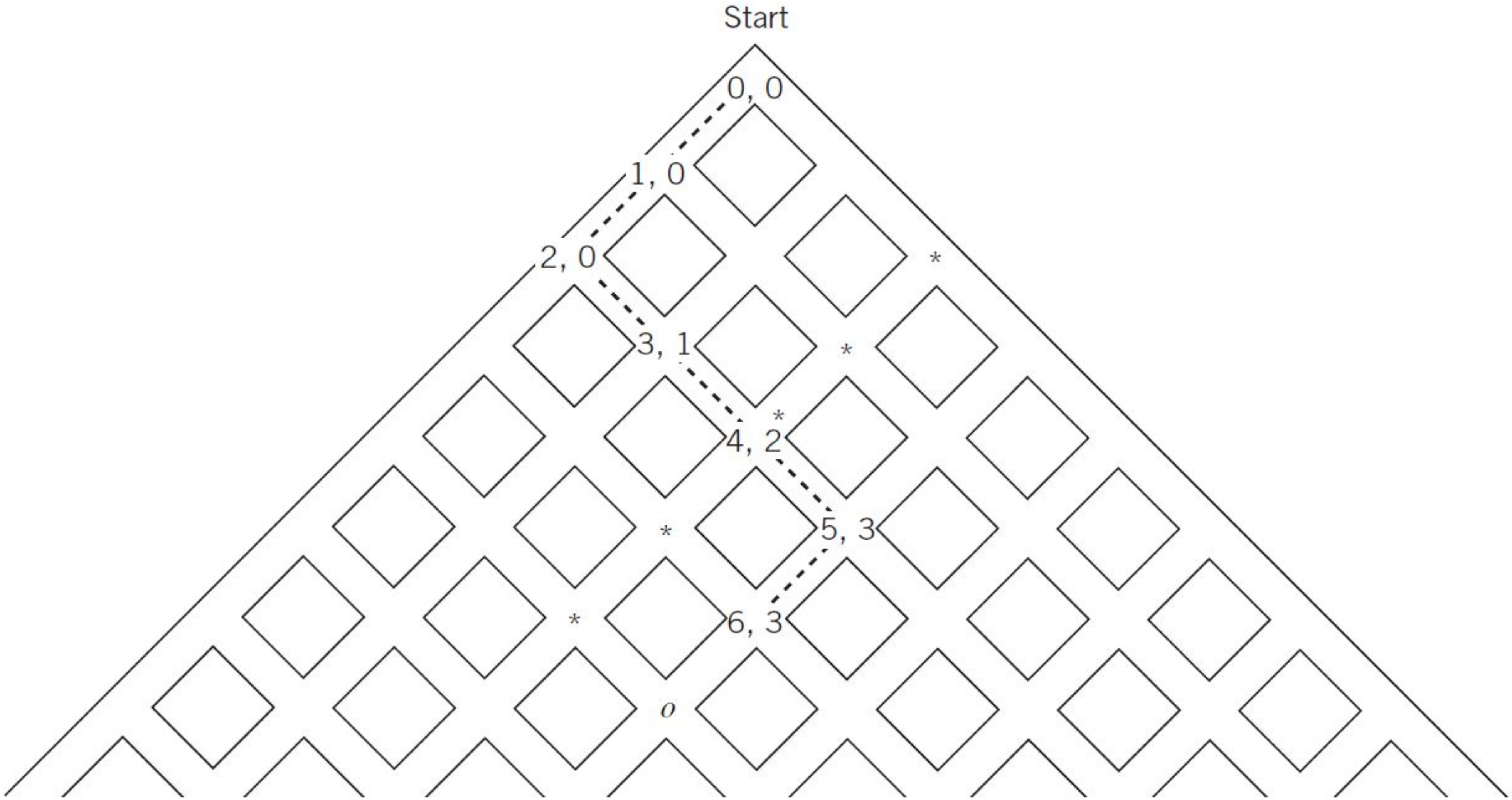
$\{a, b, c\}$



Verify identity (8) by block-walking argument.

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} \qquad (8)$$

As an example of this identity, we consider the case where r = 2 and n = 6.



From (k=2, r=2), there are two ways of reaching (7, 3)

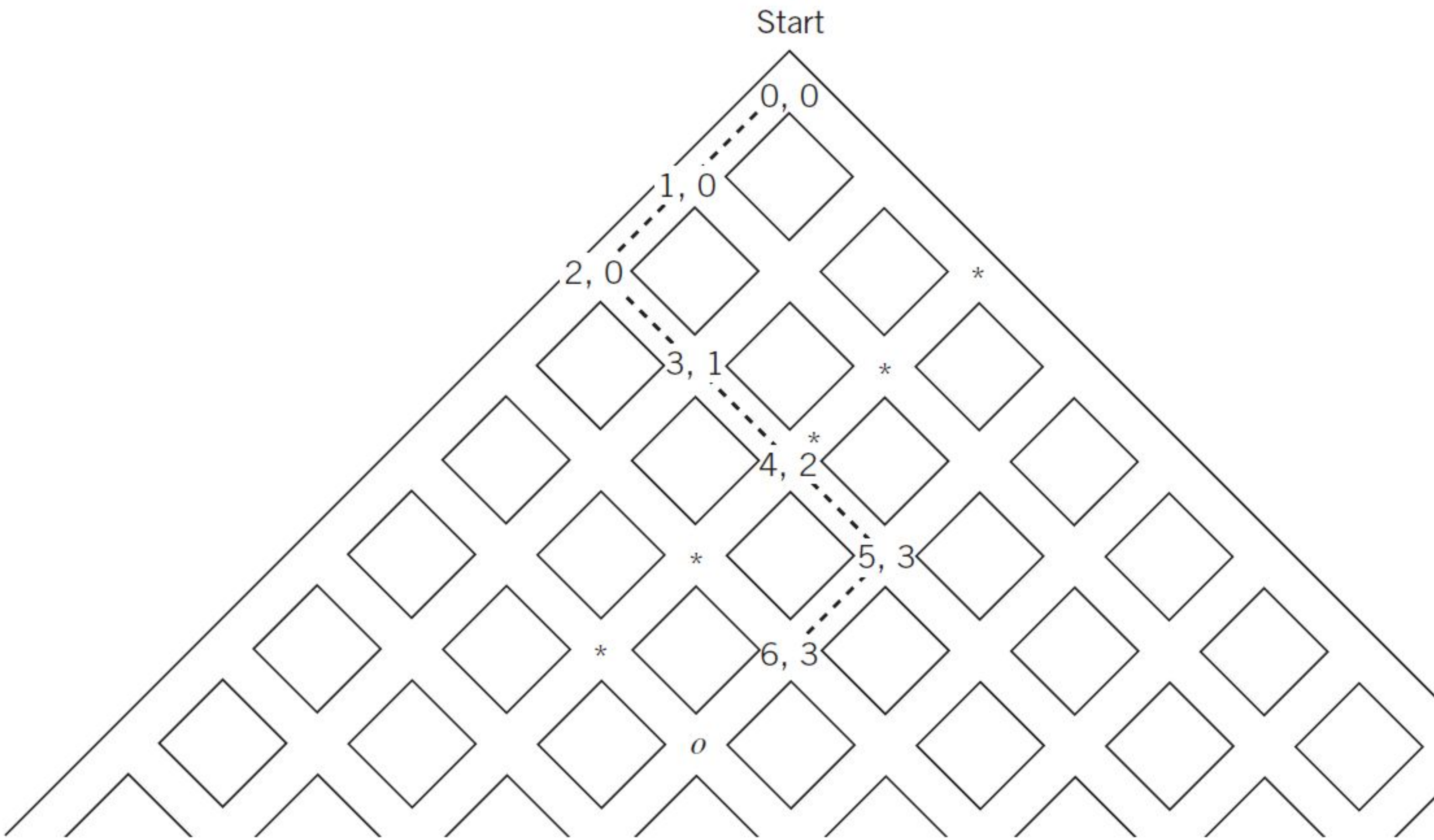
a) Starting by a left turn then left or right turns to reach (7, 3)

b) Starting by a right turn then left turns to reach (7, 3) (unique)

From (k>2, r=2), if we try to reach (7, 3) by first taking a left turn, then one of the route gets repeated, obtained in (a)

Hence, to reach (7, 3), we first consider all possible ways to reach (k>1, r=2), and then follows b).

This proves the required Identity.



The corners (k, 2), k = 2, 3, 4, 5, 6, are marked with a * in Figure 1 and corner (7, 3) is marked with an *o*.

<u>There are multiple ways to reach to the * but there is only one way to reach from * to *o*. (without repeatition)</u>

Observe that the right branches at each starred corner are the locations of last possible right branches on routes from the start (0, 0) to corner (7, 3).

After traversing one of these right branches, there is just one way to continue on to corner (7, 3), by making all remaining branches left branches.

In general, if we break all routes from (0, 0) to (n + 1, r + 1) into subcases based on the corner where the last right branch is taken, we obtain identity (8).

Verify identity (9) by a block-walking argument

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \tag{9}$$

The number of routes from (n, k) to (2n, n) is equal to number of routes from (0, 0) to (n, n − k), since both trips go a total of n blocks with n − k to the right (and k to the left).

So the number of ways to go from (0, 0) to (n, k) and then on to (2n, n) is C(n, k) × C(n, n − k).

By (2), C(n, n −k) = C(n, k), and thus the number of routes from (0, 0) to (2n, n) via (n, k) is C(n, k)^2.

Summing over all k—that is, over all intermediate corners n blocks from the start—we count all routes from (0, 0) to (2n, n).

So this sum equals C(2n, n), and identity (9) follows.

Using the following identities, evaluate the sum

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2)(n-1)n.$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \tag{6}$$

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r} \tag{7}$$

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} \tag{8}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \tag{9}$$

Evaluate the sum $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2)(n-1)n$.

The general term in this sum $(k-2)(k-1)k$ is equal to $P(k,\ 3) = k!/(k-3)!$.

Recall that the numbers of r-permutations and of r-selections differ by a factor of $r!$
That is, $C(k,\ 3) = k!/(k-3)!3! = P(k,\ 3)/3!$, or $P(k,\ 3) = 3!C(k,\ 3)$.

So the given sum can be rewritten as

$$3!\binom{3}{3} + 3!\binom{4}{3} + \cdots + 3!\binom{n}{3} = 3!\left(\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n}{3}\right)$$

By identity (8), this sum equals $3!\binom{n+1}{4}$. ■

Evaluate the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$.

A strategy for problems whose general term is not a multiple of C(n, k) or P(n, k) is to decompose the term algebraically into a sum of P(n, k)-type terms.
In this case, the general term k^2 can be written as $k^2 = k(k-1)+k$.
So the given sum can be rewritten as

$$\begin{aligned}
&[(1\times 0)+1]+[(2\times 1)+2]+[(3\times 2)+3]+\cdots+[n(n-1)+n] \\
&=[(2\times 1)+(3\times 2)+\cdots+n(n-1)]+(1+2+3+\cdots+n) \\
&=\left(2\binom{2}{2}+2\binom{3}{2}+\cdots+2\binom{n}{2}\right)+\left(\binom{1}{1}+\binom{2}{1}+\cdots+\binom{n}{1}\right) \\
&=2\binom{n+1}{3}+\binom{n+1}{2}
\end{aligned}$$

by identity (8).

Two Basic Counting Principles

Q. There are **5** different Spanish books, **6** different French books, and **8** different German books. How many ways are there to pick an (unordered) pair of two books not both in the same language?

TWO BASIC COUNTING PRINCIPLES

The Addition Principle

If there are r_1 different objects in the first set, r_2 different objects in the second set, ..., and r_m different objects in the mth set, and *if the different sets are disjoint,* then the number of ways to select an object from one of the m sets is $r_1 + r_2 + \cdots + r_m$.

The Multiplication Principle

Suppose a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in previous stages and *if the composite outcomes are all distinct,* then the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes.

How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f

a) with repetition of letters allowed?

(b) without repetition of any letter?

(c) without repetition and containing the letter e

use multiplication or additional principle

(d) with repetition and containing e

How many nonempty different collections can be formed from five (identical) apples and eight (identical) oranges?

Ans. 53

6.1: GENERATING FUNCTION MODELS

GENERATING FUNCTION

Suppose a_r is the number of ways to select r objects in a certain procedure. Then $g(x)$ is a **generating function** for a_r if $g(x)$ has the polynomial expansion

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r + \cdots + a_n x^n$$

If $a_r = C(n, r)$, what is the generating function.

How to compute the coefficient of x^5 in the expansion of $(1+x + x^2)^4$.

Find the generating function for a_r, the number of ways to select r balls from three green, three white, three blue, and three gold balls.

Find a generating function for the number of ways to select r doughnuts from five chocolate, five strawberry, three lemon, and three cherry doughnuts.

Repeat with the additional constraint that there must be at least one of each type.

Use a generating function to model the problem of counting all selections of six objects chosen from three types of objects with repetition of up to four objects of each type. Also model the problem with unlimited repetition.

6.2 CALCULATING COEFFICIENTS OF GENERATING FUNCTIONS

Table 6.1 Polynomial Expansions

(1) $\dfrac{1-x^{m+1}}{1-x} = 1 + x + x^2 + \cdots + x^m$

(2) $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$

(3) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$

(4) $(1-x^m)^n = 1 - \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + (-1)^k\binom{n}{k}x^{km} + \cdots + (-1)^n\binom{n}{n}x^{nm}$

(5) $\dfrac{1}{(1-x)^n} = 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \cdots + \binom{r+n-1}{r}x^r + \cdots$

(6) If $h(x) = f(x)g(x)$, where $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots$, then

$$h(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots$$

$$+ (a_rb_0 + a_{r-1}b_1 + a_{r-2}b_2 + \cdots + a_0b_r)x^r + \cdots$$

$$(1 + x + x^2 + x^3 + \cdots)^n \tag{7}$$

What is the coefficient of x^r in (7).

Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \cdots)^5$. What is the coefficient of x^r?

As a generating function, what does $(x^2+x^3+x^4+\cdots)^5$ represents.

It is the generating function a_r , for the number of ways to select r objects with repetition from five types with at least two of each type.
It is same as first picking two objects in each type—one way—and then counting the ways to select the remaining $r-10$ objects:
C((r −10) + 5−1, (r −10)) ways.

In above example,

we algebraically picked out an x^2 from each factor for a total of x^{10} and then found the coefficient of x^{r-10} in $(1+x+x^2+\cdots)^5$, the generating function for selection with unrestricted repetition of $r-10$ from five types.

The standard algebraic technique of extracting the highest common power of x from each factor corresponds to the "trick" used to solve the associated selection problem.
Such correspondences are a major reason for using generating functions: the algebraic techniques automatically do the combinatorial reasoning for us.

Use generating functions to find the number of ways to collect \$15 from 20 distinct people if each of the first 19 people can give a dollar (or nothing) and the twentieth person can give either \$1 or \$5 (or nothing).

The answer in Example 2 could be obtained directly by breaking the collection problem into three cases depending on how much the twentieth person gives: \$0 or \$1 or \$5. In each case, the subproblem is counting the ways to pick a subset of the other 19 people to obtain the rest of the \$15. The generating function approach automatically breaks the problem into three cases and solves each, doing all the combinatorial reasoning for us.

Using generating function, find how many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any number can go into each of the other six boxes?

How many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any number can go into each of the other six boxes?

We first count all the ways to distribute without restriction the 25 balls into the seven boxes,
$C(25+7-1, 25)$ ways,
and then subtract the distributions that violate the first box constraint, that is, distributions with at least 11 balls in the first box,
$C((25-11)+7-1, (25-11))$ (first put 11 balls in the first box and then distribute the remaining balls arbitrarily).

Again, generating functions automatically performed this combinatorial reasoning.

How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

At this stage, it is not obvious to solve it using combinatorial techniques.

Using generating function, verify the binomial identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

In how many ways can we fill a bag with n fruits such that

- The number of apples must be even
- The number of bananas must be a multiple of 5
- There can be at most four oranges
- There can be at most one pear

This problem is again extremely difficult using combinatorial approaches.

As an example, there are 7 ways to form a bag with 6 fruits:

Apples	6	4	4	2	2	0	0
Bananas	0	0	0	0	0	5	5
Oranges	0	2	1	4	3	1	0
Pears	0	0	1	0	1	0	1

6.3 PARTITIONS

A **partition** of a group of r identical objects divides the group into a collection of (unordered) subsets of various sizes.
Analogously, we define a partition of the integer r to be a collection of positive integers whose sum is r.

Compute the number of partitions of 5?

$$
\begin{aligned}
5 &= 5 \\
&= 4+1 \\
&= 3+2 \\
&= 3+1+1 \\
&= 2+2+1 \\
&= 2+1+1+1 \\
&= 1+1+1+1+1
\end{aligned}
$$

Construct a generating function for a_r , the number of partitions of the integer 5. First model it as integer-solution-to-an-equation problem.

Find the generating function for a_r , the number of ways to express r as a sum of distinct integers.

Find a generating function for a_r, the number of ways that we can choose 2¢, 3¢, and 5¢ stamps adding to a net value of r¢.

Using generating function, show that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

Find the generating function for a_r , the number of ways to express r as a sum of distinct powers of 2.

How many ways are there to express n as a sum of integers no greater than k?

How many ways are there to express n as sum of integers no greater than k, one of which is k itself?

The number of ways to partition n into nonzero parts of which the largest is k is equal to the number of ways to partition n into k nonzero parts

as in this example for n = 8 and k = 3:

$$\begin{aligned}
3+1+1+1+1+1 &\leftrightarrow 6+1+1 \\
3+2+1+1+1 &\leftrightarrow 5+2+1 \\
3+2+2+1 &\leftrightarrow 4+3+1 \\
3+3+1+1 &\leftrightarrow 4+2+2 \\
3+3+2 &\leftrightarrow 3+3+2
\end{aligned}$$

What do you observe

To exhibit this relationship, we have recourse to a visual technique for presenting partitions:

Definition 1. The *Ferrers diagram* for the partition $a_1 + a_2 + a_3 + \cdots + a_k$ for $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k > 0$ consists of k left-justified rows of equally-spaced dots with a_i dots in the ith row, for each i.

For instance, here we have a Ferrers diagram for $3 + 2 + 2 + 1$:

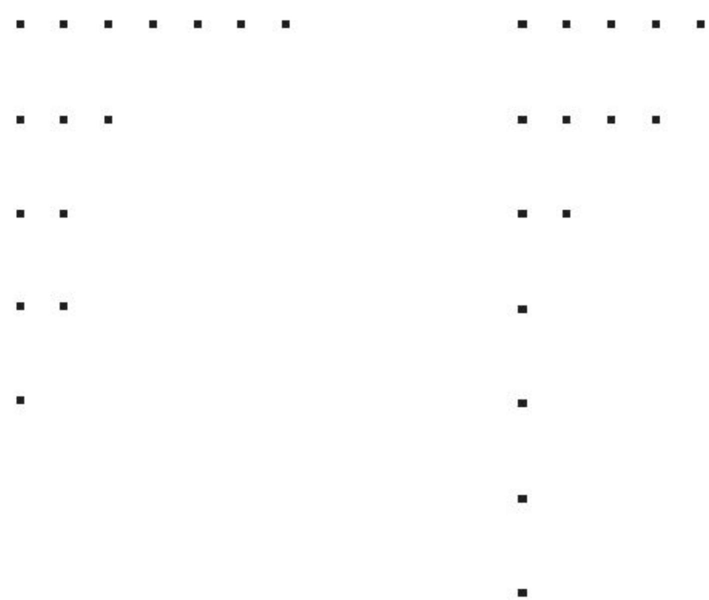
$$\begin{array}{lll}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \\
\bullet & \bullet & \\
\bullet & &
\end{array}$$

The partition 1+2+2+3+7 of 15 is shown in the Ferrers diagram in Figure 6.1a.

If we **transpose** the rows and columns of a Ferrers diagram of a partition of r, we get a Ferrers diagram of another partition of r.

This diagram is called the **conjugate** of the original Ferrers diagram.

For example, Figure 6.1b shows the conjugate of the Ferrers diagram in Figure 6.1a. Here the partition of 15 is 1+1+1+1+2+4+5.



6.1 (*a*) (*b*)

Show that the number of partitions of an integer r as a sum of m positive integers is equal to the number of partitions of r as a sum of positive integers, the largest of which is m.

If we draw a Ferrers diagram of a partition of r into m parts, then the Ferrers diagram will have m rows.
The transposition of such a diagram will have m columns, that is, the largest row will have m dots.

Thus there is a one-to-one correspondence between these two classes of partitions.

A partition is **self-conjugate** if it is equal to its conjugate, or in other words, if its Ferrers diagram is symmetric about the diagonal.
For example, the Ferrers diagram for the partition 10 = 4+3+3+1 is self-conjugate (see Figure below) .

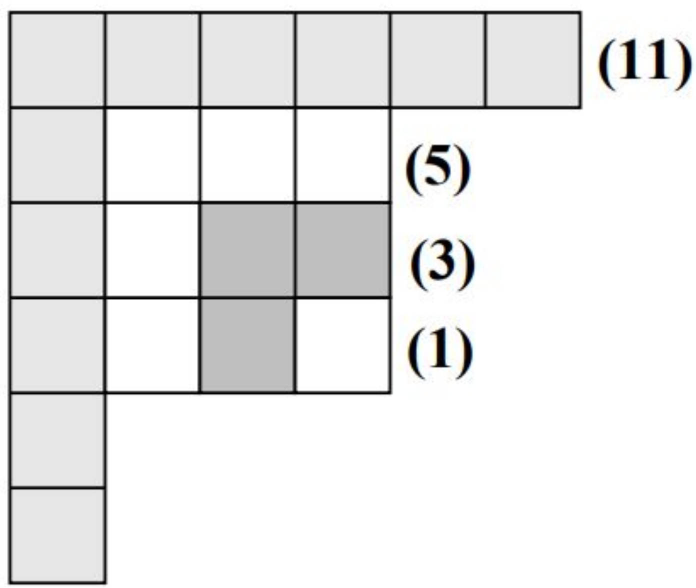
```
*                                       *
* * *                                   * * *
* * *                                   * * *
* * * *       with conjugate            * * * *
```

Prove that the number of partitions of n into parts that are both odd and distinct is equal to the number of self-conjugate partitions of n.

A generating function for the first object is

$$\prod_{j=0}^{\infty}(1+x^{2j+1}),$$

But a generating function for the latter object is not obvious.
However, using Ferrers diagrams, a bijective proof is straightforward.
The general idea is to ‘bend’ each odd, distinct part at the middle cell and then join the bent pieces together.
This yields a self-conjugate partition, a process that is clearly reversible.
As an example, the partition of 20 into the odd, distinct parts 11+5+3+1 is illustrated in Figure below.



Converting the partition $20 = 11 + 5 + 3 + 1$ into one that is self-conjugate

6.4 EXPONENTIAL GENERATING FUNCTIONS

The generating functions used in the previous sections to model selection with repetition problems are called **ordinary generating functions.**

In this section we discuss **exponential generating functions**.
They are used to model and solve problems involving arrangements with repetition.

Consider the problem of finding the number of different words (arrangements) of four letters when the letters are chosen from an unlimited supply of ***a***s, ***b***s and ***c***s, and the word must contain at least two ***a***s.

What is the generating function and corresponding equation for the given problem.

An **exponential generating function** $g(x)$ for a_r, the number of arrangements with r objects, is a function with the power series expansion

$$g(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots + a_r \frac{x^r}{r!} + \cdots$$

We build exponential generating functions in the same way that we build ordinary generating functions,

However, now each power x^r is divided by r!.

As an example, let us consider the four-letter word problem with at least two as.

We claim that the exponential generating function for the number of r-letter words formed from an unlimited number of as, bs, and cs containing at least two as is

Find the exponential generating function for a_r, the number of different arrangements of r objects chosen from four different types of objects with each type of object appearing at least two and no more than five times.

Find the generating function for a_r , the number of r arrangements without repetition of n objects. Hence, find a_r

Find the generating function for the number of ways to place r (distinct) people into three different rooms with at least one person in each room. Repeat with an even number of people in each room.

Using generating function, find the number of different arrangements of r objects chosen from unlimited supplies of n types of objects.

Using generating function, find the number of sequences of length 8 that can be formed using 1,2, or 3 a's; 2,3, or 4 b's; and 0,2, or 4 c's.

Using exponential generating function, find the number of ways to place 25 people into three rooms with at least one person in each room.

Using exponential generating function, find the number of r-digit quaternary sequences (whose digits are 0, 1, 2, and 3) with an even number of 0s and an odd number of 1s.

Find the number of ways to distribute n different objects to five different boxes if

a) An even number of objects are distributed to box 5.

b) A positive even number of objects are distributed to box 5.

6.5 A SUMMATION METHOD

In this section we show how to construct an ordinary generating function $h(x)$ whose coefficient of x^r is some specified function $p(r)$ of r, such as r^2 or $C(r, 3)$.

How to construct generating function $g^*(x)$ with $a^*_r = ra_r$ from the following generating function:

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r + \cdots$$

The next question is from where to start, i.e., what could be initial g(x).

The natural answer is: When in doubt start with the unit coefficients $a_r = 1$ of the generating function

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$$

Build a generating function $h(x)$ with $a_r = 2r^2$.

Build a generating function $h(x)$ with $a_r = (r+1)r(r-1)$.

We could multiply $(r+1)r(r-1)$ out getting $a_r = r^3 - r$, obtain generating functions for r^3 and r as an Example 1, and then subtract one generating function from the other. It is easier, however, to start with $3!(1-x)^{-4}$, whose coefficient a_r equals

$$a_r = 3!\binom{r+4-1}{r} = 3!\frac{(r+3)!}{r!3!} = \frac{(r+3)!}{r!} = (r+3)(r+2)(r+1)$$

Theorem

If $h(x)$ is a generating function where a_r is the coefficient of x^r, then $h^*(x) = h(x)/(1-x)$ is a generating function of the sums of the a_rs. That is,

$$h^*(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + \left(\sum_{i=0}^{r} a_i\right)x^r + \cdots$$

Using generating function

Evaluate the sum $2 \times 1^2 + 2 \times 2^2 + 2 \times 3^2 + \cdots + 2n^2$.

Using generating function

Evaluate the sum $3 \times 2 \times 1 + 4 \times 3 \times 2 + \cdots + (n+1)n(n-1)$.

7.1 RECURRENCE RELATION MODELS

We now consider more complex recurrence relations involving two variable relations.

Let $a_{n,\,k}$ denote the number of ways to select a subset of k objects from a set of n distinct objects. Find a recurrence relation for $a_{n,\,k}$.

Find a recurrence relation for the ways to distribute n identical balls into k distinct boxes with between two and four balls in each box. Repeat the problem with balls of three colors.

The problem with all balls identical can be solved by generating functions, but recurrence relations are the only practical approach with the extra constraint of different types of balls.

Find recurrence relations for

(a) The number of n-digit ternary sequences with an even number of 0s

(b) The number of n-digit ternary sequences with an even number of 0s and an even number of 1s

(b) The number of n-digit ternary sequences with an even number of 0s and an even number of 1s

We will need simultaneous 3 recurrence relations:

a_n : the number of n-digit ternary sequences with even 0s and even 1s;

b_n : the number of n-digit ternary sequences with even 0s and odd 1s;

c_n : the number of n-digit sequences with odd 0s and even 1s.

Why not for odd 0s and odd 1s.

Observe that $3^n - a_n - b_n - c_n$ is the number of n-digit ternary sequences with odd 0s and odd 1s.

a_n is obtained either by having a

1 for the first digit followed by (n −1)-digit sequence with even 0s and odd 1s,

or a 0 followed by (n −1)-digit sequence with odd 0s and even 1s,

or a 2 followed by an (n −1)-digit sequence with even 0s and even 1s.

Thus $a_n = b_{n-1} + c_{n-1} + a_{n-1}$.

But from where we get b_{n-1} and c_{n-1}

Similar analyses yield

$b_n = a_{n-1} + (3^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) + b_{n-1} = 3^{n-1} - c_{n-1}$ and

$c_n = a_{n-1} + (3^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) + c_{n-1} = 3^{n-1} - b_{n-1}$

The initial conditions are $a_1 = b_1 = c_1 = 1$.

To recursively compute values for a_n, we must simultaneously compute b_n and c_n.

7.1 RECURRENCE RELATION MODELS

We now consider more complex recurrence relations involving two variable relations.

Let $a_{n, k}$ denote the number of ways to select a subset of k objects from a set of n distinct objects. Find a recurrence relation for $a_{n, k}$.

Find a recurrence relation for the ways to distribute n identical balls into k distinct boxes with between two and four balls in each box. Repeat the problem with balls of three colors.

The problem with all balls identical can be solved by generating functions, but recurrence relations are the only practical approach with the extra constraint of different types of balls.

Find recurrence relations for

(a) The number of n-digit ternary sequences with an even number of 0s

(b) The number of n-digit ternary sequences with an even number of 0s and an even number of 1s

(b) The number of n-digit ternary sequences with an even number of 0s and an even number of 1s

We will need simultaneous 3 recurrence relations:

a_n : the number of n-digit ternary sequences with even 0s and even 1s;
b_n : the number of n-digit ternary sequences with even 0s and odd 1s;
c_n : the number of n-digit sequences with odd 0s and even 1s.

Why not for odd 0s and odd 1s.

Observe that $3^n - a_n - b_n - c_n$ is the number of n-digit ternary sequences with odd 0s and odd 1s.

a_n is obtained either by having a

1 for the first digit followed by (n −1)-digit sequence with even 0s and odd 1s,

or a 0 followed by (n −1)-digit sequence with odd 0s and even 1s,

or a 2 followed by an (n −1)-digit sequence with even 0s and even 1s.

Thus $a_n = b_{n-1} + c_{n-1} + a_{n-1}$.

But from where we get b_{n-1} and c_{n-1}

Similar analyses yield

$b_n = a_{n-1} + (3^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) + b_{n-1} = 3^{n-1} - c_{n-1}$ and

$c_n = a_{n-1} + (3^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) + c_{n-1} = 3^{n-1} - b_{n-1}$

The initial conditions are $a_1 = b_1 = c_1 = 1$.

To recursively compute values for a_n, we must simultaneously compute b_n and c_n.

7.1 RECURRENCE RELATION MODELS

Recurrence Relation

A recurrence relation for a sequence $A = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ is a formula that relates a_n to one or more of the preceding terms $a_0, a_1, \ldots, a_{n-1}$ in a uniform way, for any integer n greater than or equal to some initial integer k. The values of the first terms needed to start computing with a recurrence relation are called the initial conditions.

Find a recurrence relation for the number of ways to arrange n distinct objects in a row. Find the number of arrangements of eight objects.

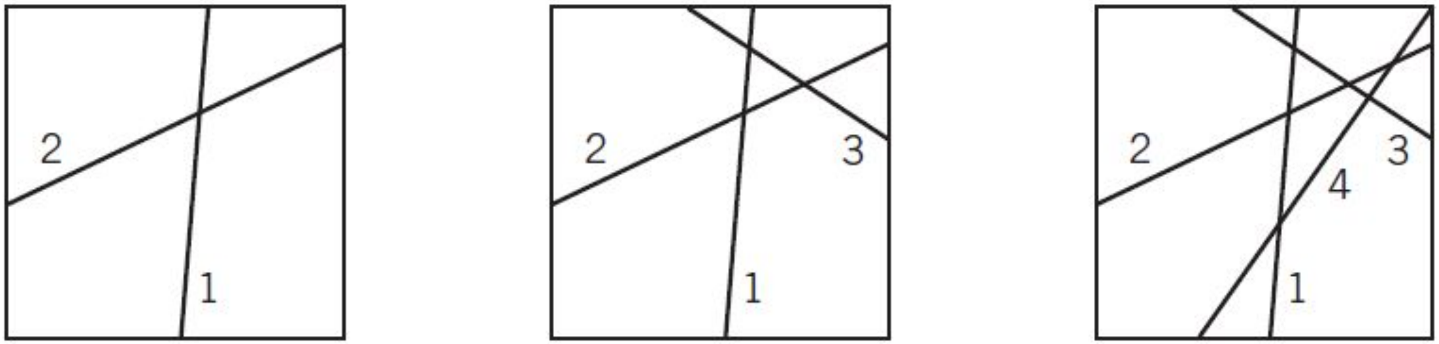
An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation for a_n, the number of different ways for the elf to ascend the n-stair staircase.

