6.2 CALCULATING COEFFICIENTS OF GENERATING FUNCTIONS

Table 6.1 Polynomial Expansions

(1) $\dfrac{1-x^{m+1}}{1-x} = 1 + x + x^2 + \cdots + x^m$

(2) $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$

(3) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$

(4) $(1-x^m)^n = 1 - \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + (-1)^k\binom{n}{k}x^{km} + \cdots + (-1)^n\binom{n}{n}x^{nm}$

(5) $\dfrac{1}{(1-x)^n} = 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \cdots + \binom{r+n-1}{r}x^r + \cdots$

(6) If $h(x) = f(x)g(x)$, where $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots$, then

$$h(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots$$

$$+ (a_rb_0 + a_{r-1}b_1 + a_{r-2}b_2 + \cdots + a_0b_r)x^r + \cdots$$

$$(1 + x + x^2 + x^3 + \cdots)^n \qquad (7)$$

What is the coefficient of x^r in (7).

Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \cdots)^5$. What is the coefficient of x^r?

As a generating function, what does $(x^2+x^3+x^4+\cdots)^5$ represents.

It is the generating function a_r, for the number of ways to select r objects with repetition from five types with at least two of each type. It is same as first picking two objects in each type—one way—and then counting the ways to select the remaining $r-10$ objects:
C((r −10) + 5−1, (r −10)) ways.

In above example,

we algebraically picked out an x^2 from each factor for a total of x^{10} and then found the coefficient of x^{r-10} in $(1+x+x^2+\cdots)^5$, the generating function for selection with unrestricted repetition of $r-10$ from five types.

The standard algebraic technique of extracting the highest common power of x from each factor corresponds to the "trick" used to solve the associated selection problem.

Such correspondences are a major reason for using generating functions: the algebraic techniques automatically do the combinatorial reasoning for us.

Use generating functions to find the number of ways to collect \$15 from 20 distinct people if each of the first 19 people can give a dollar (or nothing) and the twentieth person can give either \$1 or \$5 (or nothing).

The answer in Example 2 could be obtained directly by breaking the collection problem into three cases depending on how much the twentieth person gives: \$0 or \$1 or \$5. In each case, the subproblem is counting the ways to pick a subset of the other 19 people to obtain the rest of the \$15. The generating function approach automatically breaks the problem into three cases and solves each, doing all the combinatorial reasoning for us.

Using generating function, find how many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any number can go into each of the other six boxes?

How many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any number can go into each of the other six boxes?

We first count all the ways to distribute without restriction the 25 balls into the seven boxes,
$C(25+7-1, 25)$ ways,
and then subtract the distributions that violate the first box constraint, that is, distributions with at least 11 balls in the first box,
$C((25-11)+7-1, (25-11))$ (first put 11 balls in the first box and then distribute the remaining balls arbitrarily).

Again, generating functions automatically performed this combinatorial reasoning.

How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

At this stage, it is not obvious to solve it using combinatorial techniques.

Using generating function, verify the binomial identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

In how many ways can we fill a bag with n fruits such that

- The number of apples must be even
- The number of bananas must be a multiple of 5
- There can be at most four oranges
- There can be at most one pear

This problem is again extremely difficult using combinatorial approaches.

As an example, there are 7 ways to form a bag with 6 fruits:

Apples	6	4	4	2	2	0	0
Bananas	0	0	0	0	0	5	5
Oranges	0	2	1	4	3	1	0
Pears	0	0	1	0	1	0	1