9.3: THE CYCLE INDEX

$$N = \frac{1}{|G|} \sum_{\pi \in G} \Psi(\pi)$$

where $\Psi(\pi)$ is the number of colorings in T left fixed by π. If the set T were all 3-colorings of the corners of a 10-gon or a cube, it would seem close to impossible to determine $\Psi(\pi)$s, the number of colorings left fixed by various symmetries π of the figure. However, we shall show that $\Psi(\pi)$ can be determined easily from the structure of π. We develop the theory for this simplified calculation of $\Psi(\pi)$ in terms of 2-colorings of a square.

Observation 1: a coloring C will be left fixed by π if and only if for each corner v, the color of C at v is the same as the color at π(v) so that the symmetry leaves the color at π(v) unchanged.

Example. We know that π_3 causes corners a and c to interchange and corners b and d to interchange.

It follows that a coloring left fixed by π_3 must have the same color at corners a and c and the same color at b and d (no further conditions are needed).

With two color choices for a, c and with two color choices for b, d, we can construct $2 \times 2 = 4$ colorings that will be left fixed—namely, C_1, C_{10}, C_{11}, C_{16}. Hence $\psi(\pi_3) = 4$.

Observation 2: if π_i cyclicly permutes a subset of corners (that is, the corners form a cycle of π_i), then those corners must all be the same color in any coloring left fixed by π_i.

Example. $\pi_3 = (ac)(bd)$

For each symmetry, we need to get such a cyclic representation and then count the number of ways to assign a color to each cycle of corners.

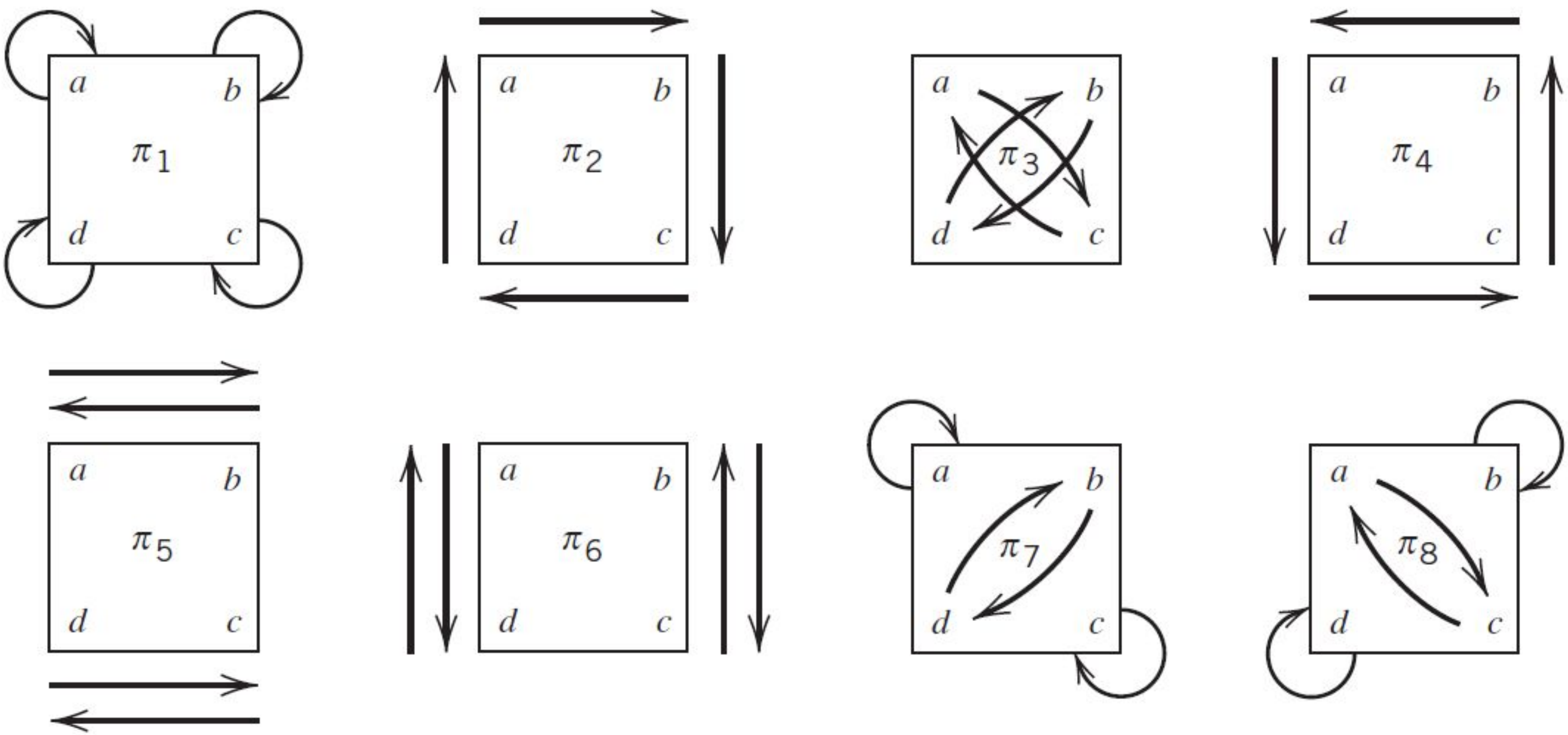
Let us also classify the cycles by their length.

