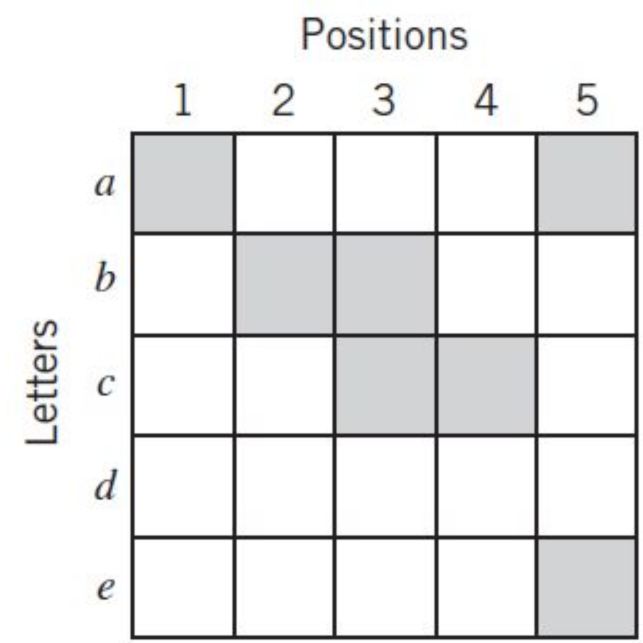
RESTRICTED POSITIONS AND ROOK POLYNOMIALS

Consider the problem of finding all arrangements of a, b, c, d, e with the restrictions indicated in Figure below, that is, a may not be put in position 1 or 5; b may not be put in 2 or 3; c not in 3 or 4; and e not in 5. A permissible arrangement can be represented by picking five unmarked squares with one square in each row and each column. For example, a permissible arrangement is (a, 2), (b, 1), (c, 5), (d, 3), (e, 4).

Count the number of permissible arrangements?
(using inclusion–exclusion formula)

N = ?



Let U be the set of all arrangements of the five letters without restrictions. So $N = 5!$. A_i?

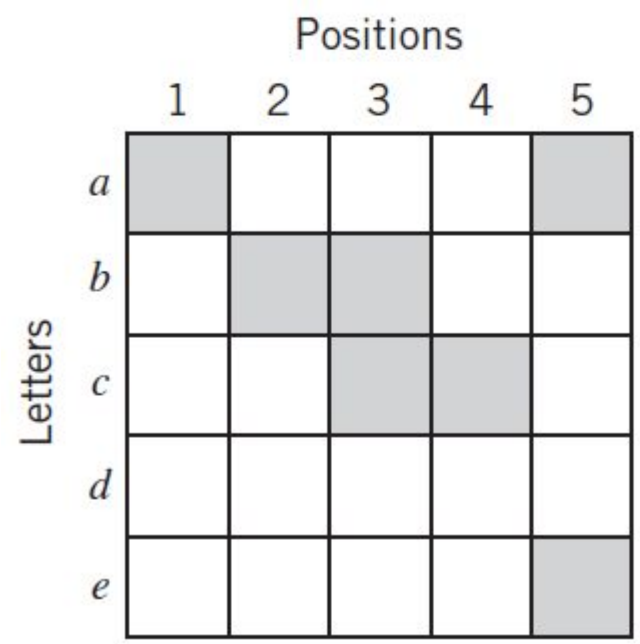
Let A_i be the set of arrangements with a forbidden letter in position i.

In terms of Figure below, A_i is the set of all collections of five squares, each in a different row and column such that the square in column i is a darkened square.

The number of permissible arrangements will then be

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$$

$N(A_i) = ?$



$N(A_1) = 1 \times 4!$

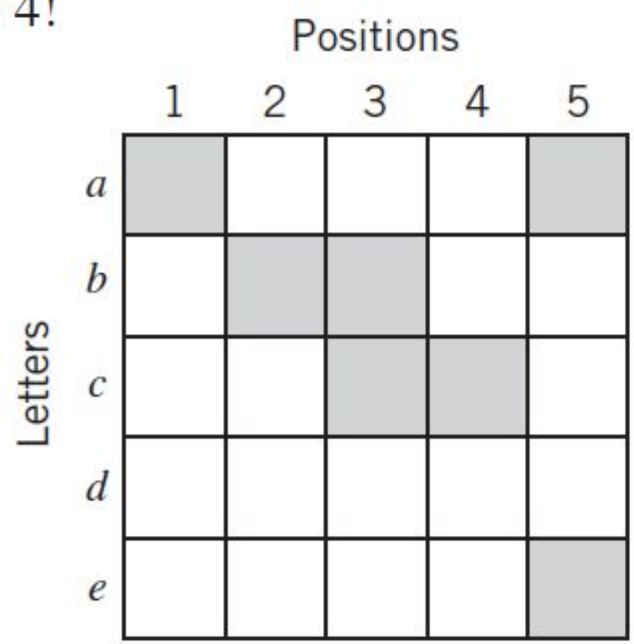
We obtain $N(A_i)$ by counting the ways to put a forbidden letter in position i times the 4! ways to arrange the remaining four letters in the other four positions (we do not worry about forbidden positions for these letters).

$N(A_2) = 1 \times 4!$, $N(A_3) = 2 \times 4!$, $N(A_4) = 1 \times 4!$, and $N(A_5) = 2 \times 4!$.