Find a recurrence relation for the number of different ways to hand out a piece of chewing gum (worth 1$) or a candy bar (worth 10$) or a doughnut (worth 20$) on successive days until n$ worth of food has been given away.

Suppose we drawn straight lines on a piece of paper so that every pair of lines intersect (but no three lines intersect at a common point). Into how many regions do these n lines divide the plane?

With one line, the paper is divided into two regions.

With two lines, we get four regions—that is, $a_2 = 4$. $\mathbf{a_3 = ?}$ $\mathbf{a_3 = a_2 + ?}$



(a) (b) (c)

Now $a_3 = 7$ (why).

3rd line cross each of the other two lines (at different points) and cuts through three of the regions formed by the first two lines, which form three new regions. Thus

Thus $a_3 = a_2 + 3 = 4 + 3 = 7$.

Let a_n be the number of regions into which the plane is divided by n lines. It means n – 1 lines are dividing the plane into a_{n-1} regions.

When n^{th} line is drawn, it intersects each of the previous lines and divide n of the previous regions into half, creating n additional regions. Thus,

$\mathbf{a_n = a_{n-1} + n}$, $n > 1$.

Find a recurrence relation for the number of binary sequences of length n with no consecutive 0's.

Find a recurrence relation for the number of binary sequences of length n with no block of three consecutive 0's.

Let a_n be the number of binary sequences of length n that contain no block of three consecutive 0's. A sequence of length n with no three consecutive 0's may start with (look for all possibilities):

: 1 and be followed by any sequence of length n – 1 that contains no three consecutive 0's,
: or it may start with a 01 and be followed by any sequence of length n – 2 that contains no three consecutive 0's,
:or it may start with a 001 and be followed by any sequence of length n – 3 that contains no three consecutive 0's.
In this case, we obtain the recurrence relation
$\mathbf{a_n = a_{n-1} + a_{n-2} + a_{n-3}}$
satisfying the initial conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.

A single coin is flipped n times. Each outcome is represented by a sequence of n digits; each digit is an H or a T. Find a recurrence relation for the number of outcomes with at least two consecutive heads.

Find a recurrence relation for a_n, the number of n-digit ternary sequences without any occurrence of the subsequence "012."

A ternary sequence is a sequence composed of 0s, 1s, and 2s.

The recurrence relations we discussed involved only one variable and a_n can be easily computed using **recursive backward substitution** or **Mathematical Induction.**

Using recursive backward substitution, the relation $a_n = a_{n-1} + n$ becomes $a_n = (a_{n-2} + n - 1) + n = \cdots = 1+2+3 + \cdot \cdot \cdot + n - 1 + n.$

Using, Mathematical Induction solve the recurrence relation

$a_n = 2a_{n-1} + 1,\ a_1 = 1.$

7.1 RECURRENCE RELATION MODELS

Recurrence Relation

A recurrence relation for a sequence $A = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ is a formula that relates a_n to one or more of the preceding terms $a_0, a_1, \ldots, a_{n-1}$ in a uniform way, for any integer n greater than or equal to some initial integer k. The values of the first terms needed to start computing with a recurrence relation are called the initial conditions.

Find a recurrence relation for the number of ways to arrange n distinct objects in a row. Find the number of arrangements of eight objects.

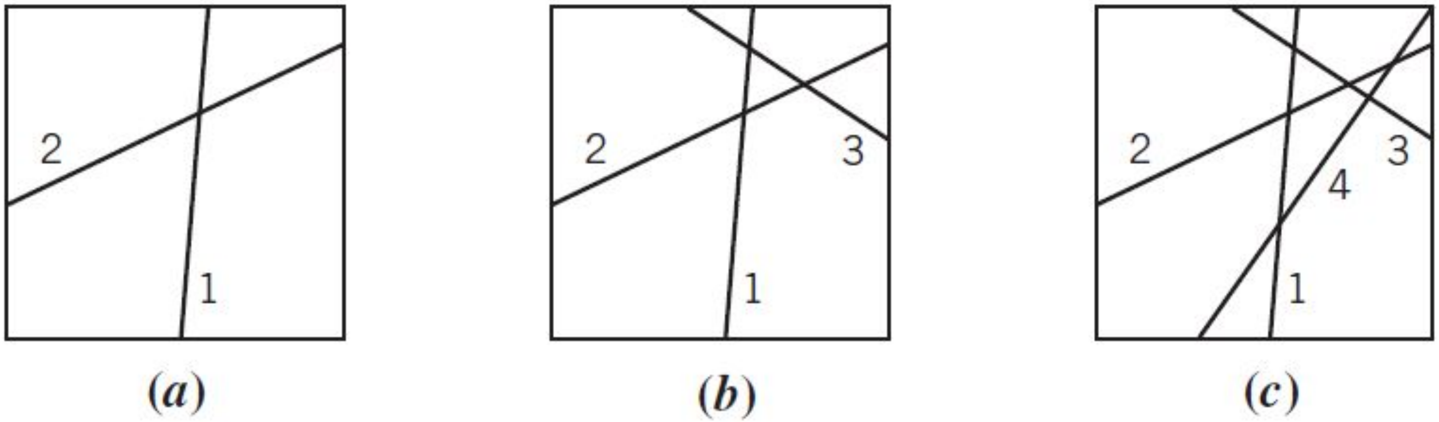
An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation for a_n, the number of different ways for the elf to ascend the n-stair staircase.

Find a recurrence relation for the number of different ways to hand out a piece of chewing gum (worth 1$) or a candy bar (worth 10$) or a doughnut (worth 20$) on successive days until n$ worth of food has been given away.

Suppose we drawn straight lines on a piece of paper so that every pair of lines intersect (but no three lines intersect at a common point). Into how many regions do these n lines divide the plane?

With one line, the paper is divided into two regions.

With two lines, we get four regions—that is, $a_2 = 4$. $\mathbf{a_3 = ?\ a_3 = a_2 + ?}$



Now $a_3 = 7$ (why).

3rd line cross each of the other two lines (at different points) and cuts through three of the regions formed by the first two lines, which form three new regions. Thus

Thus $a_3 = a_2 + 3 = 4 + 3 = 7$.

Let a_n be the number of regions into which the plane is divided by n lines. It means n – 1 lines are dividing the plane into a_{n-1} regions.

When n^{th} line is drawn, it intersects each of the previous lines and divide n of the previous regions into half, creating n additional regions. Thus,

$\mathbf{a_n = a_{n-1} + n}$, $n > 1$.

Find a recurrence relation for the number of binary sequences of length n with no consecutive 0's.

Find a recurrence relation for the number of binary sequences of length n with no block of three consecutive 0's.

Let a_n be the number of binary sequences of length n that contain no block of three consecutive 0's. A sequence of length n with no three consecutive 0's may start with (look for all possibilities):

: 1 and be followed by any sequence of length n – 1 that contains no three consecutive 0's,

: or it may start with a 01 and be followed by any sequence of length n – 2 that contains no three consecutive 0's,

:or it may start with a 001 and be followed by any sequence of length n – 3 that contains no three consecutive 0's.

In this case, we obtain the recurrence relation

$\mathbf{a_n = a_{n-1} + a_{n-2} + a_{n-3}}$

satisfying the initial conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.

A single coin is flipped n times. Each outcome is represented by a sequence of n digits; each digit is an H or a T. Find a recurrence relation for the number of outcomes with at least two consecutive heads.

Find a recurrence relation for a_n, the number of n-digit ternary sequences without any occurrence of the subsequence “012.”

A ternary sequence is a sequence composed of 0s, 1s, and 2s.

The recurrence relations we discussed involved only one variable and a_n can be easily computed using **recursive backward substitution** or **Mathematical Induction.**

Using recursive backward substitution, the relation $a_n = a_{n-1} + n$ becomes $a_n = (a_{n-2} + n - 1) + n = \cdots = 1+2+3 + \cdot\cdot\cdot + n - 1 + n$.

Using, Mathematical Induction solve the recurrence relation $a_n = 2a_{n-1} + 1$, $a_1 = 1$.

HOMOGENEOUS LINEAR RECURRENCES

Linear Recurrences

If $A = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ is a sequence, we say that A satisfies a linear recurrence relation with constant coefficients if for all $n \geq m$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m} + g(n)$, where $c_1, c_2, \ldots, c_m$ is some set of constants and $g(n)$ is a function depending on n. If $c_m \neq 0$, we say that the recurrence relation has order m, and in case $g(n) = 0$, we say the recurrence is homogeneous.

In this section we study methods for solving homogeneous recurrence relations, which usually means that we want to find a solution $a_n = f(n)$ of the recurrence that satisfies the initial conditions $a_0 = b_0$, $a_1 = b_1, \ldots, a_{m-1} = b_{m-1}$, for a given set of constants $b_0, b_1, b_2, \ldots, b_{m-1}$.

Solve $a_n = ra_{n-1}$ with $a_0 = k$.

$\mathbf{a_n = kr^n}$

In this section we show how to solve recurrence relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} \tag{1}$$

The general solution to (1) will involve a sum of individual solutions of the form $a_n = \alpha^n$.

Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ with $a_0 = a_1 = 1$.

Find a formula for the number of ways for the elf in Example 2 of Section 7.1 to climb the n stairs.

The recurrence relation obtained in Example 2 of Section 7.1 was the Fibonacci relation $a_n = a_{n-1} + a_{n-2}$, with the initial conditions $a_1 = 1$, $a_2 = 2$, or equivalently, $a_0 = a_1 = 1$. The associated characteristic equation is obtained by setting $a_n = \alpha^n$:

The Case of Repeated Roots

In the first part of this section, we saw that when the characteristic equation of a homogeneous linear recurrence relation had distinct roots, a general solution could be obtained by looking at linear combinations of m fundamental solutions, one for each characteristic root.

In this case, for each characteristic root, there was an associated fundamental solution $\mathbf{a_n = r_i^n}$.

However, when the characteristic equation has repeated roots, two or more fundamental solutions coincide. Therefore we have fewer than m fundamental solutions that are exponential functions. Thus the linear combinations of exponential functions cannot form a general solution because there are not enough fundamental solutions.

It is still possible, though, to find a general solution for a homogeneous linear recurrence even when the characteristic equation has repeated roots by looking at a slightly larger class of fundamental solutions.

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

Find a formula for a_n satisfying the relation $a_n = -2a_{n-2} - a_{n-4}$ with $a_0 = 0, a_1 = 1, a_2 = 2$, and $a_3 = 3$.

Non-HOMOGENEOUS LINEAR RECURRENCE RELATIONS

Find a particular solution of the nonhomogeneous linear recurrence $a_n = a_{n-1} + a_{n-2} + 2n$.

In this section, we discuss methods for solving inhomogeneous recurrence relations of the form of (1). The key idea in solving these relations is that a general solution for an inhomogeneous relation is made up of a general solution to the associated homogeneous relation [obtained by deleting the g(n) term] plus any one particular solution to the inhomogeneous relation.

Find a solution of the nonhomogeneous linear recurrence relation $a_n = a_{n-1} + 2a_{n-2} - 4$ satisfying the initial conditions $a_0 = 6$ and $a_1 = 7$.

Find a particular solution of the nonhomogeneous linear recurrence relation $a_n = a_{n-1} + 3a_{n-2} + a_{n-3} + 3^n$.

Find the general solution of $a_n = 2a_{n-1} + 2^n$.

Normally, when the nonhomogeneous part of a linear recurrence relation $g(n)$ is an exponential function kb^n, we can expect to obtain a particular solution that is also an exponential function. This is obviously not the case when b is a characteristic root of the associated homogeneous linear recurrence relation. However, in this case, it will still be possible to find a particular solution of a slightly different form. When b is a root of multiplicity m, we can show that there will always be a particular solution of the form $p(n) = cn^m b^n$.

Solve the recurrence relation $a_n = 3a_{n-1} - 4n + 3 \times 2^n$ to find its general solution. Also find the solution when $a_1 = 8$.

INCLUSION–EXCLUSION
(COUNTING WITH VENN DIAGRAMS)

If a school has 100 students with 50 students taking French, 40 students taking Latin, and 20 students taking both languages, how many students take no language?

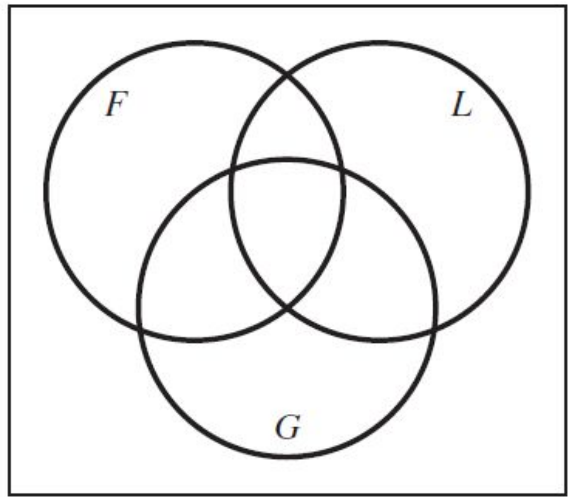
$$N(\overline{F} \cap \overline{L}) = N - N(F \cup L) = N - N(F) - N(L) + N(F \cap L) \tag{2}$$

Formula (2) is the 2-set version of the general n-set inclusion–exclusion formula that we present in the next section. It is called an inclusion–exclusion formula because first we include the whole set, N; then we exclude (subtract) the single sets F and L; and then we include (add) the 2-set intersection $F \cap L$. With more sets, this alternating inclusion and exclusion process will continue several rounds.

How many arrangements of the digits 0, 1, 2, . . . , 9 are there in which the first digit is greater than 1 and the last digit is less than 8?

For the inclusion–exclusion formula, we need to define sets that represent the complements of the original constraints we are given.

What would be $N(\overline{F} \cap \overline{L} \cap \overline{G})$



$$N(\overline{F} \cap \overline{L} \cap \overline{G}) = N - [N(F) + N(L) + N(G)] + [N(F \cap L) + N(L \cap G) + N(F \cap G)] - N(F \cap L \cap G) \quad (4)$$

For general sets A_1, A_2, A_3, we rewrite (4) as

$$N(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3) = N - \sum_i N(A_i) + \sum_{ij} N(A_i \cap A_j) - N(A_1 \cap A_2 \cap A_3) \quad (5)$$

How many n-digit ternary (0, 1, 2) sequences are there with at least one 0, at least one 1, and at least one 2? How many n-digit ternary sequences with at least one void (missing digit)?

Now we turn to the second part of this problem involving n-digit ternary sequences with at least one void.

The phrase "at least" is used in a very different way here than it was used in the first part. At least one void means a void of the digit 0 or a void of the digit 1 or a void of digit 2.

In terms of the A_i defined above, we want to count the union of the A_i's—namely, $N(A_0 \cup A_1 \cup A_2)$.

$N(A_0 \cup A_1 \cup A_2) = N - N(\overline{A}_0 \cap \overline{A}_1 \cap \overline{A}_2) = 3^n - (3^n - 3 \times 2^n + 3)$

$$= 3 \times 2^n - 3 \quad \blacksquare$$

Here is an example of a counting problem that cannot be solved by methods developed in the three previous chapters.

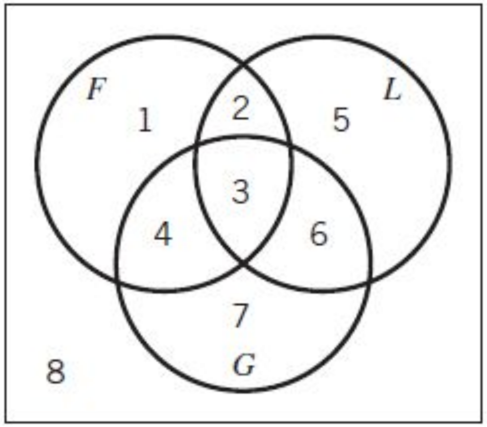
How many positive integers ≤70 are relatively prime to 70?
("relatively prime to 70" means "have no common divisors with 70.")

Let U be the set of integers between 1 and 70.
The prime divisors of 70 are 2, 5, and 7. We want to count the number of integers ≤70 that do not have 2 or 5 or 7 as divisors.
What would be A_1 A_2 A_3
Let A_1 be the set of integers in U that are evenly divisible by 2, or equivalently, integers in U that are multiples of 2; A_2 be integers evenly divisible by 5; and A_3 be integers evenly divisible by 7.
Then the number of positive integers ≤70 that are relatively prime to 70 equals $N(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3)$

Suppose there are 100 students in a school and there are 40 students taking each language, French, Latin, and German. Twenty students are taking only French, 20 only Latin, and 15 only German. In addition, 10 students are taking French and Latin. How many students are taking all three languages? No language?

Here Inclusion-exclusion formula cannot be applied directly.

We draw the Venn diagram for this problem and number each region as shown in Figure below.
Let N_i denote the number of students in region i, for i = 1, 2, . . . , 8.



Given

$N_1 = 20$, $N_5 = 20$, $N_7 = 15$, $N_2 + N_3 = 10$.

The set of students taking French is F, which consists of regions 1, 2, 3, and 4. So N_4?

$40 = N(F) = N_1 + (N_2 + N_3) + N_4 = 20 + (10) + N_4$, or $N_4 = 10$

Similarly, set L consists of regions 2, 3, 5, and 6, and so N_6?

$40 = N(L) = (N_2 + N_3) + N_5 + N_6 = (10) + 20 + N_6$, implying $N_6 = 10$.

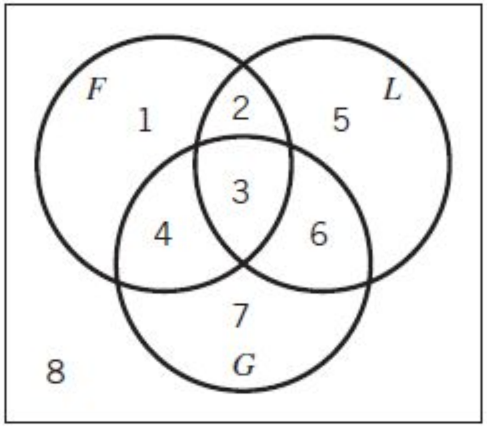
Since G consists of regions 3, 4, 6, and 7, and since we were given $N_7 = 15$ and have just found that $N_4 = N_6 = 10$, then $40 = N(G) = N_3 + N_4 + N_6 + N_7 = N_3 + 10 + 10 + 15$ or $N_3 = 5$.

But region 3 is the subset $F \cap L \cap G$ of students taking all 3 languages.

Thus, there are five trilingual students.

Now $N_2 + N_3 = 10$ and N_3 was found to be 5;

thus $N_2 = 5$.



$$N_8 = N(\overline{F \cap L \cap G}) \text{ is the number of students taking no language.}$$

Since all regions total to N, then

$$N_8 = N - \sum_{i=1}^{7} N_i = 100 - (20 + 5 + 5 + 10 + 20 + 10 + 15)$$
$$= 100 - 85 = 15 \quad \blacksquare$$

INCLUSION–EXCLUSION FORMULA

In this section we generalize the inclusion–exclusion formula for counting $N(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3)$ to n sets $A_1, A_2, \ldots, A_n$. To simplify notation, we will omit the intersection symbol "$\cap$" in expressions and write intersected sets as a product. For example, $A_1 \cap A_2 \cap A_3$ would be written $A_1 A_2 A_3$. Using this new notation, the number of elements in none of the sets $A_1, A_2, \ldots, A_n$ will be written $N(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n)$.

Theorem 1 **Inclusion–Exclusion Formula**

Let $A_1, A_2, \cdots A_n$, be n sets in a universe $\mathcal{U}$ of N elements. Let S_k denote the sum of the sizes of all k-tuple intersections of the A_is. Then

$$N(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n \qquad (1)$$

To clarify the definition of the S_ks, $S_1 = \sum_i N(A_i)$, $S_2 = \sum_{ij} N(A_i A_j)$, S_k is the sum of the $N(A_{j_1} A_{j_2} \cdots A_{j_k})$s for all sets of k A_j's, and finally $S_n = N(A_1 A_2 \cdots A_n)$. We

$$N(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3) = N - \sum_i N(A_i) + \sum_{ij} N(A_i \cap A_j) - N(A_1 \cap A_2 \cap A_3)$$

Corollary

Let A_1, $A_2 \cdots A_n$ be sets in the universe $\mathcal{U}$. Then

$$\begin{aligned} N(A_1 \cup A_2 \cup \cdots \cup A_n) = {} & S_1 - S_2 + S_3 \\ & - \cdots + (-1)^{k-1} S_k + \cdots + (-1)^{n-1} S_n \end{aligned} \tag{4}$$

How many ways are there to select a 6-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

For n = 4

$$N(\overline{A}_1\overline{A}_2\cdots\overline{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n$$

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4) = \binom{52}{6} - 4\binom{39}{6} + 6\binom{26}{6} - 4\binom{13}{6} + 0$$

How many ways are there to distribute r distinct objects into five (distinct) boxes with at least one empty box?

We do not need to determine $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$ in this problem, because it does not concern outcomes where some property does not hold for all boxes. Rather, this is a union problem, using the corollary's formula.

It is easy to mistake union problems, which use phrases such as "with at least one empty box," with standard inclusion–exclusion problems, which use phrases such as "at least one object in every box."

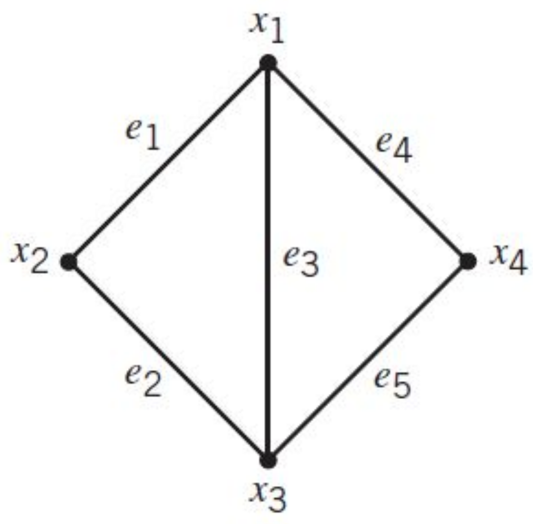
Let U be all distributions of r distinct objects into five boxes.
Let A_i be the set of distributions with a void in box i. Then the required number of distributions with at least one void is $N(A_1 \cup A_2 \cup \cdots A_5)$.

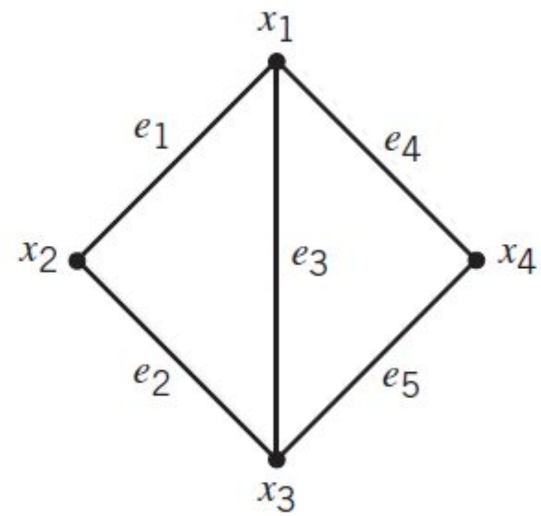
N = ? $N(A_i) = ?$ $N(A_i A_j) = ?$ $S_1 =?$ $S_2 =?$ $S_3 =?$ $S_4 = ?$ $S_5 =?$

How many different integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20 \quad 0 \le x_i \le 8$$

How many ways are there to color the four vertices in the graph shown in Figure below with n colors (such that vertices with a common edge must be different colors)?




x_1
e_1
e_4
x_2
e_3
x_4
e_2
e_5
x_3

$$
\begin{aligned}
N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= n^4 - \binom{5}{1} n^3 + \binom{5}{2} n^2 \\
&\quad - \left[2n^2 + \left(\binom{5}{3} - 2\right) \times n\right] + \binom{5}{4} n - n \\
&= n^4 - 5n^3 + 8n^2 - 4n \quad \blacksquare
\end{aligned}
$$

This expression is the chromatic polynomial of the given graph.

What is the probability that if n people randomly reach into a dark closet to retrieve their hats, no person will pick his own hat?

The problem treated in Example 4 is equivalent to asking for all permutations of the sequence $1, 2, \ldots, n$ such that no number is left fixed, that is, no number i is still in the ith position. Such rearrangements of a sequence are called **derangements.** The symbol D_n is used to denote the number of derangements of n integers. From Example 4, we have

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx n! e^{-1}$$

The number of arrangements of n different objects where no object is in its natural position is given by

$$\begin{aligned} D(n) &= n! - n(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \cdots + (-1)^n C(n,n)0! \\ &= n!(1 - 1/1! + 1/2! - 1/3! + \cdots + (-1)^n/n!) \end{aligned}$$

COUNTING RESTRICTED ARRANGEMENTS

As an application of the Principle of Inclusion-Exclusion, we consider some problems involving restricted arrangements.

Derangements

Consider the permutations of the set $\{1, 2, 3, 4, \ldots, n\}$. In some of these arrangements, which we call **derangements**, none of the n integers appear in their natural position, that is, 1 is not the first integer, 2 is not the second integer, 3 is not the third integer, and so on. We denote the number of derangements of n objects by $D(n)$. We will use the Principle of Inclusion-Exclusion to derive a formula for $D(n)$, which is valid for any positive integer n.

$D(2) = 1$	$D(3) = 2$	$D(4) = 9$		
21	231	2341	2143	4312
	312	3142	3412	2413
		4123	4321	3421

A math professor has typed five letters to five different persons and has also addressed five different envelopes for these letters. At the end of a particularly long day, he absentmindedly stuffs the five letters into the five envelopes at random. What is the probability that no letter is stuffed into the right envelope?

$$\begin{aligned} D(5) &= 5! - C(5,1)4! + C(5,2)3! - C(5,3)2! + C(5,4)1! - C(5,5)0! \\ &= 5!(1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5!) = 44 \end{aligned}$$

The probability that an arrangement of five letters is a derangement is thus $D(5)/5! = 44/120$. $\square$

There are seven different pairs of gloves in a drawer. When seven children go out to play, each selects a left-hand glove and a right-hand glove at random.

a) In how many ways can the selection be made so that no child selects a matching pair of gloves?

b) In how many ways can the selection be made so that exactly one child selects a matching pair?

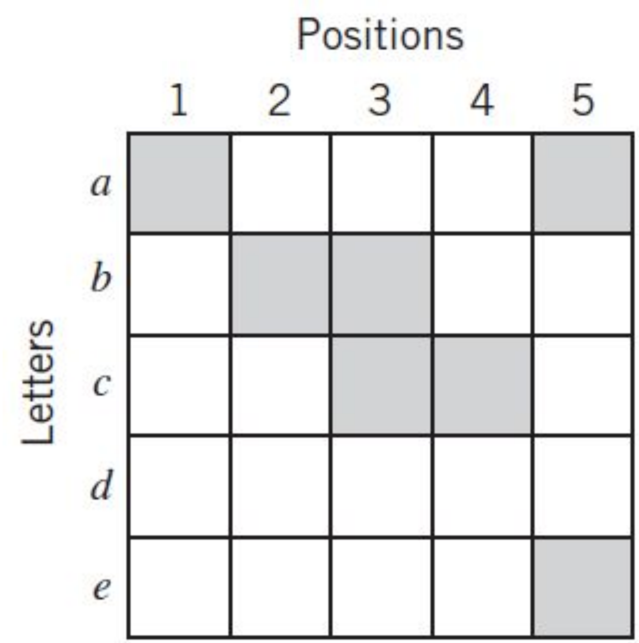
c) In how many ways can the selection be made so that at least two children select matching pairs?

RESTRICTED POSITIONS AND ROOK POLYNOMIALS

Consider the problem of finding all arrangements of a, b, c, d, e with the restrictions indicated in Figure below, that is, a may not be put in position 1 or 5; b may not be put in 2 or 3; c not in 3 or 4; and e not in 5. A permissible arrangement can be represented by picking five unmarked squares with one square in each row and each column. For example, a permissible arrangement is (a, 2), (b, 1), (c, 5), (d, 3), (e, 4).

Count the number of permissible arrangements?
(using inclusion–exclusion formula)

N = ?



Let U be the set of all arrangements of the five letters without restrictions. So $N = 5!$. A_i?

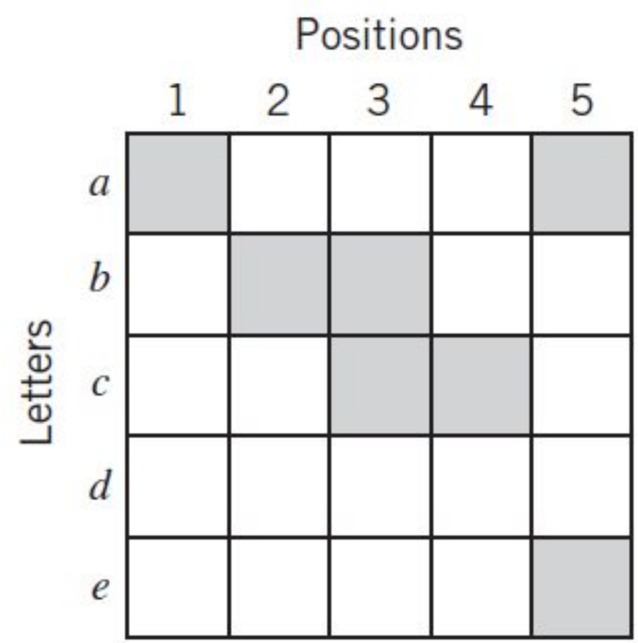
Let A_i be the set of arrangements with a forbidden letter in position i.

In terms of Figure below, A_i is the set of all collections of five squares, each in a different row and column such that the square in column i is a darkened square.

The number of permissible arrangements will then be

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$$

$N(A_i) = ?$



$N(A_1) = 1 \times 4!$

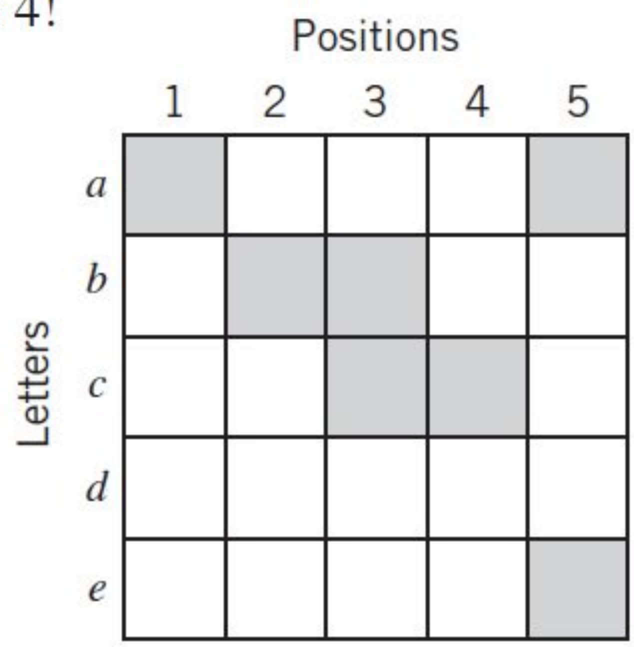
We obtain $N(A_i)$ by counting the ways to put a forbidden letter in position i times the 4! ways to arrange the remaining four letters in the other four positions (we do not worry about forbidden positions for these letters).

$N(A_2) = 1 \times 4!$, $N(A_3) = 2 \times 4!$, $N(A_4) = 1 \times 4!$, and $N(A_5) = 2 \times 4!$.

