5.5 BINOMIAL IDENTITIES

Why the numbers *C(n, r)* are called *binomial coefficients ?*

Consider the polynomial expression $(a + x)^3$.

$$(a+x)(a+x)(a+x) = aaa + aax + axa + axx + xaa + xax + xxa + xxx$$

Why there are 8 terms on RHS

There are 3 positions and each position has 2 choices (with repeation).

Collecting similar terms, we reduce the right-hand side of this expansion to

$$a^3 + 3a^2x + 3ax^2 + x^3 \tag{1}$$

How many of the formal products in the expansion of $(a + x)^3$ contain k xs and $(3 - k)$ as?
This question is equivalent to asking for the coefficient of $a^{3-k} x^k$ in (1).

Since formal products are just three-letter sequences of as and xs, we are simply asking for the number of all three-letter sequences with k xs and $(3 - k)$ as.

The answer is C(3, k) and so the reduced expansion for $(a + x)^3$ can be written as

$$\binom{3}{0}a^3 + \binom{3}{1}a^2x + \binom{3}{2}ax^2 + \binom{3}{3}x^3$$

By the same argument, we see that the coefficient of $a^{n-k}x^k$ in $(a+x)^n$ will be equal to the number of n-letter sequences formed by k xs and $(n-k)$ as, that is, $C(n, k)$. If we set $a = 1$, we have the following theorem.

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

Write the Equivalent Identity

The number of ways to select a subset of k objects out of a set of n objects is equal to the number of ways to select a group of n − k of the objects to set aside (the objects not in the subset).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \tag{2}$$

Prove the following Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (3)$$

LHS: represents C(n, k) committees of k people chosen from a set of n people

To prove (3), we need to classify the C(n, k) committees of k people into two categories, depending on whether or not the committee contains a given person P.
If P is not a part of the committee,
there are C(n − 1, k) ways to form the committee from the other n − 1 people.
On the other hand, if P is on the committee,
the problem reduces to choosing the k − 1 remaining members of the committee from the other n − 1 people. This can be done C(n − 1, k − 1) ways.

Thus C(n, k) = C(n − 1, k) + C(n − 1, k − 1).

The above proof is a useful interpretation of binomial coefficients known as **committee selection model.**

Use committee selection model to show that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \tag{4}$$

The left-hand side of (4) counts the ways to select a group of k people chosen from a set of n people and then to select a subset of m leaders within the group of k people.
Here, we are first choosing k members and then m members from k members.

Equivalently, as counted on the right side, we could first select the subset of m leaders from the set of n people and then select the remaining k − m members of the group from the remaining n − m people.
We first selected m members which means we still need to select k – m members out of n – m members.

