9.2: BURNSIDE'S THEOREM

We now develop a theory for counting the number of different (nonequivalent) 2-colorings of the square.
More generally, in a set T of colorings of the corners (or edges or faces) of some figure, we seek the number **N** of equivalence classes of **T** induced by a group **G** of symmetries of this figure.

Consider seatings of 4 people around a round table. It can be done in 4! Ways. If only cyclic rotations are allowed, compute the number of equivalence classes (cyclicly nonequivalent seatings)?

There are 4 cyclic rotations of the seatings, and each equivalence class consists of 4 seatings.
Thus, the number of equivalence classes (cyclicly nonequivalent seatings) is N = 4!/4 = 3!

1234, 1243, 1324, 1342, 1423, 1432.

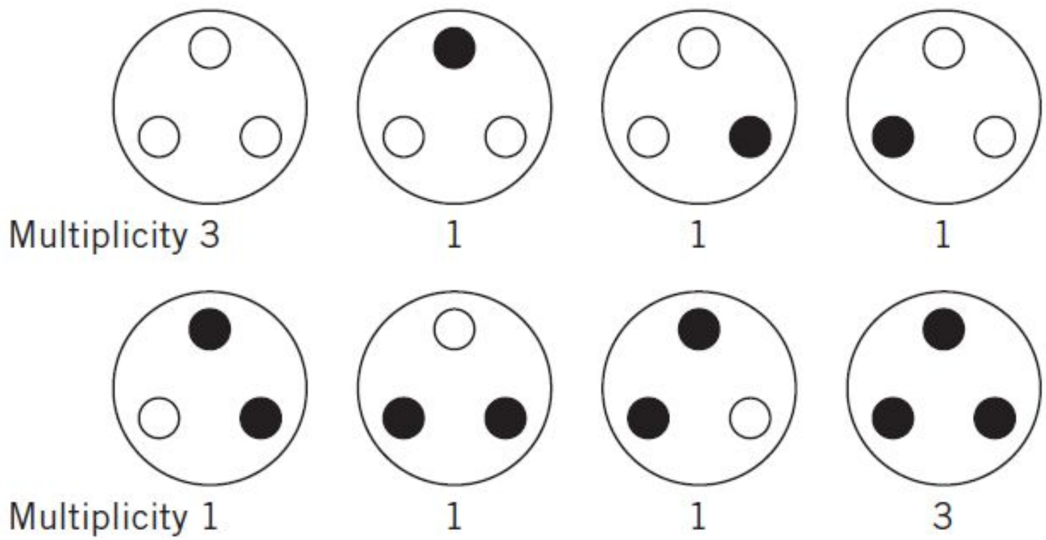
If every equivalence class is like this with s colorings, then

$$sN = c: \quad \text{(number of symmetries)} \times \text{(number of equivalence classes)}$$
$$= \text{(total number of colorings)}$$

Solving for N, we have $N = c/s$.