Collecting terms, we obtain

$$S_1 = \sum_{i=1}^{5} N(A_i) = 1 \times 4! + 1 \times 4! + 2 \times 4! + 1 \times 4! + 2 \times 4!$$
$$= (1+1+2+1+2)4! = 7 \times 4!$$

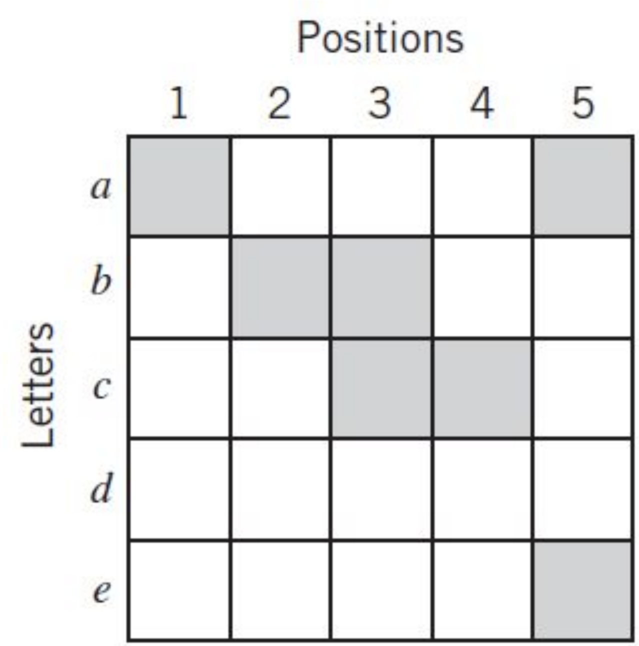
Here (1+1+2+1+2) = 7 is the number of the darkened squares in the Figure below.



S_1 = (number of darkened squares) × 4!
for any restricted-positions problem with a 5×5 family of darkened squares similar to Figure below.

$N(A_iA_j)$ = ? S_2 = ?

$N(A_iA_j)$ is the number of ways to put (different) forbidden letters in positions i and j times the 3! ways to arrange the remaining three letters.

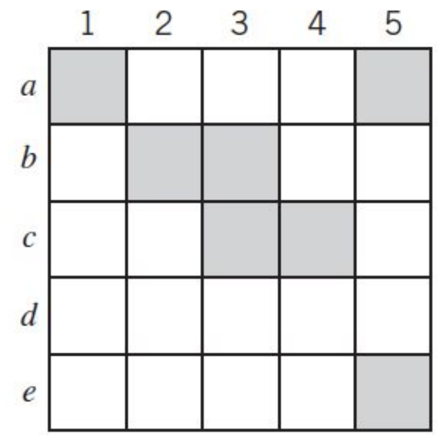


$N(A_1A_2) = 1 \times 3!$ $\quad N(A_1A_3) = 2 \times 3!$ $\quad N(A_1A_4) = 1 \times 3!$

$N(A_1A_5) = 1 \times 3!$ $\quad N(A_2A_3) = 1 \times 3!$ $\quad N(A_2A_4) = 1 \times 3!$

$N(A_2A_5) = 2 \times 3!$ $\quad N(A_3A_4) = 1 \times 3!$ $\quad N(A_3A_5) = 4 \times 3!$

$N(A_4A_5) = 2 \times 3!$



Collecting terms, we obtain

$$S_2 = \sum_{ij} N(A_iA_j) = (1+2+1+1+1+1+2+1+4+2)3! = 16 \times 3!$$

$S_3 = ?, S_4 = ?, S_5 = ?$

The number 16 counts the ways to select two darkened squares, each in a different row and column. Generalizing, we will have

$$S_k = \begin{pmatrix} \text{number of ways to pick } k \text{ darkened squares} \\ \text{each in a different row and column} \end{pmatrix} \times (5-k)! \tag{1}$$

Since letter d's row has no darkened squares, there is no way to pick five darkened squares, each in a different row and column.

Thus $S_5 = 0$.

On the other hand, it is tedious to compute S_3 and S_4 case by case.

Hence, we discuss a theory for determining the number of ways to pick k darkened squares, each in a different row and column.

This darkened squares selection problem can be restated in terms of a recreational mathematics question about a chess-like game.

A chess piece called a **rook** can capture any opponent's piece on the chessboard in the same row or column as the rook (provided there are no intervening pieces).

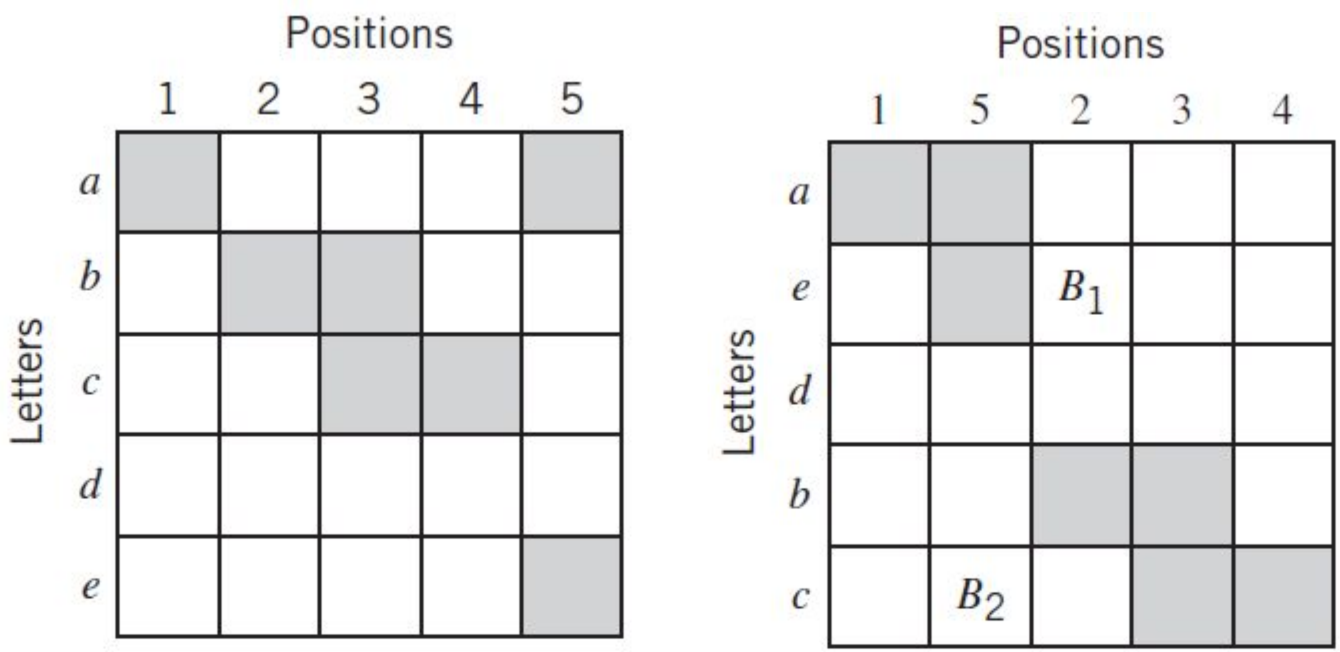
Counting the number of ways to place k mutually non-capturing rooks on this board of darkened squares is equivalent to our original sub-problem of counting the number of ways to pick k darkened squares.

Now we will develop two breaking-up operations to help us count non-capturing rooks on a given board *B*.

The first operation applies to a board B that can be decomposed into **disjoint sub-boards** B_1 and B_2,—that is, sub-boards involving different sets of rows and columns.

Often a board has to be properly rearranged before the disjoint nature of the two sub-boards can be seen.

Let B be the board of darkened squares in Figure below, let B_1 be the three darkened squares in rows a and e, and let B_2 be the four darkened squares in rows b and c.



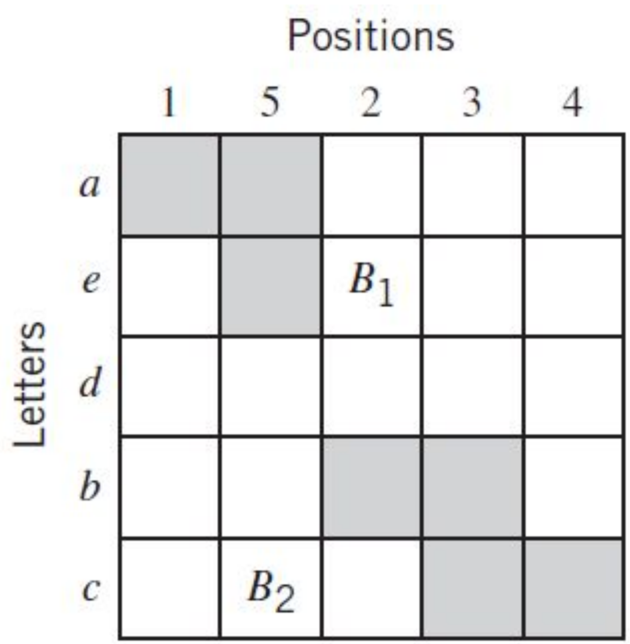
Define $r_k(B)$ to be the number of ways to place k non-capturing rooks on board B, $r_k(B_1)$ the number of ways to place k non-capturing rooks on sub-board B_1, and $r_k(B_2)$ the number of ways to place k non-capturing rooks on sub-board B_2.

$r_1(B_1) = ?$ $r_1(B_2) = ?$

There are three ways to place one rook on sub-board B_1 in Figure below, since B_1 has three squares, and similarly four ways to place one rook on sub-board B_2.

Then $r_1(B_1) = 3$ and $r_1(B_2) = 4$.

$r_2(B_1) = ?$ $r_2(B_2) = ?$

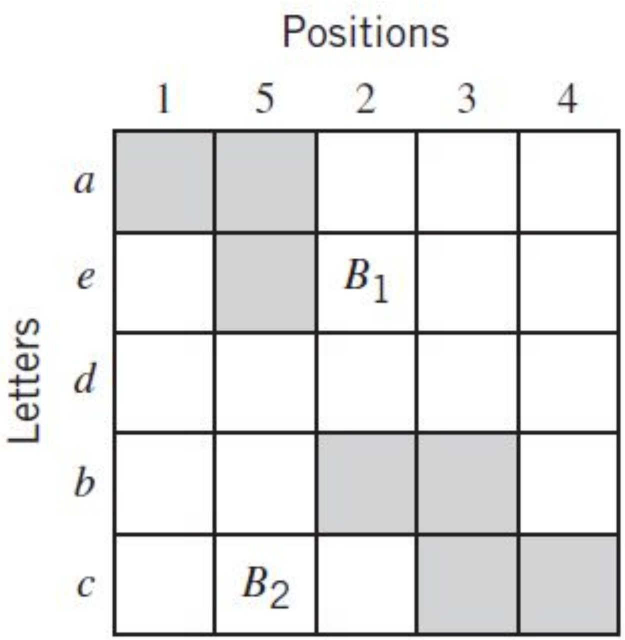


$r_2(B_1) = 1$ and $r_2(B_2) = 3$.
$r_3(B_1) = ?$ $r_3(B_2) = ?$

$r_k(B_1) = r_k(B_2) = 0$ for $k \geq 3$, since each sub-board has only two rows.
It will be convenient to define $r_0 = 1$ for all boards.

Observe that (why)

$$r_2(B) = r_2(B_1) + r_1(B_1)r_1(B_2) + r_2(B_2)$$



Observe next that since B_1 and B_2 are disjoint, placing, say, two noncapturing rooks on the whole board B can be broken into three cases: placing two noncapturing rooks on B_1 (and none on B_2), placing one rook on each subboard, or placing two noncapturing rooks on B_2. Thus we see that

$$r_2(B) = r_2(B_1) + r_1(B_1)r_1(B_2) + r_2(B_2)$$

or, using that fact that $r_0(B_2) = r_0(B_1) = 1$,

$$\begin{aligned} r_2(B) &= r_2(B_1)r_0(B_2) + r_1(B_1)r_1(B_2) + r_0(B_1)r_2(B_2) \\ &= 1 \times 1 + 3 \times 4 + 1 \times 3 = 16 \end{aligned} \tag{2}$$

Recall that 16 is the number obtained earlier when summing all $N(A_i A_j)$ to count all ways to pick two darkened squares each in a different row and column.

Lemma

If B is a board of darkened squares that decomposes into the two disjoint subboards B_1 and B_2, then

$$r_k(B) = r_k(B_1)r_0(B_2) + r_{k-1}(B_1)r_1(B_2) + \cdots + r_0(B_1)r_k(B_2) \tag{3}$$

We define the **rook polynomial** $R(x, B)$ of the board B of darkened squares to be

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

Remember that $r_0(B) = 1$ for all B.

$R(x, B_1) = ?$ $R(x, B_2) = ?$

$$R(x, B_1) = 1 + 3x + 1x^2 \quad \text{and} \quad R(x, B_2) = 1 + 4x + 3x^2$$

$R(x, B) = ?$

Moreover, by the correspondence between (3) and the formula for the product of two generating functions, we see that $r_k(B)$, the coefficient of x^k in the rook polynomial $R(x, B)$ of the full board, is simply the coefficient of x^k in the product $R(x, B_1)R(x, B_2)$. That is,

$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) = (1 + 3x + 1x^2)(1 + 4x + 3x^2) \\ &= 1 + [(3 \times 1) + (1 \times 4)]x + [(1 \times 1) + (3 \times 4) + (1 \times 3)]x^2 \\ &\quad + [(1 \times 4) + (3 \times 3)]x^3 + (1 \times 3)x^4 \\ &= 1 + 7x + 16x^2 + 13x^3 + 3x^4 \end{aligned}$$

Theorem 1

If B is a board of darkened squares that decomposes into the two disjoint subboards B_1 and B_2 then

$$R(x, B) = R(x, B_1)R(x, B_2)$$

$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$

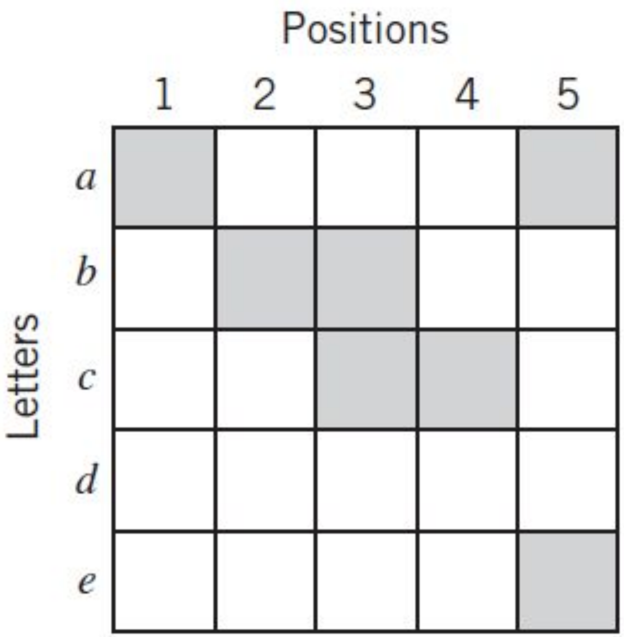
$$\begin{aligned} N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= N - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= 5! - r_1(B)4! + r_2(B)3! - r_3(B)2! + r_4(B)1! - r_5(B)0! \\ &= 5! - 7 \times 4! + 16 \times 3! - 13 \times 2! + 3 \times 1! - 0 \times 0! \end{aligned}$$

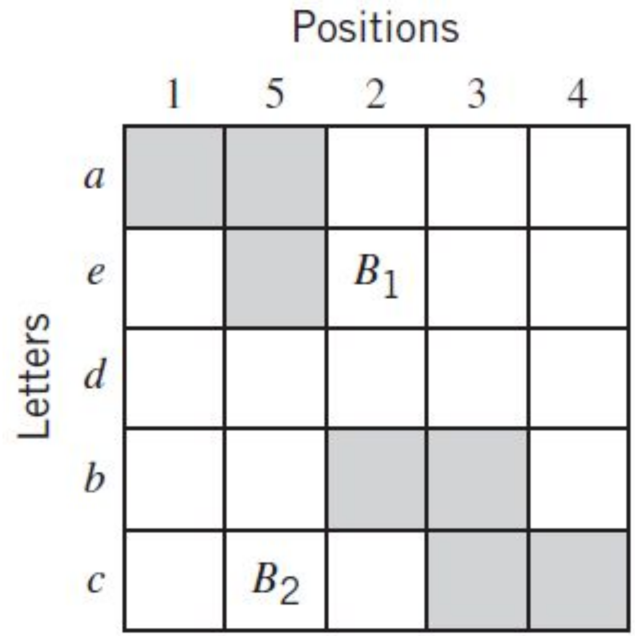
Theorem 2

The number of ways to arrange n distinct objects when there are restricted positions is equal to

$$\begin{aligned} &n! - r_1(B)(n-1)! + r_2(B)(n-2)! + \cdots + (-1)^k r_k(B)(n-k)! \\ &\quad + \cdots + (-1)^n r_n(B)0! \end{aligned} \tag{4}$$

where the $r_k(B)$s are the coefficients of the rook polynomial $R(x, B)$ for the board B of forbidden positions.





Count the number of permissible arrangements?

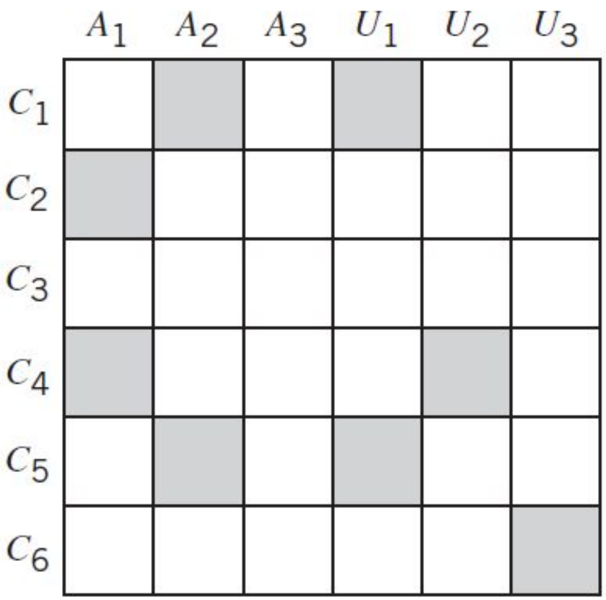
$$R(x, B_1) = 1 + 3x + 1x^2 \quad \text{and} \quad R(x, B_2) = 1 + 4x + 3x^2$$

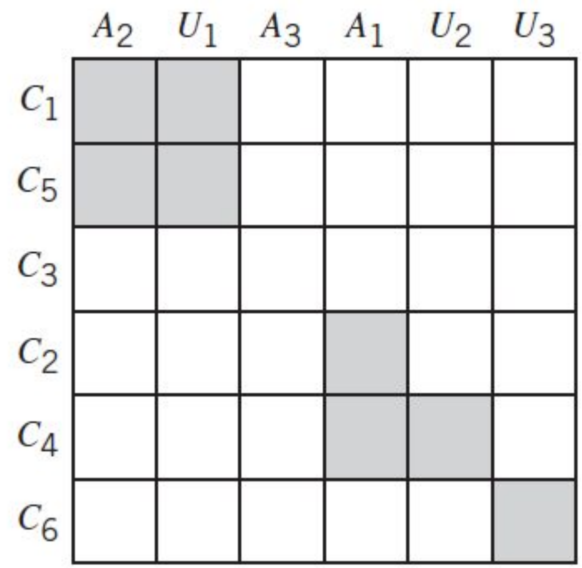
$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) = (1 + 3x + 1x^2)(1 + 4x + 3x^2) \\ &= 1 + 7x + 16x^2 + 13x^3 + 3x^4 \end{aligned}$$

$$\begin{aligned} N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= N - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= 5! - r_1(B)4! + r_2(B)3! - r_3(B)2! + r_4(B)1! - r_5(B)0! \\ &= 5! - 7 \times 4! + 16 \times 3! - 13 \times 2! + 3 \times 1! - 0 \times 0! \\ &= 120 - 168 + 96 - 26 + 3 - 0 = 25 \end{aligned}$$

How many ways are there to send six different birthday cards, denoted C_1, C_2, C_3, C_4, C_5, C_6, to three aunts and three uncles, denoted A_1, A_2, A_3, U_1, U_2, U_3, if aunt A_1 would not like cards C_2 and C_4; if A_2 would not like C_1 or C_5; if A_3 likes all cards; if U_1 would not like C_1 or C_5; if U_2 would not like C_4; and if U_3 would not like C_6?

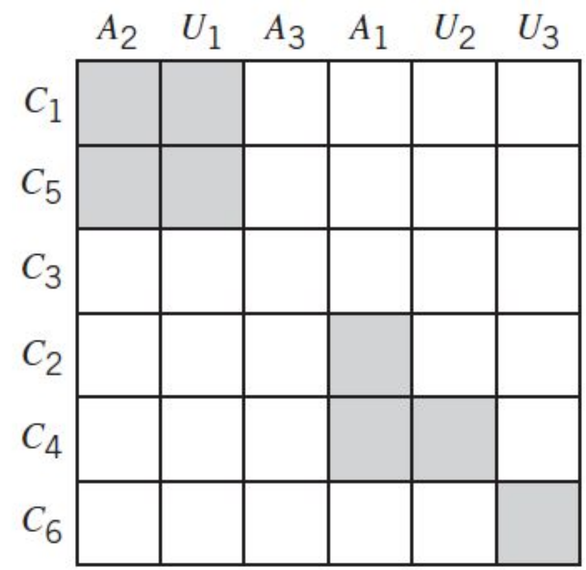
First draw the board and compute the Rook Polynomial.





Thus the original board B of darkened squares decomposes into the two disjoint subboards, B_1 in rows C_1 and C_5, and B_2 in rows C_2, C_4, and C_6. Actually B_2 itself decomposes into two disjoint subboards B_2' and B_2'', where B_2'' is the single square (C_6, U_3). By inspection, we see that

$$R(x, B_1) = 1 + 4x + 2x^2$$
$$R(x, B_2) = R(x, B_2')R(x, B_2'') = (1 + 3x + x^2)(1 + x)$$



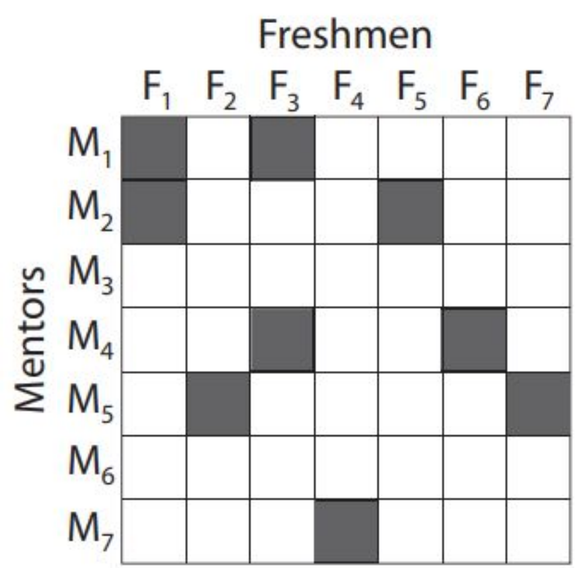
$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) \\ &= (1 + 4x + 2x^2)(1 + 3x + x^2)(1 + x) \\ &= 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5 \end{aligned}$$

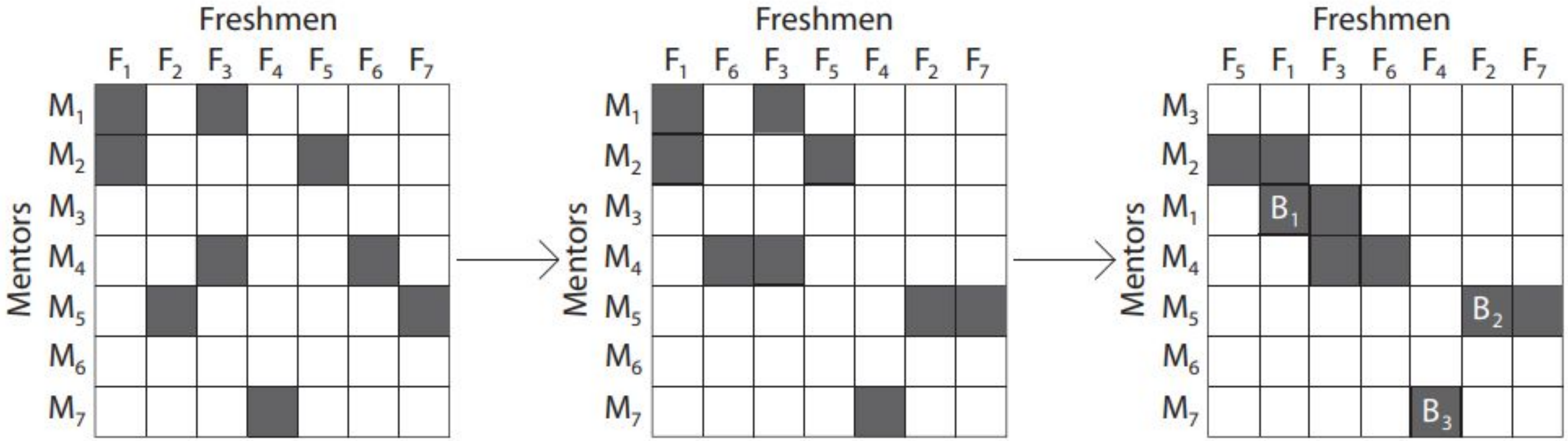
Then the answer to the card-mailing problem is

$$\begin{aligned} &\sum_{k=0}^{6} (-1)^k r_k(B)(6 - k)! \\ &\quad = 6! - 8 \times 5! + 22 \times 4! - 25 \times 3! + 12 \times 2! - 2 \times 1! + 0 \times 0! \\ &\quad = 720 - 960 + 528 - 150 + 24 - 2 + 0 = 160 \ \blacksquare \end{aligned}$$

At a university, seven freshmen, $F_1, F_2, F_3, F_4, F_5, F_6$ and F_7, enter the same academic program. Their department head, eager to retain these new students, wants to assign each incoming freshman a mentor from among the upperclassmen of the program. Seven mentors are chosen, $M_1, M_2, M_3, M_4, M_5, M_6$ and M_7, but there are some scheduling conflicts. M_1 cannot work with F_1 or F_3, M_2 cannot work with F_1 or F_5, M_4 cannot work with F_3 or F_6, M_5 cannot work with F_2 or F_7, and M_7 cannot work with F_4. In how many ways can the department head assign the mentors so that each incoming freshman has a different mentor?

First draw the board and compute the Rook Polynomial.





Interchanging columns F_2 and F_6, F_4 and F_5, F_1 and F_5, F_1 and F_6, and rows M_1 and M_3 yields a decomposition of the original board into the three subboards displayed in the final board of Fig.(1.4). Now we set about calculating the $r_k(B_i)$'s for these subboards, and arrive at the following: $r_1(B_1) = 6$, $r_2(B_1) = 10$, $r_3(B_1) = 4$; $r_1(B_2) = 2$; $r_1(B_3) = 1$. Thus we arrive with the following rook polynomials for B_1, B_2 and B_3:

$$R(x, B_1) = 1 + 6x + 10x^2 + 4x^3$$

$$R(x, B_2) = 1 + 2x$$

$$R(x, B_3) = 1 + x$$

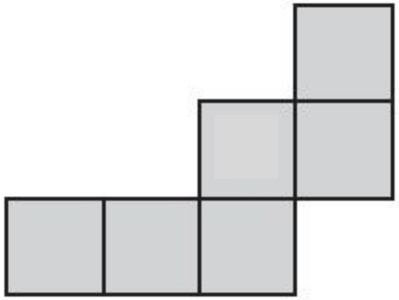
Multiplying these rook polynomials yields:

$$
\begin{gathered}
R(x, B) = R(x, B_1)R(x, B_2)R(x, B_3) = \\
(1 + 6x + 10x^2 + 4x^3)(1 + 2x)(1 + x) = \\
(1 + 6x + 10x^2 + 4x^3)(1 + 3x + 2x^2) = \\
1 + 3x + 2x^2 + 6x + 18x^2 + 12x^3 + 10x^2 + 30x^3 + 20x^4 + 4x^3 + 12x^4 + 8x^5 = \\
1 + 9x + 30x^2 + 46x^3 + 32x^4 + 8x^5.
\end{gathered}
$$

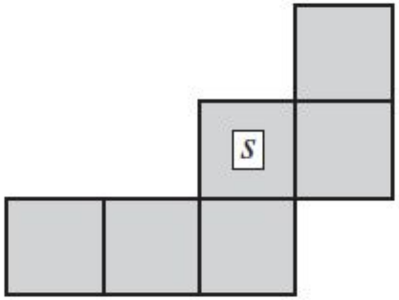
The required answer is

$$
\begin{gathered}
7! - 9 \times 6! + 30 \times 5! - 46 \times 4! + 32 \times 3! - 8 \times 2! + 0 \times 1! - 0 \times 0! = \\
5,040 - 6,480 + 3,600 - 1,104 + 192 - 16 = 1,232.
\end{gathered}
$$

Thus there are 1,232 ways to assign each freshman his or her own mentor, in accordance with the given restrictions.■



Determine the coefficients of $R(x, B)$



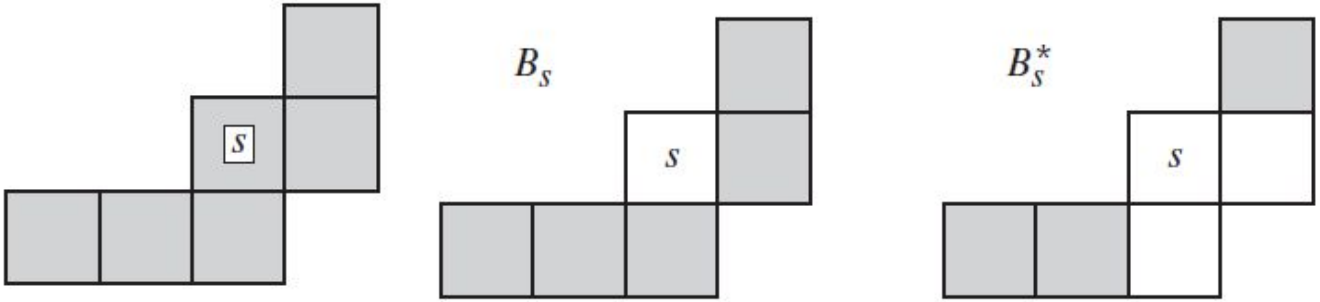
It is a problem of determining the coefficients of $R(x, B)$ when the board B does not decompose into two disjoint sub-boards.

Let us break the problem of determining $r_k(B)$ into two cases, depending on whether or not a certain square s is one of the squares chosen for the k non-capturing rooks.

How the board can be split now?

Let B_s be the board obtained from B by deleting square s (if square s is not chosen), and

let B^*_s be the board obtained from B by deleting square s plus all squares in the same row or column as s (if square s is chosen).



If s is the square indicated in Figure above, then B_s and B^*_s are shown below.

$r_k(B)$?