It will prove convenient to encode a symmetry's cycle information in the form of a product containing one x_1 for each cycle of length 1, one x_2 for each cycle of size 2, and so forth.

This expression is called the **cycle structure representation of a** symmetry.

The cycle structure representation of π_2 (90 degree), π_3 (180 degree) and π_1(identity)?

For π_2, its x_4, since it consists of one 4-cycle: π_2 = (abcd).
For π_3, its $x_2^{\ 2}$, since it consists of two 2-cycles: π_3 = (ac)(bd)
For π_1, its $x_1^{\ 4}$



(*i*) Motion π_i	(*ii*) Colorings Left Fixed by π_i	(*iii*) Cycle Structure Representation
π_1	16—all colorings	x_1^4
π_2	2—C_1, C_{16}	x_4
π_3	2—$C_1, C_{10}, C_{11}, C_{16}$	x_2^2
π_4	2—C_1, C_{16}	x_4
π_5	2—C_1, C_6, C_8, C_{16}	x_2^2
π_6	2—$C_1, C_7, C_9 C_{16}$	x_2^2
π_7	8—C_1, C_2, C_4, C_{10} $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2 x_2$
π_8	8—C_1, C_3, C_5, C_{10} $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2 x_2$

Observation 3:

For any symmetry π of any figure, *the number of colorings left fixed will be given by setting each* x_j *equal to* 2 *(or, in general, the number of colors available) in the cycle structure representation of* π, *that is,*

$$\Psi(\pi) = 2^{\text{number of cycles in } \pi}$$

To obtain the number of different 2-colorings of the floating square with Burnside's Theorem, we sum the numbers in column (ii) of Figure 9.7 and divide by 8:

$$\frac{1}{|G|}\sum_{\pi \in G} \Psi(\pi) = \frac{1}{8}(16+2+4+2+4+4+8+8) = \frac{1}{8}(48) = 6$$

There is a slightly simpler way to get this result. First, algebraically sum the cycle structure representations of each symmetry, collecting like terms together, and then divide by 8. From column (iii) of Figure 9.7, we obtain $\frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2)$. This expression is called the **cycle index** $P_G(x_1, x_2, \ldots, x_k)$ for a group G of symmetries. By setting each $x_i = 2$ in this cycle index—that is, $P_G(2, 2, \ldots, 2)$—we get the same answer.