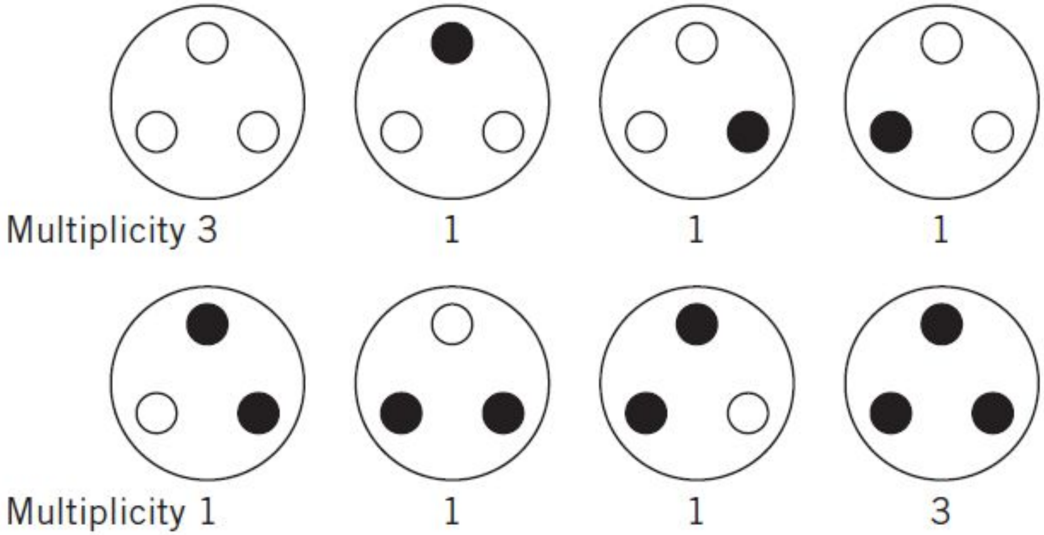
Suppose we have a small round table with three positions for chairs (each 120◦ apart), and white and black chairs are available. There are $2^3 = 8$ ways to place a white or black chair in each position. See Figure below.

Compute N?
Number of Symmetry (only rotation is allowed)



There are **three** cyclic rotations of the table possible, 0◦, 120◦, and 240◦. We have **c = 8** "colorings" and s = 3 symmetries, but the number of equivalence classes cannot be N = 8/3, a fraction!
(the answer must be 4, see figure below)

Here an arrangement of three black chairs (or three white chairs) forms an equivalence class **by itself**. I.E. Any rotation maps this arrangement of three black chairs into itself, that is, **leaves it fixed**.