Collecting terms, we obtain

$$S_1 = \sum_{i=1}^{5} N(A_i) = 1 \times 4! + 1 \times 4! + 2 \times 4! + 1 \times 4! + 2 \times 4!$$
$$= (1 + 1 + 2 + 1 + 2)4! = 7 \times 4!$$

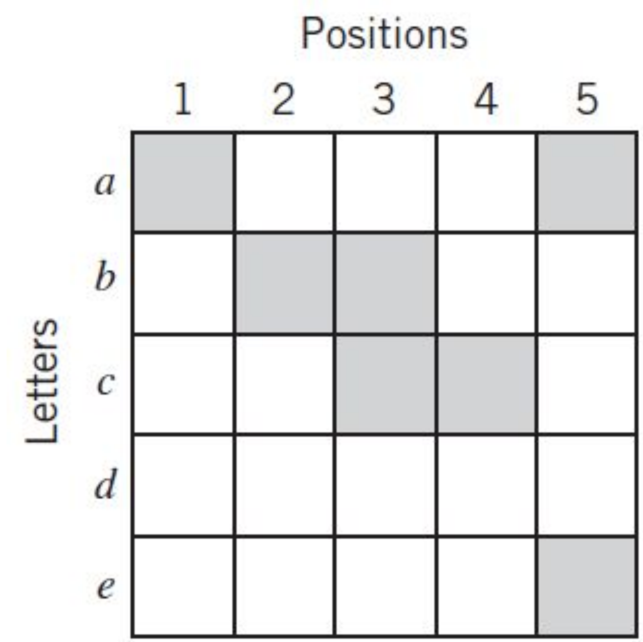
Here (1+1+2+1+2) = 7 is the number of the darkened squares in the Figure below.



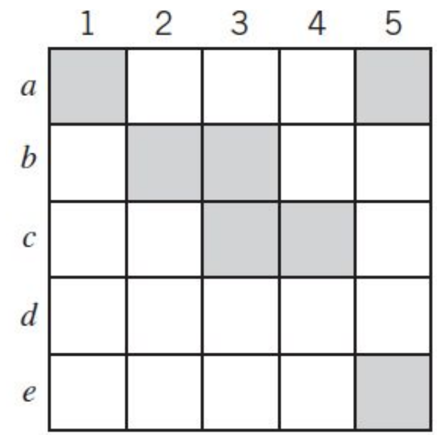
S_1 = (number of darkened squares) × 4!
for any restricted-positions problem with a 5×5 family of darkened squares similar to Figure below.

$N(A_iA_j)$ = ? S_2 = ?

$N(A_iA_j)$ is the number of ways to put (different) forbidden letters in positions i and j times the 3! ways to arrange the remaining three letters.



$N(A_1A_2) = 1 \times 3!$ $\quad N(A_1A_3) = 2 \times 3!$ $\quad N(A_1A_4) = 1 \times 3!$
$N(A_1A_5) = 1 \times 3!$ $\quad N(A_2A_3) = 1 \times 3!$ $\quad N(A_2A_4) = 1 \times 3!$
$N(A_2A_5) = 2 \times 3!$ $\quad N(A_3A_4) = 1 \times 3!$ $\quad N(A_3A_5) = 4 \times 3!$
$N(A_4A_5) = 2 \times 3!$



Collecting terms, we obtain

$$S_2 = \sum_{ij} N(A_iA_j) = (1+2+1+1+1+1+2+1+4+2)3! = 16 \times 3!$$

$S_3 = ?$, $S_4 = ?$, $S_5 = ?$

The number 16 counts the ways to select two darkened squares, each in a different row and column. Generalizing, we will have

$$S_k = \begin{pmatrix} \text{number of ways to pick } k \text{ darkened squares} \\ \text{each in a different row and column} \end{pmatrix} \times (5-k)! \tag{1}$$

Since letter d's row has no darkened squares, there is no way to pick five darkened squares, each in a different row and column.

Thus $S_5 = 0$.

On the other hand, it is tedious to compute S_3 and S_4 case by case.

Hence, we discuss a theory for determining the number of ways to pick k darkened squares, each in a different row and column.

This darkened squares selection problem can be restated in terms of a recreational mathematics question about a chess-like game.

A chess piece called a **rook** can capture any opponent's piece on the chessboard in the same row or column as the rook (provided there are no intervening pieces).

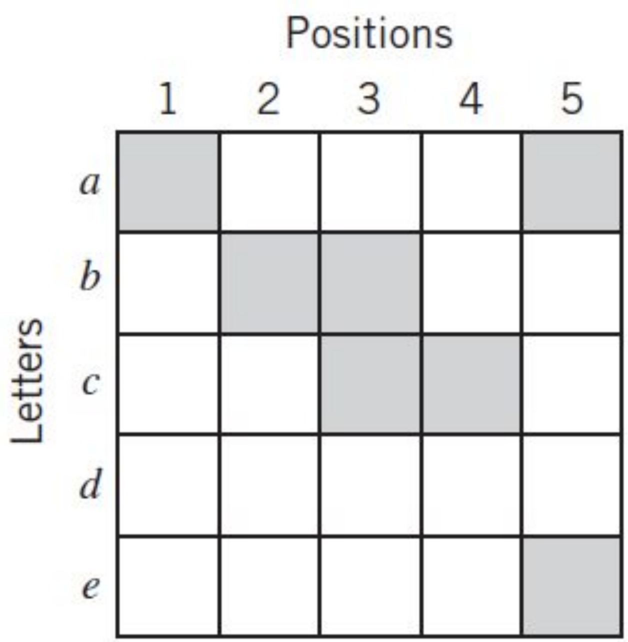
Counting the number of ways to place k mutually non-capturing rooks on this board of darkened squares is equivalent to our original sub-problem of counting the number of ways to pick k darkened squares.

