6.5 A SUMMATION METHOD

In this section we show how to construct an ordinary generating function $h(x)$ whose coefficient of x^r is some specified function $p(r)$ of r, such as r^2 or $C(r, 3)$.

How to construct generating function $g^*(x)$ with $a^*_r = ra_r$ from the following generating function:

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_r x^r + \cdots$$

The next question is from where to start, i.e., what could be initial g(x).

The natural answer is: When in doubt start with the unit coefficients $a_r = 1$ of the generating function

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$$

Build a generating function $h(x)$ with $a_r = 2r^2$.

Build a generating function $h(x)$ with $a_r = (r+1)r(r-1)$.

We could multiply $(r+1)r(r-1)$ out getting $a_r = r^3 - r$, obtain generating functions for r^3 and r as an Example 1, and then subtract one generating function from the other. It is easier, however, to start with $3!(1-x)^{-4}$, whose coefficient a_r equals

$$a_r = 3!\binom{r+4-1}{r} = 3!\frac{(r+3)!}{r!3!} = \frac{(r+3)!}{r!} = (r+3)(r+2)(r+1)$$

Theorem

If $h(x)$ is a generating function where a_r is the coefficient of x^r, then $h^*(x) = h(x)/(1-x)$ is a generating function of the sums of the a_rs. That is,

$$h^*(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + \left(\sum_{i=0}^{r} a_i\right)x^r + \cdots$$

Using generating function

Evaluate the sum $2 \times 1^2 + 2 \times 2^2 + 2 \times 3^2 + \cdots + 2n^2$.

Using generating function

Evaluate the sum $3 \times 2 \times 1 + 4 \times 3 \times 2 + \cdots + (n+1)n(n-1)$.