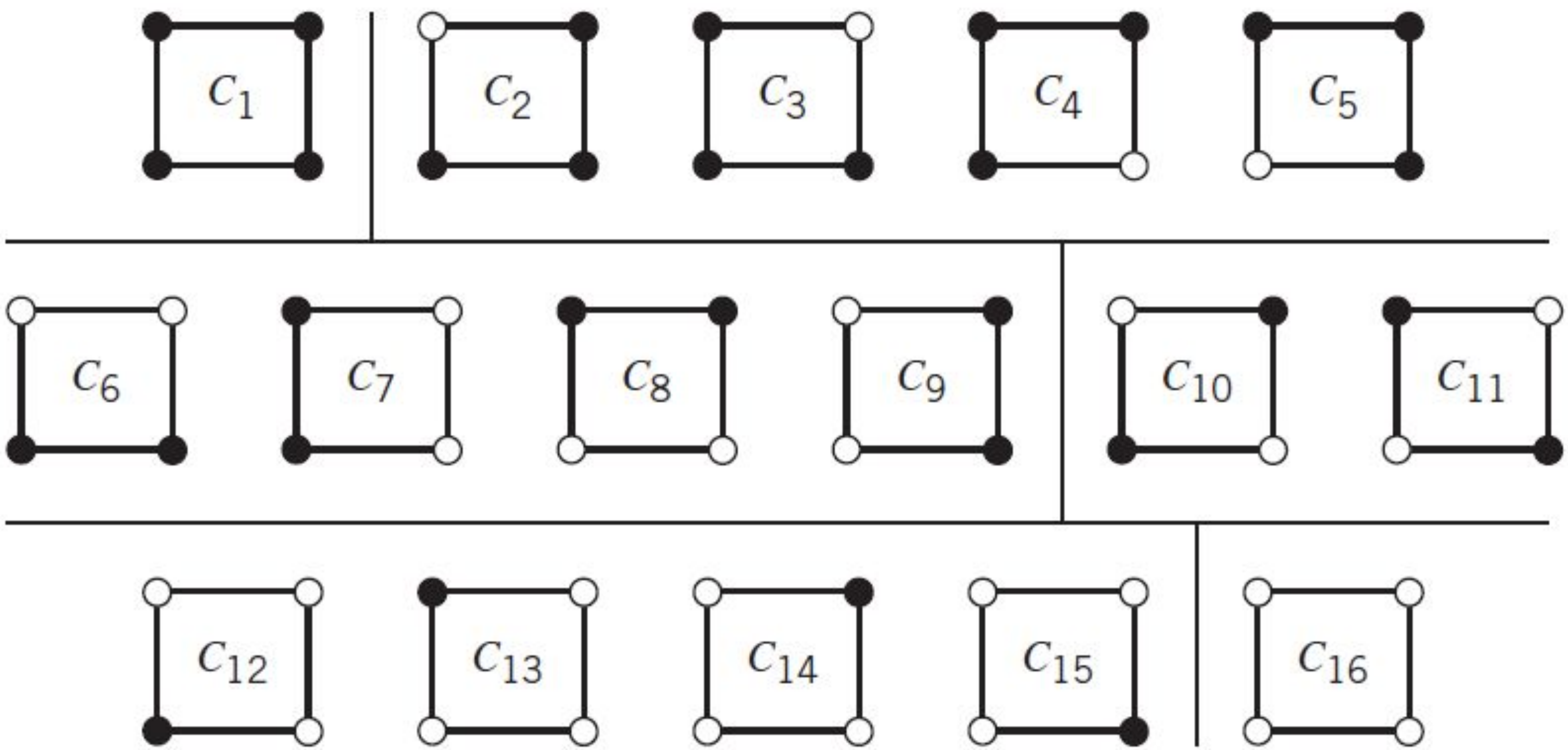


We need to correct the numerator in the formula N = c/s by adding the **multiplicities** of an arrangement.

Since two symmetries, along with the 0° symmetry, leave the all-black-chair arrangement fixed and similarly for