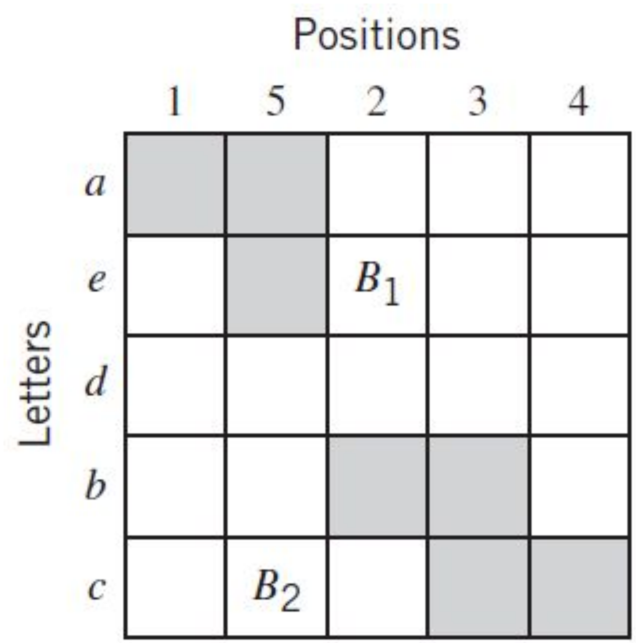
Now we will develop two breaking-up operations to help us count non-capturing rooks on a given board *B*.

The first operation applies to a board B that can be decomposed into **disjoint sub-boards** B_1 and B_2,—that is, sub-boards involving different sets of rows and columns.

Often a board has to be properly rearranged before the disjoint nature of the two sub-boards can be seen.

Let B be the board of darkened squares in Figure below, let B_1 be the three darkened squares in rows a and e, and let B_2 be the four darkened squares in rows b and c.





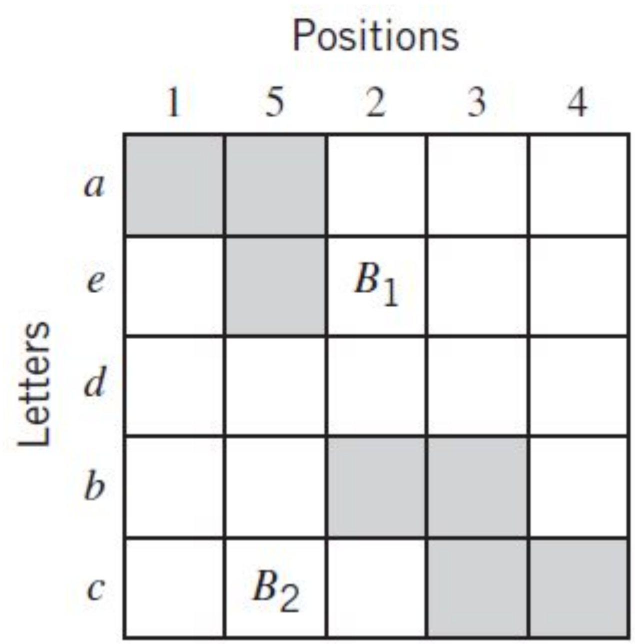
Define $r_k(B)$ to be the number of ways to place k non-capturing rooks on board B, $r_k(B_1)$ the number of ways to place k non-capturing rooks on sub-board B_1, and $r_k(B_2)$ the number of ways to place k non-capturing rooks on sub-board B_2.

$r_1(B_1) = ?$ $r_1(B_2) = ?$

There are three ways to place one rook on sub-board B_1 in Figure below, since B_1 has three squares, and similarly four ways to place one rook on sub-board B_2.

Then $r_1(B_1) = 3$ and $r_1(B_2) = 4$.

$r_2(B_1) = ?$ $r_2(B_2) = ?$

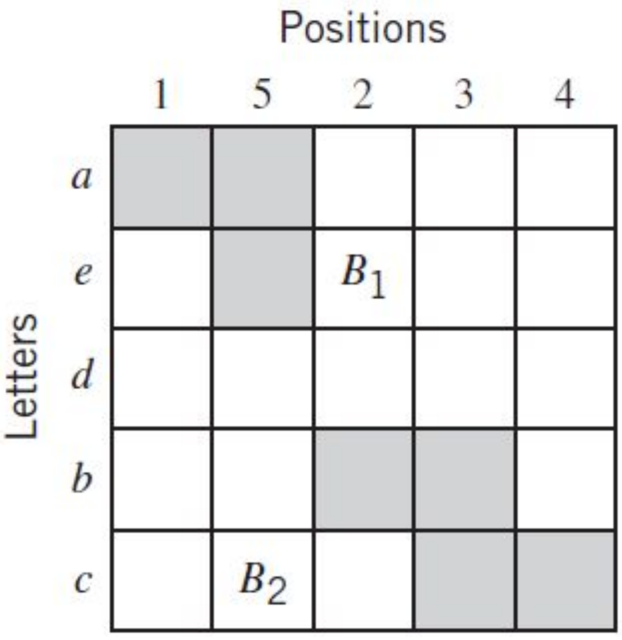


$r_2(B_1) = 1$ and $r_2(B_2) = 3$.
$r_3(B_1) = ?$ $r_3(B_2) = ?$

$r_k(B_1) = r_k(B_2) = 0$ for $k \geq 3$, since each sub-board has only two rows.
It will be convenient to define $r_0 = 1$ for all boards.