If square s is not used, we must place k noncapturing rooks on B_s. If square s is used, then we must place $k-1$ noncapturing rooks on B_s^*. Hence we conclude that

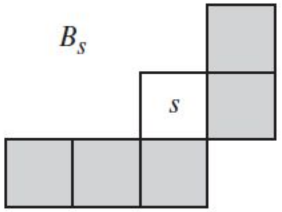
$$r_k(B) = r_k(B_s) + r_{k-1}(B_s^*) \tag{5}$$

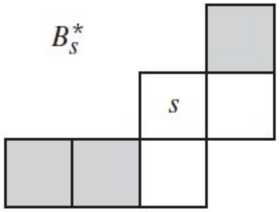
Using the generating function methods introduced in Section 7.5 for turning a recurrence relation into a generating function, we obtain from (5)

$$\begin{aligned} R(x, B) = \sum_k r_k(B)x^k &= \sum_k r_k(B_s)x^k + \sum_k r_{k-1}(B_s^*)x^k \\ &= \sum_k r_k(B_s)x^k + x\sum_h r_h(B_s^*)x^h \\ &= R(x, B_s) + xR(x, B_s^*) \end{aligned}$$

R(x, Bs) = ?, R(x, B^*_s) = ?

$$R(x, B_s) = (1+3x)(1+2x) = 1+5x+6x^2$$
$$R(x, B_s^*) = (1+2x)(1+x) = 1+3x+2x^2$$





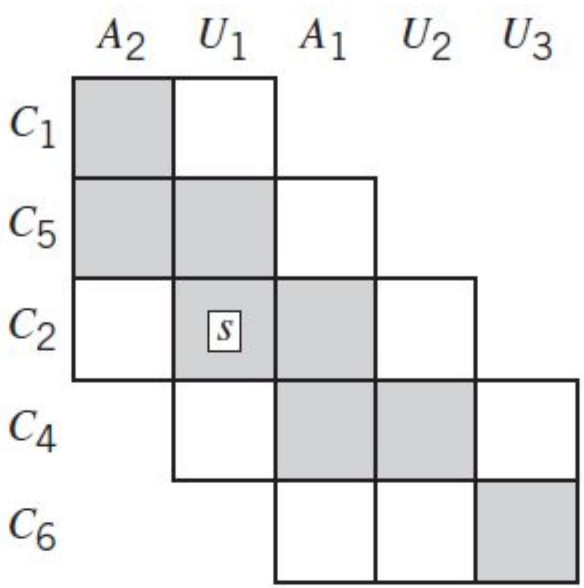
$$R(x, B) = R(x, B_s) + xR(x, B_s^*) = (1+5x+6x^2) + x(1+3x+2x^2)$$
$$= 1+6x+9x^2+2x^3$$

Theorem 3

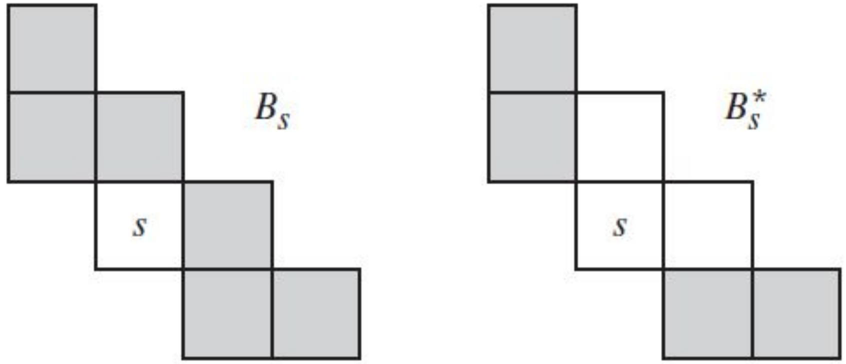
Let B be any board of darkened squares. Let s be one of the squares of B, and let B_s and B_s^* be as defined above. Then

$$R(x, B) = R(x, B_s) + xR(B_s^*)$$

Compute R(x, B)



The square in the bottom right corner is t, which is disjoint from the other squares, and the remaining board is B_1.



$$R(x, B_s) = (1 + 3x + x^2)(1 + 3x + x^2) = 1 + 6x + 11x^2 + 6x^3 + x^4$$
$$R(x, B_s^*) = (1 + 2x)(1 + 2x) = 1 + 4x + 4x^2$$

Then

$$\begin{aligned} R(x, B_1) = R(x, B_s) + x\,R(x, B_s^*) &= (1 + 6x + 11x^2 + 6x^3 + x^4) \\ &\quad + x(1 + 4x + 4x^2) \\ &= 1 + 7x + 15x^2 + 10x^3 + x^4 \end{aligned}$$

and

$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, t) \\ &= (1 + 7x + 15x^2 + 10x^3 + x^4)(1 + x) \\ &= 1 + 8x + 22x^2 + 25x^3 + 11x^4 + x^5 \end{aligned}$$

The number of ways to send birthday cards is

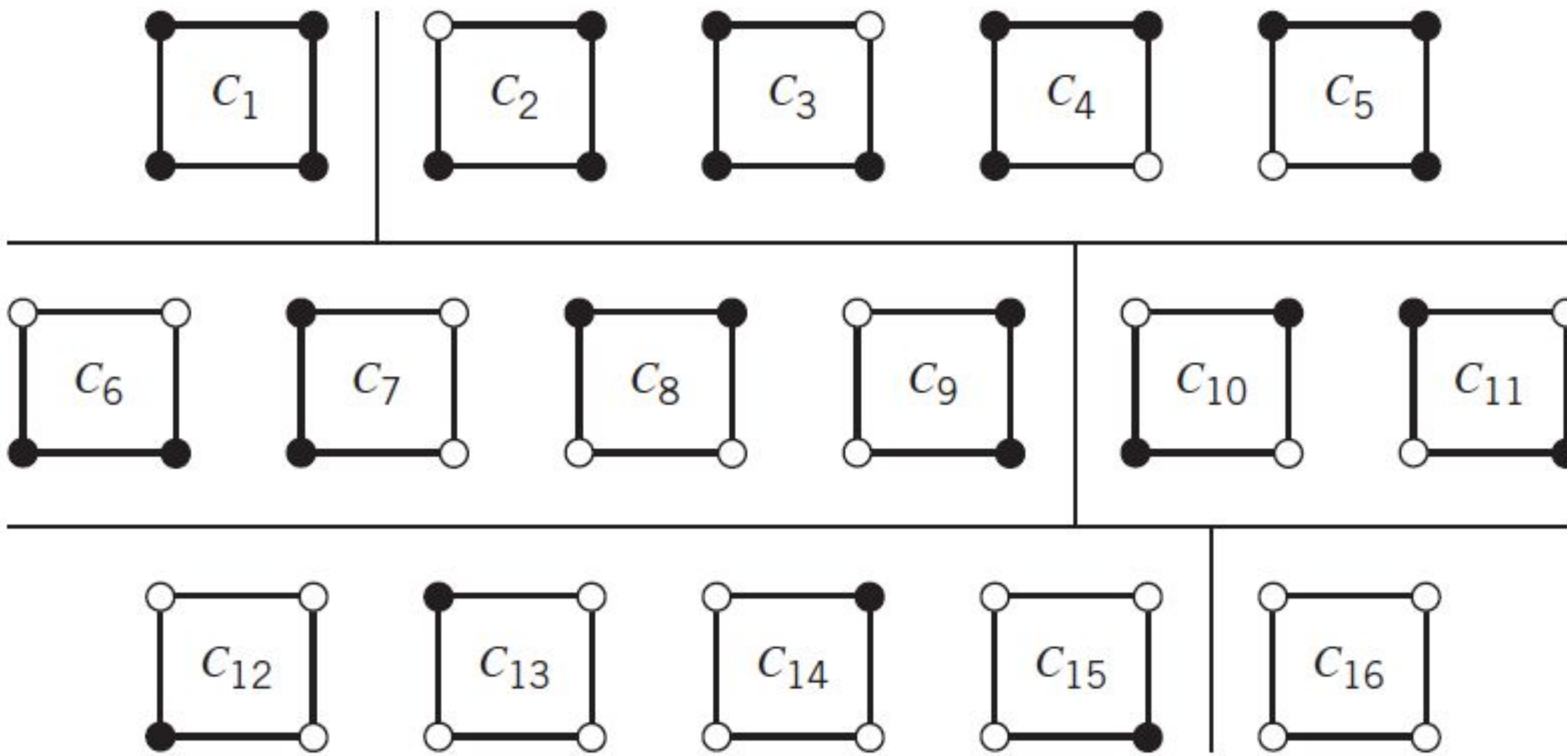
$$6! - 8 \times 5! + 22 \times 4! - 25 \times 3! + 11 \times 2! - 1 \times 1! + 0 \times 0! = 159$$

POLYA'S ENUMERATION FORMULA

EQUIVALENCE AND SYMMETRY GROUPS

Consider a fixed square. Color its corners with black and white colors only. How many arrangements are possible, considering each corner to be distinct.

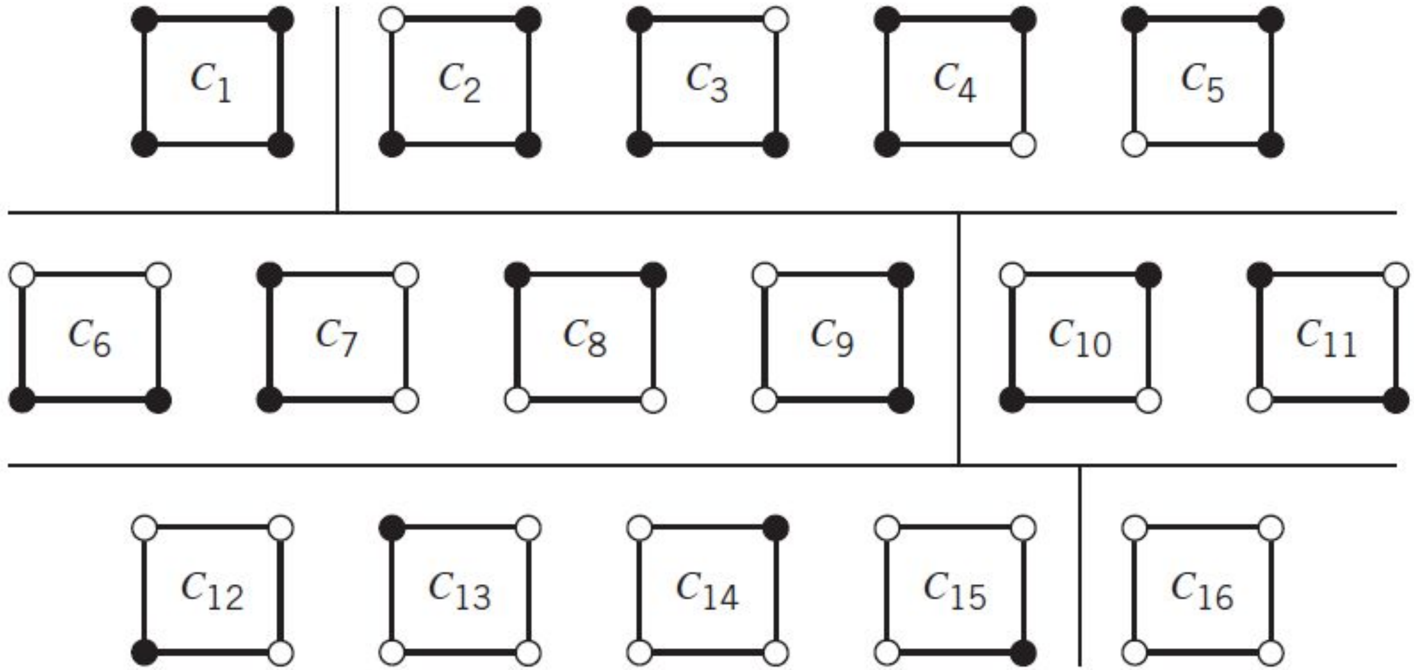
$2^4 = 16$ black–white colorings are there for a fixed square.



We can partition these 16 colorings into subsets of colorings that are equivalent when the square is floating.

How many such subsets are there?

There are six such subsets (see the groupings of colored squares shown in Figure below), and so there are **six** different 2-colorings of the floating square.



We seek a theory and formula to explain why there are six such distinct 2-colorings of the square. Note that the six subsets of equivalent colorings vary in size.

To define the partition of a set into subsets of equivalent elements, we first define the general concept of the equivalence of two elements a and b. We write this equivalence as $a \sim b$. The fundamental properties of an **equivalence relation** are

(i) Transitivity: $a \sim b, b \sim c \Rightarrow a \sim c$

(ii) Reflexivity: $a \sim a$

(iii) Symmetry: $a \sim b \Rightarrow b \sim a$

All other properties of equivalence can be derived from these three. Any binary relation with these three properties is called an equivalence relation. Such a relation defines a partition into subsets of mutually equivalent elements called **equivalence classes.**

Consider a set of numbers, differing by an even number. Is it an equivalence relation? What are corresponding equivalence classes.

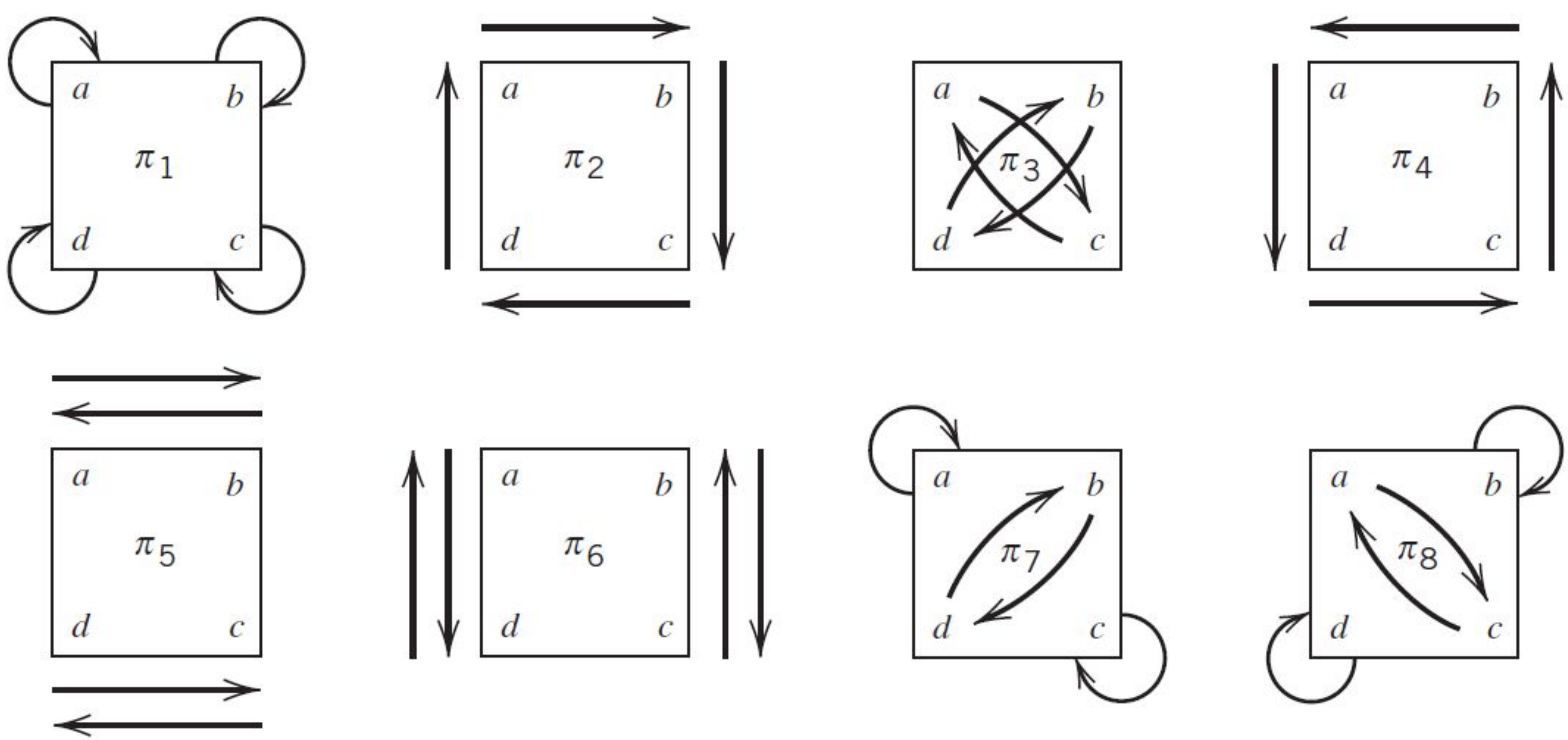
The even numbers form one equivalence class and the odd numbers the other class.

How many **symmetries** does square has?

In a square, the symmetries are because of **rotations and reflections**.

The rotations are $\boldsymbol{\pi_2}$ = 90◦ rotation, $\boldsymbol{\pi_3}$ = 180◦ rotation, $\boldsymbol{\pi_4}$ = 270◦ rotation, and $\boldsymbol{\pi_1}$ = 360◦ (or 0◦) rotation (the rotations are about the center of the square).

The reflections are $\boldsymbol{\pi_5}$ = reflection about the vertical axis, $\boldsymbol{\pi_6}$ = reflection about the horizontal axis, $\boldsymbol{\pi_7}$ = reflection about opposite corners a and c, and $\boldsymbol{\pi_8}$ = reflection about opposite corners b and d.



Can you compute the number of symmetries of an n-gon, for n even?

In a regular n-gon, the smallest rotation is $(360/n)^\circ$.
Any multiple of this $(360/n)^\circ$ rotation is again a rotation, and so there are **n rotations** in all.
There are two types of reflections for a regular even n-gon:
flipping about the middles of two opposite sides and flipping about two opposite corners.

Since there are n/2 pairs of opposite sides and n/2 pairs of opposite corners, a regular even n-gon will have $n/2+n/2 = n$ reflections.

Summing rotations and reflections, we find that a regular even n-gon has **2n symmetries**.

Describe the symmetries of a pentagon, and more generally, of an n-gon for odd n.

As noted in Example above, any regular n-gon has n rotational symmetries.
A pentagon will have five rotational symmetries of 0°, 72°, 144°, 216°, and 288°.
However, the reflections about opposite sides or opposite corners do not exist in the pentagon.
Instead, we reflect about an axis of symmetry running from one corner to the middle of an opposite side.
There are five such reflections, for a total of 10 symmetries.

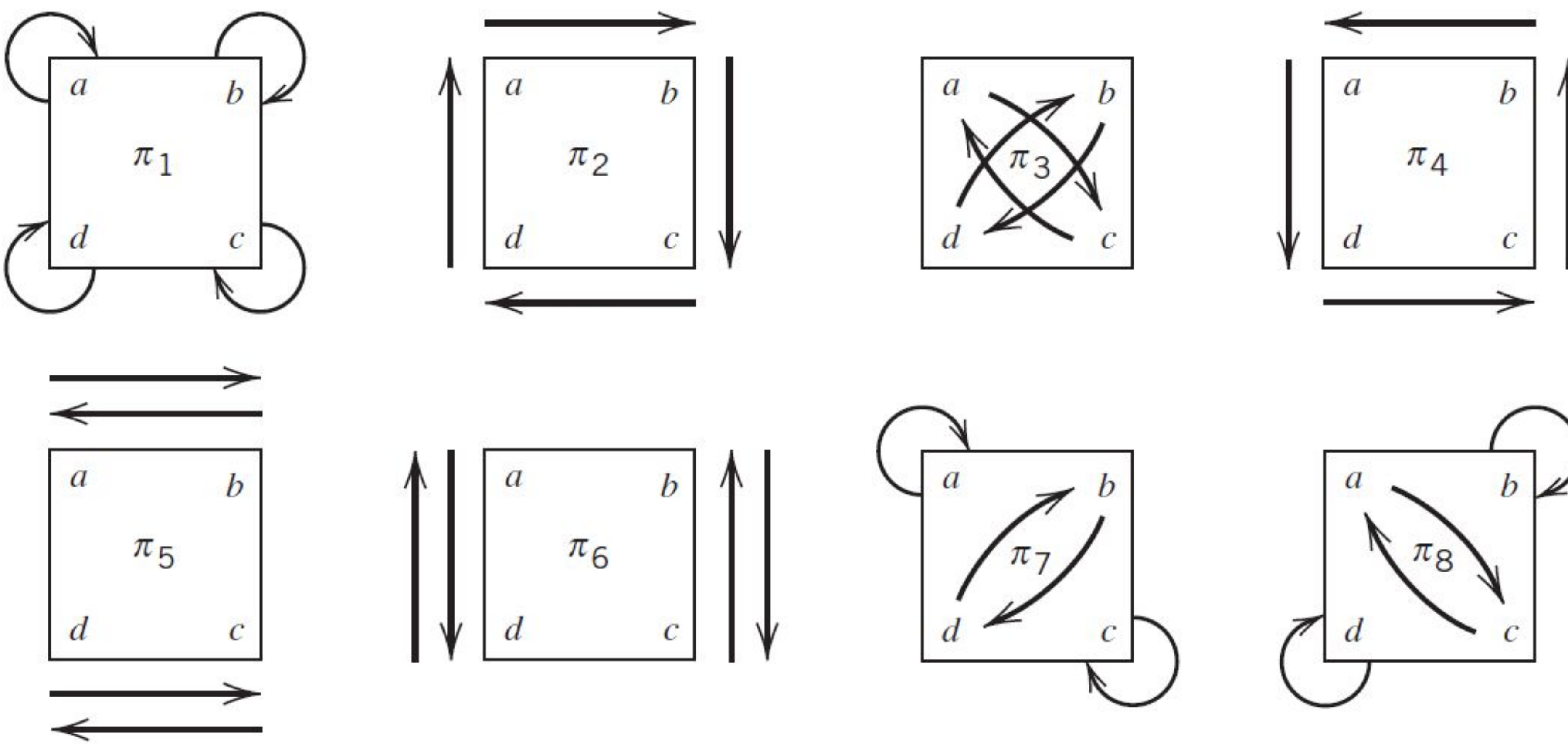
In a regular odd n-gon, there are n such flips, along with n rotations. Summing rotations and reflections, we find that a regular odd n-gon also has **2n symmetries**.

The symmetries of a square are naturally characterized by the way they permute the corners of the square. Thus, the 180∘ rotation π_3 (see Figure below) can be described as the corner permutation:

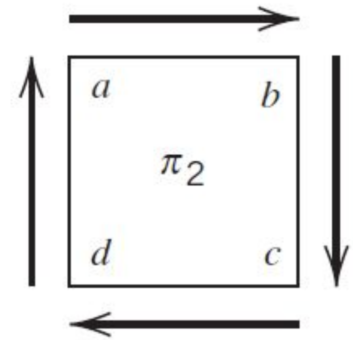
a ⟶ c, b ⟶ d, c ⟶ a, d ⟶ b;

in tabular form, we write

$$\begin{bmatrix} a & b & c & d \\ c & d & a & b \end{bmatrix}$$



Similarly, the 90◦ rotation π_2 can be described as
a ⟶ b, b ⟶ c, c ⟶ d, d ⟶ a, or a ⟶b ⟶ c ⟶ d ⟶ a.

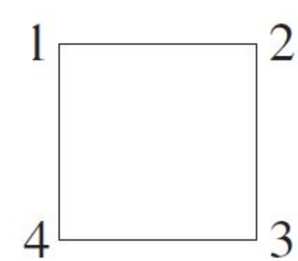


A permutation of the form $x_1 \to x_2 \to x_3 \cdots \to x_n \to x_1$ is called a cyclic permutation or **cycle.** Thus, π_2 is a cycle of length 4. Cycles are usually written in the form $(x_1 x_2 x_3 \ldots x_n x_1)$. So $\pi_2 = (abcd)$. Any permutation can be expressed as a product of disjoint cycles (proof of this claim is an exercise). For example, $\pi_3 = (ac)(bd)$, $\pi_4 = (adcb)$, and $\pi_7 = (a)(bd)(c)$.

Find the group of permutations that describe the symmetries of the square shown below.

$$\rho = (1234); \quad \rho^2 = (13)(24); \quad \rho^3 = (1432); \quad \rho^4 = e = (1)(2)(3)(4)$$

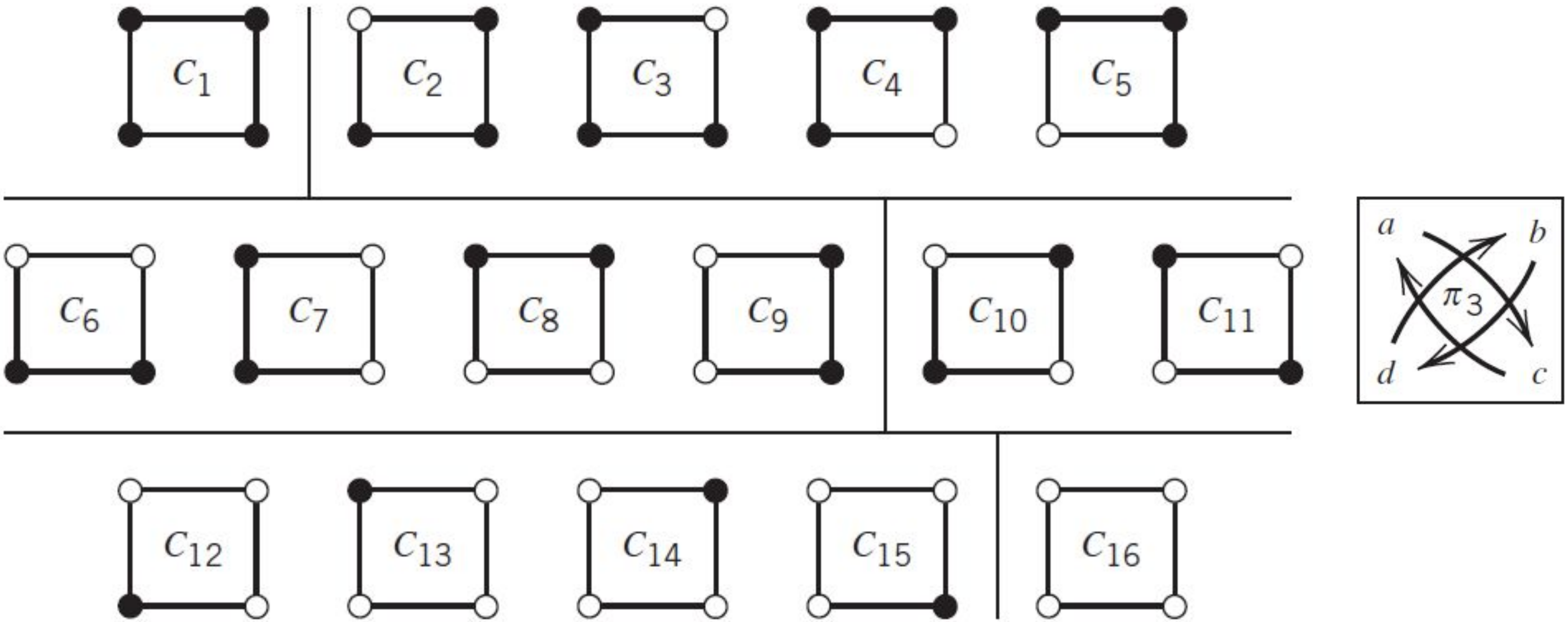
We will use α, β, γ, and δ to represent the listed reflections.



$$\alpha = (24); \quad \beta = (13); \quad \gamma = (12)(34); \quad \delta = (14)(23)$$

In permuting the corners, the symmetries create permutations of the colorings of the corners. For example, if C_i is the ith square in Figure 9.1, then π_3 is the following permutation of colorings:

$$\pi_3 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_4 & C_5 & C_2 & C_3 & C_8 & C_9 & C_6 & C_7 & C_{10} & C_{11} & C_{14} & C_{15} & C_{12} & C_{13} & C_{16} \end{pmatrix} \tag{1}$$



The point is that while a symmetry π_i is easily visualized by how it moves the corners of the square, what we are really interested in is the way π_i takes one coloring into another (making them equivalent). Thus, we formally define our coloring equivalence as follows:

$$\begin{gathered}\text{Colorings C and } C' \text{ are } \textit{equivalent}, \ C \sim C', \\ \text{if there exists a symmetry } \pi_i \text{ such that } \pi_i(C) = C' \end{gathered} \tag{2}$$

The properties of the set G of symmetries that interest us are the ones that make the relation $C \sim C'$ in Eq. (2) an equivalence relation. These properties of G are (here $\pi_i \cdot \pi_j$ means *applying motion π_i followed by motion π_j*):

1. *Closure:* If $\pi_i, \pi_j \in G$, then $\pi_i \cdot \pi_j \in G$;
2. *Identity:* G contains an identity motion π_{I} such that $\pi_{\mathrm{I}} \cdot \pi_i = \pi_i$ and $\pi_i \cdot \pi_{\mathrm{I}} = \pi_i$;
3. *Inverses:* For each $\pi_i \in G$, there exists an inverse in G, denoted π_i^{-1}, such that $\pi_i^{-1} \cdot \pi_i = \pi_{\mathrm{I}}$ and $\pi_i \cdot \pi_i^{-1} = \pi_{\mathrm{I}}$;

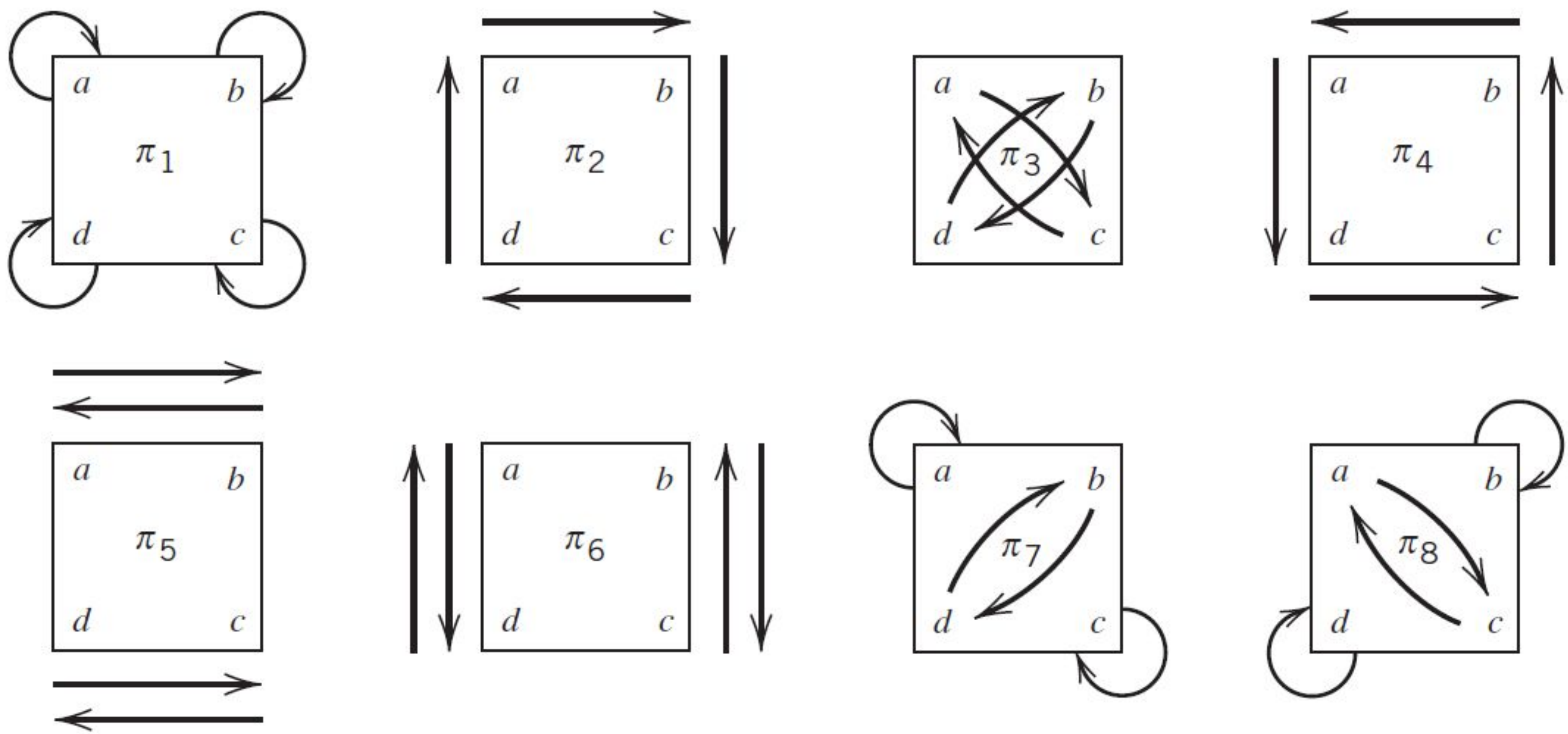
1. *Closure:* If $\pi_i, \pi_j \in G$, then $\pi_i \cdot \pi_j \in G$; $\pi_2 \cdot \pi_5 = ?$

π_2 (a)(b)(c)(d) = (d)(a)(b)(c) and π_5 (d)(a)(b)(c) = (a)(d)(c)(b) = π_7

2. *Identity:* G contains an identity motion π_I such that $\pi_I \cdot \pi_i = \pi_i$ and $\pi_i \cdot \pi_I = \pi_i$;

π_I is π_1

3. *Inverses:* $\pi_2^{-1} =$ π_2 (a)(b)(c)(d) = (d)(a)(b)(c), π_4 (d)(a)(b)(c) = (a)(b)(c)(d) implies $\pi_2^{-1} = \pi_4$.