Suppose that instead of two colors, we had three colors. Then the same reasoning applies, but now there are three choices for the color of the corners in each cycle. If a symmetry has k cycles, then it will leave 3^k 3-colorings of the square fixed, and the number of different 3-colorings will be $P_G(3, 3, \ldots, 3)$. More generally, for any m, $P_G(m, m, \ldots, m)$ will be the number of nonequivalent m-colorings of an unoriented square. The argument used to derive this coloring counting formula with the cycle

Theorem

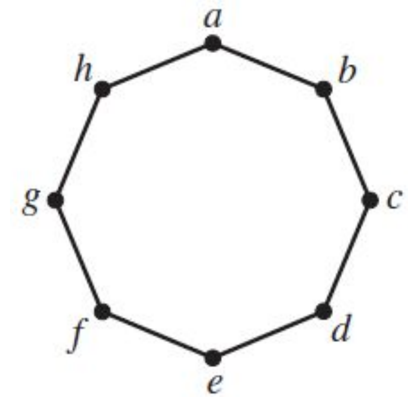
Let S be a nonempty set of elements and G be a group of symmetries of S that acts to induce an equivalence relation on the set of m-colorings of S. Then the number of nonequivalent m-colorings of S is given by $P_G(m, m, \ldots, m)$.

Use this Theorem to solve the following question of last section:

Suppose a necklace can be made from beads of three colors—black, white, and red. How many different necklaces with 3 beads are there?

the rotations are of 0°, 120°, and 240° with cycle structure representations of x_1^3, x_3, and x_3, respectively. Thus, $P_G = \frac{1}{3}(x_1^3 + 2x_3)$. The number of 3-colored strings of three beads is $P_G(3, 3, 3) = \frac{1}{3}(3^3 + 2 \times 3) = 11$. More generally, the number of m-colored necklaces of three beads is $P_G(m, m, m) = \frac{1}{3}(m^3 + 2m)$.

Suppose a necklace can be made from beads of three colors—black, white, and red. How many different necklaces with 8 beads are there?



The rotations are of 0°, 45°, 90°, 135°, 180°, 225°, 270°, and 315°.

The 0° rotation consists of eight 1-cycles.

The 45° rotation is the cyclic permutation (abcdefgh).

The 90° rotation has the cycle decomposition (aceg)(bdfh).

The 135° rotation is the cyclic permutation (adgbehcf).

The 180° rotation has the cyclic decomposition (ae)(bf)(cg)(dh).

The cycle structure representations are thus 0° rotation, x_1^8 45° rotation, x_8, 90° rotation, x_4^2; 135° rotation, x_8; and 180° rotation, x_4^2.

The 225°, 270°, and 315° rotations are reverse rotations of 135°, 90°, 45°, respectively, and have the corresponding cycle structure representations. Collecting terms, we obtain

$$P_G = \frac{1}{8}\left(x_1^8 + 4x_8 + 2x_4^2 + x_2^4\right)$$

The number of different m-colored necklaces of eight beads is

$$\frac{1}{8}(m^8 + 4m + 2m^2 + m^4)$$

For $m = 3$, we have

$$\frac{1}{8}(3^8 + 4 \times 3 + 2 \times 3^2 + 3^4) = \frac{1}{8}(6561 + 12 + 18 + 81) = 834 \ \blacksquare$$

Compute the number of colorings of pentagon using k colors (by computing the corresponding cycle index).

Pentagon has 10 symmetries: 5 rotation and 5 reflection.

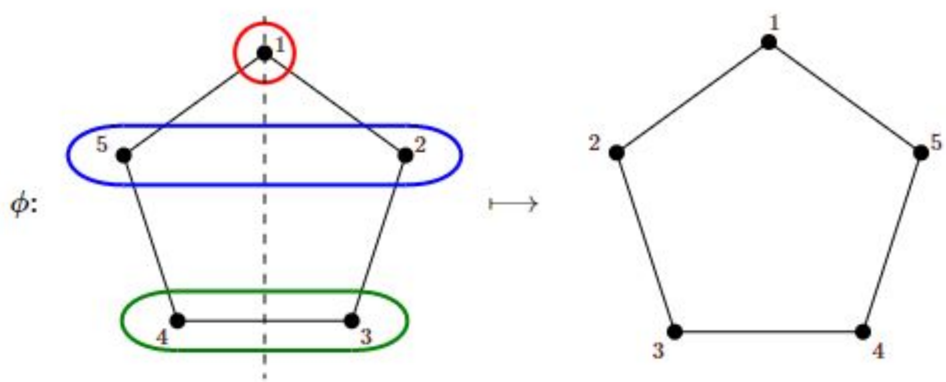
For identity, the cycle structure representation is x_1^5