the all-white, then counting arrangements in below with multiplicities we have the correct answer

N = (3+1+1+1+1+1+1+3)/3 = 12/3 = 4

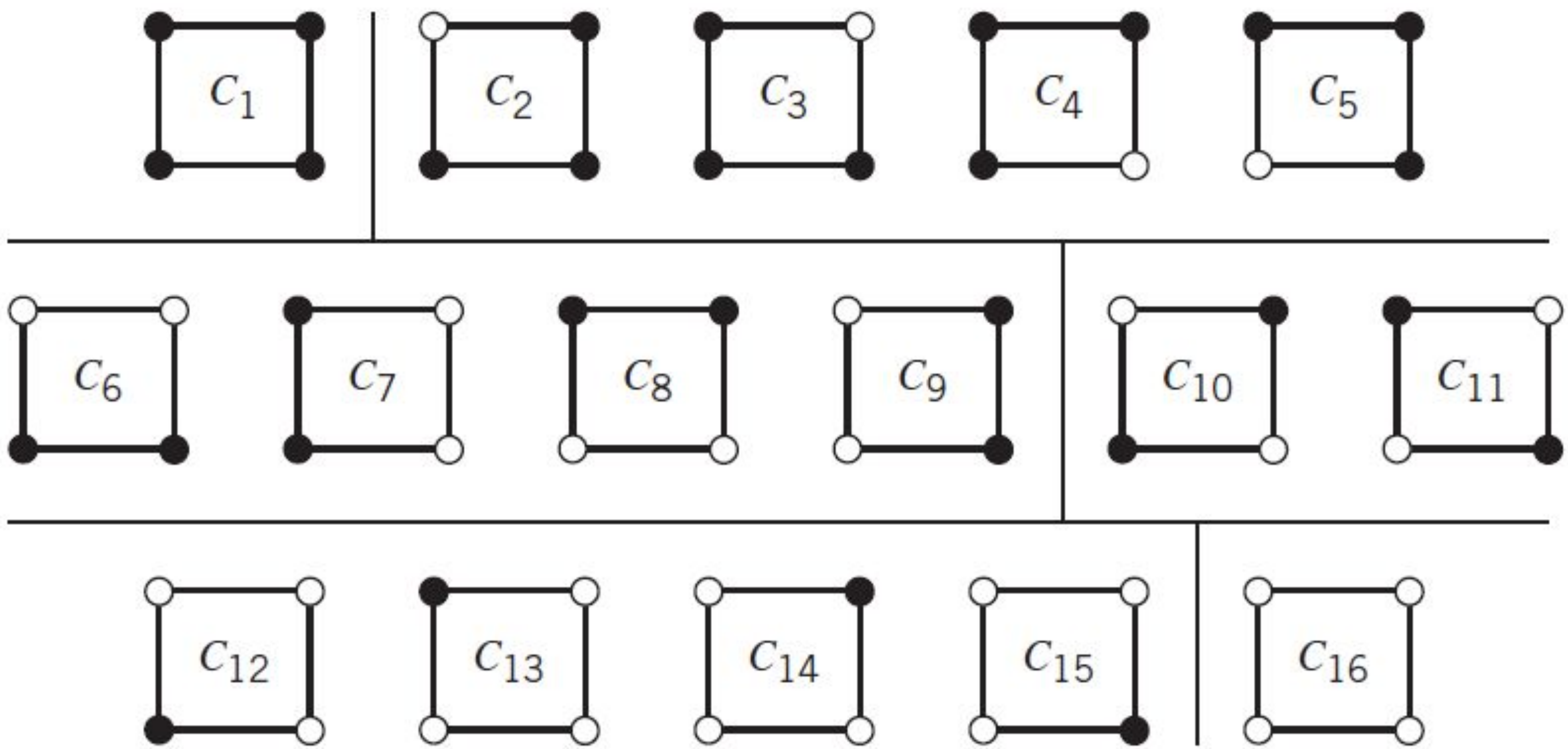
The "multiplicity" correction is even more complicated for 2-colorings of the square.