Observe that closure makes our coloring relation $\sim$ satisfy transitivity

For suppose $C \sim C'$ and $C' \sim C''$. Since $C \sim C'$, there must exist $\pi_i \in G$ such that $\pi_i(C) = C'$. Similarly, there is a $\pi_j \in G$ such that $\pi_j(C') = C''$. Then by closure, there exists $\pi_k = \pi_i \cdot \pi_j \in G$ with $\pi_k(C) = (\pi_i \cdot \pi_j)(C) = C''$. Thus $C \sim C''$. Similarly, properties (2) and (3) of the symmetries imply that our coloring relation satisfies properties (ii) and (iii) of an equivalence relation, respectively

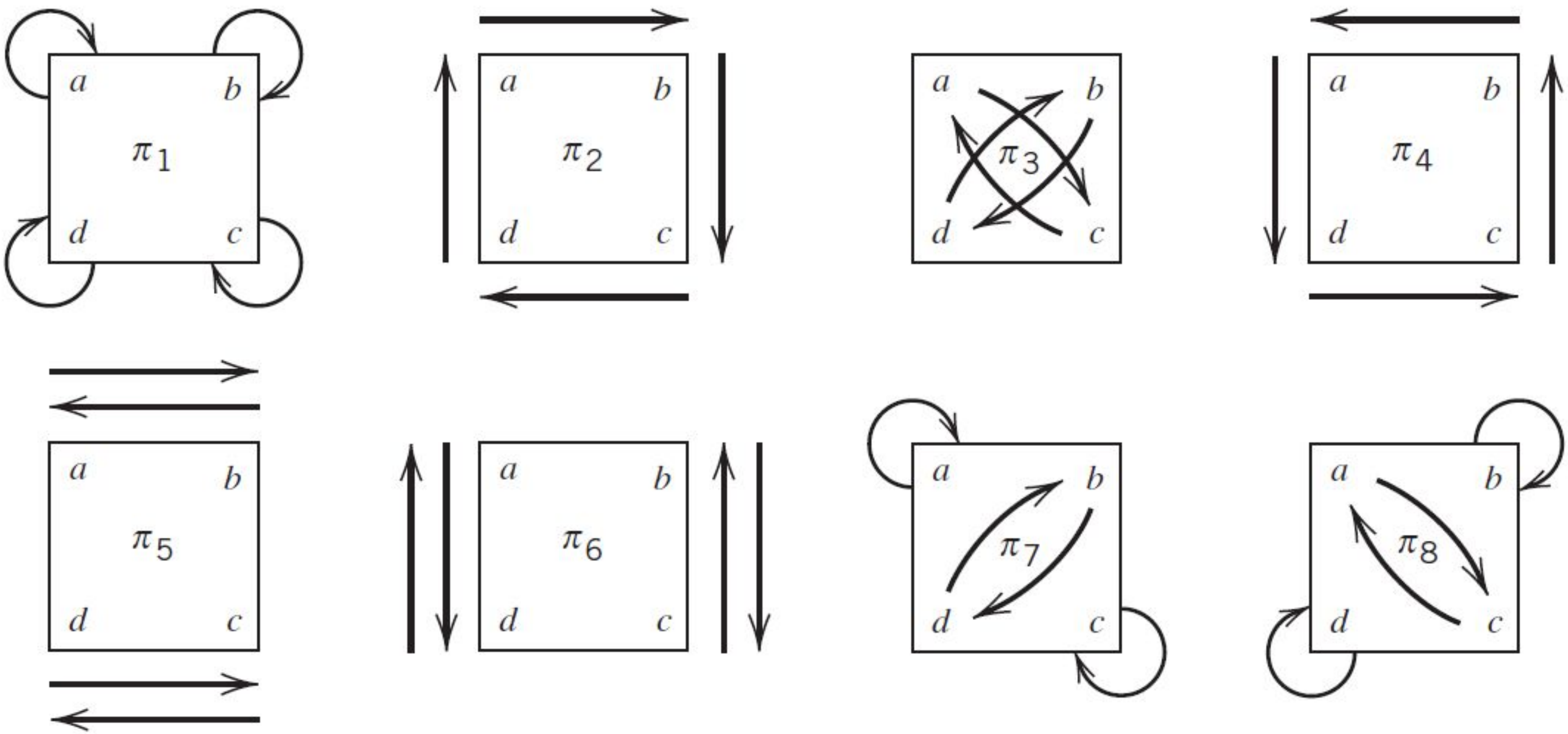
A collection G of mathematical objects with a binary operation is called a **group** if it satisfies properties 1, 2, and 3 along with the associativity property—$(\pi_i \cdot \pi_j) \cdot \pi_k = \pi_i \cdot (\pi_j \cdot \pi_k)$. Thus we have the following theorem.

Theorem

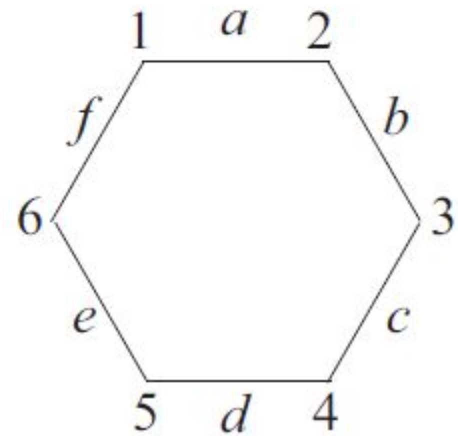
Let G be a group of permutations of the set S (corners of a square) and T be any collection of colorings of S (2-colorings of the corners). Then G induces a partition of T into equivalence classes with the relation $C \sim C' \Leftrightarrow$ some $\pi \in G$ takes C to C'.

Find two symmetries of the square π_i, π_j such that $\pi_i \cdot \pi_j \neq \pi_j \cdot \pi_i$ (this means the group of symmetries of a square is noncommutative).

$\pi_2 \cdot \pi_5$ is not equal to $\pi_5 \cdot \pi_2$



Find the group of permutations that describe the symmetries of the following figure



We have the rotation $\rho = (123456)$ representing a $60°$ clockwise rotation of the figure, and the powers $\rho^2 = (135)(246)$, $\rho^3 = (14)(25)(36)$, $\rho^4 = (153)(264)$, $\rho^4 = (165432)$, and $\rho^6 = e$. Now we list the six reflections; they are $\alpha = (12)(36)(45)$, $\beta = (13)(46)$, $\gamma = (23)(14)(56)$, $\delta = (15)(24)$, $\epsilon = (16)(25)(34)$, and $\phi = (26)(35)$.

Lemma

For any two permutations π_i, π_j in a group G, there exists a unique permutation $\pi_k = \pi_i^{-1} \cdot \pi_j$ in G such that $\pi_i \cdot \pi_k = \pi_j$.

Proof

First we show that $\pi_i \cdot \pi_k = \pi_j$. Since $\pi_k = \pi_i^{-1} \cdot \pi_j$,

$$\begin{aligned}\pi_i \cdot \pi_k = \pi_i \cdot \left(\pi_i^{-1} \cdot \pi_j\right) &= \left(\pi_i \cdot \pi_i^{-1}\right) \cdot \pi_j \quad \text{(by associativity)} \\ &= \pi_{\mathrm{I}} \cdot \pi_j = \pi_j\end{aligned}$$

as claimed. Next we show that π_k is unique. Suppose there also exists a permutation π_k' such that $\pi_i \cdot \pi_k' = \pi_j$. Then $\pi_i \cdot \pi_k = \pi_i \cdot \pi_k'$. Multiplying the equation by π_i^{-1}, we have

$$\begin{aligned}\pi_i^{-1} \cdot (\pi_i \cdot \pi_k) = \pi_i^{-1} \cdot (\pi_i \cdot \pi_k') &\Rightarrow (\pi_i^{-1} \cdot \pi_i) \cdot \pi_k = \left(\pi_i^{-1} \cdot \pi_i\right) \cdot \pi_k' \\ &\Rightarrow \pi_{\mathrm{I}} \cdot \pi_k = \pi_{\mathrm{I}} \cdot \pi_k' \Rightarrow \pi_k = \pi_k' \ \blacklozenge\end{aligned}$$

9.2: BURNSIDE'S THEOREM

We now develop a theory for counting the number of different (nonequivalent) 2-colorings of the square.

More generally, in a set T of colorings of the corners (or edges or faces) of some figure, we seek the number **N** of equivalence classes of **T** induced by a group **G** of symmetries of this figure.

Consider seatings of 4 people around a round table. It can be done in 4! Ways. If only cyclic rotations are allowed, compute the number of equivalence classes (cyclicly nonequivalent seatings)?

There are 4 cyclic rotations of the seatings, and each equivalence class consists of 4 seatings.

Thus, the number of equivalence classes (cyclicly nonequivalent seatings) is N = 4!/4 = 3!

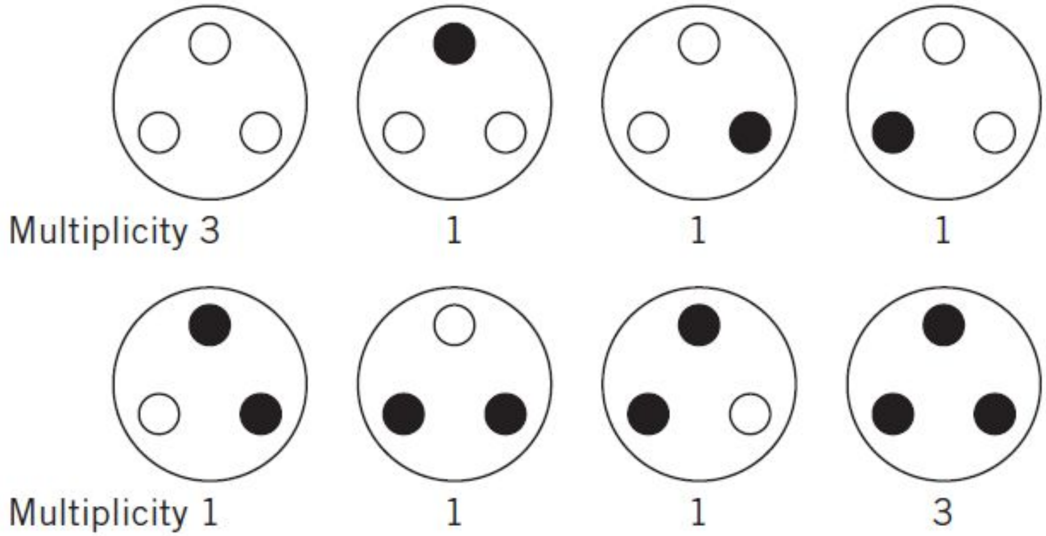
1234, 1243, 1324, 1342, 1423, 1432.

If every equivalence class is like this with s colorings, then

$$sN = c: \quad \text{(number of symmetries)} \times \text{(number of equivalence classes)}$$
$$= \text{(total number of colorings)}$$

Solving for N, we have $N = c/s$.

Suppose we have a small round table with three positions for chairs (each 120◦ apart), and white and black chairs are available. There are $2^3 = 8$ ways to place a white or black chair in each position. See Figure below.



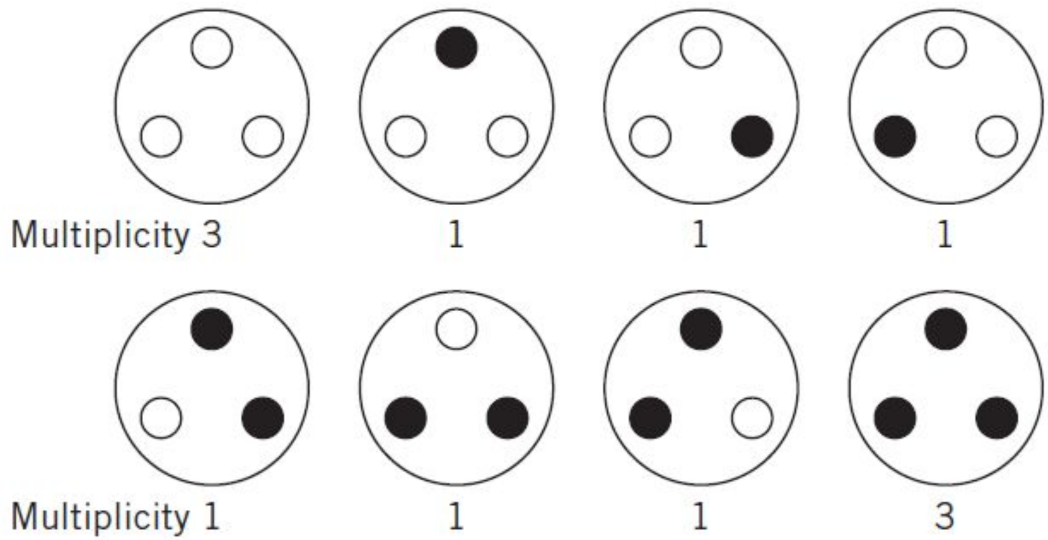
Compute N?

Number of Symmetry (only rotation is allowed)

There are **three** cyclic rotations of the table possible, 0◦, 120◦, and 240◦. We have **c = 8** "colorings" and s = 3 symmetries, but the number of equivalence classes cannot be N = 8/3, a fraction!

(the answer must be 4, see figure below)

Here an arrangement of three black chairs (or three white chairs) forms an equivalence class **by itself**. I.E. Any rotation maps this arrangement of three black chairs into itself, that is, **leaves it fixed**.

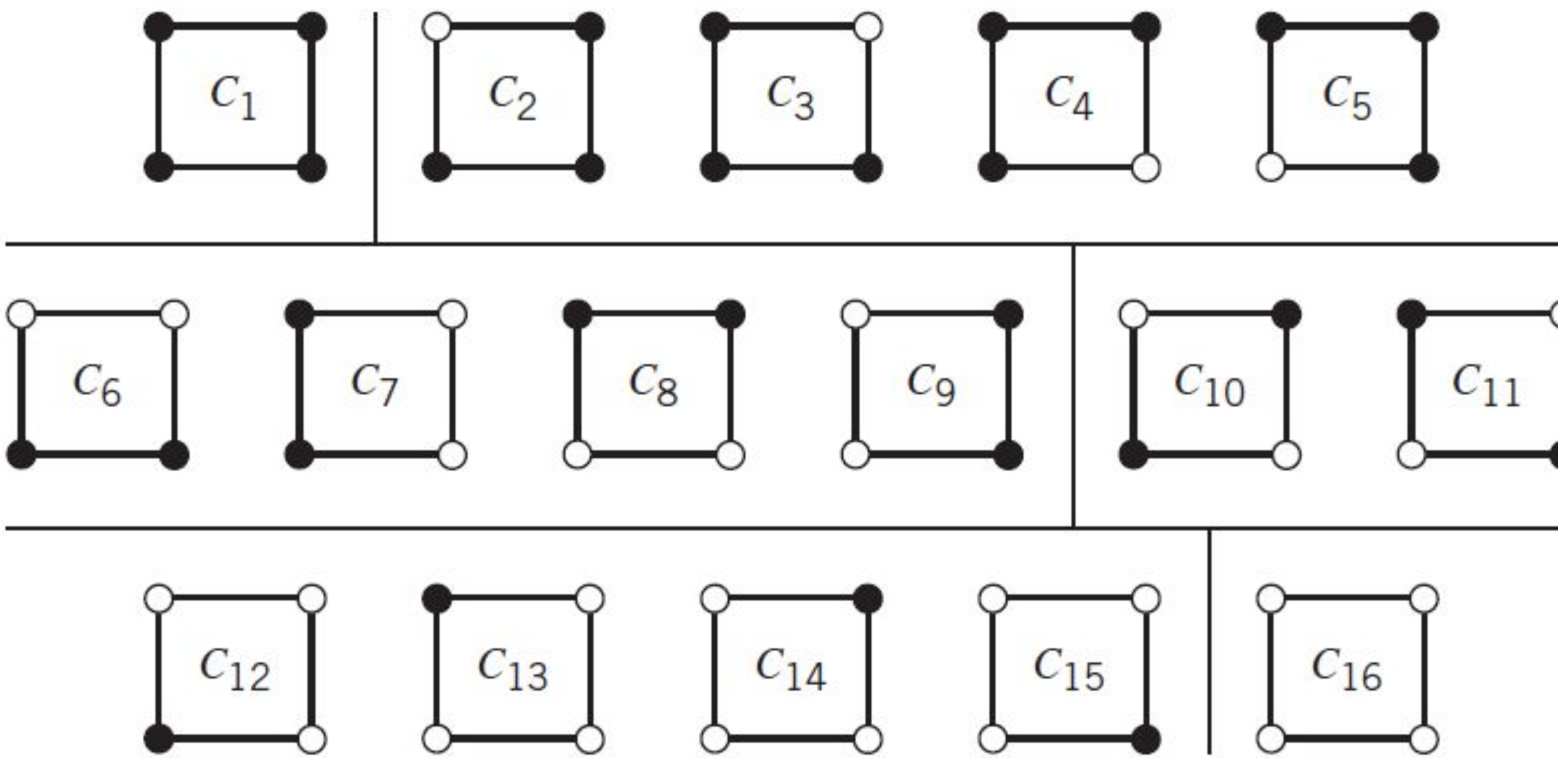


We need to correct the numerator in the formula N = c/s by adding the **multiplicities** of an arrangement.

Since two symmetries, along with the 0° symmetry, leave the all-black-chair arrangement fixed and similarly for the all-white, then counting arrangements in below with multiplicities we have the correct answer

N = (3+1+1+1+1+1+1+3)/3 = 12/3 = 4

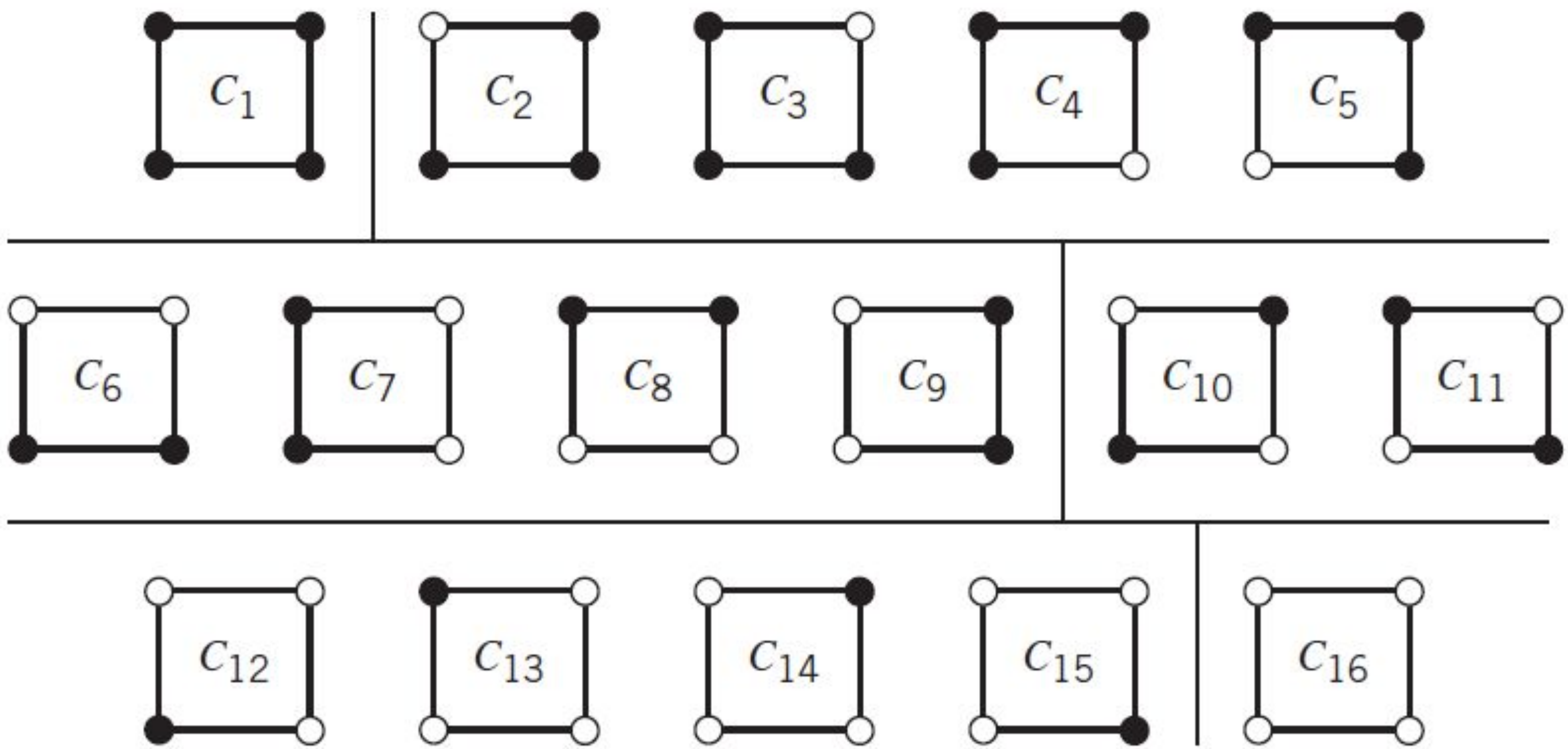
The “multiplicity” correction is even more complicated for 2-colorings of the square.



Observation 1: Several **π**s, besides the identity symmetry π_1, may leave a coloring C_i **fixed**—that is, $\pi(C_i) = C_i$. For example, the coloring C_{10} is fixed by symmetries π_1, π_3, π_7, π_8

Observation 2: If C_k is another coloring in C_i's equivalence class, there may be several πs all taking C_i to C_k.

C_{10} is mapped to C_{11} by symmetries π_2, π_4, π_5, π_6.

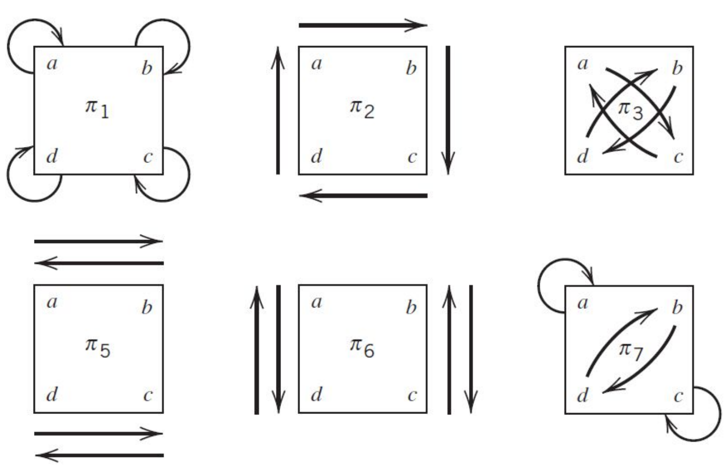


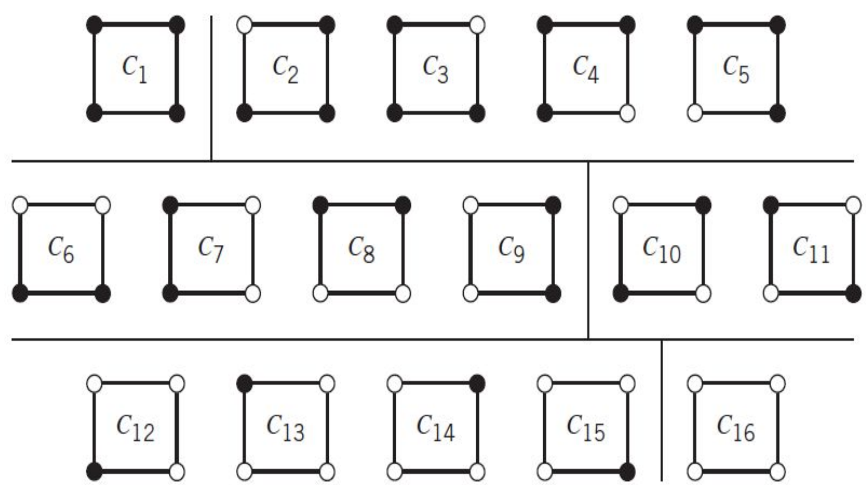
For any two permutations π_i, π_j in a group G, there exists a unique permutation $\pi_k = \pi_i^{-1} \cdot \pi_j$ in G such that $\pi_i \cdot \pi_k = \pi_j$.

By the lemma in Section 9.1, the symmetries $\pi_2, \pi_4, \pi_5, \pi_6$ taking C_{10} to C_{11} can be written in the form $\pi = \pi_2 \cdot \pi'$, where π' is a symmetry that leaves C_{11} fixed, or else π_2 followed by π' would not take C_{10} to C_{11}. For example,

$$\pi_5 = \pi_2 \cdot \pi_8 \colon \begin{pmatrix} a\,b\,c\,d \\ b\,a\,d\,c \end{pmatrix} = \begin{pmatrix} a\,b\,c\,d \\ b\,c\,d\,a \end{pmatrix} \cdot \begin{pmatrix} a\,b\,c\,d \\ c\,b\,a\,d \end{pmatrix}$$

Similarly $\pi_2 = \pi_2 \cdot \pi_1, \pi_4 = \pi_2 \cdot \pi_3, \pi_6 = \pi_2 \cdot \pi_7$. Conversely, given any π^* that leaves C_{11} fixed, $\pi_2 \cdot \pi^*$ takes C_{10} to C_{11} and so $\pi_2 \cdot \pi^*$ must be one of $\pi_2, \pi_4, \pi_5, \pi_6$. Thus there is a $1-1$ correspondence between the πs that take C_{10} to C_{11} and the πs that leave C_{11} fixed.





Therefore, to count the colorings in an equivalence class E with appropriate multiplicities (i.e., coloring C_{11} has multiplicity 4 since four different πs take C_{10} to C_{11}), it suffices to sum over the colorings in E the **number of πs that leave each coloring fixed**.

In the case of the equivalence class consisting of C_{10} and C_{11}, each of C_{10} and C_{11} have multiplicity 4, so that the size of their equivalence class including multiplicities is 4 + 4 = 8 (= s, the number of symmetries), as required.

In general, when multiplicities are counted, each equivalence class E will have **s** elements.

If $\phi(x)$ denotes the number of πs that leave the coloring x fixed, then $\sum_{x \in E} \phi(x) = s$.

Theorem **(Burnside, 1897)**

Let G be a group of permutations of the set S (corners of a square). Let T be any collection of colorings of S (2-colorings of the corners) that is closed under G. Then the number N of equivalence classes is

$$N = \frac{1}{|G|} \sum_{x \in T} \phi(x)$$

or

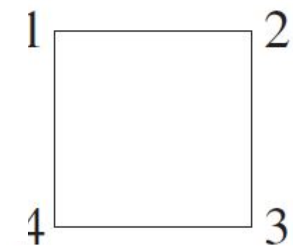
$$N = \frac{1}{|G|} \sum_{\pi \in G} \Psi(\pi) \qquad (*)$$

where $|G|$ is the number of permutations and $\Psi(\pi)$ is the number of colorings in T left fixed by π.

Determine the number of ways in which the four corners of a square can be colored with two colors. (It is permissible to use a single color on all four corners.)

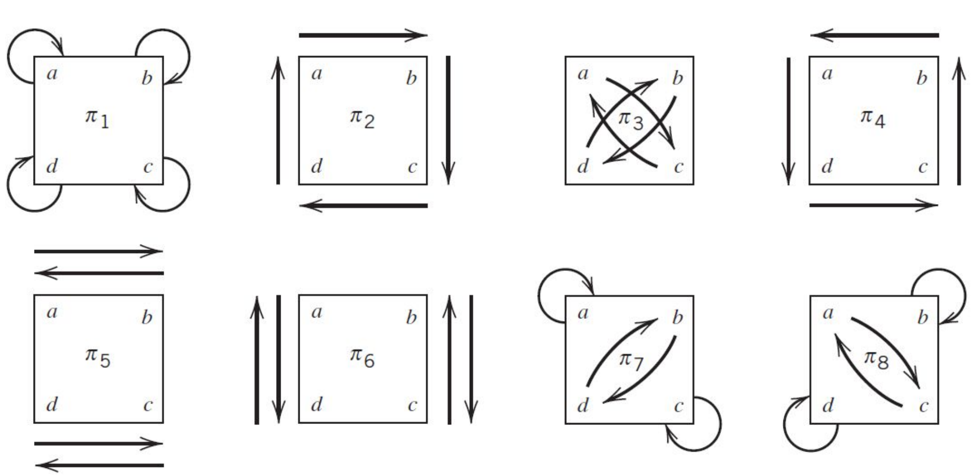
Let S be the set of all colorings. Clearly, |S| = 2^4 = 16.

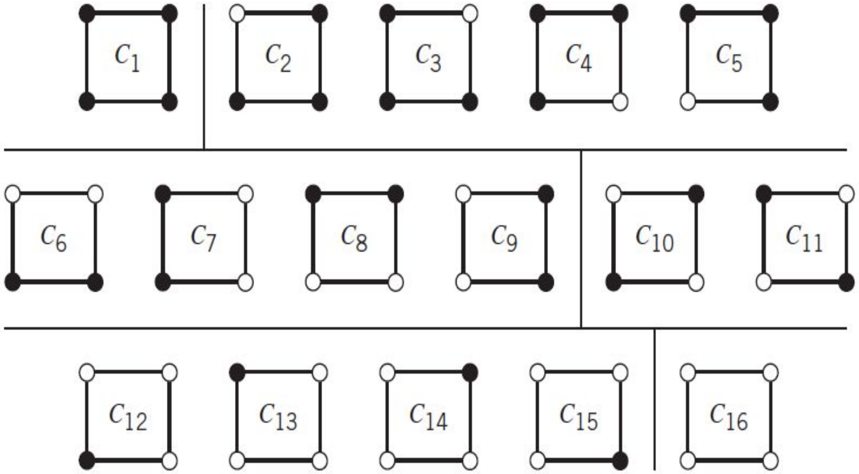
$$\rho = (1234); \quad \rho^2 = (13)(24); \quad \rho^3 = (1432); \quad \rho^4 = e = (1)(2)(3)(4)$$



We will use α, β, γ, and δ to represent the listed reflections.

$$\alpha = (24); \quad \beta = (13); \quad \gamma = (12)(34); \quad \delta = (14)(23)$$





In case of π_1, $\psi(0^\circ)$ = ?.

$\psi(0^\circ) = 16$.

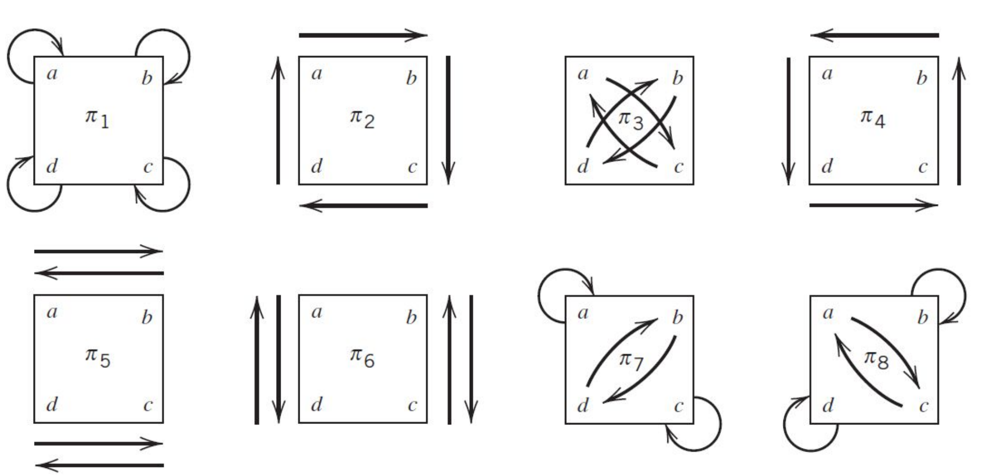
In case of π_2, $\psi(90^\circ)$ = ?