Observe that (why)

$$r_2(B) = r_2(B_1) + r_1(B_1)r_1(B_2) + r_2(B_2)$$



Observe next that since B_1 and B_2 are disjoint, placing, say, two noncapturing rooks on the whole board B can be broken into three cases: placing two noncapturing rooks on B_1 (and none on B_2), placing one rook on each subboard, or placing two noncapturing rooks on B_2. Thus we see that

$$r_2(B) = r_2(B_1) + r_1(B_1)r_1(B_2) + r_2(B_2)$$

or, using that fact that $r_0(B_2) = r_0(B_1) = 1$,

$$\begin{aligned} r_2(B) &= r_2(B_1)r_0(B_2) + r_1(B_1)r_1(B_2) + r_0(B_1)r_2(B_2) \\ &= 1 \times 1 + 3 \times 4 + 1 \times 3 = 16 \end{aligned} \tag{2}$$

Recall that 16 is the number obtained earlier when summing all $N(A_i A_j)$ to count all ways to pick two darkened squares each in a different row and column.

Lemma

If B is a board of darkened squares that decomposes into the two disjoint subboards B_1 and B_2, then

$$r_k(B) = r_k(B_1)r_0(B_2) + r_{k-1}(B_1)r_1(B_2) + \cdots + r_0(B_1)r_k(B_2) \tag{3}$$

We define the **rook polynomial** $R(x, B)$ of the board B of darkened squares to be

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

Remember that $r_0(B) = 1$ for all B.

$R(x, B_1) = ?$ $R(x, B_2) = ?$

$$R(x, B_1) = 1 + 3x + 1x^2 \quad \text{and} \quad R(x, B_2) = 1 + 4x + 3x^2$$

$R(x, B) = ?$

Moreover, by the correspondence between (3) and the formula for the product of two generating functions, we see that $r_k(B)$, the coefficient of x^k in the rook polynomial $R(x, B)$ of the full board, is simply the coefficient of x^k in the product $R(x, B_1)R(x, B_2)$. That is,

$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) = (1 + 3x + 1x^2)(1 + 4x + 3x^2) \\ &= 1 + [(3 \times 1) + (1 \times 4)]x + [(1 \times 1) + (3 \times 4) + (1 \times 3)]x^2 \\ &\quad + [(1 \times 4) + (3 \times 3)]x^3 + (1 \times 3)x^4 \\ &= 1 + 7x + 16x^2 + 13x^3 + 3x^4 \end{aligned}$$

Theorem 1

If B is a board of darkened squares that decomposes into the two disjoint subboards B_1 and B_2 then

$$R(x, B) = R(x, B_1)R(x, B_2)$$

$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$

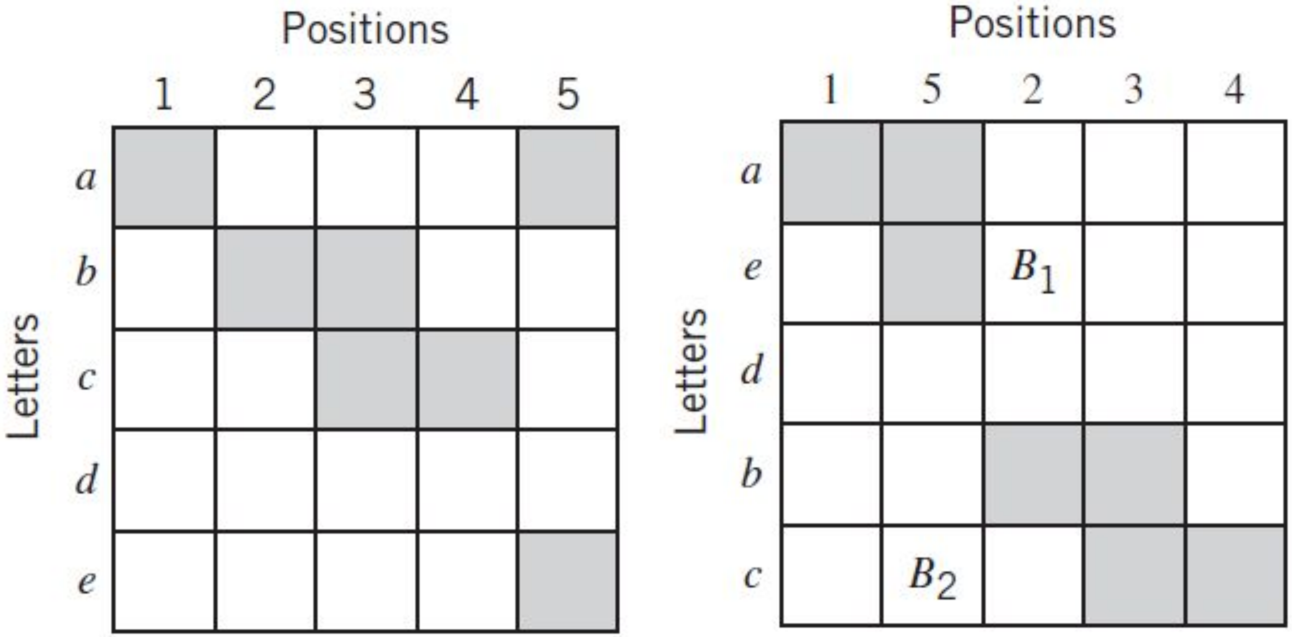
$$\begin{aligned} N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= N - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= 5! - r_1(B)4! + r_2(B)3! - r_3(B)2! + r_4(B)1! - r_5(B)0! \\ &= 5! - 7 \times 4! + 16 \times 3! - 13 \times 2! + 3 \times 1! - 0 \times 0! \end{aligned}$$