Observation 1: Several **π**s, besides the identity symmetry π_1, may leave a coloring C_i **fixed**—that is, $\pi(C_i) = C_i$. For example, the coloring C_{10} is fixed by symmetries $\pi_1, \pi_3, \pi_7, \pi_8$

Observation 2: If C_k is another coloring in C_i's equivalence class, there may be several πs all taking C_i to C_k.

C_{10} is mapped to C_{11} by symmetries $\pi_2, \pi_4, \pi_5, \pi_6$.

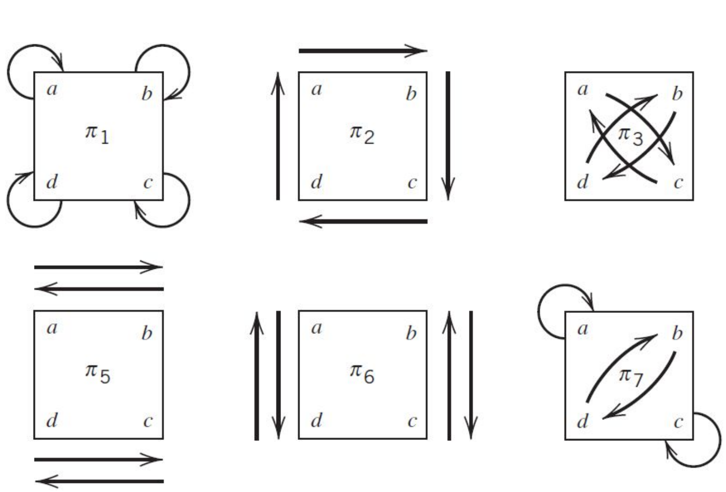


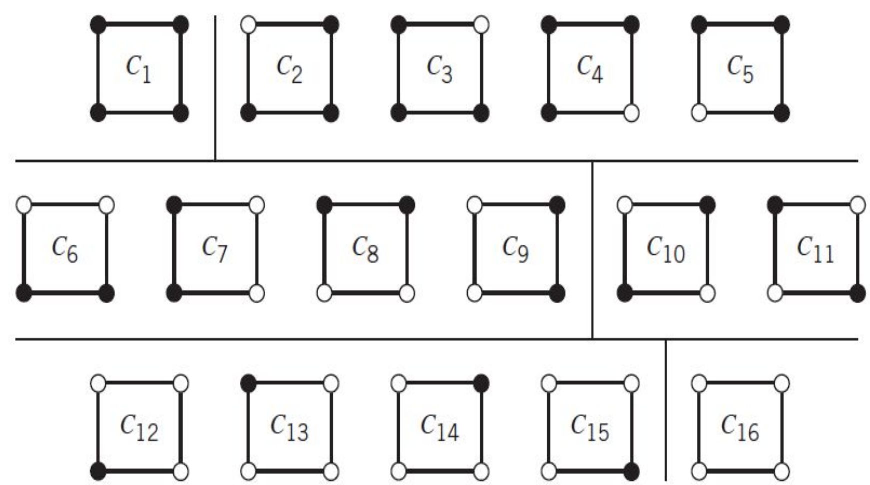
For any two permutations π_i, π_j in a group G, there exists a unique permutation $\pi_k = \pi_i^{-1} \cdot \pi_j$ in G such that $\pi_i \cdot \pi_k = \pi_j$.

By the lemma in Section 9.1, the symmetries $\pi_2, \pi_4, \pi_5, \pi_6$ taking C_{10} to C_{11} can be written in the form $\pi = \pi_2 \cdot \pi'$, where π' is a symmetry that leaves C_{11} fixed, or else π_2 followed by π' would not take C_{10} to C_{11}. For example,

$$\pi_5 = \pi_2 \cdot \pi_8 \colon \begin{pmatrix} a\,b\,c\,d \\ b\,a\,d\,c \end{pmatrix} = \begin{pmatrix} a\,b\,c\,d \\ b\,c\,d\,a \end{pmatrix} \cdot \begin{pmatrix} a\,b\,c\,d \\ c\,b\,a\,d \end{pmatrix}$$

Similarly $\pi_2 = \pi_2 \cdot \pi_1, \pi_4 = \pi_2 \cdot \pi_3, \pi_6 = \pi_2 \cdot \pi_7$. Conversely, given any π^* that leaves C_{11} fixed, $\pi_2 \cdot \pi^*$ takes C_{10} to C_{11} and so $\pi_2 \cdot \pi^*$ must be one of $\pi_2, \pi_4, \pi_5, \pi_6$. Thus there is a $1-1$ correspondence between the πs that take C_{10} to C_{11} and the πs that leave C_{11} fixed.





Therefore, to count the colorings in an equivalence class E with appropriate multiplicities (i.e., coloring C_{11} has multiplicity 4 since four different πs take C_{10} to C_{11}), it suffices to sum over the colorings in E the **number of πs that leave each coloring fixed**.

In the case of the equivalence class consisting of C_{10} and C_{11}, each of C_{10} and C_{11} have multiplicity 4, so that the size of their equivalence class including multiplicities is 4 + 4 = 8 (= s, the number of symmetries), as required.

In general, when multiplicities are counted, each equivalence class E will have **s** elements.

If $\phi(x)$ denotes the number of πs that leave the coloring x fixed, then $\sum_{x \in E} \phi(x) = s$.

Theorem **(Burnside, 1897)**

Let G be a group of permutations of the set S (corners of a square). Let T be any collection of colorings of S (2-colorings of the corners) that is closed under G. Then the number N of equivalence classes is

$$N = \frac{1}{|G|} \sum_{x \in T} \phi(x)$$

or

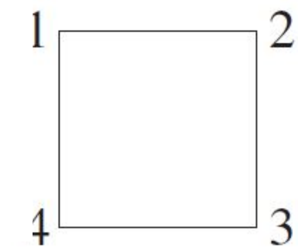
$$N = \frac{1}{|G|} \sum_{\pi \in G} \Psi(\pi) \tag{*}$$

where $|G|$ is the number of permutations and $\Psi(\pi)$ is the number of colorings in T left fixed by π.

