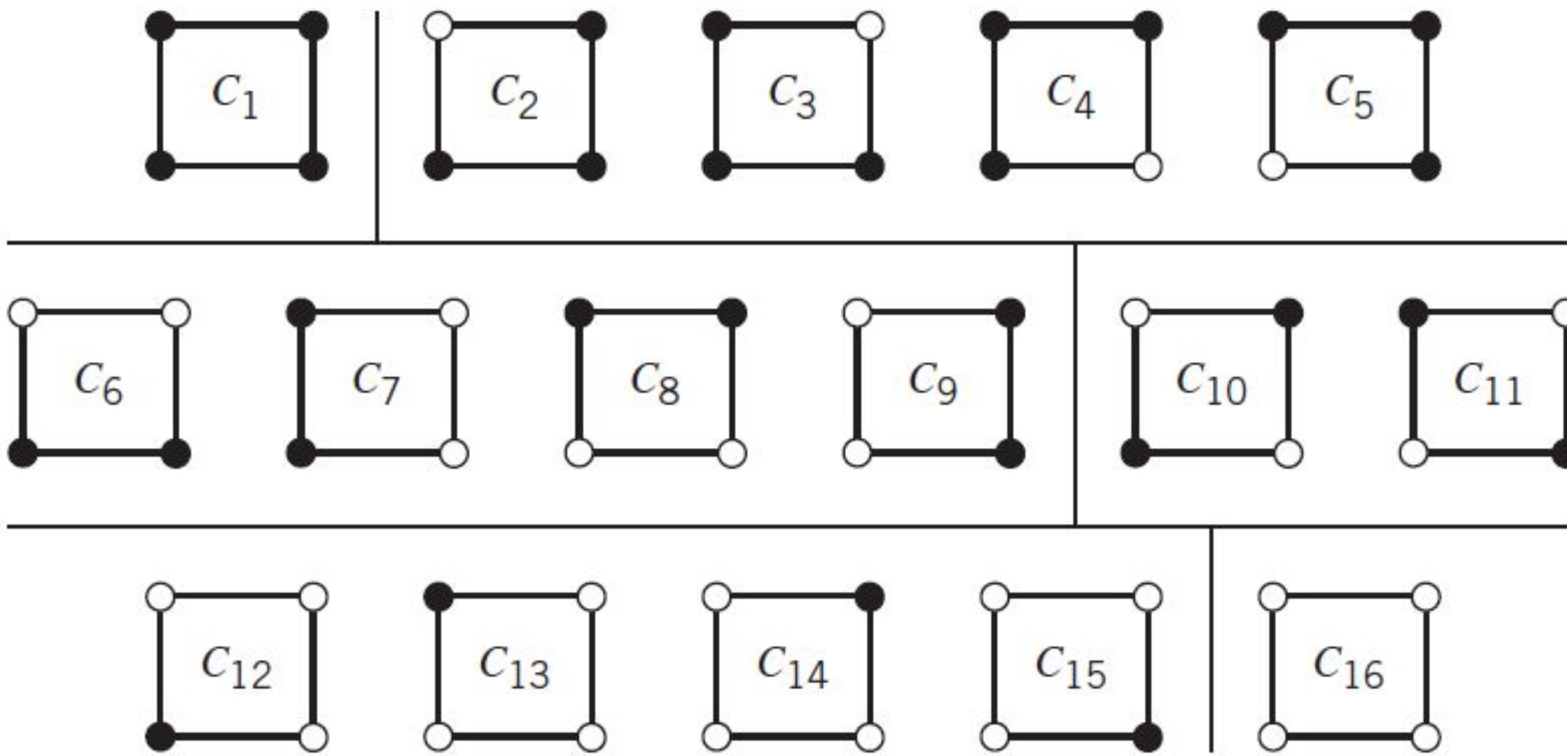
POLYA'S ENUMERATION FORMULA

EQUIVALENCE AND SYMMETRY GROUPS

Consider a fixed square. Color its corners with black and white colors only. How many arrangements are possible, considering each corner to be distinct.

