The Pigeonhole Principle

Pigeonhole Principle: Simple Form

If $n + 1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.

Example 1: Among 13 people there are 2 who have their birthdays in the same month.

Q1. There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple has been selected?

$n + 1$

Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m. Less formally, there exist consecutive a's in the sequence $a_1, a_2, \ldots, a_m$ whose sum is divisible by m.

To see this, consider the m sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \cdots + a_m.$$

A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, to avoid tiring himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of (consecutive) days during which the chess master will have played *exactly* 21 games.

Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number of games played on the first, second, and third days, and so on. The sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence[3] since at least one game is played each day. Moreover, $a_1 \geq 1$, From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that, among the integers chosen, there are two such that one of them is divisible by the other.

Consider a subset of natural number $A = \{n \in N | 1 \leq n \leq 50\}$ and $|A| = 10$. Prove that there exists two subsets B, C of A such that $|B| = |C| = 4$ and $\sum_{i=1}^{4} b_i = \sum_{i=1}^{4} c_i$ where $b_i \in B$ and $c_i \in C$.

A Theorem of Ramsey

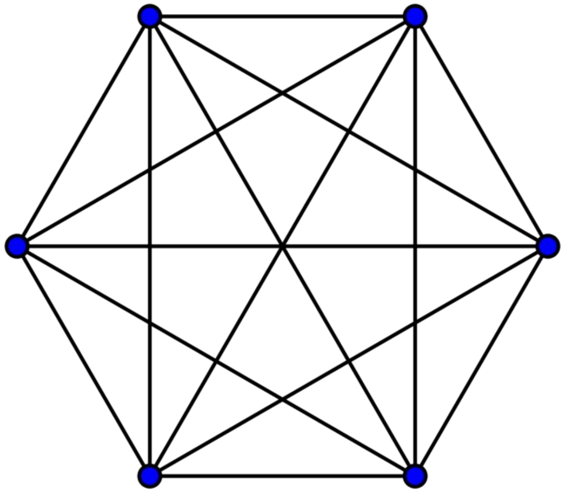
Frank Ramsey was born in 1903 and died in 1930 when he was not quite 27 years of age. In spite of his premature death, he laid the foundation for what is now called *Ramsey theory.*

The following is the most popular and easily understood instance of Ramsey's theorem:

> Of six (or more) people, either there are three, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.

One way to prove this result is to examine all the different ways in which six people can be acquainted and unacquainted. This is a tedious task, but nonetheless one that can be accomplished with a little fortitude. There is, however, a simple and elegant proof that avoids consideration of cases. Before giving this proof, we formulate the result more abstractly as

$$K_6 \to K_3, K_3 \qquad \text{(read } K_6 \textit{ arrows } K_3, K_3\text{).} \tag{3.1}$$

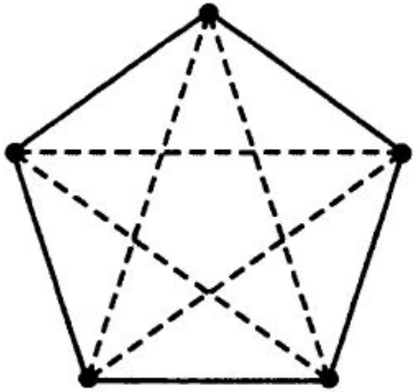


We distinguish between acquainted pairs and unacquainted pairs by coloring edges **red** for acquainted and **blue** for unacquainted.
Three mutually acquainted people now means
" a K_3 each of whose edges is colored red: a red K_3."
Similarly, three mutually unacquainted people form a blue K_3.
We can now explain the expression

$K_6 \rightarrow K_3, K_3$ is the assertion that no matter how the edges of K_6 are colored with the colors red and blue, there is always a red K_3 or a blue K_3

Prove or disprove: $K_5 \longrightarrow K_3, K_3$

The assertion $K_5 \longrightarrow K_3, K_3$ is false. This is because there is some way to color the edges of K_5 without creating a red K_3 or a blue K_5.
This is shown in Figure below, where the edges of the pentagon (the solid edges) are the red edges and the edges of the inscribed pentagram (the dashed edges) are the blue edges.