Determine the number of ways in which the four corners of a square can be colored with two colors. (It is permissible to use a single color on all four corners.)

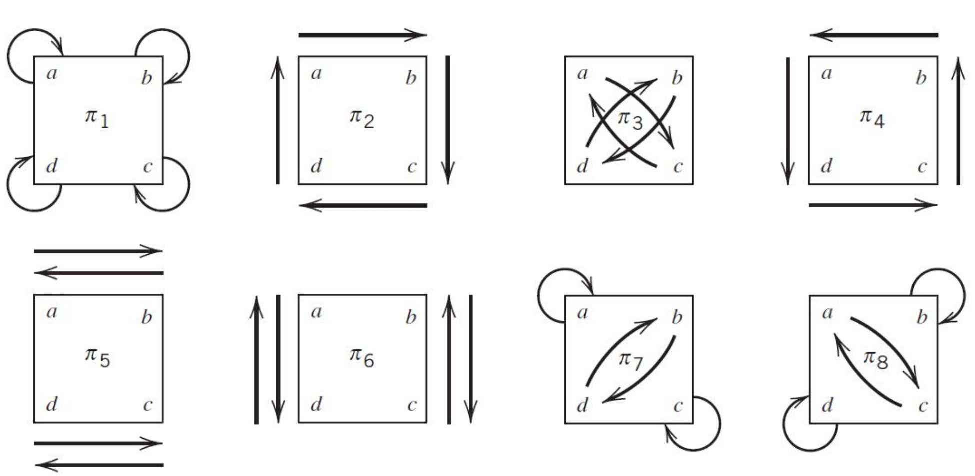
Let S be the set of all colorings. Clearly, $|S| = 2^4 = 16$.

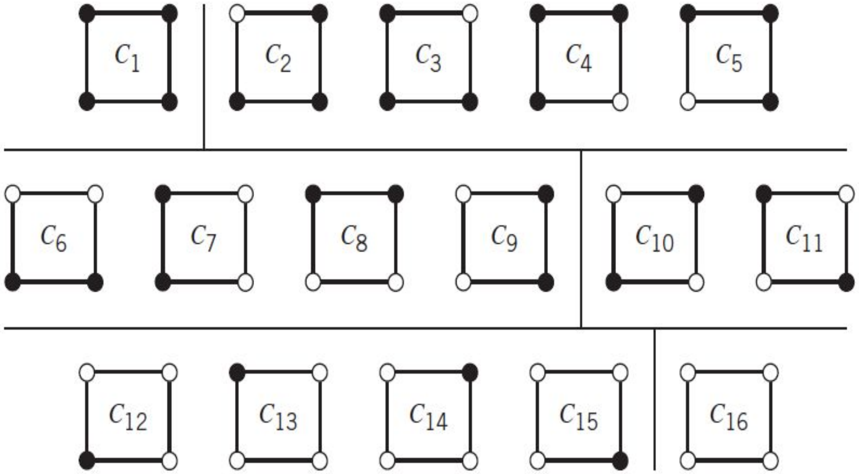
$$\rho = (1234); \quad \rho^2 = (13)(24); \quad \rho^3 = (1432); \quad \rho^4 = e = (1)(2)(3)(4)$$



We will use α, β, γ, and δ to represent the listed reflections.

$$\alpha = (24); \quad \beta = (13); \quad \gamma = (12)(34); \quad \delta = (14)(23)$$





In case of π_1, $\psi(0^\circ)$ = ?.

$\psi(0^\circ) = 16$.