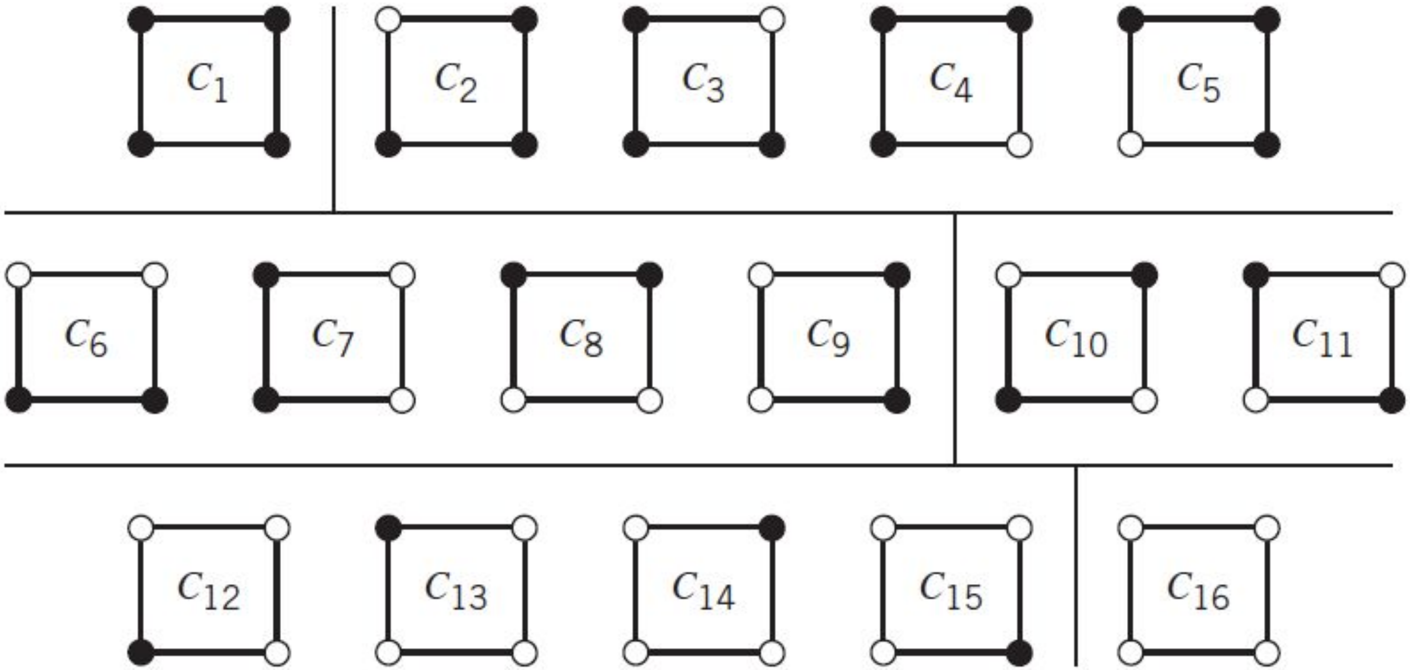
$2^4 = 16$ black–white colorings are there for a fixed square.



We can partition these 16 colorings into subsets of colorings that are equivalent when the square is floating.

How many such subsets are there?

There are six such subsets (see the groupings of colored squares shown in Figure below), and so there are **six** different 2-colorings of the floating square.



We seek a theory and formula to explain why there are six such distinct 2-colorings of the square. Note that the six subsets of equivalent colorings vary in size.

To define the partition of a set into subsets of equivalent elements, we first define the general concept of the equivalence of two elements a and b. We write this equivalence as $a \sim b$. The fundamental properties of an **equivalence relation** are

(i) Transitivity: $a \sim b, b \sim c \Rightarrow a \sim c$

(ii) Reflexivity: $a \sim a$

(iii) Symmetry: $a \sim b \Rightarrow b \sim a$

All other properties of equivalence can be derived from these three. Any binary relation with these three properties is called an equivalence relation. Such a relation defines a partition into subsets of mutually equivalent elements called **equivalence classes.**

Consider a set of numbers, differing by an even number. Is it an equivalence relation? What are corresponding equivalence classes.