Show that

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \blacksquare \qquad (5)$$

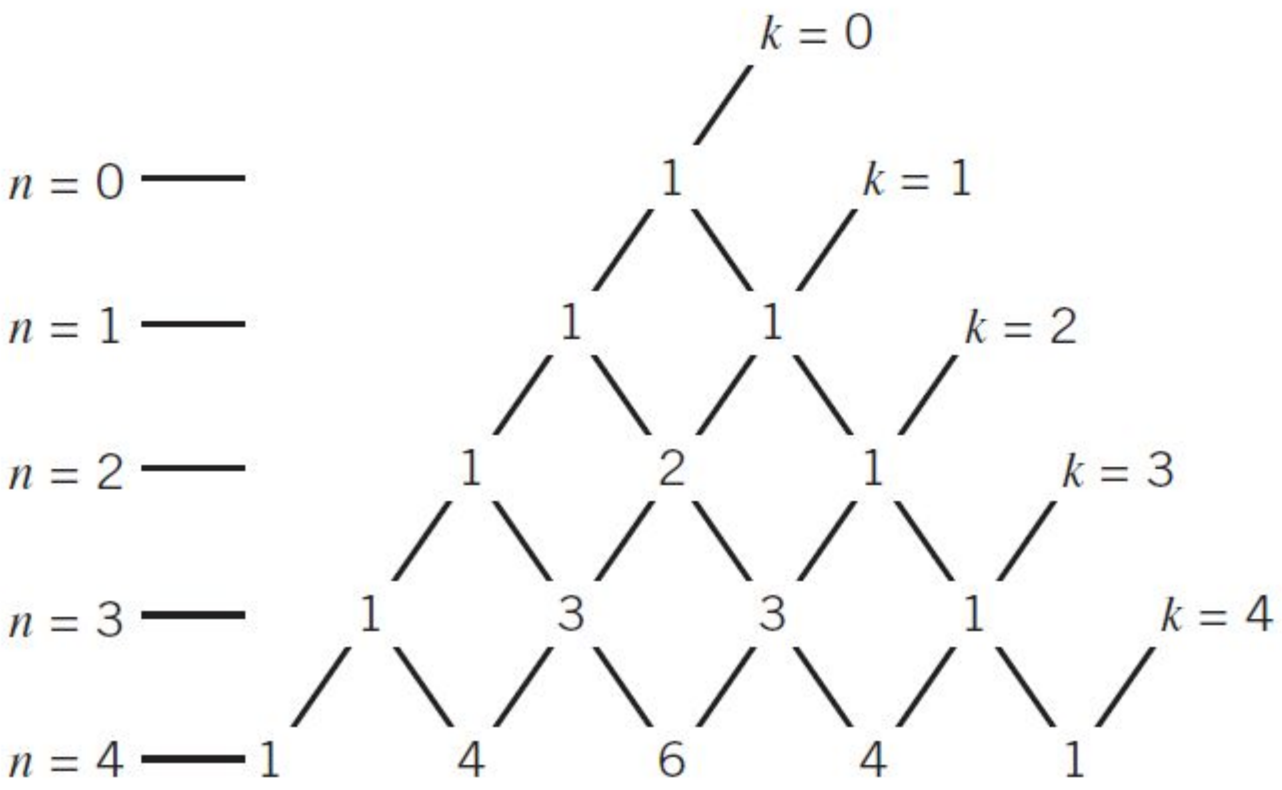
It is the special form of (4) when m = 1.

$$k\binom{n}{k} = n\binom{n-1}{k-1} \quad \text{or} \quad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Using (3) and the fact that C(n, 0) = C(n, n) = 1 for all nonnegative n, we can recursively build successive rows in the following table of binomial coefficients, called **Pascal's triangle**.
Each number in this table, except the first and last numbers in a row, is the sum of the two neighboring numbers in the preceding row.

Table of binomial coefficients: kth number in row n is $\binom{n}{k}$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$



Pascal's triangle has the following combinatorial interpretation.

Consider the ways a person can traverse the blocks in the map of streets shown in Fig below. The person begins at the top of the network, at the spot marked (0, 0), and moves down the network making a choice at each intersection to go right or left.

We label each street corner in the network with a pair (*n, k*), where n is the number of blocks traversed from (0, 0) and k is the number of times the person chose the right branch at intersections.