The cycle structure representations for all other rotations are x_5

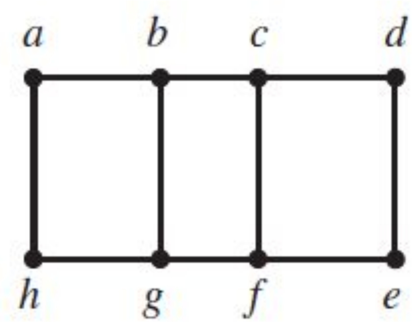
The cycle structure representations for each reflection is $x_1 x_2^2$

Thus, $P_G = 1/10\ (x_1^5 + 4x_5 + 5x_1 x_2^2)$

Substituting $x_i = k$, we have $P_G = 1/10\ (k^5 + 4k + 5k^3)$



Find the number of different m-colorings of the vertices of the following figure.



The symmetries are:

0◦, 180◦, about vertical and horizontal axis.

The 0◦ rotation consists of 8 1-cycles.

The 180◦ rotation has the cyclic decomposition (ae)(bf)(cg)(dh).

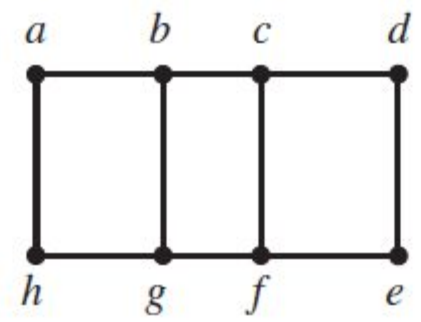
Symmetry about vertical axis has the cyclic decomposition (ad)(he)(bc)(gf).

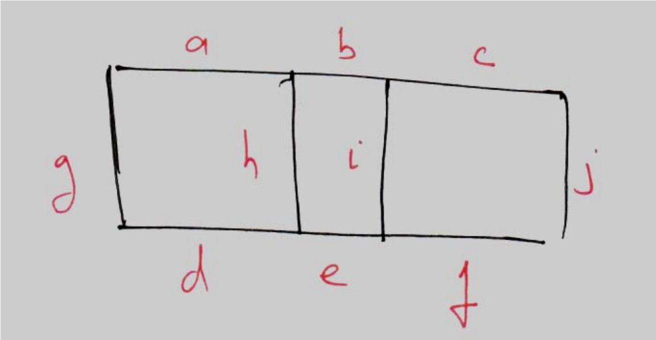
Symmetry about horizontal axis has the cyclic decomposition (ah)(bg)(cf)(de).

$P_G = ¼ (x_1^8 + 3x_2^4)$

The number of different m-colorings is ¼ $(m^8 + 3m^4)$

Find the number of different m-colorings of the edges of the following figure.



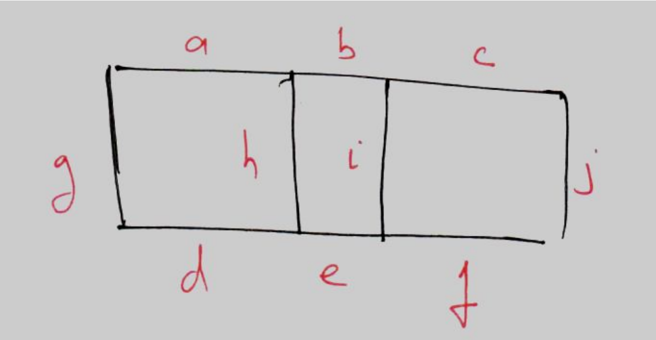


The symmetries are:

0◦, 180◦, about vertical and horizontal axis.

