Section 7.5

Using generating funct. solve the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2},\ n \geq 2,\ a_0 = 2,\ a_1 = 2. \quad — (1)$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \quad — (2) \Rightarrow g(x) - a_0 - a_1 x = \sum_{n=2} a_n x^n$$

(1) $\times\ x^n$ & take $\sum_{n=2}^{\infty}$

$$\sum_{n\geq 2} a_n x^n = 2 \sum_{n\geq 2} a_{n-1} x^n + 3 \sum_{n\geq 2} a_{n-2} x^n$$

$$g(x) - a_0 - a_1 x = 2x\left(g(x) - a_0\right) + 3x^2 g(x) \quad -(3)$$

$$\left(\sum_{n\geq 2} a_{n-1} x^n = x \sum_{n\geq 2} a_{n-1} x^{n-1} = x \sum_{n=1}^{\infty} a_n x^n\right.$$

$$= x\left(g(x) - a_0\right)$$

$$\left(\sum_{n\geq 2} a_{n-2} x^n = x^2 \sum_{n\geq 2} a_{n-2} x^{n-2} = x^2 \sum_{n=0}^{\infty} a_n x^n\right.$$

$$g(x)\left(1 - 2x - 3x^2\right) = a_0 + a_1 x - 2x a_0$$

$$a_0 = 2,\quad a_1 = 2$$

$$g(x) = \frac{2(x-1)}{(3x-1)(x+1)} = \frac{1}{x+1} + \frac{1}{1-3x}$$

Coeffn of x^n is

$$a_n = (-1)^n + 3^n$$

Q. $a_n = a_{n-1} + n, \quad a_0 = 1$

$$\sum_{n\geq 1} a_n x^n = \sum_{n\geq 1} a_{n-1} x^n + \sum_{n\geq 1} n x^n$$

$$g(x) = \sum_{n=0} a_n x^n$$

$$g(x) - a_0 = x \sum_{n\geq 1} a_{n-1} x^{n-1} + \sum_{n=0}^{\infty} n x^n$$

$$g(x) - 1 = x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n x^n$$

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \cdots$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x + 2x^2 + \cdots + n x^n + \cdots$$

$$g(x) - 1 = x\,g(x) + \frac{x}{(1-x)^2}$$

$$g(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

Coeffⁿ of x^n in $\frac{1}{1-x} = 1$

Coeffⁿ of x^n in $\frac{x}{(1-x)^3}$ = Coeffⁿ of x^{n-1} in $\frac{1}{(1-x)^3}$

$$= C\left((n-1)+3-1,\ n-1\right)$$

$$= C(n+1,\ n-1) = C(n+1, 2)$$

$$a_n = 1 + C(n+1, 2)$$

Q. Let $g_n(x)$ be a family of g.s.

$$g_n(x) = a_{n,0} + a_{n,1}x + \cdots + a_{n,n}x^n$$

satisfying $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$

with $a_{n,0} = a_{n,n} = 1$, $a_{n,k} = 0$, $k > n$

Using g.s. compute $a_{n,k}$.

$$\sum_{k=1}^{n} a_{n,k} x^k = \sum_{k=1} a_{n-1,k} x^k + \sum_{k=1} a_{n-1,k-1} x^k$$

$$g_n(x) - 1 = \underline{g_{n-1}(x) - 1} + x \underbrace{\sum_{h=0}^{n-1} a_{n-1,h} x^h}_{x\, g_{n-1}(x)}$$

$$g_n - 1 = g_{n-1} - 1 + x\, g_{n-1}$$

$$g_n(x) = (1+x)\, g_{n-1}(x) = (1+x)^n g_0(x)$$

$$g_0(x) = a_{0,0} = 1 \Rightarrow g_n(x) = (1+x)^n$$

$$a_{n,k} = C(n,k)$$

Find a recurrence relation for a_n, the number of ways to place parentheses to multiply the n numbers $k_1 \times k_2 \times \cdots \times k_n$. Hence solve it using generating function

$a_2 = 1$ $\quad (k_1 \times k_2)$

$a_3 = 2$ $\quad (k_1 \times k_2) \times k_3, \quad k_1 \times (k_2 \times k_3)$

$a_0 = 0, \quad a_1 = 1.$

$$(k_1 \times k_2 \times \cdots \times k_i) \times (k_{i+1} \times \cdots \times k_n)$$

$i = 1, 2, \ldots, n-1$

$a_i \qquad\qquad a_{n-i}$

$$a_n = \sum_{i=1}^{n-1} a_i a_{n-i}$$

$$= \underbrace{a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1}$$

coeff of x^n in $g(x)$ — coeff of x^n in $g(x) \times g(x)$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n \geq 2} (a_1 a_{n-1} + \cdots + a_{n-1} a_1) x^n$$

$$g(x) - x = (g(x))^2$$

$$g(x)^2 - g(x) + x = 0$$

$$g(x) = \frac{1}{2}\left(1 \pm \sqrt{1-4x}\right)$$

$$g(0) = 0 \Rightarrow g(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

Q1. $a_n = 8a_{n-1} + 10^{n-1}$, $a_0 = 1$, $n \geq 1$

$$g(x) = \frac{1 - 9x}{(1-8x)(1-10x)}$$

Q2. $\sum_{i=0}^{n} i^5$

$a_n = a_{n-1} + n^5$, $a_0 = 0$

$a_n = k$ (g.s. for hom)

$p(n) = p_0 + p_1 n + \cdots + p_5 n^5$ $\times$

$p(n) = p_0 n + p_1 n^2 + \cdots + p_5 n^6$

$a_n = 2a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

$a_n = a_{n-1} a_{n-2}$, $a_0 = a_1 = x$

$\log_x a_n = \log_x a_{n-1} + \log_x a_{n-2}$

$\downarrow$

b_n, $b_0 = \log_x a_0 = \log_x x = 1$, $b_1 = 1$