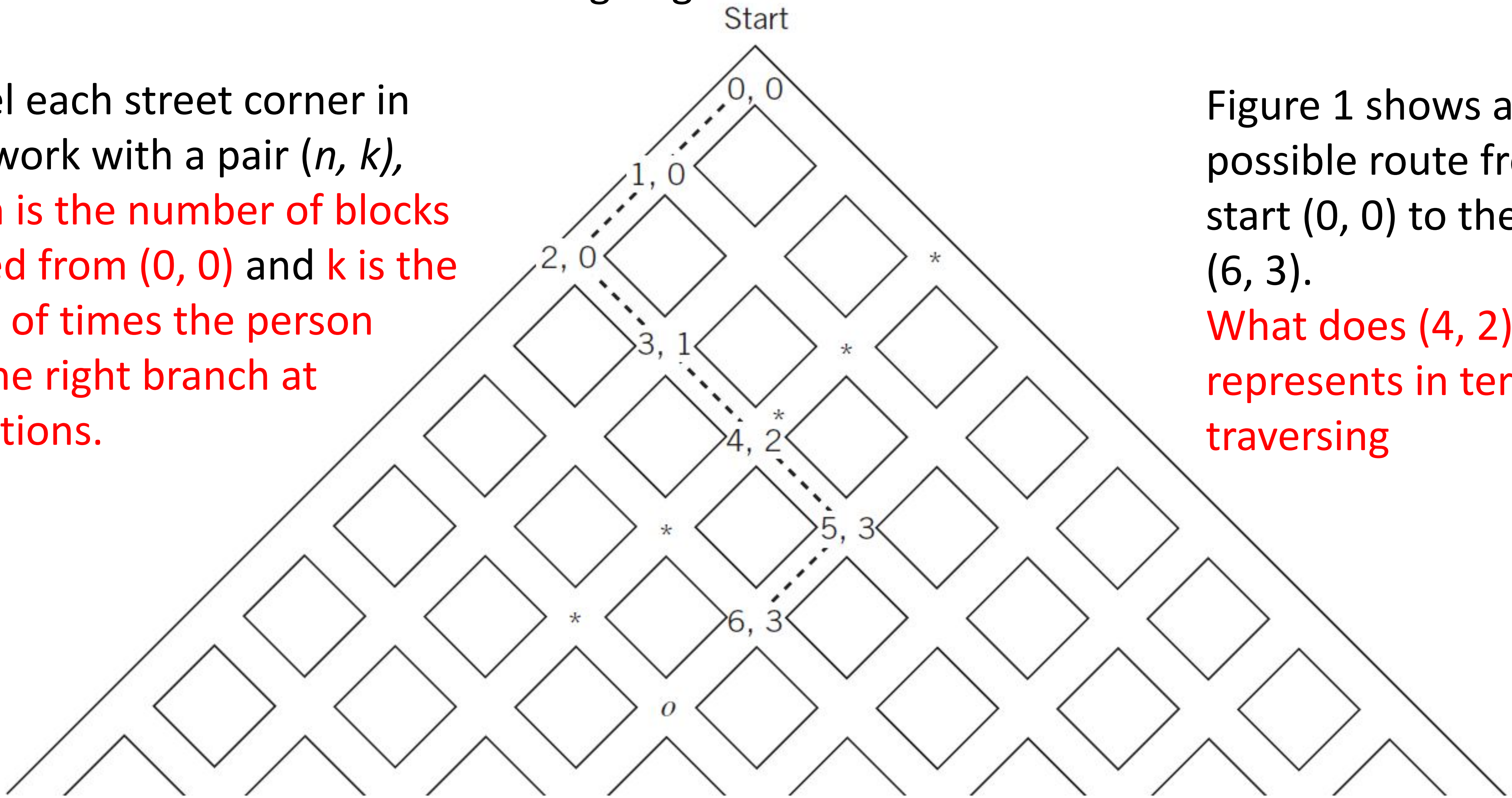


Figure 1 shows a possible route from the start (0, 0) to the corner (6, 3).

What does (4, 2) represents in terms of traversing

To go to corner (6, 3) following the route shown in Figure 1, we have the sequence of turns LLRRRL.

Any route to corner (n, k) can be written as a list of the branches (left or right) chosen at the successive corners on the path from (0, 0) to (n, k). Such a list is a sequence of k Rs (right branches) and (n − k) Ls (left branches).

What is the number of possible routes from the start (0, 0) to corner (n, k).

Let s(n, k) be the number of possible routes from (0, 0) to (n, k).

This is the number of sequences of k Rs and (n − k) Ls,

and hence s(n, k) = C(n, k).

Using "block-walking" model for binomial coefficients prove identity (3).