The 0◦ rotation consists of 10 1-cycles.

Symmetry about vertical axis has the cyclic decomposition (b)(e)(hi)(gj)(ac)(df).



Symmetry about horizontal axis has the cyclic decomposition (ad)(be)(cf)(g)(h)(i)(j).

The 180◦ rotation has the cyclic decomposition (af)(be)(cd)(hi)(gj)

$P_G = ¼\,(x_1^{10} + x_2^4 x_1^2 + x_2^3 x_1^4 + x_2^5)$

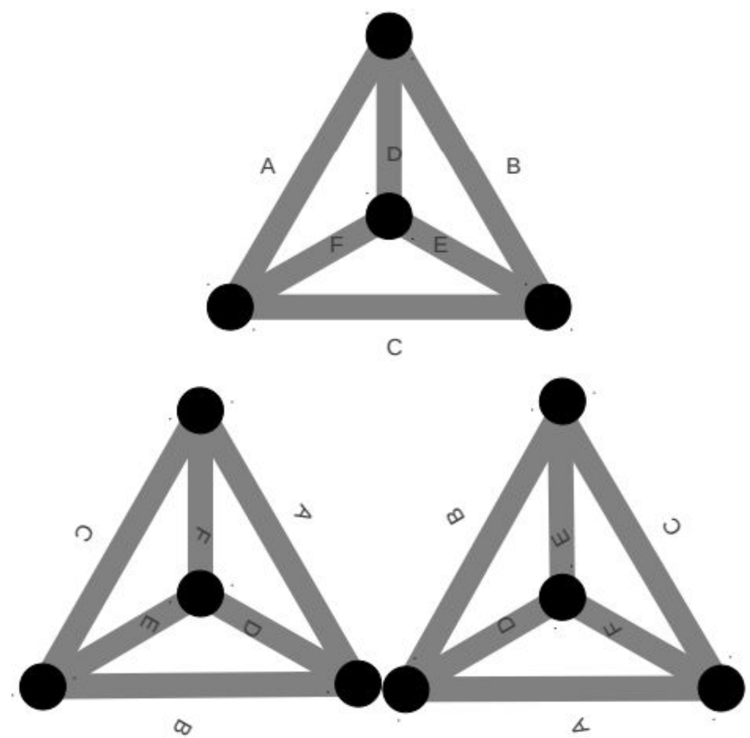
The number of different m-colorings is ¼ ($m^8 + m^6 + m^7 + m^5$)

Use the theorem to determine the number of 3-colorings of the four corners of a floating tetrahedron.

Symmetries?

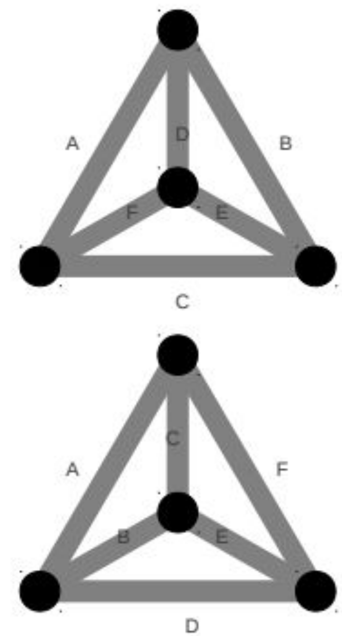
There exists 12 symmetries of the Tetrahedron:
the 0° revolution,
the eight revolutions of 120° and 240° about a corner and the middle of the opposite face (there are 4 corners and at a time **one corner is fixed**),

Rotation of tetrahedron by 120 degrees and 240 degrees



and the three revolutions of 180◦ about the middle of opposite edges (changing the position of a corner).