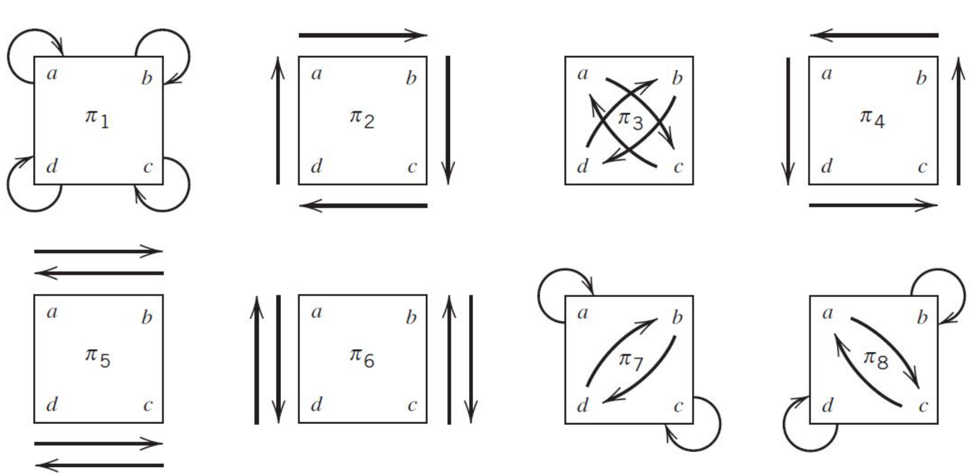
In case of π_2, $\psi(90^\circ)$ = ?

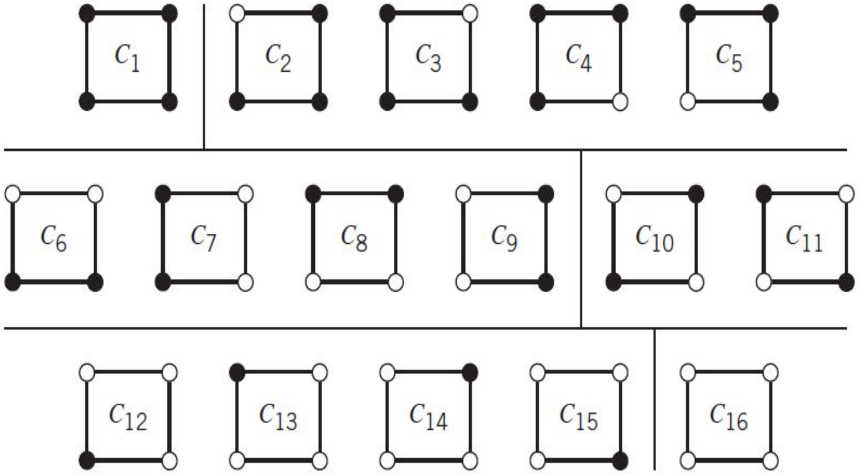
A coloring with only one color is fixed under rotation, therefore, $\psi(90^\circ) = 2$, since there are two colorings.

Also, $\psi(90^\circ) = \psi(270^\circ)$.

In case of π_3, two opposite vertices are moved to each other.

So the coloring with the same color for two non-adjacent vertices are fixed. $\psi(180^\circ) = 4$.





Because of a similar reason, π_5 the horizontal flip (14)(23) or π_6 vertical flip (12)(34) has 4 fixed points set.

In case of π_7 (13)(2)(4), 1 and 3 must have same colors, i.e., 2 possibilities and for 2, 4, each has 2 possibilities. In total 8 possibilities.

By Burnside's theorem, (16 + 2 + 4 + 2 + 4 + 4 + 8 + 8) / |G| = 48 / 8 = 6, i.e., there are 6 ways to color.

(*a*) (*b*) (*c*)

A baton is painted with equal-sized cylindrical bands. Each band can be painted black or white. If the baton is unoriented as when spun in the air, how many different 2-colorings of the baton are possible if the baton has **(a)** 2 bands? **(b)** 3 bands? **(c)** 4 bands?

First is to identify the number of symmetries.