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n \tag{6}$$

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r} \tag{7}$$

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1} \tag{8}$$

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n} \tag{9}$$

$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r} \tag{10}$$

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k}=\binom{m+n}{m+r} \tag{11}$$

$$\sum_{k=n-s}^{m-r}\binom{m-k}{r}\binom{n+k}{s}=\binom{m+n+1}{r+s+1} \tag{12}$$

Here $C(n, r) = 0$ if $0 \leq n < r$.

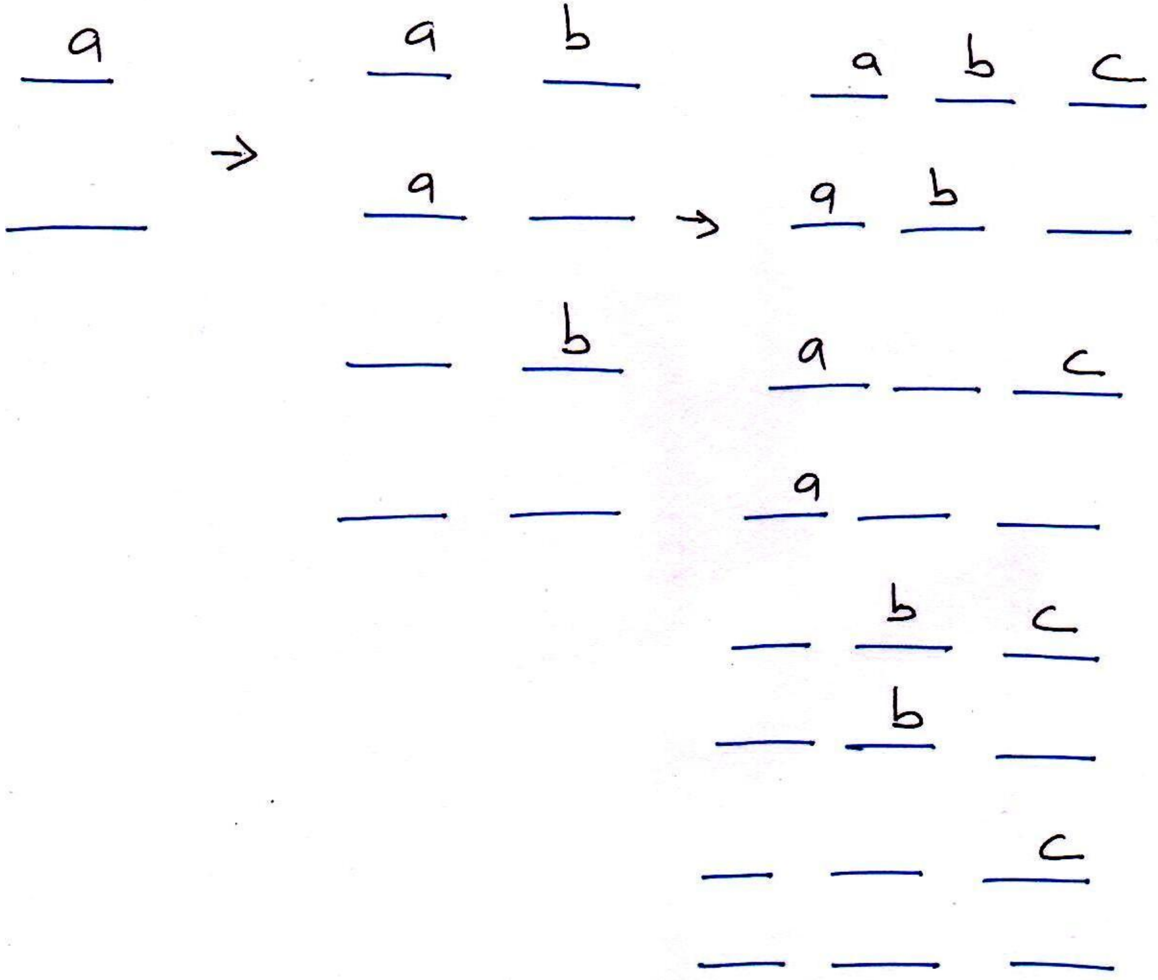
These identities can be explained by the "committee" type of combinatorial argument or by "block-walking arguments."