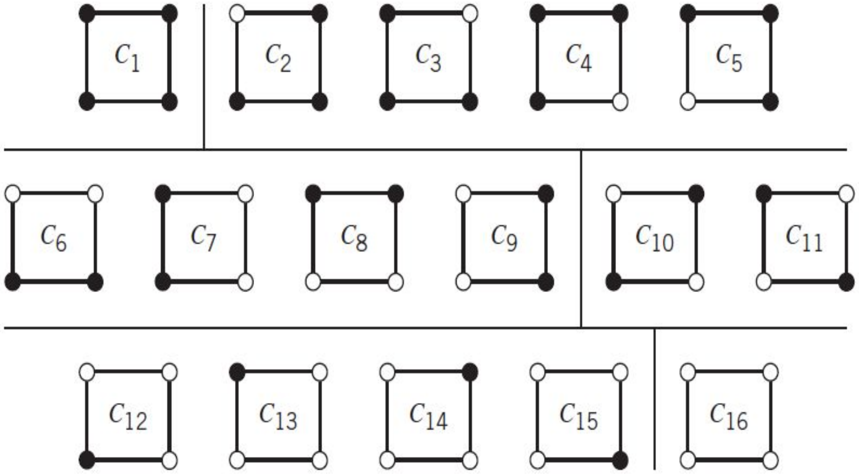
A coloring with only one color is fixed under rotation, therefore, $\psi(90^\circ) = 2$, since there are two colorings.

Also, $\psi(90^\circ) = \psi(270^\circ)$.

In case of π_3, two opposite vertices are moved to each other.

So the coloring with the same color for two non-adjacent vertices are fixed. $\psi(180^\circ) = 4$.





Because of a similar reason, π_5 the horizontal flip (14)(23) or π_6 vertical flip (12)(34) has 4 fixed points set.

In case of π_7 (13)(2)(4), 1 and 3 must have same colors, i.e., 2 possibilities and for 2, 4, each has 2 possibilities. In total 8 possibilities.

By Burnside's theorem, (16 + 2 + 4 + 2 + 4 + 4 + 8 + 8) / |G| = 48 / 8 = 6, i.e., there are 6 ways to color.

(*a*) (*b*) (*c*)

A baton is painted with equal-sized cylindrical bands. Each band can be painted black or white. If the baton is unoriented as when spun in the air, how many different 2-colorings of the baton are possible if the baton has **(a)** 2 bands? **(b)** 3 bands? **(c)** 4 bands?

First is to identify the number of symmetries.

Irrespective of the number of bands, there are two symmetries of a baton: π_1 is a 0° revolution of the baton—π_1 is the identity symmetry—and π_2 is a 180° revolution of the baton.

$[\Psi(\pi_1) + \Psi(\pi_2)]$

(a) For the 2-band baton, the set of 2-colorings left fixed by π_1 is all 2-colorings of the baton. There are $2^2 = 4$ 2-colorings, and so $\Psi(\pi_1) = 4$. The set of 2-colorings left fixed by π_2 consists of the all-black and all-white coloring, and so $\Psi(\pi_2) = 2$. By Burnside's theorem, the number of different colorings is $\frac{1}{2}[\Psi(\pi_1) + \Psi(\pi_2)] = \frac{1}{2}(4 + 2) = 3$.

(1, 1), (2, 2), (1, 2)

(b) For the 3-band baton, all 2^3 2-colorings are left fixed by π_1, and so $\Psi(\pi_1) = 2^3 = 8$. The set of 2-colorings left fixed by π_2 can have any color in the middle band (band 2) and a common color in the two end bands, and so $\Psi(\pi_2) = 2 \times 2 = 4$. The number of different colorings is $\frac{1}{2}[\Psi(\pi_1) + \Psi(\pi_2)] = \frac{1}{2}(8 + 4) = 6$.

(1, 2, 3), (2, 1, 3), (1, 3, 2), (1, 1, 1), (2, 2, 2), (3, 3, 3).

(c) For the 4-band baton, all 2^4 2-colorings are left fixed by π_1, and so $\Psi(\pi_1) = 2^4 = 16$. The set of 2-colorings left fixed by π_2 have a common color for the end bands and a common color for the inner bands, so $\Psi(\pi_2) = 2 \times 2 = 4$. The number of different colorings is $\frac{1}{2}[\Psi(\pi_1) + \Psi(\pi_2)] = \frac{1}{2}(16 + 4) = 10$. ■

How many different 3-colorings of the bands of an n-band baton are there if the baton is unoriented.

The symmetries of the baton are a 0◦ revolution and a 180◦ revolution. There are 3^n colorings of the fixed baton and so
$\psi(0◦) = 3^n$.
The number of colorings left fixed by a 180◦ spin depends on whether n is **even or odd**.

If n is even, each of the $n/2$ bands on one half of the baton can be any color—$3^{n/2}$ choices—and then for the coloring to be fixed by a 180° spin, each of the symmetrically opposite bands must be the corresponding color. So $\Psi(180°) = 3^{n/2}$ and we have from formula (*): $N = \frac{1}{2}(3^n + 3^{n/2})$.

To enumerate batons left fixed by a 180° spin when n is odd, we can use any color for the "odd" band in the middle of the baton—three choices. Each of the $(n-1)/2$ bands on one side of the middle band can be any color—$3^{(n-1)/2}$ choices—and again the other $(n-1)/2$ bands must be colored symmetrically. So $\Psi((180°) = 3 \times 3^{(n-1)/2} = 3^{(n+1)/2}$ and $N = \frac{1}{2}(3^n + 3^{(n+1)/2})$. ■

Suppose a necklace can be made from beads of three colors—black, white, and red. How many different necklaces with 3 beads are there? (beads are allowed to move freely about the circle but flips are not allowed)

Rotations?

There are $3^3 = 27$ 3-colorings of a 3-bead necklace, and three rotations of $0°$, $120°$, $240°$. The $0°$ rotation leaves all colorings fixed, and so $\Psi(0°) = 27$. The $120°$ rotation cannot fix colorings in which some color occurs at only one corner. It follows that the $120°$ rotation fixes just the monochromatic colorings. Thus, $\Psi(120°) = 3$. The $240°$ rotation is a reverse $120°$ rotation, and so $\Psi(240°) = 3$. By formula (*), we have

$$N = \frac{1}{3}(27 + 3 + 3) = 11 \quad \blacksquare$$

9.3: THE CYCLE INDEX

$$N = \frac{1}{|G|} \sum_{\pi \in G} \Psi(\pi)$$

where $\Psi(\pi)$ is the number of colorings in T left fixed by π. If the set T were all 3-colorings of the corners of a 10-gon or a cube, it would seem close to impossible to determine $\Psi(\pi)$s, the number of colorings left fixed by various symmetries π of the figure. However, we shall show that $\Psi(\pi)$ can be determined easily from the structure of π. We develop the theory for this simplified calculation of $\Psi(\pi)$ in terms of 2-colorings of a square.

Observation 1: a coloring C will be left fixed by π if and only if for each corner v, the color of C at v is the same as the color at π(v) so that the symmetry leaves the color at π(v) unchanged.

Example. We know that π_3 causes corners a and c to interchange and corners b and d to interchange.

It follows that a coloring left fixed by π_3 must have the same color at corners a and c and the same color at b and d (no further conditions are needed).

With two color choices for a, c and with two color choices for b, d, we can construct $2 \times 2 = 4$ colorings that will be left fixed—namely, C_1, C_{10}, C_{11}, C_{16}. Hence $\psi(\pi_3) = 4$.

Observation 2: if π_i cyclicly permutes a subset of corners (that is, the corners form a cycle of π_i), then those corners must all be the same color in any coloring left fixed by π_i.

Example. $\pi_3 = (ac)(bd)$

For each symmetry, we need to get such a cyclic representation and then count the number of ways to assign a color to each cycle of corners.

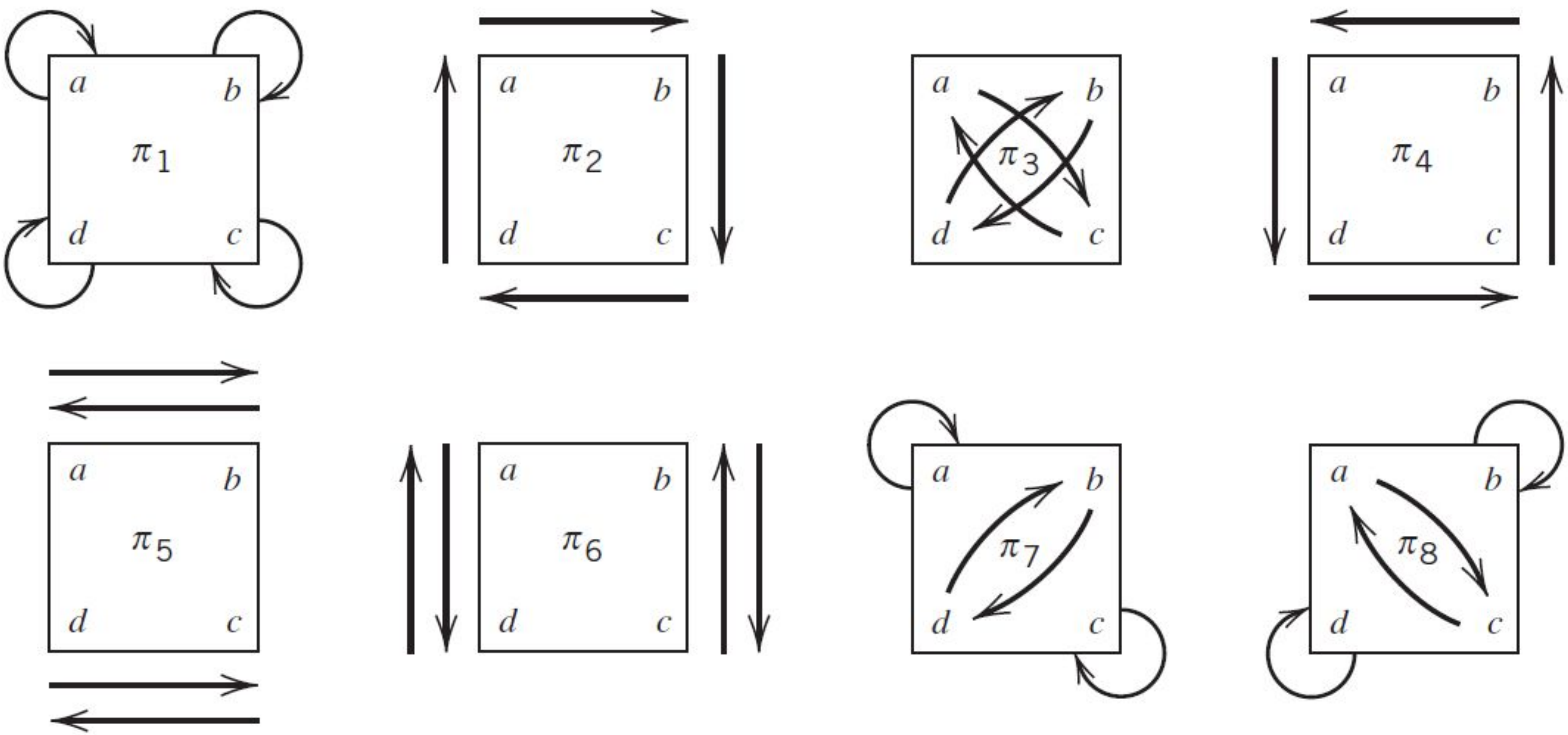
Let us also classify the cycles by their length.

It will prove convenient to encode a symmetry's cycle information in the form of a product containing one x_1 for each cycle of length 1, one x_2 for each cycle of size 2, and so forth.

This expression is called the **cycle structure representation of a** symmetry.

The cycle structure representation of π_2 (90 degree), π_3 (180 degree) and π_1(identity)?

For π_2, its x_4, since it consists of one 4-cycle: π_2 = (abcd).
For π_3, its $x_2^{\ 2}$, since it consists of two 2-cycles: π_3 = (ac)(bd)
For π_1, its $x_1^{\ 4}$



(*i*) Motion π_i	(*ii*) Colorings Left Fixed by π_i	(*iii*) Cycle Structure Representation
π_1	16—all colorings	x_1^4
π_2	2—C_1, C_{16}	x_4
π_3	2—$C_1, C_{10}, C_{11}, C_{16}$	x_2^2
π_4	2—C_1, C_{16}	x_4
π_5	2—C_1, C_6, C_8, C_{16}	x_2^2
π_6	2—$C_1, C_7, C_9 C_{16}$	x_2^2
π_7	8—C_1, C_2, C_4, C_{10} $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2 x_2$
π_8	8—C_1, C_3, C_5, C_{10} $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2 x_2$

Observation 3:

For any symmetry π of any figure, *the number of colorings left fixed will be given by setting each x_j equal to* 2 *(or, in general, the number of colors available) in the cycle structure representation of π, that is,*

$$\Psi(\pi) = 2^{\text{number of cycles in } \pi}$$

To obtain the number of different 2-colorings of the floating square with Burnside's Theorem, we sum the numbers in column (ii) of Figure 9.7 and divide by 8:

$$\frac{1}{|G|}\sum_{\pi \in G} \Psi(\pi) = \frac{1}{8}(16+2+4+2+4+4+8+8) = \frac{1}{8}(48) = 6$$

There is a slightly simpler way to get this result. First, algebraically sum the cycle structure representations of each symmetry, collecting like terms together, and then divide by 8. From column (iii) of Figure 9.7, we obtain $\frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2)$. This expression is called the **cycle index** $P_G(x_1, x_2, \ldots, x_k)$ for a group G of symmetries. By setting each $x_i = 2$ in this cycle index—that is, $P_G(2, 2, \ldots, 2)$—we get the same answer.

Suppose that instead of two colors, we had three colors. Then the same reasoning applies, but now there are three choices for the color of the corners in each cycle. If a symmetry has k cycles, then it will leave 3^k 3-colorings of the square fixed, and the number of different 3-colorings will be $P_G(3, 3, \ldots, 3)$. More generally, for any m, $P_G(m, m, \ldots, m)$ will be the number of nonequivalent m-colorings of an unoriented square. The argument used to derive this coloring counting formula with the cycle

Theorem

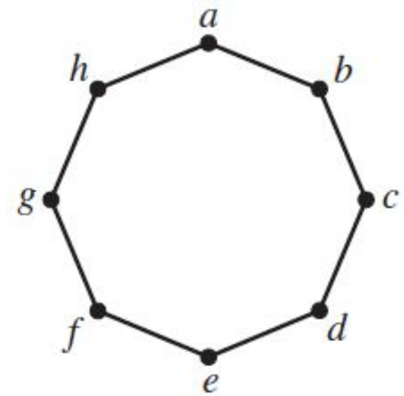
Let S be a nonempty set of elements and G be a group of symmetries of S that acts to induce an equivalence relation on the set of m-colorings of S. Then the number of nonequivalent m-colorings of S is given by $P_G(m, m, \ldots, m)$.

Use this Theorem to solve the following question of last section:

Suppose a necklace can be made from beads of three colors—black, white, and red. How many different necklaces with 3 beads are there?

the rotations are of 0°, 120°, and 240° with cycle structure representations of x_1^3, x_3, and x_3, respectively. Thus, $P_G = \frac{1}{3}(x_1^3 + 2x_3)$. The number of 3-colored strings of three beads is $P_G(3, 3, 3) = \frac{1}{3}(3^3 + 2 \times 3) = 11$. More generally, the number of m-colored necklaces of three beads is $P_G(m, m, m) = \frac{1}{3}(m^3 + 2m)$.

Suppose a necklace can be made from beads of three colors—black, white, and red. How many different necklaces with 8 beads are there?



The rotations are of

0◦, 45◦, 90◦, 135◦, 180◦, 225◦, 270◦, and 315◦.

The 0◦ rotation consists of eight 1-cycles.

The 45◦ rotation is the cyclic permutation (abcdefgh).

The 90◦ rotation has the cycle decomposition (aceg)(bdfh).

The 135◦ rotation is the cyclic permutation (adgbehcf).

The 180◦ rotation has the cyclic decomposition (ae)(bf)(cg)(dh).

The cycle structure representations are thus 0◦ rotation, x_1^8 45◦ rotation, x_8, 90◦ rotation, x_4^2; 135◦ rotation, x_8; and 180◦ rotation, x_4^2.

The 225◦, 270◦, and 315◦ rotations are reverse rotations of 135◦, 90◦, 45◦, respectively, and have the corresponding cycle structure representations. Collecting terms, we obtain

$$P_G = \frac{1}{8}\left(x_1^8 + 4x_8 + 2x_4^2 + x_2^4\right)$$

The number of different m-colored necklaces of eight beads is

$$\frac{1}{8}(m^8 + 4m + 2m^2 + m^4)$$

For $m = 3$, we have

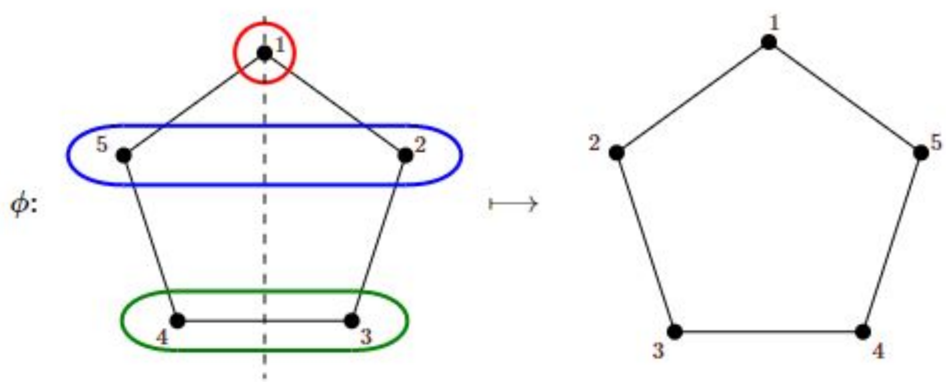
$$\frac{1}{8}(3^8 + 4 \times 3 + 2 \times 3^2 + 3^4) = \frac{1}{8}(6561 + 12 + 18 + 81) = 834 \ \blacksquare$$

Compute the number of colorings of pentagon using k colors (by computing the corresponding cycle index).

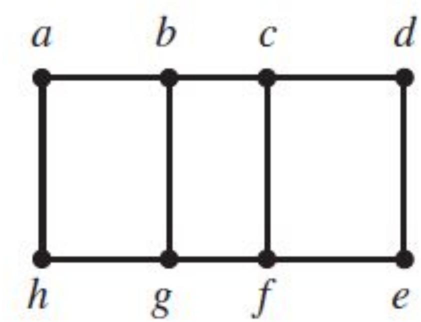
Pentagon has 10 symmetries: 5 rotation and 5 reflection.

For identity, the cycle structure representation is $x_1^{\,5}$
The cycle structure representations for all other rotations are x_5
The cycle structure representations for each reflection is
$x_1 x_2^{\,2}$
Thus, $P_G = 1/10\ (x_1^{\,5} + 4x_5 + 5x_1x_2^{\,2})$

Substituting $x_i = k$, we have $P_G = 1/10\ (k^5 + 4k + 5k^3)$



Find the number of different m-colorings of the vertices of the following figure.



The symmetries are:

0◦, 180◦, about vertical and horizontal axis.

The 0◦ rotation consists of 8 1-cycles.

The 180◦ rotation has the cyclic decomposition (ae)(bf)(cg)(dh).

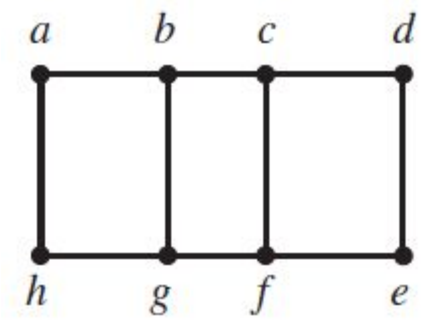
Symmetry about vertical axis has the cyclic decomposition (ad)(he)(bc)(gf).

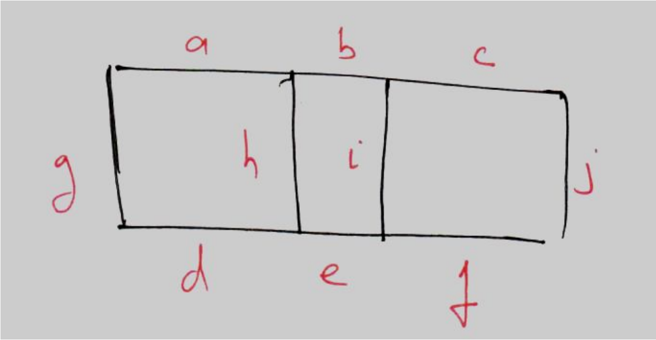
Symmetry about horizontal axis has the cyclic decomposition (ah)(bg)(cf)(de).

$P_G = \frac{1}{4}(x_1^8 + 3x_2^4)$

The number of different m-colorings is $\frac{1}{4}(m^8 + 3m^4)$

Find the number of different m-colorings of the edges of the following figure.



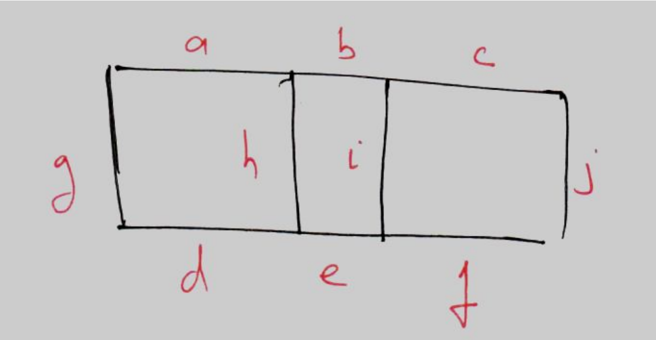


The symmetries are:

0◦, 180◦, about vertical and horizontal axis.

The 0◦ rotation consists of 10 1-cycles.

Symmetry about vertical axis has the cyclic decomposition (b)(e)(hi)(gj)(ac)(df).



Symmetry about horizontal axis has the cyclic decomposition (ad)(be)(cf)(g)(h)(i)(j).

The 180◦ rotation has the cyclic decomposition (af)(be)(cd)(hi)(gj)

$P_G = ¼\ (x_1^{10} + x_2^4 x_1^2 + x_2^3 x_1^4 + x_2^5)$

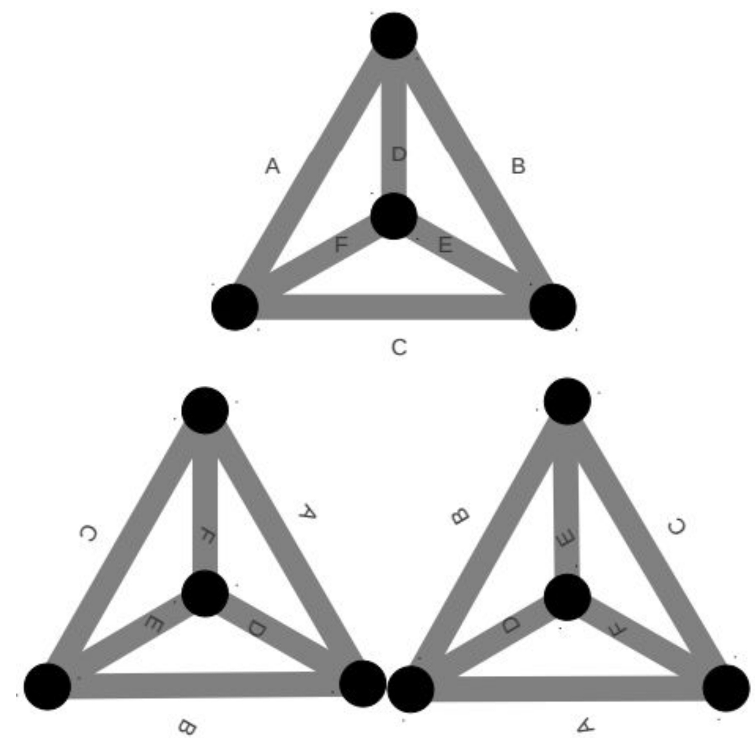
The number of different m-colorings is ¼ ($m^8 + m^6 + m^7 + m^5$)

Use the theorem to determine the number of 3-colorings of the four corners of a floating tetrahedron.

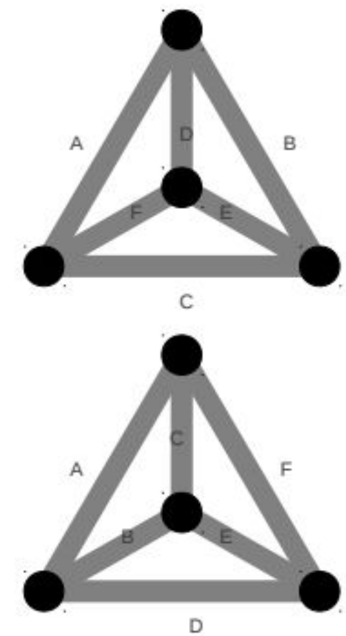
Symmetries?

There exists 12 symmetries of the Tetrahedron:
the 0° revolution,
the eight revolutions of 120° and 240° about a corner and the middle of the opposite face (there are 4 corners and at a time **one corner is fixed**),

Rotation of tetrahedron by 120 degrees and 240 degrees



and the three revolutions of 180◦ about the middle of opposite edges (changing the position of a corner).



Rotation of tetrahedron by 180 degrees about edges A and E

The 0◦ revolution has the cycle structure representation x_1^4.

The 120◦ revolution about corner ***a*** and the middle of face bcd has the cyclic decomposition (a)(bcd) and its cycle structure representation is x_1x_3.

By symmetry, the other 120◦ and 240◦ revolutions have this same cycle structure representation.

The 180◦ revolution about the middle of edges ab and cd has the cyclic decomposition (ab)(cd) and its cycle structure representation is x_2^2.

By symmetry, the other 180◦ revolutions have the same cycle structure representation.
Thus we have

$$P_G = \frac{1}{12}\left(x_1^4 + 8x_1x_3 + 3x_2^2\right)$$

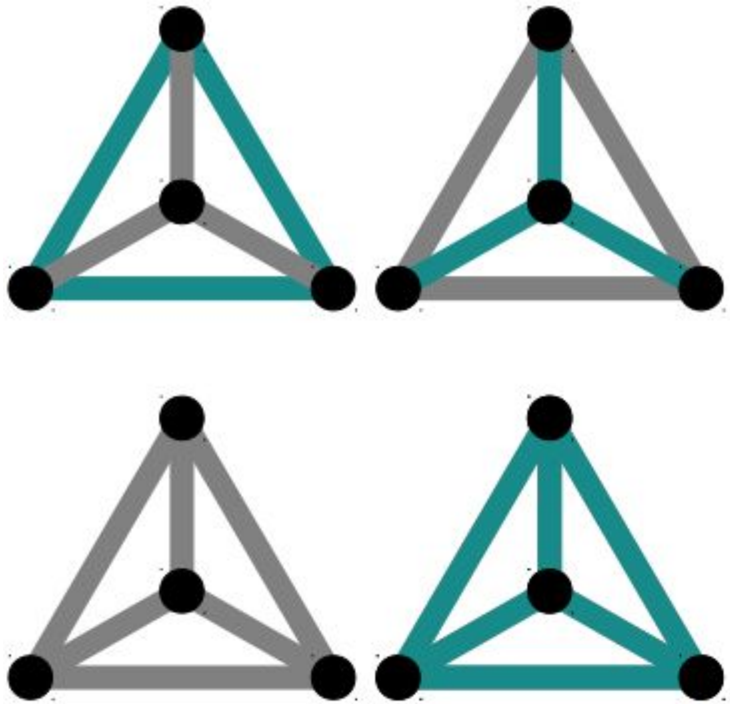
The number of different corner 3-colorings is

$$\begin{aligned} P_G(3,3,3,3) &= \frac{1}{12}(3^4 + 8 \times 3 \times 3 + 3 \times 3^2) \\ &= \frac{1}{12}(81 + 72 + 27) = 15 \ \blacksquare \end{aligned}$$

Compute cycle index for the edge colorings of a tetrahedron.

The identity leaves all 6 edges fixed and has structure representation x_1^6

Four 120 degree rotations about a corner and the middle of the opposite face give two cycles of length three.
They cyclically permute the edges incident to that corner and also the edges bounding the opposite face, so the cycle structure representation is x_3^2.
The four colorations this fixes are shown in Figure below.



Colorations of the tetrahedron fixed by rotations of 120 or 240 degrees

Three 180 degree rotations about opposite edges leave the two edges fixed. The other four edges are left in cycles of length 2. Thus we have the structure $x_1^2 x_2^2$.

$$P_G = \frac{1}{12}(x_1^6 + 8x_3^2 + 3x_1^2x_2^2).$$

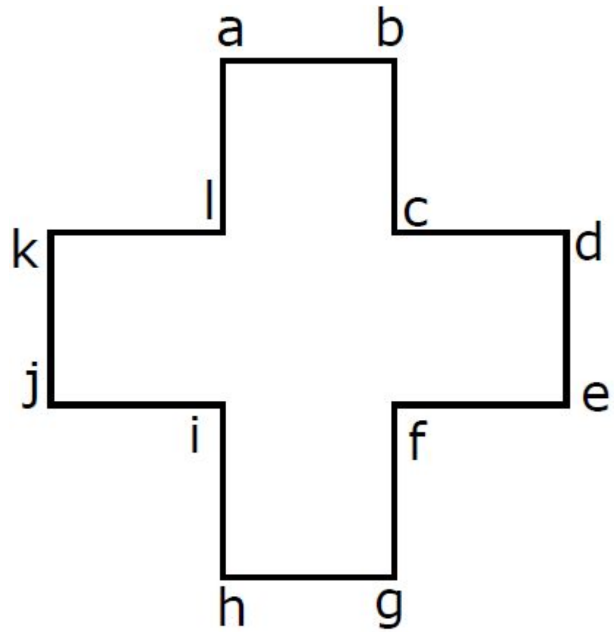
Compute the number of ways to color the vertices of a rhombus (which is not a square) with k colors.

¼ $(x_1^4 + x_2^2 + 2x_1^2x_2)$

Compute the number of ways to color the vertices of an equilateral triangle with 5 colors while using at least 2 colors.

6 symmetries are there.
For 0 degree, $5^3 – 5 = 120$.
For 120 and 240, $5 – 5 = 0$ and for 3 flips, $5^2 – 5 = 20$.
$1/6\ (120 + 0 + 3.20) = 30$.

Find the number of different k-colorings of the vertices of the following figure



For 0 degree, x_1^{12}

For 90=270 degree, x_4^{3}

For 180 degree, x_2^{6}

For horizontal and vertical reflections, x_2^{6}

For diagonal reflections, $x_1^{2}x_2^{5}$

POLYA’S FORMULA

(*i*) Motion π_i	(*ii*) Colorings Left Fixed by π_i	(*iii*) Cycle Structure Representation
π_1	16—all colorings	x_1^4
π_2	2—C_1, C_{16}	x_4
π_3	2—$C_1, C_{10}, C_{11}, C_{16}$	x_2^2
π_4	2—C_1, C_{16}	x_4
π_5	2—C_1, C_6, C_8, C_{16}	x_2^2
π_6	2—$C_1, C_7, C_9 C_{16}$	x_2^2
π_7	8—C_1, C_2, C_4, C_{10} $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2 x_2$
π_8	8—C_1, C_3, C_5, C_{10} $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2 x_2$

We are now ready to address our ultimate goal of a formula for the **pattern inventory**.

The pattern inventory is a generating function that tells how many colorings of an unoriented figure there are using different possible collections of colors.

For black–white colorings of the unoriented square, the pattern inventory is

$1b^4 + 1b^3w + 2b^2w^2 + 1bw^3 + 1w^4$.

For example, the term $1b^3w$ tells us that there is one nonequivalent coloring with three black (b) corners and one white (w) corner. Our aim is to compute the coefficient of $b^{4-k}w^k$ in the pattern inventory. For computing the coefficient of $b^{4-k}w^k$, refer to the Table below.