Theorem 2

The number of ways to arrange n distinct objects when there are restricted positions is equal to

$$\begin{aligned} & n! - r_1(B)(n-1)! + r_2(B)(n-2)! + \cdots + (-1)^k r_k(B)(n-k)! \\ & \quad + \cdots + (-1)^n r_n(B)0! \end{aligned} \tag{4}$$

where the $r_k(B)$s are the coefficients of the rook polynomial $R(x, B)$ for the board B of forbidden positions.



Count the number of permissible arrangements?

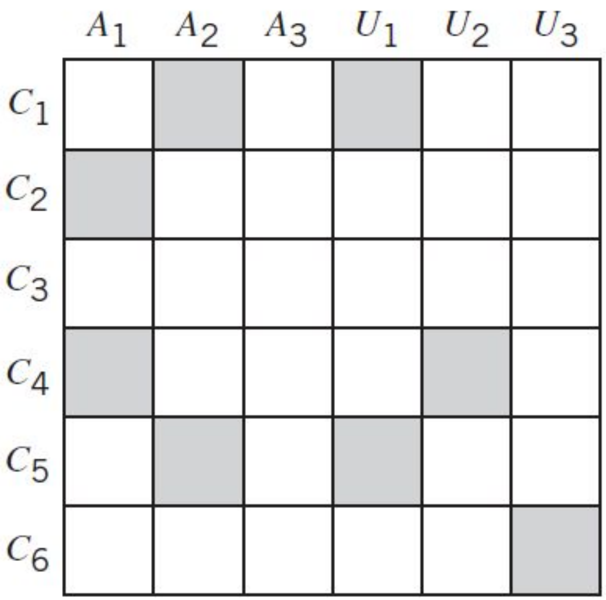
$$R(x, B_1) = 1 + 3x + 1x^2 \quad \text{and} \quad R(x, B_2) = 1 + 4x + 3x^2$$

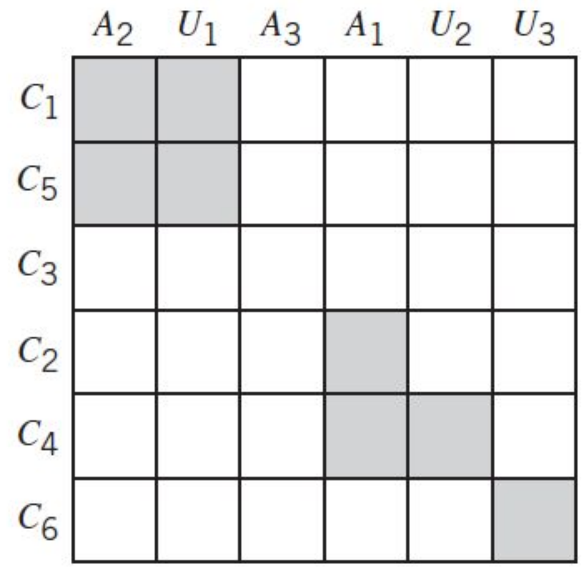
$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) = (1 + 3x + 1x^2)(1 + 4x + 3x^2) \\ &= 1 + 7x + 16x^2 + 13x^3 + 3x^4 \end{aligned}$$

$$\begin{aligned} N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= N - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= 5! - r_1(B)4! + r_2(B)3! - r_3(B)2! + r_4(B)1! - r_5(B)0! \\ &= 5! - 7 \times 4! + 16 \times 3! - 13 \times 2! + 3 \times 1! - 0 \times 0! \\ &= 120 - 168 + 96 - 26 + 3 - 0 = 25 \end{aligned}$$

How many ways are there to send six different birthday cards, denoted C_1, C_2, C_3, C_4, C_5, C_6, to three aunts and three uncles, denoted A_1, A_2, A_3, U_1, U_2, U_3, if aunt A_1 would not like cards C_2 and C_4; if A_2 would not like C_1 or C_5; if A_3 likes all cards; if U_1 would not like C_1 or C_5; if U_2 would not like C_4; and if U_3 would not like C_6?

First draw the board and compute the Rook Polynomial.