Using committee argument prove

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n \tag{6}$$

1^{st} 2^{nd} 3^{rd} person

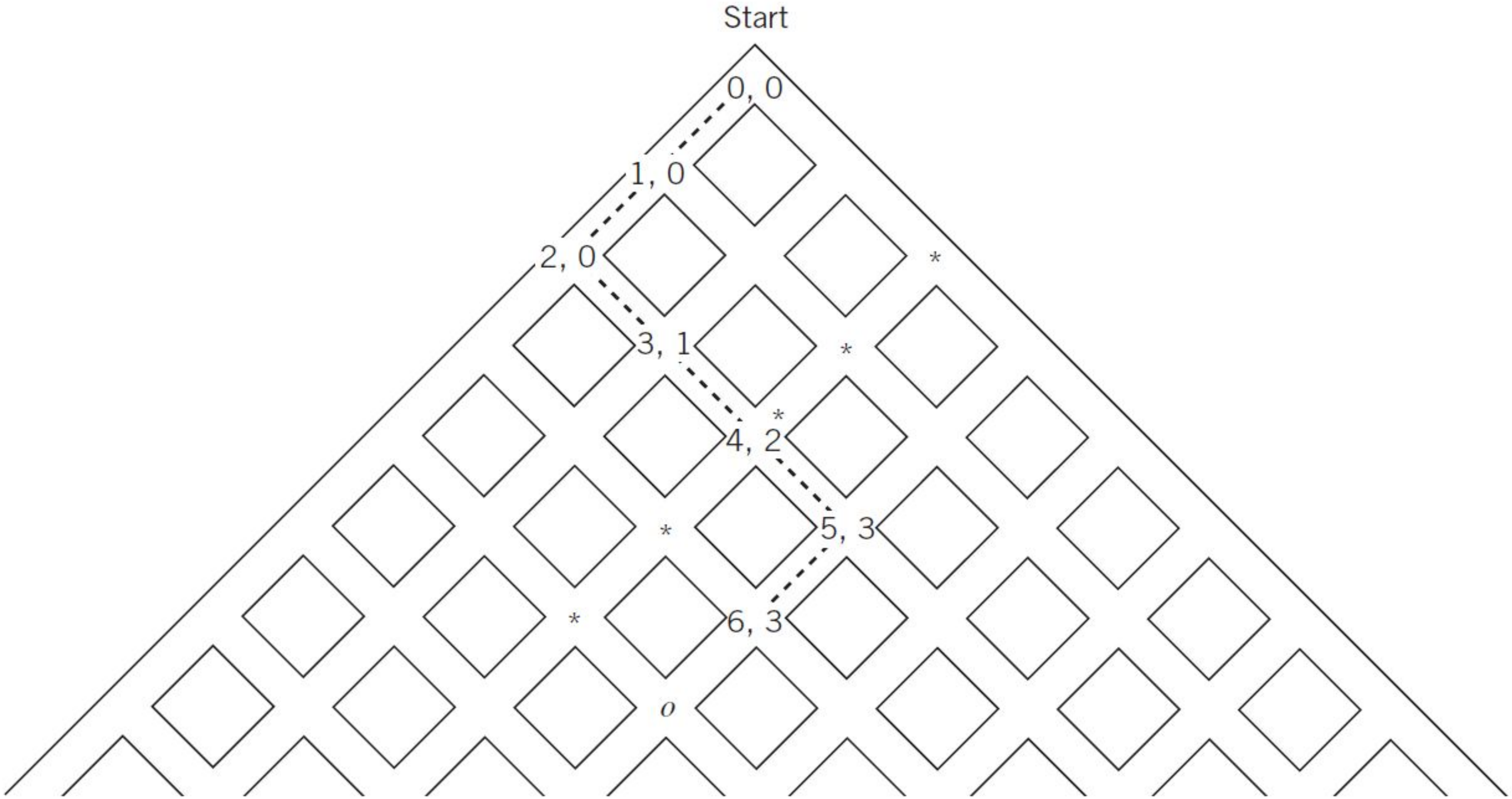
$\{a, b, c\}$



Verify identity (8) by block-walking argument.