(i) Motion π_i	(ii) Colorings Left Fixed by π_i	(iii) Cycle Structure Representation	(iv) Inventory of Colorings Left Fixed by π_i					
π_1	16—all colorings	x_1^4	$(b+w)^4$	$= 1b^4$	$+4b^3w$	$+6b^2w^2$	$+4bw^3$	$+1w^4$
π_2	2—C_1, C_{16}	x_4	(b^4+w^4)	$= 1b^4$				$+1w^4$
π_3	2—$C_1, C_{10}, C_{11}, C_{16}$	x_2^2	$(b^2+w^2)^2$	$= 1b^4$		$+2b^2w^2$		$+1w^4$
π_4	2—C_1, C_{16}	x_4	(b^4+w^4)	$= 1b^4$				$+1w^4$
π_5	2—C_1, C_6, C_8, C_{16}	x_2^2	$(b^2+w^2)^2$	$= 1b^4$		$+2b^2w^2$		$+1w^4$
π_6	2—C_1, C_7, C_9, C_{16}	x_2^2	$(b^2+w^2)^2$	$= 1b^4$		$+2b^2w^2$		$+1w^4$
π_7	8—C_1, C_2, C_4, C_{10} $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2x_2$	$(b+w)^2(b^2+w^2)$	$= 1b^4$	$+2b^3w$	$+2b^2w^2$	$+2bw^3$	$+1w^4$
π_8	8—C_1, C_3, C_5, C_{10} $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2x_2$	$(b+w)^2(b^2+w^2)$	$= 1b^4$	$+2b^3w$	$+2b^2w^2$	$+2bw^3$	$+1w^4$

In the first row of column (iv), we write a polynomial whose coefficients give the numbers of 2-colorings in each T_k left fixed by π_1, then in the second row of the table we write a polynomial for the numbers of 2-colorings in each T_k left fixed by π_2, then by π_3, and so forth. Then we total up the b^4 term in each row (the number of 2-colorings with four blacks) and divide by 8 to get the coefficient of b^4 in the pattern inventory, total up the b^3w term in each row and divide by 8 to get the coefficient of b^3w, and so forth.

Since the action of π_1 leaves all C_s fixed, the first row's coefficients are 1, 4, 6, 4, 1. We write $b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4$; this is an *inventory of fixed colorings*. For π_1, the inventory of fixed colorings is an inventory of all colorings. Observe that this inventory is simply $(b+w)^4 = (b+w)(b+w)(b+w)(b+w)$, **why**

Since each corner can have any color, black or white

For π_2, the inventory is $b^4 + w^4$.

If all corners have same color, then only coloring is fixed by π_2

The motion π_3 has two 2-cycles (ac) and (bd). Each 2-cycle uses two blacks or two whites in a fixed coloring. Hence the inventory of a cycle of size two is $b^2 + w^2$. The possibilities with two such cycles have the inventory $(b^2 + w^2)(b^2 + w^2)$.

The inventory of fixed colorings for π_i will be a product of factors $(b^j + w^j)$, one factor for each j-cycle of the π_i. So we need to know the number of cycles in π_i of each size. But this is exactly the information encoded in the cycle structure representation. Indeed, setting $x_j = (b^j + w^j)$ in the representation yields precisely the inventory of fixed colorings for π_i. By this method we compute the rest of the inventories of fixed colorings.

If three colors, black, white, and red, were permitted, each cycle of size j would have an inventory of $(b^j + w^j + r^j)$ in a fixed coloring. So we would set $x_j = (b^j + w^j + r^j)$ in P_G. The preceding argument applies for any number of colors and any figure. In greater generality we have the following theorem.

Theorem **(Polya's Enumeration Formula)**

Let S be a set of elements and G be a group of permutations of S that acts to induce an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using two colors is given by the generating function $P_G((b+w), (b^2+w^2), (b^3+w^3), \ldots, (b^k+w^k))$. The inventory using colors $c_1, c_2, \ldots, c_m$ is

$$P_G\left(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, \ldots, \sum_{j=1}^{m} c_j^k\right)$$

Determine the pattern inventory for 3-bead necklaces distinct under rotations using black and white beads.

From Example 1 of Section 9.3, we know $P_G = \frac{1}{3}(x_1^3 + 2x_3)$. Substituting $x_j = (b^j + w^j)$, we get

$$\begin{aligned}\frac{1}{3}[(b+w)^3 + 2(b^3+w^3)] &= \frac{1}{3}[(b^3+3b^2w+3bw^2+w^3)+(2b^3+2w^3)] \\ &= \frac{1}{3}(3b^3+3b^2w+3bw^2+3w^3) \\ &= b^3+b^2w+bw^2+w^3\end{aligned}$$

Find the number of 7-bead necklaces distinct under rotations using three black and four white beads.

We need to determine the coefficient of b^3w^4 in the pattern inventory. Each rotation, except the 0° rotation, is a cyclic permutation

so $P_G = \frac{1}{7}(x_1^7 + 6x_7)$. The pattern inventory is $\frac{1}{7}[(b+w)^7 + 6(b^7+w^7)]$.

Since the factor $6(b^7 + w^7)$ in the pattern inventory contributes nothing to the b^3w^4 term, we can neglect it. Thus the number of 3-black, 4-white necklaces is simply

$$\frac{1}{7}[(\text{coefficient of } b^3w^4 \text{ in } (b+w)^7] = \frac{1}{7}\binom{7}{3} \quad \blacksquare$$

Compute the number of 6-bead distinct necklaces using 3 black, 2 white and 1 grey beads (consider both rotation and reflection).

$1/12\ (x_1^6 + 4x_2^3 + 2x_3^2 + 2x_6 + 3x_1^2x_2^2)$

Replace x_i by $b^i + w^i + g^i$

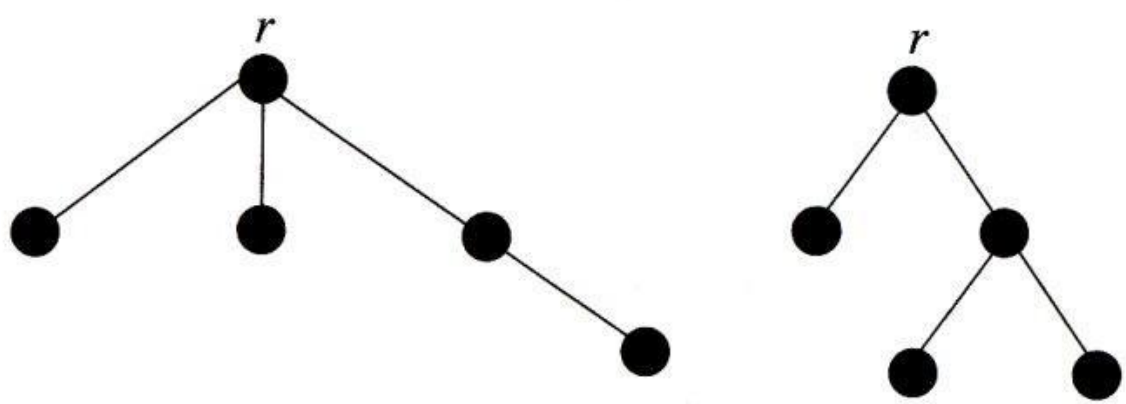
Coefficient of b^3w^2g is $1/12(\ C(6; 3,2,1) + 3.2.2) = 6$.

Catalon Numbers

Rooted Trees

Let T be a tree and $r \in V(T)$. A *rooted tree* is the ordered pair (T, r). The vertex r is called the *root* of T.

Example: Are the following trees isomorphic

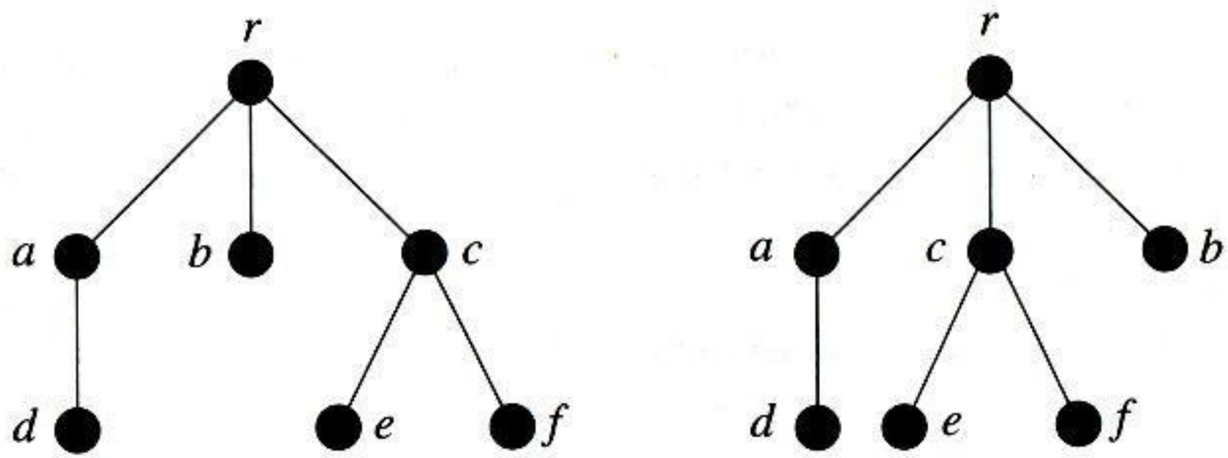


Two rooted trees, isomorphic as trees but not as rooted trees.

Ordered rooted tree

A rooted tree (T, r) in which the left/right order of every set of siblings is specified is called an *ordered rooted tree*.

Example

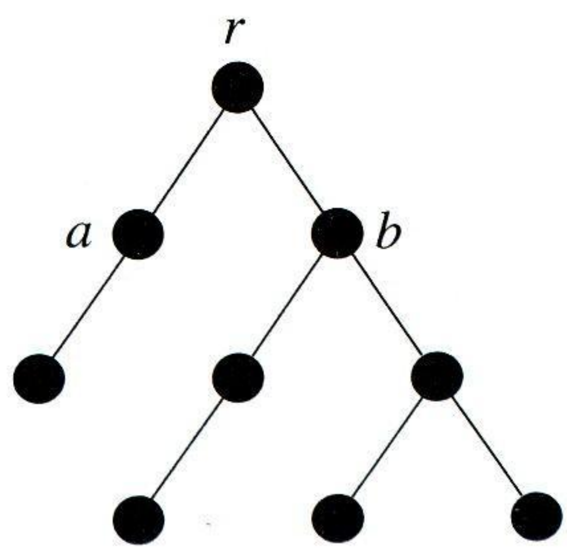


(a) (b)

Two distinct ordered rooted trees that are identical when viewed as rooted trees.

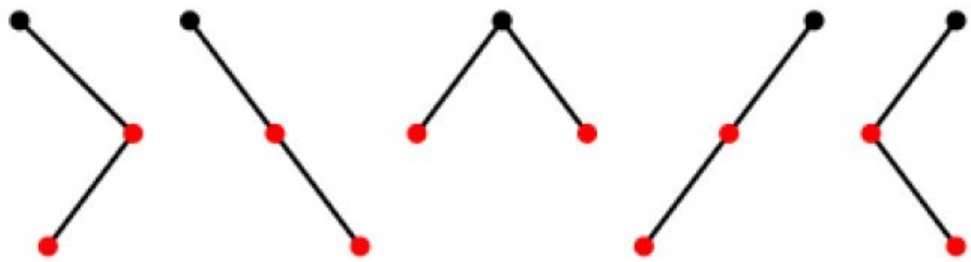
Binary Trees

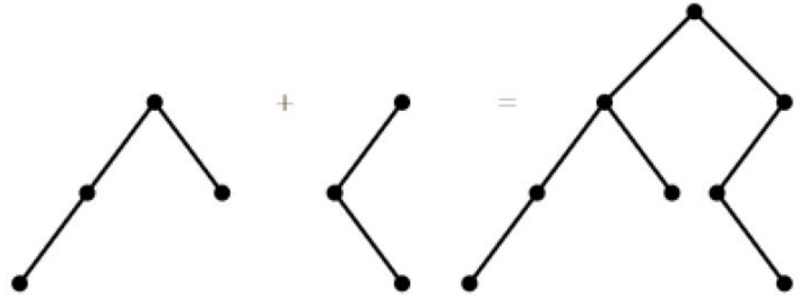
A *binary tree* is an ordered rooted tree in which each vertex has at most two children. Each child of a vertex is called either the *left* child or the *right* child. A subtree rooted at the left (right) child of a vertex u is known as u's *left* (*right*) *subtree*. By convention, the single vertex tree is considered a "trivial" binary tree.



How many different rooted binary trees are there with n vertices?

Let us denote this number by $\mathbf{C_n}$; these are the **Catalan numbers.**
For convenience, we allow a rooted binary tree to be empty, and let C_0 = 1. Then it is easy to see that C_1 = 1 and C_2 = 2.
Compute C_3.

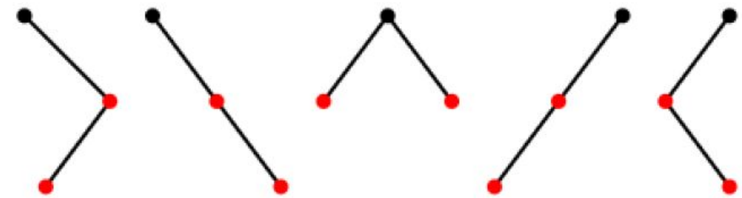




Any rooted binary tree on at least one vertex can be viewed as two binary trees joined into a new tree by introducing a new root vertex and making the children of this root the two roots of the original trees.

since we know that $C_0 = C_1 = 1$ and $C_2 = 2$,

$$C_3 = C_0C_2 + C_1C_1 + C_2C_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5,$$



Now we can write

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

Now we use a generating function to find a formula for C_n. Let $f = \sum_{i=0}^{\infty} C_i x^i$. Now consider f^2: the coefficient of the term x^n in the expansion of f^2 is $\sum_{i=0}^{n} C_i C_{n-i}$, corresponding to all possible ways to multiply terms of f to get an x^n term:

$$C_0 \cdot C_n x^n + C_1 x \cdot C_{n-1} x^{n-1} + C_2 x^2 \cdot C_{n-2} x^{n-2} + \cdots + C_n x^n \cdot C_0.$$

Now we recognize this as precisely the sum that gives C_{n+1}, so $f^2 = \sum_{n=0}^{\infty} C_{n+1}x^n$. If we multiply this by x and add 1 (which is C_0) we get exactly f again, that is, $xf^2 + 1 = f$ or $xf^2 - f + 1 = 0$; here 0 is the zero function, that is, $xf^2 - f + 1$ is 0 for all x. Using the quadratic formula,

$$f = \frac{1 \pm \sqrt{1-4x}}{2x},$$

as long as $x \neq 0$. It is not hard to see that as x approaches 0,

$$\frac{1 + \sqrt{1-4x}}{2x}$$

goes to infinity while

$$\frac{1 - \sqrt{1-4x}}{2x}$$

goes to 1. Since we know $f(0) = C_0 = 1$, this is the f we want.

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

$$
\begin{aligned}
(1-4z)^{1/2} &= 1 - \frac{\left(\frac{1}{2}\right)}{1}4z + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2\cdot 1}(4z)^2 - \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2\cdot 1}(4z)^3 + \\
& \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4\cdot 3\cdot 2\cdot 1}(4z)^4 - \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{5\cdot 4\cdot 3\cdot 2\cdot 1}(4z)^5 + \cdots
\end{aligned}
$$

We can get rid of many powers of 2 and combine things to obtain:

$$
(1-4z)^{1/2} = 1 - \frac{1}{1!}2z - \frac{1}{2!}4z^2 - \frac{3\cdot 1}{3!}8z^3 - \frac{5\cdot 3\cdot 1}{4!}16z^4 - \frac{7\cdot 5\cdot 3\cdot 1}{5!}32z^5 - \cdots \tag{9}
$$

From Equations 9 and 8:

$$
f(z) = 1 + \frac{1}{2!}2z + \frac{3\cdot 1}{3!}4z^2 + \frac{5\cdot 3\cdot 1}{4!}8z^3 + \frac{7\cdot 5\cdot 3\cdot 1}{5!}16z^4 + \cdots \tag{10}
$$

The terms that look like $7\cdot 5\cdot 3\cdot 1$ are a bit troublesome. They are like factorials, except they are missing the even numbers. But notice that $2^2\cdot 2! = 4\cdot 2$, that $2^3\cdot 3! = 6\cdot 4\cdot 2$, that $2^4\cdot 4! = 8\cdot 6\cdot 4\cdot 2$, et cetera. Thus $(7\cdot 5\cdot 3\cdot 1)\cdot 2^4 4! = 8!$. If we apply this idea to Equation 10 we can obtain:

$$
f(z) = 1 + \frac{1}{2}\left(\frac{2!}{1!1!}\right)z + \frac{1}{3}\left(\frac{4!}{2!2!}\right)z^2 + \frac{1}{4}\left(\frac{6!}{3!3!}\right)z^3 + \frac{1}{5}\left(\frac{8!}{4!4!}\right)z^4 + \cdots = \sum_{i=0}^{\infty}\frac{1}{i+1}\binom{2i}{i}z^i.
$$

From this we can conclude that the i^{th} Catalan number is given by the formula

$$C_i = \frac{1}{i+1}\binom{2i}{i}.$$

1.4 Polygon Triangulation

If you count the number of ways to triangulate a regular polygon with $n+2$ sides, you also obtain the Catalan numbers. Figure 2 illustrates the triangulations for polygons having 3, 4, 5 and 6 sides.

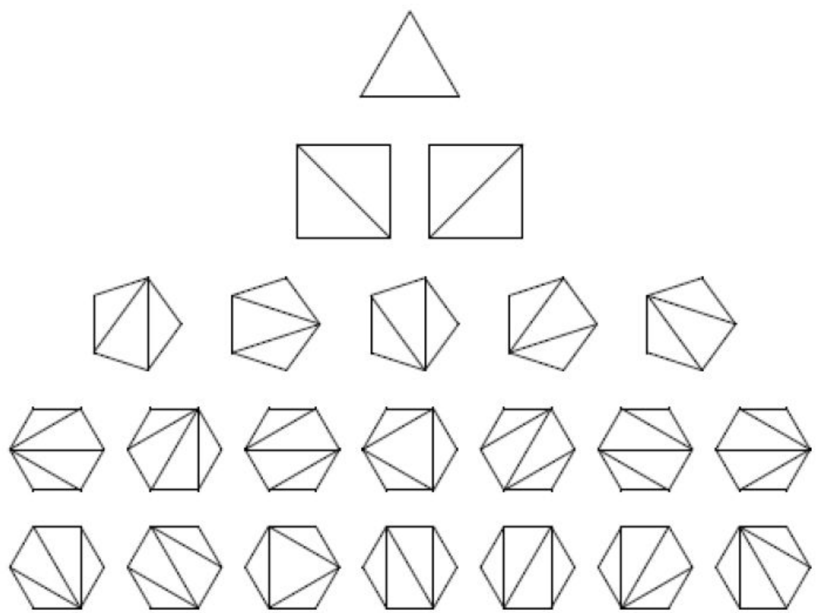
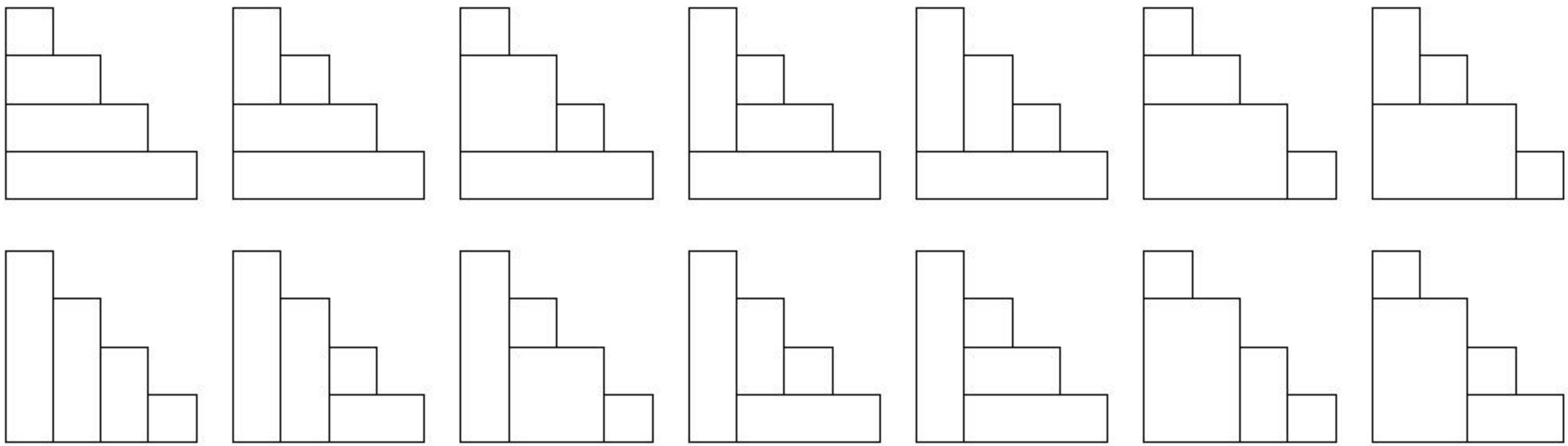


Figure 2: Polygon Triangulations

Given a "triangular" region composed of n blocks on a side, in how many different ways can the region be tiled with exactly 4 rectangles?



Counting the number of Non-Isomorphic Graphs

Count the number of graphs with n vertices.

There are **C(n, 2)** possible edges, each of which can be included or excluded. Thus, there are $2^{C(n, 2)}$ labeled graphs on n vertices.

A more difficult problem arises when we want to start counting **non-isomorphic graphs** on n vertices.
Since any vertex can be mapped to any other vertex, the symmetric group S_4 acts on the vertices.

To account for edges, we move from the symmetric group S_4 to its **pair group** $S_4^{\,2}$ (we need non-isomorphic graphs when n = 4 which depends on the edges between the 4 vertices). The objects that $S_4^{\,2}$ permutes are the 2-element subsets of {1, 2, 3, 4}.

Consider n = 4 and S_4. First we will review the permutations in S_4 and their cycle structures as follows.

1. **The permutation (1)(2)(3)(4) has four cycles of length one**, giving the term s_1^4.
2. **The permutation (1)(2)(34) has structure $s_1^2 s_2$.**
   There are six of these permutations, for each of the six possible pairs of vertices: (12),(13),(14),(23),(24), and (34).
3. **The permutation (1)(234) has structure $s_1 s_3$.**
   There are eight possible differing cycles of length three, and thus eight permutations with this structure. The other seven cycles of length three are (123),(132),(134),(143),(124),(142), and (243).
4. **The permutations (12)(34),(13)(24), and (14)(23)** have structure s_2^2.
5. **The permutation (1234) has structure s_4.**
   The other five with this structure include (1243), (1423), (1342),(1324), (1432).

Thus

$$Z(S_4) = \frac{1}{24}(s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 + 6s_4).$$

To evaluate permutations in $S_4^{(2)}$ we will switch from permuting vertices to permuting *pairs* of vertices, since we are trying to eventually count edges. For notation, we will let $\overline{ij}$ be the pair of vertices i and j, and $\overline{ij} = \overline{ji}$.

1. If our term in S_4 is s_1^4, the corresponding term in $S_4^{(2)}$ is s_1^6. The new permutation is $(\overline{12})(\overline{13})(\overline{14})(\overline{23})(\overline{24})(\overline{34})$.

2. If our term in S_4 is $s_1^2 s_2$, the corresponding term in $S_4^{(2)}$ is $s_1^2 s_2^2$. The permutation $(1)(2)(34)$ would become $(\overline{12})\ (\overline{34})\ (\overline{14}\ \overline{13})\ (\overline{23}\ \overline{24})$.

3. If our term in S_4 is s_1s_3, the corresponding term in $S_4^{(2)}$ is s_3^2. The permutation $(1)(243)$ would become $(\overline{12}\ \overline{14}\ \overline{13})(\overline{23}\ \overline{24}\ \overline{34})$.

4. If our term in S_4 is s_2^2, the corresponding term in $S_4^{(2)}$ is $s_1^2s_2^2$. The permutation $(12)(34)$ would become $(\overline{12})(\overline{34})(\overline{24}\ \overline{13})(\overline{23}\ \overline{14})$.

5. If our term in S_4 is s_4, the corresponding term in $S_4^{(2)}$ is s_2s_4. The permutation (1432) would become $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{41})$.

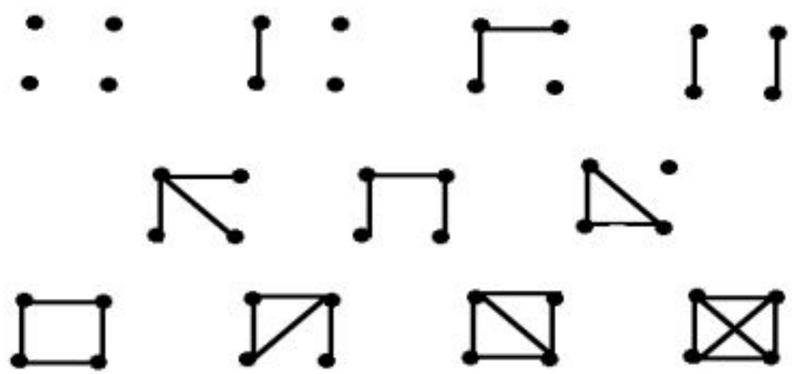
Thus

$$\begin{aligned} Z(S_4^{(2)}) &= \frac{1}{24}(s_1^6 + 6s_1^2s_2^2 + 8s_3^2 + 3s_1^2s_2^2 + 6s_2s_4) \\ &= \frac{1}{24}(s_1^6 + 9s_1^2s_2^2 + 8s_3^2 + 6s_2s_4). \end{aligned}$$

To apply Polya's Theorem and count the number of different graphs, we need a weight structure. If the edge exists, it will have weight 1, and it if does not, it will have weight 0, giving f(x) = 1 + x as the **generating function** for the set of colors.

$$
\begin{aligned}
g_4(x) &= Z(S_4^{(2)}, 1+x) \\
&= \frac{1}{24}((x+1)^6 + 9(x+1)^2(x^2+1)^2 + 8(x^3+1)^2 + 6(x^2+1)(x^4+1)) \\
&= x^6 + x^5 + 2x^4 + 3x^3 + 2x^2 + x + 1.
\end{aligned}
$$

There are **eleven** distinct graphs on four vertices including one graph each with six edges, five edges, one edge, or no edges, two graphs each with four edges or two edges, and three graphs with three edges.



Now we will find the number of graphs on five vertices.

First consider the cycle index for S_5. We will review the permutations in S_5 and their cycle structures. Also, $|S_5| = 5! = 120$

1. The permutation (1)(2)(3)(4)(5) has five cycles of length one, giving the term s_1^5.
2. The permutation (12)(3)(4)(5) has structure $s_1^3 s_2$. There are ten of these permutations, which can be found by all the combinations of five objects into pairs, i.e. $\binom{5}{2}$.
3. The permutation (123)(4)(5) has structure $s_1^2 s_3$. There are twenty, which can be counted by first recognizing the ten combinations for the cycle of length two, and then for each remaining cycle of length three there are two possibilities for their arrangement.
4. The permutation (1234)(5) has structure $s_1 s_4$. There are six different ways that four vertices can be permuted, and there are five choices for our cycle of length one. Thus there are thirty of these permutations.
5. The permutation (12)(34)(5) has structure $s_1 s_2^2$. From the previous example with four vertices, we saw that there were three possible permutations with the same structure as (12)(34). Now we have five possibilities for which number we choose to be the cycle of length one, so we have a total of fifteen permutations with this structure.
6. The permutation (123)(45) has structure $s_2 s_3$. There are twenty, which can be counted in the same way as permutation 3.
7. Finally, the permutation (12345) has structure s_5 and there are $4! = 24$ of them.

Thus

$$Z(S_5) = \frac{1}{120}(s_1^5 + 10s_1^3s_2 + 20s_1^2s_3 + 30s_1s_4 + 15s_1s_2^2 + 20s_2s_3 + 24s_5). \tag{22}$$

Now we will examine each structure in $S_5^{(2)}$.

1. The permutation (1)(2)(3)(4)(5) becomes $(\overline{12})(\overline{13})(\overline{14})(\overline{15})(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})$, giving the term s_1^{10}. This again makes intuitive sense, because there are a total of ten edges on a graph of five vertices.

2. The permutation (12)(3)(4)(5) becomes $(\overline{12})(\overline{34})(\overline{35})(\overline{45})(\overline{15}\ \overline{25})(\overline{14}\ \overline{24})(\overline{13}\ \overline{23})$, giving the term $s_1^4s_2^3$. The cycles of length two are depicted by the red, blue, and purple edges in Figure 27.

3. The permutation (123)(4)(5) becomes $(\overline{45})(\overline{12}\ \overline{23}\ \overline{31})(\overline{14}\ \overline{24}\ \overline{34})(\overline{15}\ \overline{25}\ \overline{35})$, giving the term $s_1s_3^3$, as in Figure 28.

4. The permutation (1234)(5) becomes $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{14})(\overline{15}\ \overline{25}\ \overline{35}\ \overline{45})$, giving the term $s_2s_4^2$, as in Figure 29.

5. The permutation (12)(34)(5) becomes $(\overline{12})(\overline{34})(\overline{14}\ \overline{23})(\overline{13}\ \overline{24})(\overline{15}\ \overline{25})(\overline{35}\ \overline{45})$, giving structure $s_1^2s_2^4$. These cycles are shown in Figure 30, where the gray edges are fixed.

6. The permutation (123)(45) becomes $(\overline{45})(\overline{13})(\overline{12}\ \overline{23})(\overline{14}\ \overline{35}\ \overline{24}\ \overline{15}\ \overline{3425})$. The cycle structure is then $s_1s_3s_6$, as reflected in Figure 31.

7. Finally, the permutation (12345) would become $(\overline{12}\ \overline{23}\ \overline{34}\ \overline{45}\ \overline{51})(\overline{13}\ \overline{24}\ \overline{35}\ \overline{41}\ \overline{52})$, giving the term s_5^2. The cycles are shown in Figure 32.

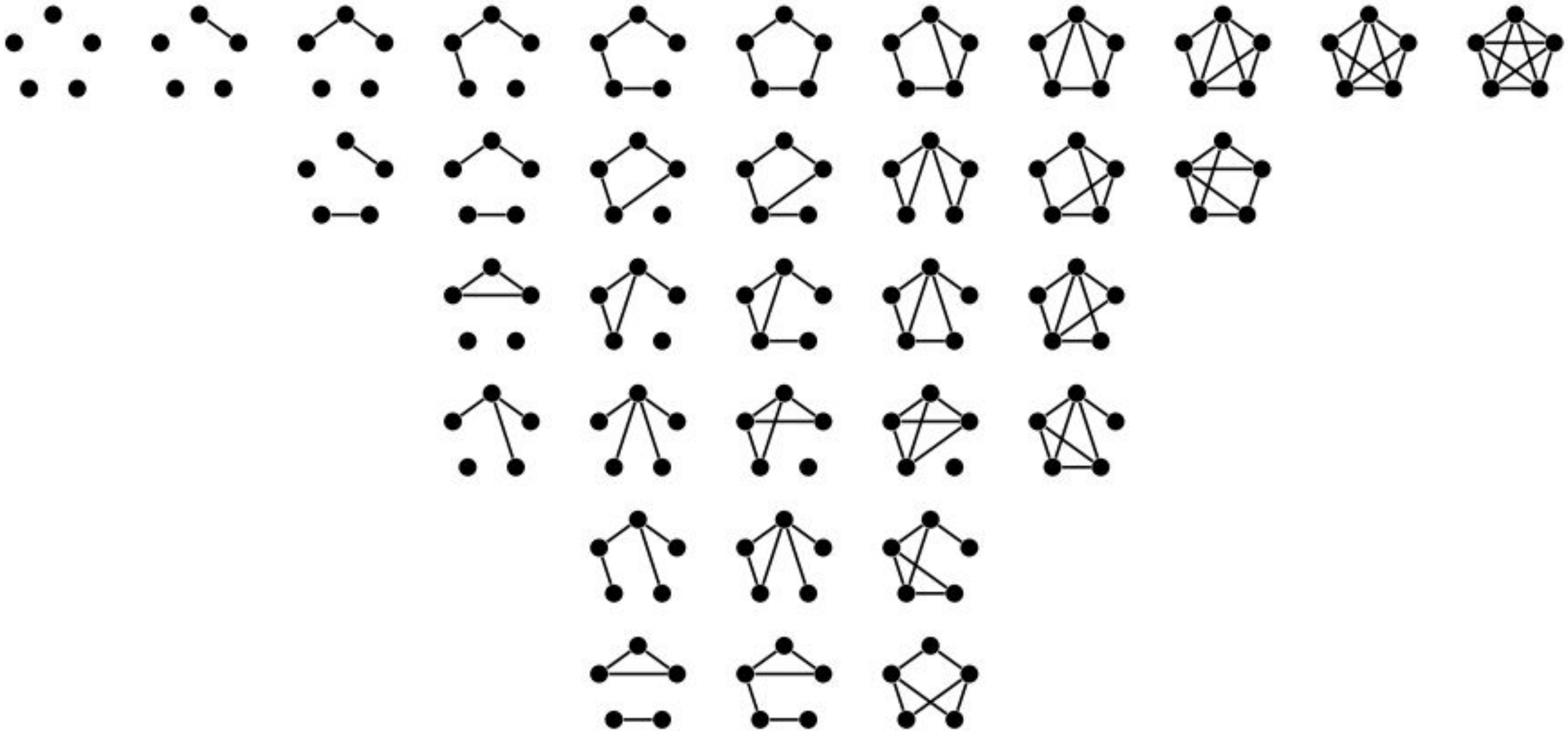
Thus

$$Z(S_5^{(2)}) = \frac{1}{120}(s_1^{10} + 10s_1^4 s_2^3 + 20s_1 s_3^3 + 30s_2 s_4^2 + 15s_1^2 s_2^4 + 20s_1 s_3 s_6 + 24s_5^2). \tag{23}$$

Finally, we can substitute $x + 1$ in this cycle index and use WolframAlpha to give

$$\begin{aligned}
g_5(x) &= Z(S_5^{(2)}, x+1) \\
&= \frac{1}{120}((x+1)^{10} + 10(x+1)^4(x^2+1)^3 + 20(x+1)(x^3+1)^3 + 30(x^2+1)(x^4+1)^2 \\
&\quad + 15(x+1)^2(x^2+1)^4 + 20(x+1)(x^3+1)(x^6+1) + 24(x^5+1)^2) \\
&= x^{10} + x^9 + 2x^8 + 4x^7 + 6x^6 + 6x^5 + 6x^4 43x^3 + 2x^2 + x + 1,
\end{aligned} \tag{24}$$

which is, unsurprisingly, the same result we acheived using Theorem 8. There is 1 graph with each of zero, one, nine, or ten edges, 2 graphs with each of two or eight edges, 4 graphs with each of three or seven edges, and 6 graphs with each of five, six, or seven edges. This gives a total of 34 graphs on five vertices.