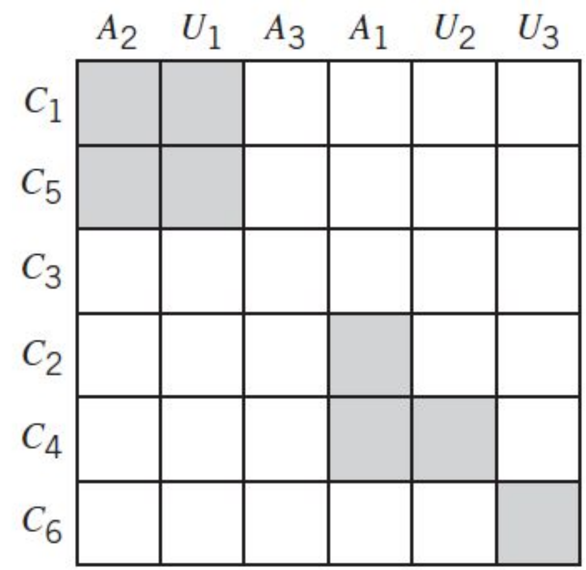




Thus the original board B of darkened squares decomposes into the two disjoint subboards, B_1 in rows C_1 and C_5, and B_2 in rows C_2, C_4, and C_6. Actually B_2 itself decomposes into two disjoint subboards B_2' and B_2'', where B_2'' is the single square (C_6, U_3). By inspection, we see that

$$R(x, B_1) = 1 + 4x + 2x^2$$

$$R(x, B_2) = R(x, B_2')R(x, B_2'') = (1 + 3x + x^2)(1 + x)$$



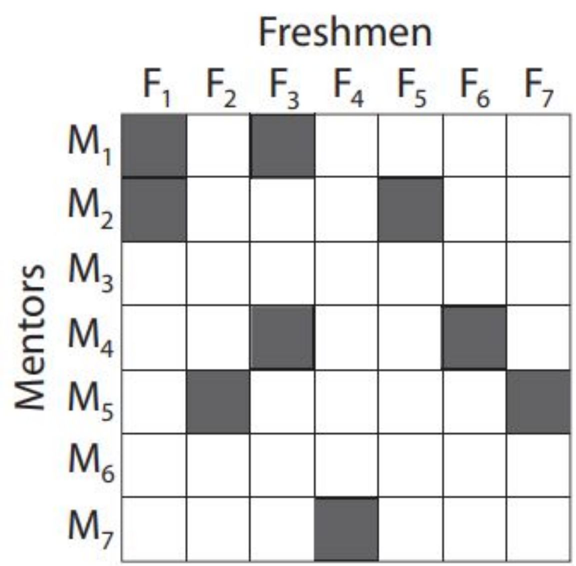
$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) \\ &= (1 + 4x + 2x^2)(1 + 3x + x^2)(1 + x) \\ &= 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5 \end{aligned}$$

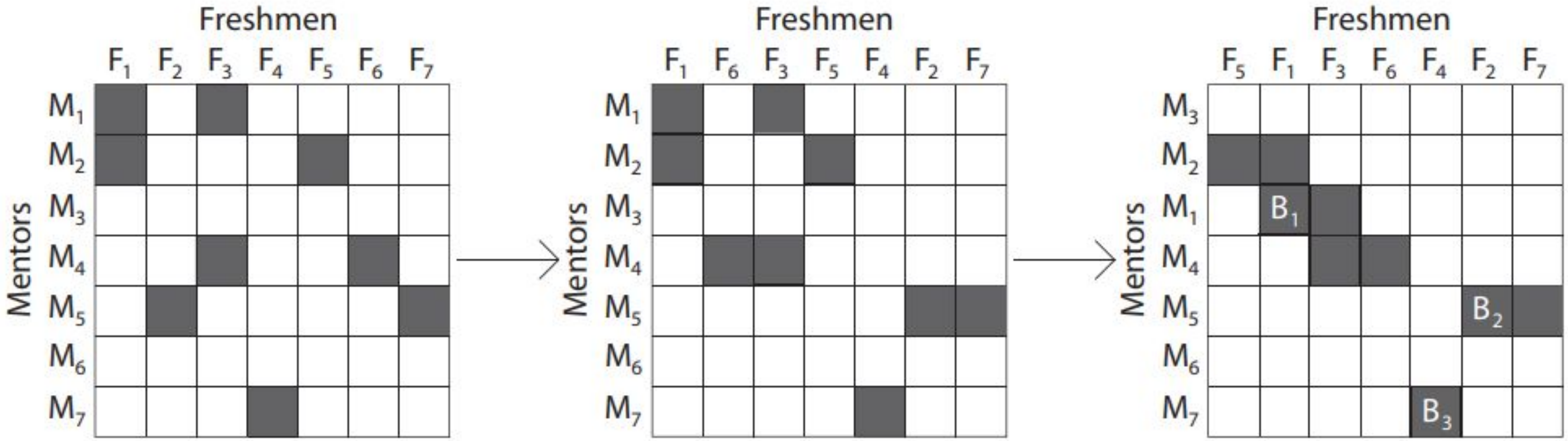
Then the answer to the card-mailing problem is

$$\begin{aligned} &\sum_{k=0}^{6} (-1)^k r_k(B)(6 - k)! \\ &\quad = 6! - 8 \times 5! + 22 \times 4! - 25 \times 3! + 12 \times 2! - 2 \times 1! + 0 \times 0! \\ &\quad = 720 - 960 + 528 - 150 + 24 - 2 + 0 = 160 \ \blacksquare \end{aligned}$$