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1} \qquad (8)$$

As an example of this identity, we consider the case where r = 2 and n = 6.



From (k=2, r=2), there are two ways of reaching (7, 3)

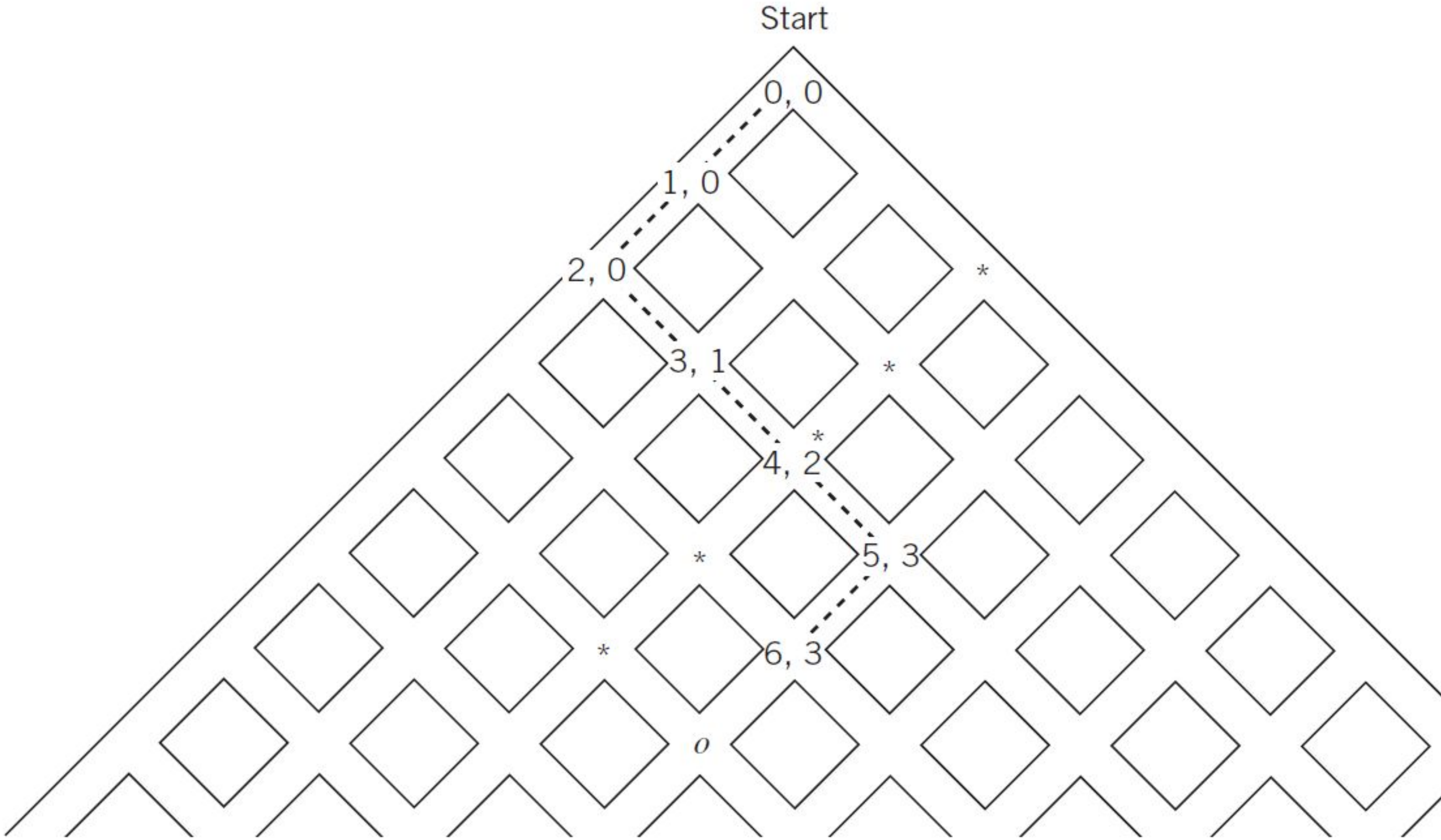
a) Starting by a left turn then left or right turns to reach (7, 3)