Rotation of tetrahedron by 180 degrees about edges A and E



The 0◦ revolution has the cycle structure representation x_1^4.
The 120◦ revolution about corner ***a*** and the middle of face bcd has the cyclic decomposition (a)(bcd) and its cycle structure representation is x_1x_3.
By symmetry, the other 120◦ and 240◦ revolutions have this same cycle structure representation.

The 180◦ revolution about the middle of edges ab and cd has the cyclic decomposition (ab)(cd) and its cycle structure representation is x_2^2.

By symmetry, the other 180◦ revolutions have the same cycle structure representation.
Thus we have

$$P_G = \frac{1}{12}\left(x_1^4 + 8x_1x_3 + 3x_2^2\right)$$

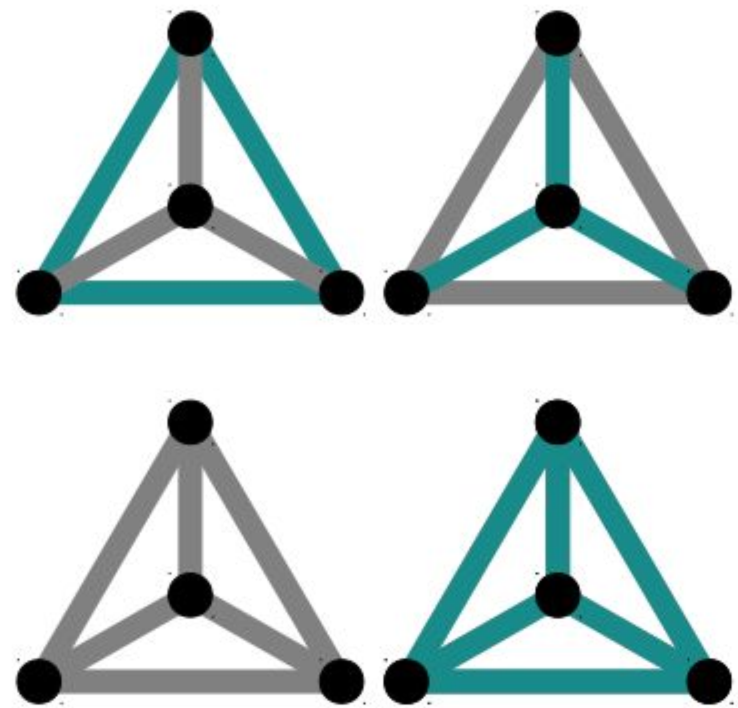
The number of different corner 3-colorings is

$$\begin{aligned} P_G(3,3,3,3) &= \frac{1}{12}(3^4 + 8 \times 3 \times 3 + 3 \times 3^2) \\ &= \frac{1}{12}(81 + 72 + 27) = 15 \quad \blacksquare \end{aligned}$$

Compute cycle index for the edge colorings of a tetrahedron.

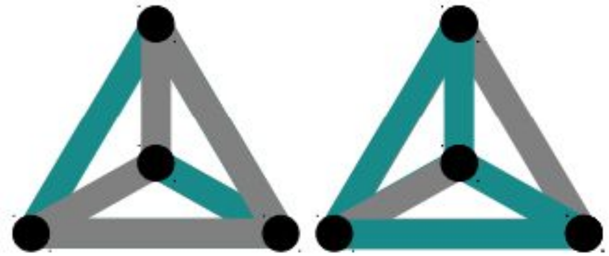
The identity leaves all 6 edges fixed and has structure representation x_1^6

Four 120 degree rotations about a corner and the middle of the opposite face give two cycles of length three.
They cyclically permute the edges incident to that corner and also the edges bounding the opposite face, so the cycle structure representation is x_3^2.
The four colorations this fixes are shown in Figure below.



Colorations of the tetrahedron fixed by rotations of 120 or 240 degrees

Three 180 degree rotations about opposite edges leave the two edges fixed. The other four edges are left in cycles of length 2. Thus we have the structure $x_1^2x_2^2$.



$$P_G = \frac{1}{12}(x_1^6 + 8x_3^2 + 3x_1^2x_2^2).$$