We now state and prove Ramsey's theorem, although still not in its full generality.

Theorem 3.3.1 *If $m \geq 2$ and $n \geq 2$ are integers, then there is a positive integer p such that*

$$K_p \rightarrow K_m, K_n.$$

In words, Ramsey's theorem asserts that given m and n there is a positive integer p such that, if the edges of K_p are colored red or blue, then either there is a red K_m or there is a blue K_n. The existence of either a red K_m or a blue K_n is guaranteed, no matter how the edges of K_p are colored. If $K_p \rightarrow K_m, K_n$, then $K_q \rightarrow K_m, K_n$ for every integer $q \geq p$. The *Ramsey number* $r(m,n)$ is the smallest integer p such that $K_p \rightarrow K_m, K_n$. Thus *Ramsey's theorem asserts the existence of the number* $r(m,n)$. By interchanging the colors red and blue, we see that

$$r(m,n) = r(n,m).$$

The facts that $K_6 \rightarrow K_3, K_3$ and $K_5 \not\rightarrow K_3, K_3$ imply that

$$r(3,3) = 6.$$

Prove that r(2, n) = n.

In a similar way, we show that $r(m, 2) = m$. The numbers $r(2, n)$ and $r(m, 2)$ with $m, n \geq 2$ are the *trivial Ramsey numbers*.

$$
\begin{aligned}
&r(3,3) = 6,\\
&r(3,4) = r(4,3) = 9,\\
&r(3,5) = r(5,3) = 14,\\
&r(3,6) = r(6,3) = 18,\\
&r(3,7) = r(7,3) = 23,\\
&r(3,8) = r(8,3) = 28,\\
&r(3,9) = r(9,3) = 36,\\
&40 \leq r(3,10) = r(10,3) \leq 43,\\
&r(4,4) = 18,\\
&r(4,5) = r(5,4) = 25,\\
&35 \leq r(4,6) = r(6,4) \leq 41\\
&43 \leq r(5,5) \leq 49\\
&58 \leq r(5,6) = r(6,5) \leq 87\\
&102 \leq r(6,6) \leq 165.
\end{aligned}
$$

Notice that the fact that $r(3,10)$ lies between 40 and 43 implies that

$$K_{43} \to K_3, K_{10}$$

and

$$K_{39} \not\to K_3, K_{10}.$$

Thus, there is no way to color the edges of K_{43} without creating either a red K_3 or a blue K_{10}; there is a way to color the edges of K_{39} without creating either a red K_3 or a blue K_{10}, but neither of these conclusions is known to be true for $K_{40}, K_{41},$ and K_{42}.

Ramsey's theorem generalizes to any number of colors. We give a very brief introduction. If n_1, n_2, and n_3 are integers greater than or equal to 2, then there exists an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, K_{n_3}.$$

In words, if each of the edges of K_p is colored red, blue, or green, then either there is a red K_{n_1} or a blue K_{n_2} or a green K_{n_3}. The smallest integer p for which this assertion holds is the Ramsey number $r(n_1, n_2, n_3)$. The only nontrivial Ramsey number of this type that is known is

$$r(3,3,3) = 17.$$

Thus $K_{17} \to K_3, K_3, K_3$ but $K_{16} \not\to K_3, K_3, K_3$. The Ramsey numbers $r(n_1, n_2, \ldots, n_k)$ are defined in a similar way, and Ramsey's theorem in its full generality for pairs asserts that these numbers exist; that is, there is an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}.$$

Let G be a complete graph of order six and H be a path of length 4. Consider the edge coloring of G with two colors white and black. Prove or disprove: In every coloring of G, there is a white copy of H or a black copy of H.