At a university, seven freshmen, $F_1, F_2, F_3, F_4, F_5, F_6$ and F_7, enter the same academic program. Their department head, eager to retain these new students, wants to assign each incoming freshman a mentor from among the upperclassmen of the program. Seven mentors are chosen, $M_1, M_2, M_3, M_4, M_5, M_6$ and M_7, but there are some scheduling conflicts. M_1 cannot work with F_1 or F_3, M_2 cannot work with F_1 or F_5, M_4 cannot work with F_3 or F_6, M_5 cannot work with F_2 or F_7, and M_7 cannot work with F_4. In how many ways can the department head assign the mentors so that each incoming freshman has a different mentor?

First draw the board and compute the Rook Polynomial.





Interchanging columns F_2 and F_6, F_4 and F_5, F_1 and F_5, F_1 and F_6, and rows M_1 and M_3 yields a decomposition of the original board into the three subboards displayed in the final board of Fig.(1.4). Now we set about calculating the $r_k(B_i)$'s for these subboards, and arrive at the following: $r_1(B_1) = 6$, $r_2(B_1) = 10$, $r_3(B_1) = 4$; $r_1(B_2) = 2$; $r_1(B_3) = 1$. Thus we arrive with the following rook polynomials for B_1, B_2 and B_3:

$$R(x, B_1) = 1 + 6x + 10x^2 + 4x^3$$

$$R(x, B_2) = 1 + 2x$$

$$R(x, B_3) = 1 + x$$

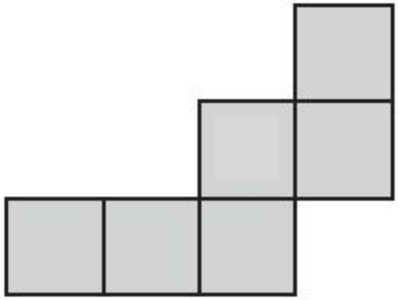
Multiplying these rook polynomials yields:

$$R(x,B) = R(x,B_1)R(x,B_2)R(x,B_3) =$$
$$(1+6x+10x^2+4x^3)(1+2x)(1+x) =$$
$$(1+6x+10x^2+4x^3)(1+3x+2x^2) =$$
$$1+3x+2x^2+6x+18x^2+12x^3+10x^2+30x^3+20x^4+4x^3+12x^4+8x^5 =$$
$$1+9x+30x^2+46x^3+32x^4+8x^5.$$

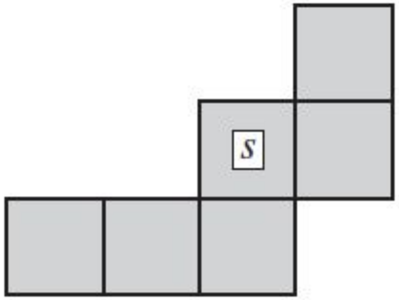
The required answer is

$$7! - 9\times 6! + 30\times 5! - 46\times 4! + 32\times 3! - 8\times 2! + 0\times 1! - 0\times 0! =$$
$$5,040 - 6,480 + 3,600 - 1,104 + 192 - 16 = 1,232.$$

Thus there are 1,232 ways to assign each freshman his or her own mentor, in accordance with the given restrictions.■



Determine the coefficients of R(x, B)



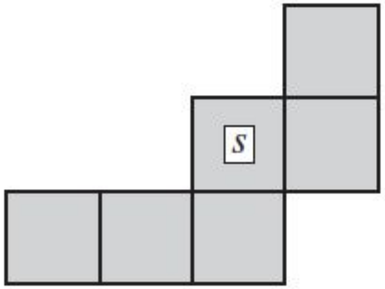
It is a problem of determining the coefficients of R(x, B) when the board B does not decompose into two disjoint sub-boards.

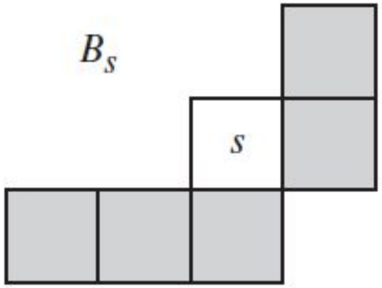
Let us break the problem of determining $r_k(B)$ into two cases, depending on whether or not a certain square s is one of the squares chosen for the k non-capturing rooks.

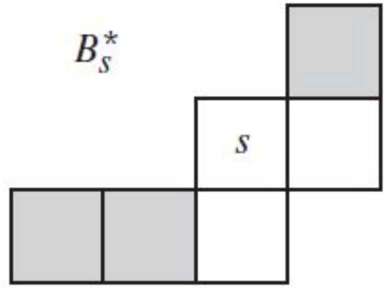
How the board can be split now?

Let B_s be the board obtained from B by deleting square s (if square s is not chosen), and

let B^*_s be the board obtained from B by deleting square s plus all squares in the same row or column as s (if square s is chosen).







If s is the square indicated in Figure above, then B_s and B^*_s are shown below.
$r_k(B)$?