b) Starting by a right turn then left turns to reach (7, 3) (unique)

From (k>2, r=2), if we try to reach (7, 3) by first taking a left turn, then one of the route gets repeated, obtained in (a)

Hence, to reach (7, 3), we first consider all possible ways to reach (k>1, r=2), and then follows b).

This proves the required Identity.



The corners (k, 2), k = 2, 3, 4, 5, 6, are marked with a * in Figure 1 and corner (7, 3) is marked with an *o*.

<u>There are multiple ways to reach to the * but there is only one way to reach from * to *o*. (without repeatition)</u>

Observe that the right branches at each starred corner are the locations of last possible right branches on routes from the start (0, 0) to corner (7, 3).

After traversing one of these right branches, there is just one way to continue on to corner (7, 3), by making all remaining branches left branches.

In general, if we break all routes from (0, 0) to (n + 1, r + 1) into subcases based on the corner where the last right branch is taken, we obtain identity (8).

Verify identity (9) by a block-walking argument

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \qquad (9)$$

The number of routes from (n, k) to (2n, n) is equal to number of routes from (0, 0) to (n, n − k), since both trips go a total of n blocks with n − k to the right (and k to the left).

So the number of ways to go from (0, 0) to (n, k) and then on to (2n, n) is C(n, k) × C(n, n − k).

By (2), C(n, n −k) = C(n, k), and thus the number of routes from (0, 0) to (2n, n) via (n, k) is $C(n, k)^2$.

Summing over all k—that is, over all intermediate corners n blocks from the start—we count all routes from (0, 0) to (2n, n).

So this sum equals C(2n, n), and identity (9) follows.

Using the following identities, evaluate the sum

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2)(n-1)n.$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \tag{6}$$

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r} \tag{7}$$

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} \tag{8}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \tag{9}$$

Evaluate the sum $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2)(n-1)n$.

The general term in this sum $(k-2)(k-1)k$ is equal to $P(k, 3) = k!/(k-3)!$.

Recall that the numbers of r-permutations and of r-selections differ by a factor of $r!$ That is, $C(k, 3) = k!/(k-3)!3! = P(k, 3)/3!$, or $P(k, 3) = 3!C(k, 3)$.

So the given sum can be rewritten as

$$3!\binom{3}{3} + 3!\binom{4}{3} + \cdots + 3!\binom{n}{3} = 3!\left(\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n}{3}\right)$$

By identity (8), this sum equals $3!\binom{n+1}{4}$. ■

Evaluate the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$.

A strategy for problems whose general term is not a multiple of C(n, k) or P(n, k) is to decompose the term algebraically into a sum of P(n, k)-type terms.
In this case, the general term k^2 can be written as $k^2 = k(k-1)+k$.
So the given sum can be rewritten as

$$\begin{aligned}
&[(1 \times 0) + 1] + [(2 \times 1) + 2] + [(3 \times 2) + 3] + \cdots + [n(n-1) + n] \\
&= [(2 \times 1) + (3 \times 2) + \cdots + n(n-1)] + (1 + 2 + 3 + \cdots + n) \\
&= \left(2\binom{2}{2} + 2\binom{3}{2} + \cdots + 2\binom{n}{2}\right) + \left(\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1}\right) \\
&= 2\binom{n+1}{3} + \binom{n+1}{2}
\end{aligned}$$

by identity (8).