Irrespective of the number of bands, there are two symmetries of a baton: π_1 is a 0° revolution of the baton—π_1 is the identity symmetry—and π_2 is a 180° revolution of the baton.

$[\Psi(\pi_1) + \Psi(\pi_2)]$

(a) For the 2-band baton, the set of 2-colorings left fixed by π_1 is all 2-colorings of the baton. There are $2^2 = 4$ 2-colorings, and so $\Psi(\pi_1) = 4$. The set of 2-colorings left fixed by π_2 consists of the all-black and all-white coloring, and so $\Psi(\pi_2) = 2$. By Burnside's theorem, the number of different colorings is $\frac{1}{2}[\Psi(\pi_1) + \Psi(\pi_2)] = \frac{1}{2}(4 + 2) = 3$.

(1, 1), (2, 2), (1, 2)

(b) For the 3-band baton, all 2^3 2-colorings are left fixed by π_1, and so $\Psi(\pi_1) = 2^3 = 8$. The set of 2-colorings left fixed by π_2 can have any color in the middle band (band 2) and a common color in the two end bands, and so $\Psi(\pi_2) = 2 \times 2 = 4$. The number of different colorings is $\frac{1}{2}[\Psi(\pi_1) + \Psi(\pi_2)] = \frac{1}{2}(8+4) = 6$.

(1, 2, 3), (2, 1, 3), (1, 3, 2), (1, 1, 1), (2, 2, 2), (3, 3, 3).

(c) For the 4-band baton, all 2^4 2-colorings are left fixed by π_1, and so $\Psi(\pi_1) = 2^4 = 16$. The set of 2-colorings left fixed by π_2 have a common color for the end bands and a common color for the inner bands, so $\Psi(\pi_2) = 2 \times 2 = 4$. The number of different colorings is $\frac{1}{2}[\Psi(\pi_1) + \Psi(\pi_2)] = \frac{1}{2}(16+4) = 10$. ■

How many different 3-colorings of the bands of an n-band baton are there if the baton is unoriented.

The symmetries of the baton are a 0◦ revolution and a 180◦ revolution. There are 3^n colorings of the fixed baton and so
$\psi(0◦) = 3^n$.
The number of colorings left fixed by a 180◦ spin depends on whether n is **even or odd**.

If n is even, each of the $n/2$ bands on one half of the baton can be any color—$3^{n/2}$ choices—and then for the coloring to be fixed by a 180° spin, each of the symmetrically opposite bands must be the corresponding color. So $\Psi(180°) = 3^{n/2}$ and we have from formula (*): $N = \frac{1}{2}(3^n + 3^{n/2})$.

To enumerate batons left fixed by a 180° spin when n is odd, we can use any color for the "odd" band in the middle of the baton—three choices. Each of the $(n-1)/2$ bands on one side of the middle band can be any color—$3^{(n-1)/2}$ choices—and again the other $(n-1)/2$ bands must be colored symmetrically. So $\Psi((180°) = 3 \times 3^{(n-1)/2} = 3^{(n+1)/2}$ and $N = \frac{1}{2}(3^n + 3^{(n+1)/2})$. ■

Suppose a necklace can be made from beads of three colors—black, white, and red. How many different necklaces with 3 beads are there?
(beads are allowed to move freely about the circle but flips are not allowed)

Rotations?

There are $3^3 = 27$ 3-colorings of a 3-bead necklace, and three rotations of 0°, 120°, 240°. The 0° rotation leaves all colorings fixed, and so $\Psi(0^\circ) = 27$. The 120° rotation cannot fix colorings in which some color occurs at only one corner. It follows that the 120° rotation fixes just the monochromatic colorings. Thus, $\Psi(120^\circ) = 3$. The 240° rotation is a reverse 120° rotation, and so $\Psi(240^\circ) = 3$. By formula (*), we have

$$N = \frac{1}{3}(27 + 3 + 3) = 11 \quad \blacksquare$$