The Pigeonhole Principle

Pigeonhole Principle: Simple Form

If $n + 1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.

Example 1: Among 13 people there are 2 who have their birthdays in the same month.

Q1. There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple has been selected?

$n + 1$

Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m. Less formally, there exist consecutive a's in the sequence $a_1, a_2, \ldots, a_m$ whose sum is divisible by m.

To see this, consider the m sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \cdots + a_m.$$

A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, to avoid tiring himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of (consecutive) days during which the chess master will have played *exactly* 21 games.

Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number of games played on the first, second, and third days, and so on. The sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence[3] since at least one game is played each day. Moreover, $a_1 \geq 1$, From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that, among the integers chosen, there are two such that one of them is divisible by the other.

Consider a subset of natural number $A = \{n \in N | 1 \leq n \leq 50\}$ and $|A| = 10$. Prove that there exists two subsets B, C of A such that $|B| = |C| = 4$ and $\sum_{i=1}^{4} b_i = \sum_{i=1}^{4} c_i$ where $b_i \in B$ and $c_i \in C$.

A Theorem of Ramsey

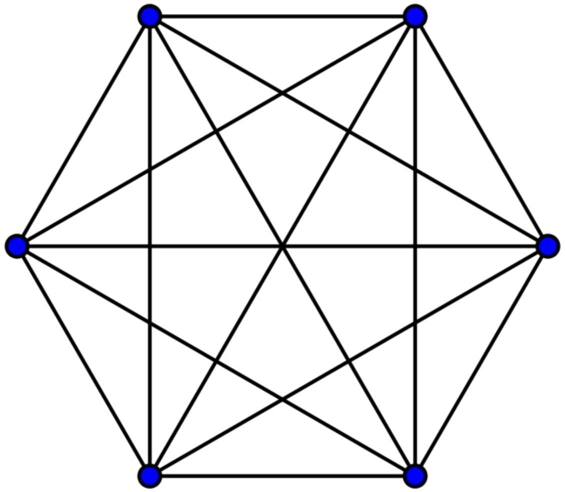
Frank Ramsey was born in 1903 and died in 1930 when he was not quite 27 years of age. In spite of his premature death, he laid the foundation for what is now called *Ramsey theory.*

The following is the most popular and easily understood instance of Ramsey's theorem:

> Of six (or more) people, either there are three, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.

One way to prove this result is to examine all the different ways in which six people can be acquainted and unacquainted. This is a tedious task, but nonetheless one that can be accomplished with a little fortitude. There is, however, a simple and elegant proof that avoids consideration of cases. Before giving this proof, we formulate the result more abstractly as

$$K_6 \rightarrow K_3, K_3 \qquad \text{(read } K_6 \textit{ arrows } K_3, K_3\text{)}. \tag{3.1}$$

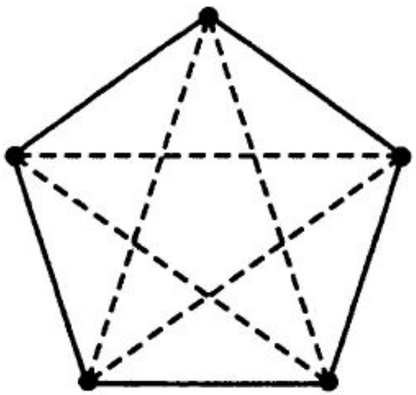


We distinguish between acquainted pairs and unacquainted pairs by coloring edges **red** for acquainted and **blue** for unacquainted.
Three mutually acquainted people now means
" a K_3 each of whose edges is colored red: a red K_3."
Similarly, three mutually unacquainted people form a blue K_3.
We can now explain the expression

$K_6 \longrightarrow K_3, K_3$ is the assertion that no matter how the edges of K_6 are colored with the colors red and blue, there is always a red K_3 or a blue K_3

Prove or disprove: $K_5 \longrightarrow K_3, K_3$

The assertion $K_5 \longrightarrow K_3, K_3$ is false. This is because there is some way to color the edges of K_5 without creating a red K_3 or a blue K_5.
This is shown in Figure below, where the edges of the pentagon (the solid edges) are the red edges and the edges of the inscribed pentagram (the dashed edges) are the blue edges.



We now state and prove Ramsey's theorem, although still not in its full generality.

Theorem 3.3.1 *If $m \geq 2$ and $n \geq 2$ are integers, then there is a positive integer p such that*

$$K_p \to K_m, K_n.$$

In words, Ramsey's theorem asserts that given m and n there is a positive integer p such that, if the edges of K_p are colored red or blue, then either there is a red K_m or there is a blue K_n. The existence of either a red K_m or a blue K_n is guaranteed, no matter how the edges of K_p are colored. If $K_p \to K_m, K_n$, then $K_q \to K_m, K_n$ for every integer $q \geq p$. The *Ramsey number* $r(m, n)$ is the smallest integer p such that $K_p \to K_m, K_n$. Thus *Ramsey's theorem asserts the existence of the number* $r(m, n)$. By interchanging the colors red and blue, we see that

$$r(m, n) = r(n, m).$$

The facts that $K_6 \to K_3, K_3$ and $K_5 \not\to K_3, K_3$ imply that

$$r(3, 3) = 6.$$

Prove that r(2, n) = n.

In a similar way, we show that $r(m,2) = m$. The numbers $r(2,n)$ and $r(m,2)$ with $m, n \geq 2$ are the *trivial Ramsey numbers.*

$$
\begin{aligned}
&r(3,3) = 6,\\
&r(3,4) = r(4,3) = 9,\\
&r(3,5) = r(5,3) = 14,\\
&r(3,6) = r(6,3) = 18,\\
&r(3,7) = r(7,3) = 23,\\
&r(3,8) = r(8,3) = 28,\\
&r(3,9) = r(9,3) = 36,\\
&40 \leq r(3,10) = r(10,3) \leq 43,\\
&r(4,4) = 18,\\
&r(4,5) = r(5,4) = 25,\\
&35 \leq r(4,6) = r(6,4) \leq 41\\
&43 \leq r(5,5) \leq 49\\
&58 \leq r(5,6) = r(6,5) \leq 87\\
&102 \leq r(6,6) \leq 165.
\end{aligned}
$$

Notice that the fact that $r(3,10)$ lies between 40 and 43 implies that

$$K_{43} \rightarrow K_3, K_{10}$$

and

$$K_{39} \not\rightarrow K_3, K_{10}.$$

Thus, there is no way to color the edges of K_{43} without creating either a red K_3 or a blue K_{10}; there is a way to color the edges of K_{39} without creating either a red K_3 or a blue K_{10}, but neither of these conclusions is known to be true for $K_{40}, K_{41},$ and K_{42}.

Ramsey's theorem generalizes to any number of colors. We give a very brief introduction. If n_1, n_2, and n_3 are integers greater than or equal to 2, then there exists an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, K_{n_3}.$$

In words, if each of the edges of K_p is colored red, blue, or green, then either there is a red K_{n_1} or a blue K_{n_2} or a green K_{n_3}. The smallest integer p for which this assertion holds is the Ramsey number $r(n_1, n_2, n_3)$. The only nontrivial Ramsey number of this type that is known is

$$r(3,3,3) = 17.$$

Thus $K_{17} \to K_3, K_3, K_3$ but $K_{16} \not\to K_3, K_3, K_3$. The Ramsey numbers $r(n_1, n_2, \ldots, n_k)$ are defined in a similar way, and Ramsey's theorem in its full generality for pairs asserts that these numbers exist; that is, there is an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}.$$

Let G be a complete graph of order six and H be a path of length 4. Consider the edge coloring of G with two colors white and black. Prove or disprove: In every coloring of G, there is a white copy of H or a black copy of H.

Section 7.5

Using generating func? solve the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}, \quad n \geq 2, \quad a_0 = 2, \; a_1 = 2. \quad \text{— (1)}$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{— (2)} \quad \Rightarrow g(x) - a_0 - a_1 x = \sum_{n=2} a_n x^n$$

(1) $\times\, x^n$ & take $\sum_{n=2}^{\infty}$

$$\sum_{n \geq 2} a_n x^n = 2 \sum_{n \geq 2} a_{n-1} x^n + 3 \sum_{n \geq 2} a_{n-2} x^n$$

$$g(x) - a_0 - a_1 x = 2x\left(g(x) - a_0\right) + 3x^2 g(x) \quad \text{— (3)}$$

$$\left(\sum_{n \geq 2} a_{n-1} x^n = x \sum_{n \geq 2} a_{n-1} x^{n-1} = x \sum_{n=1}^{\infty} a_n x^n \right.$$

$$= x\left(g(x) - a_0\right)$$

$$\left(\sum_{n \geq 2} a_{n-2} x^n = x^2 \sum_{n \geq 2} a_{n-2} x^{n-2} = x^2 \sum_{n=0}^{\infty} a_n x^n \right.$$

$$g(x)\left(1 - 2x - 3x^2\right) = a_0 + a_1 x - 2x a_0$$

$a_0 = 2, \quad a_1 = 2$

$$g(x) = \frac{2(x-1)}{(3x-1)(x+1)} = \frac{1}{x+1} + \frac{1}{1-3x}$$

Coeffn of x^n is

$$a_n = (-1)^n + 3^n$$

Q. $a_n = a_{n-1} + n, \quad a_0 = 1$

$$\sum_{n\geq 1} a_n x^n = \sum_{n\geq 1} a_{n-1} x^n + \sum_{n\geq 1} n x^n$$

$$g(x) = \sum_{n=0} a_n x^n$$

$$g(x) - a_0 = x \sum_{n\geq 1} a_{n-1} x^{n-1} + \sum_{n=0}^{\infty} n x^n$$

$$g(x) - 1 = x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n x^n$$

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \cdots$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x + 2x^2 + \cdots + n x^n + \cdots$$

$$g(x) - 1 = x\,g(x) + \frac{x}{(1-x)^2}$$

$$g(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

Coeffn of x^n in $\frac{1}{1-x} = 1$

Coeffn of x^n in $\frac{x}{(1-x)^3}$ = Coeffn of x^{n-1} in $\frac{1}{(1-x)^3}$

$$= C((n-1)+3-1,\ n-1)$$

$$= C(n+1,\ n-1) = C(n+1, 2)$$

$$a_n = 1 + C(n+1, 2)$$

Q. Let $g_n(x)$ be a family of g.s.

$$g_n(x) = a_{n,0} + a_{n,1}x + \cdots + a_{n,n}x^n$$

satisfying $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$

with $a_{n,0} = a_{n,n} = 1$, $a_{n,k} = 0$, $k > n$

Using g.s. compute $a_{n,k}$.

$$\sum_{k=1}^{n} a_{n,k} x^k = \sum_{k=1} a_{n-1,k} x^k + \sum_{k=1} a_{n-1,k-1} x^k$$

$$g_n(x) - 1 = \underline{g_{n-1}(x) - 1} + x \underbrace{\sum_{h=0}^{n-1} a_{n-1,h} x^h}_{x\, g_{n-1}(x)}$$

$$g_n - 1 = g_{n-1} - 1 + x g_{n-1}$$

$$g_n(x) = (1+x) g_{n-1}(x) = (1+x)^n g_0(x)$$

$$g_0(x) = a_{0,0} = 1 \Rightarrow g_n(x) = (1+x)^n$$

$$a_{n,k} = C(n,k)$$

Find a recurrence relation for a_n, the number of ways to place parentheses to multiply the n numbers $k_1 \times k_2 \times \cdots \times k_n$. Hence solve it using generating function.

$a_2 = 1$ $\quad (k_1 \times k_2)$

$a_3 = 2$ $\quad \underline{(k_1 \times k_2) \times k_3}, \quad k_1 \times (k_2 \times k_3)$

$a_0 = 0, \quad a_1 = 1.$

$$(k_1 \times k_2 \times \cdots \times k_i) \times (k_{i+1} \times \cdots \times k_n)$$

$i = 1, 2, \ldots, n-1$

$a_i \qquad\qquad a_{n-i}$

$$a_n = \sum_{i=1}^{n-1} a_i a_{n-i}$$

$$= \underbrace{a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1}$$

coeff of x^n in $g(x)$ $\qquad$ coeff of x^n in $g(x) \times g(x)$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n \geq 2} (a_1 a_{n-1} + \cdots + a_{n-1} a_1) x^n$$

$$g(x) - x = (g(x))^2$$

$$g(x)^2 - g(x) + x = 0$$

$$g(x) = \frac{1}{2}\left(1 \pm \sqrt{1-4x}\right)$$

$$g(0) = 0 \Rightarrow g(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

Q1. $a_n = 8a_{n-1} + 10^{n-1}, \quad a_0 = 1, \; n \geq 1$

$$g(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

Q2. $\displaystyle\sum_{i=0}^{n} i^5$

$a_n = a_{n-1} + n^5, \quad a_0 = 0$

$a_n = k$ (g.s. for hom)

$p(n) = p_0 + p_1 n + \cdots + p_5 n^5$ X

$p(n) = p_0 n + p_1 n^2 + \cdots + p_5 n^6$

$a_n = 2a_{n-1} + 3a_{n-2}, \quad a_0 = 1, \; a_1 = 2$

$a_n = a_{n-1} a_{n-2}, \quad a_0 = a_1 = x$

$\log_x a_n = \log_x a_{n-1} + \log_x a_{n-2}$

$\downarrow$

b_n, $\quad b_0 = \log_x a_0 = \log_x x = 1, \; b_1 = 1$

Stirling Numbers

Stirling numbers of the second kind S(n, k)

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Compute $\left\{ 4 \atop 2 \right\}$

$\{1,2,3\} \cup \{4\}$	$\{1,2,4\} \cup \{3\}$	$\{1,3,4\} \cup \{2\}$	$\{2,3,4\} \cup \{1\}$
$\{1,2\} \cup \{3,4\}$	$\{1,3\} \cup \{2,4\}$	$\{1,4\} \cup \{2,3\}$	

Hence $\left\{ 4 \atop 2 \right\} = 7$

Compute S(n, k) for k = 0, 1, n , n – 1, 2

$k = 0$ We can partition a set into no nonempty parts if and only if the set is empty.

That is: $\left\{ {n \atop 0} \right\} = [n = 0]$.

$k = 1$ We can partition a set into one nonempty part if and only if the set is nonempty.

$k = n$ If $n > 0$, the only way to partition a set with n elements into n nonempty parts, is to put every element by itself.

That is: $\left\{ {n \atop n} \right\} = 1$. (This also matches the case $n = 0$.)

$k = n - 1$ Choosing a partition of a set with n elements into $n - 1$ nonempty subsets, is the same as choosing the two elements that go together.

That is: $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

$k=2$ Let X be a set with two or more elements.

- Each partition of X into two subsets is identified by two ordered pairs $(A, X \setminus A)$ for $A \subseteq X$.
- There are 2^n such pairs, but $(\emptyset, X)$ and $(X, \emptyset)$ do not satisfy the nonemptiness condition.
- Then $\left\{ {n \atop 2} \right\} = \dfrac{2^n - 2}{2} = 2^{n-1} - 1$ for $n \geqslant 2$.

In general, $\left\{ {n \atop 2} \right\} = (2^{n-1} - 1)[n \geqslant 1]$

Compute $\left\{ {n \atop k} \right\}$

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty groups, then decide which group to add the nth element to.

2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty groups out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}, \qquad \text{for } n > 0,$$

that yields the following triangle:

Calculate the value $\left\{ {4 \atop 2} \right\}$ by applying the recurrence above.

$$\left\{ {4 \atop 2} \right\} = \left\{ {3 \atop 1} \right\} + 2\left\{ {3 \atop 2} \right\} = \left\{ {2 \atop 0} \right\} + 1\left\{ {2 \atop 1} \right\} + 2\left(\left\{ {2 \atop 1} \right\} + 2\left\{ {2 \atop 2} \right\}\right)$$

We use $S(2,0) = 0$ and $S(2,1) = S(2,2) = 1$ to simplify further.

$$\left\{ {4 \atop 2} \right\} = 0 + \left\{ {2 \atop 1} \right\} + 2\left\{ {2 \atop 1} \right\} + 4(1) = 3\left\{ {2 \atop 1} \right\} + 4$$

So, $\left\{ {4 \atop 2} \right\} = 7$.

To extend the table or verify any entry in it, choose an element. Multiply that element by the number at the top of its column, and add to it the element on its left (or 0 if there is no element to the left). The result should be the number below the chosen element.

$n\backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	3	1	0	0	0
4	0	1	7	6	1	0	0
5	0	1	15	25	10	1	0
6	0	1	31	90	65	15	1

How many ways are there to factor the number $2 \cdot 3 \cdot 5 \cdot 11 \cdot 23 = 7590$ into a product of two factors each greater than 1? How many for three such factors?

Consider the set {2, 3, 5, 11, 23}. Now we need to break this set into 2 sets which can be done in S{5, 2} = 15 ways.
S{5, 3} = 25.

Angered by all the candies being given to children at the beginning of this section, a group of indignant parents take five distinct pamphlets on the dangers of childhood obesity to give to two textbook authors. In how many ways may this be done if the two authors are considered distinct? In how many ways if the authors are considered identical? In either case, no author escapes being given at least one pamphlet, and there is only one copy of each of the five pamphlets.

Here $n = 5$ refers to the pamphlets, and $k = 2$ refers to the authors. From the table, $\left\{ {5 \atop 2} \right\} = 15$, so if the authors are distinct the answer is $2! \times 15$ or 30, and if the authors are identical we get 15. $\square$

Q. Properly chastened, the authors meet with three indignant parents and offer them a collection of six distinct snacks. Assuming the three parents are distinct, in how many ways can this be done if each parent is to receive at least one snack? In how many ways if a parent may refuse a snack (all snacks are distributed)? In how many ways if we do not require that all snacks be distributed, and any parent may refuse a snack?

The first problem is just a matter of looking at the table; $n = 6$, $k = 3$, so we get $3! \times \left\{ {6 \atop 3} \right\} = 6 \times 90 = 540$.

The second problem must be done in pieces. If all three parents accept a snack, 540 is still the answer; if one parent refuses, there are three ways to choose a parent to refuse and $2! \times \left\{ {6 \atop 2} \right\} = 62$ ways to distribute the snacks to the other two parents, so we get another 186 possibilities. Then if two parents refuse, we get another three possibilities (three choices of the parent to accept all six snacks, or if you like $\binom{3}{1} \times \left\{ {6 \atop 1} \right\} = 3$). Adding these, we find a total of 729 possible distributions of snacks.

The third problem can be solved by considering the trash can as a fourth potential snack recipient.

If all four possible recipients get a snack, we have $4! \times \left\{ {6 \atop 4} \right\} = 1560$. If one recipient (parent or trash can) gets no snack, we count $\binom{4}{1} \times 3! \times \left\{ {6 \atop 3} \right\} = 2160$. Two snackless recipients yields $\binom{4}{2} \times 2! \times \left\{ {6 \atop 2} \right\} = 372$. Finally, if all the snacks go to one recipient, there are four possibilities. Adding these numbers yields 4096 ways. Realizing that this answer is just 4^6, we see that we could have saved ourselves much trouble by saying: Associate with each snack a number from $\{0, 1, 2, 3\}$ according to which recipient receives it. Placing these numbers in order gives us a six-digit base-4 number, of which there are 4^6. $\square$

We use the notation $[r]_k = r!/(r-k)! = r(r-1)\ldots(r-k+1)$, the *falling factorial* notation.

(A Stirling Number Formula)

$$r^n = \sum_{k=0}^{r} \left\{ {n \atop k} \right\} [r]_k$$

Proof. Suppose we have r recipients of n objects, and, although all the objects must be distributed, some recipients may get no object. If all r recipients get an object, we have $r! \times \left\{ {n \atop r} \right\}$, which is the $k = r$ summand of the formula. For the case in which one recipient gets no object, we count $\binom{r}{1} \times (r-1)! \times \left\{ {n \atop r-1} \right\}$, which is equivalent to the $k = r-1$ summand. In general, if i recipients get no objects, we have $\binom{r}{i} \times (r-i)! \times \left\{ {n \atop r-i} \right\} = \frac{r!}{i!} \left\{ {n \atop r-i} \right\}$

which is the $r-i$th summand. On the other hand, the quantity we are counting is the number of n-digit base-r integers, of which there are r^n. $\square$

Write the polynomial x^3 as a linear combination of the polynomials 1, x, $x(x-1)$, and $x(x-1)(x-2)$.

In the preceding theorem, substitute 3 for n and x for r; this gives us

$$x^3 = \sum_{k=0}^{3} \left\{ \begin{matrix} 3 \\ k \end{matrix} \right\} \frac{x!}{(x-k)!}$$

and we simplify this to get $x^3 = 0 \cdot 1 + 1 \cdot x + 3 \cdot x(x-1) + 1 \cdot x(x-1)(x-2)$, or $x^3 = x + 3x(x-1) + x(x-1)(x-2)$. Even though the recurrence requires that x be an integer, the identity holds for any value of x. □

We have three distinct snacks and wish to put them into bags. How many ways can this be done?

There is exactly one way to do this with three bags (each snack into its own bag); three ways to do this with two bags (depending on which snack goes into a bag by itself); and one way with one bag. Thus $B_3 = 5$. □

We define the **Bell number** B_n to be the number of ways to partition a set of n distinct objects.

$$B_n = \sum_{i=0}^{n} \left\{ {n \atop i} \right\}.$$

How many ways are there of writing $2 \times 3 \times 5 \times 7 = 210$ as a product of distinct integers?

$B_4 = 15.$

(Bell Number Recurrence)

$$B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$$

Note that B_0 is 1.

Find B_6.

The recurrence gives $B_6 = B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 + B_5$. We may compute these by summing rows in the triangle, or by recursively applying Theorem 7.6 to get $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, and $B_5 = 52$. This gives us $B_6 = 1 + 5 + 20 + 50 + 75 + 52 = 203$.

Stirling numbers of the first kind s(n, k) $\left[{n \atop k}\right]$

The *falling factorial polynomial* of degree n is

$$(x)_n = x(x-1)(x-2)(x-3)\cdots(x-n+1),$$

a polynomial of degree n in one indeterminate x. If we evaluate the polynomial at m, we get the number of n-permutations chosen from a set of size m:

$$(m)_n = m(m-1)(m-2)(m-3)\cdots(m-n+1) = \frac{m!}{(m-n)!} = P(m,n).$$

Here are the first few of these polynomials.

$$\begin{aligned}
&(x)_0 = 1 \text{ (the empty product!)}\\
&(x)_1 = x\\
&(x)_2 = x(x-1) = x^2 - x\\
&(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x\\
&(x)_4 = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x\\
&(x)_5 = x(x-1)(x-2)(x-3)(x-4) = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x
\end{aligned}$$

The coefficients appearing in $(x)_n$ are called *Stirling numbers of the first kind*. The coefficient of x^k in $(x)_n$ is denoted $s(n,k)$, thus

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k.$$

$$\left[{m \atop n} \right] = \left[{m-1 \atop n-1} \right] - (m-1) \left[{m-1 \atop n} \right]$$

Proof. We begin by using the definition and applying a bit of algebra.

$$\sum_{j=0}^{m} \left[{m \atop j} \right] x^j = x(x-1)\dots(x-m+1) = [x]_m = (x-m+1)[x]_{m-1}$$

Replacing the falling factorial on the right with the equivalent sum gives us:

$$\sum_{j=0}^{m} \left[{m \atop j} \right] x^j = (x-m+1) \sum_{j=0}^{m-1} \left[{m-1 \atop j} \right] x^j$$

and with the distributive law we obtain

$$\sum_{j=0}^{m} \left[{m \atop j} \right] x^j = x \sum_{j=0}^{m-1} \left[{m-1 \atop j} \right] x^j - (m-1) \sum_{j=0}^{m-1} \left[{m-1 \atop j} \right] x^j.$$

What is the coefficient of x^n on each side of the equation above? We get $\left[{m \atop n} \right]$ on the left and $\left[{m-1 \atop n-1} \right] - (m-1) \left[{m-1 \atop n} \right]$ on the right. Since the two polynomials are equal, the coefficients are equal for each n, establishing the theorem. $\square$

As with Stirling numbers of the second kind, we may present them in the form of a table and refer to it, shown below. To extend the table or verify any entry in it, choose an element. Multiply that element by the number n at the left of its row, and subtract it from the element on its left (or 0 if there is no element to the left). The result should be the number below the chosen element.

$n\backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	-1	1	0	0	0	0
3	0	2	-3	1	0	0	0
4	0	-6	11	-6	1	0	0
5	0	24	-50	35	-10	1	0
6	0	-120	274	-225	85	-15	1

(Unsigned Stirling Number Recurrence)

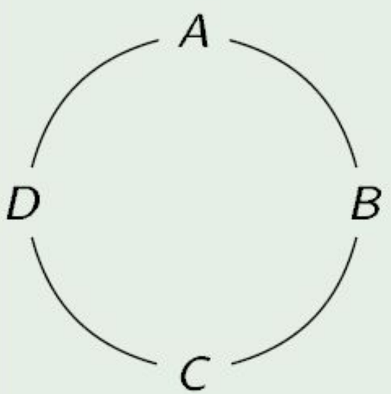
$$\left|\begin{bmatrix} m \\ n \end{bmatrix}\right| = \left|\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right| + (m-1)\left|\begin{bmatrix} m-1 \\ n \end{bmatrix}\right|$$

The absolute value of $s(n,k)$ is denoted $|s(n,k)|$ and is called an *unsigned Stirling number of the first kind.* The signs alternate, so $s(n,k) = (-1)^{n-k}|s(n,k)|$.

Unsigned Stirling numbers of the first kind

The Stirling number of the first kind $\left[n \atop k \right]$, read "n cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Circle is a cyclic arrangement



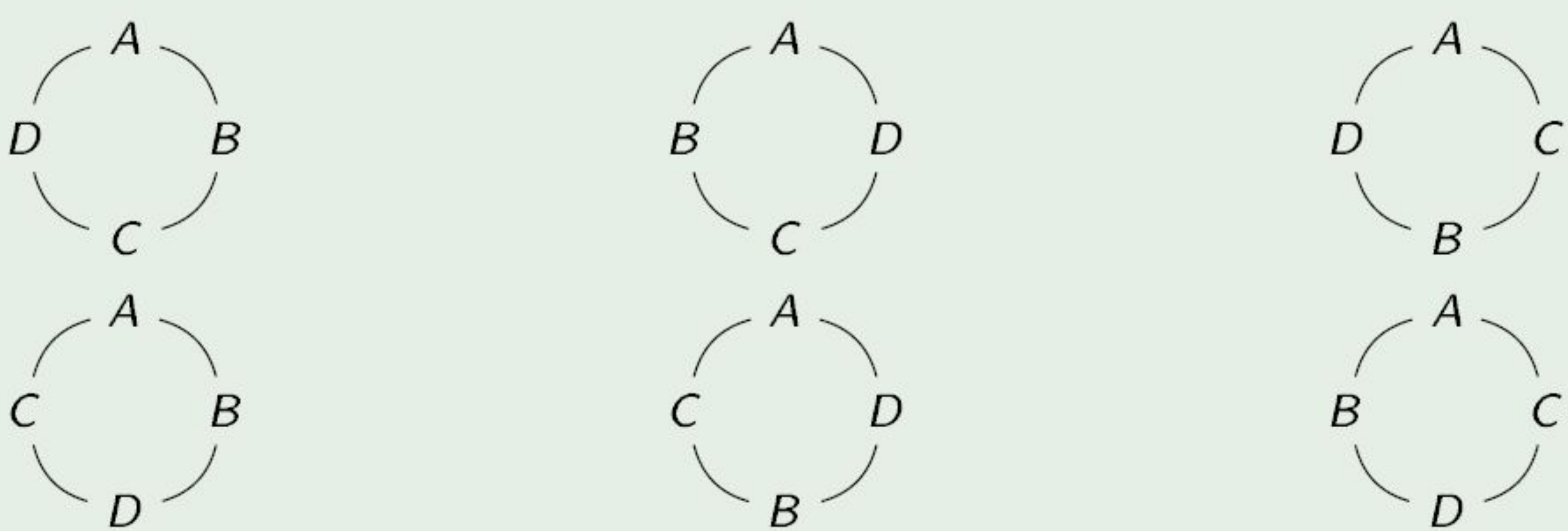
- Circle can be written as $[A, B, C, D]$;
- It means that $[A, B, C, D] = [B, C, D, A] = [C, D, A, B] = [D, A, B, C]$;
- It is not same as $[A, B, D, C]$ or $[D, C, B, A]$.

Compute $\left[4 \atop 2 \right]$

$[1,2,3]\ \ [4]$	$[1,2,4]\ \ [3]$	$[1,3,4]\ \ [2]$	$[2,3,4]\ \ [1]$
$[1,3,2]\ \ [4]$	$[1,4,2]\ \ [3]$	$[1,4,3]\ \ [2]$	$[2,4,3]\ \ [1]$
$[1,2]\ \ [3,4]$	$[1,3]\ \ [2,4]$	$[1,4]\ \ [2,3]$	

Hence $\left[{4 \atop 2}\right] = 11$

Compute $\left[{n \atop k}\right]$ for k = 1, n, n – 1



$k = 1$ To arrange one circle of n objects: choose the order, and forget which element was the first. That is: $\left[{n \atop 1}\right] = n!/n = (n-1)!$.

$k = n$ Every circle is the singleton and there is just one partition into circles. That is, $\begin{bmatrix} n \\ n \end{bmatrix} = 1$ for any n:

$$[1] \quad [2] \quad [3] \quad [4]$$

$k = n-1$ The partition into circles consists of $n-2$ singletons and one pair. So $\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$, the number of ways to choose a pair:

$$[1,2] \;\; [3] \;\; [4] \qquad [1,3] \;\; [2] \;\; [4] \qquad [1,4] \;\; [2] \;\; [3]$$

$$[2,3] \;\; [1] \;\; [4] \qquad [2,4] \;\; [1] \;\; [3] \qquad [3,4] \;\; [1] \;\; [2]$$

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty cycles, then decide which element to put the nth one *after*.
2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty cycles out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right], \qquad \text{for } n > 0,$$

that yields the following triangle:

Compute the number of ways to seat 4 people around 2 circular tables With no table left empty.