If square s is not used, we must place k noncapturing rooks on B_s. If square s is used, then we must place $k-1$ noncapturing rooks on B_s^*. Hence we conclude that

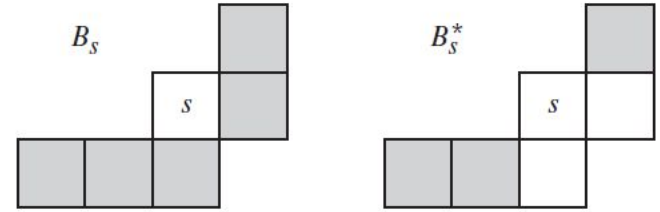
$$r_k(B) = r_k(B_s) + r_{k-1}(B_s^*) \tag{5}$$

Using the generating function methods introduced in Section 7.5 for turning a recurrence relation into a generating function, we obtain from (5)

$$\begin{aligned} R(x, B) = \sum_k r_k(B)x^k &= \sum_k r_k(B_s)x^k + \sum_k r_{k-1}(B_s^*)x^k \\ &= \sum_k r_k(B_s)x^k + x\sum_h r_h(B_s^*)x^h \\ &= R(x, B_s) + xR(x, B_s^*) \end{aligned}$$

R(x, Bs) = ?, R(x, B^*_s) = ?

$$R(x, B_s) = (1+3x)(1+2x) = 1+5x+6x^2$$
$$R(x, B_s^*) = (1+2x)(1+x) = 1+3x+2x^2$$



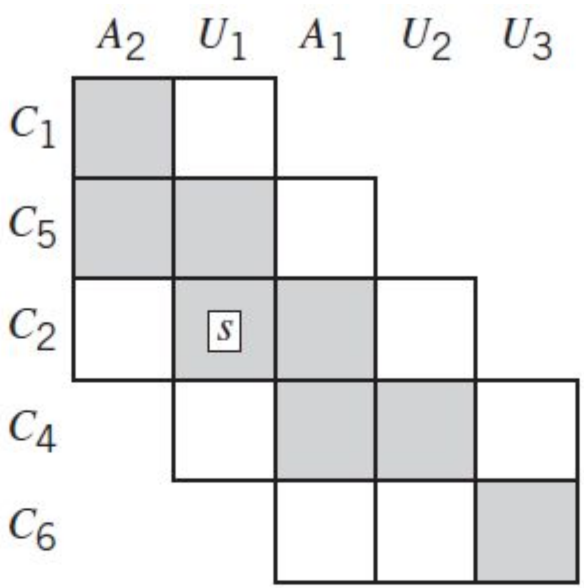
$$R(x, B) = R(x, B_s) + xR(x, B_s^*) = (1+5x+6x^2) + x(1+3x+2x^2)$$
$$= 1+6x+9x^2+2x^3$$

Theorem 3

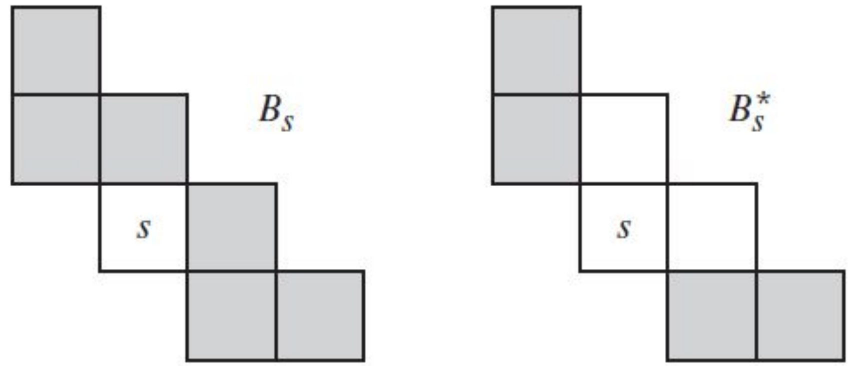
Let B be any board of darkened squares. Let s be one of the squares of B, and let B_s and B_s^* be as defined above. Then

$$R(x, B) = R(x, B_s) + xR(B_s^*)$$

Compute R(x, B)



The square in the bottom right corner is t, which is disjoint from the other squares, and the remaining board is B_1.



$$R(x, B_s) = (1 + 3x + x^2)(1 + 3x + x^2) = 1 + 6x + 11x^2 + 6x^3 + x^4$$
$$R(x, B_s^*) = (1 + 2x)(1 + 2x) = 1 + 4x + 4x^2$$

Then

$$\begin{aligned} R(x, B_1) = R(x, B_s) + x\,R(x, B_s^*) &= (1 + 6x + 11x^2 + 6x^3 + x^4) \\ &\quad + x(1 + 4x + 4x^2) \\ &= 1 + 7x + 15x^2 + 10x^3 + x^4 \end{aligned}$$

and

$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, t) \\ &= (1 + 7x + 15x^2 + 10x^3 + x^4)(1 + x) \\ &= 1 + 8x + 22x^2 + 25x^3 + 11x^4 + x^5 \end{aligned}$$

The number of ways to send birthday cards is

$$6! - 8 \times 5! + 22 \times 4! - 25 \times 3! + 11 \times 2! - 1 \times 1! + 0 \times 0! = 159$$