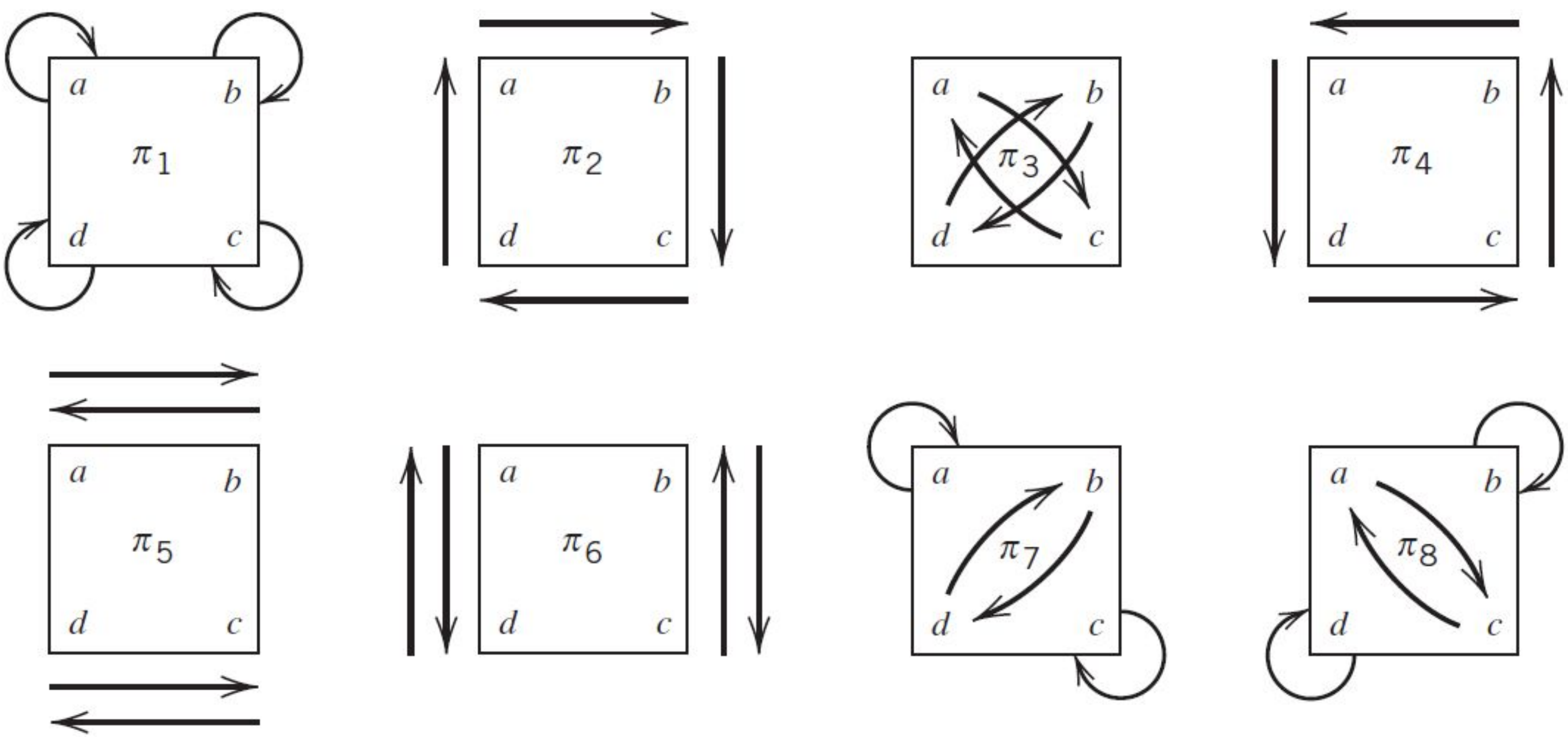
The even numbers form one equivalence class and the odd numbers the other class.

How many **symmetries** does square has?

In a square, the symmetries are because of **rotations and reflections**.

The rotations are $\boldsymbol{\pi_2} = 90^\circ$ rotation, $\boldsymbol{\pi_3} = 180^\circ$ rotation, $\boldsymbol{\pi_4} = 270^\circ$ rotation, and $\boldsymbol{\pi_1} = 360^\circ$ (or 0°) rotation (the rotations are about the center of the square).

The reflections are $\boldsymbol{\pi_5}$ = reflection about the vertical axis, $\boldsymbol{\pi_6}$ = reflection about the horizontal axis, $\boldsymbol{\pi_7}$ = reflection about opposite corners a and c, and $\boldsymbol{\pi_8}$ = reflection about opposite corners b and d.



Can you compute the number of symmetries of an n-gon, for n even?

In a regular n-gon, the smallest rotation is $(360/n)^\circ$.
Any multiple of this $(360/n)^\circ$ rotation is again a rotation, and so there are **n rotations** in all.
There are two types of reflections for a regular even n-gon:
flipping about the middles of two opposite sides and flipping about two opposite corners.

Since there are $n/2$ pairs of opposite sides and $n/2$ pairs of opposite corners, a regular even n-gon will have $n/2+n/2 = n$ reflections.

Summing rotations and reflections, we find that a regular even n-gon has **2n symmetries**.

Describe the symmetries of a pentagon, and more generally, of an n-gon for odd n.

As noted in Example above, any regular n-gon has n rotational symmetries.
A pentagon will have five rotational symmetries of 0°, 72°, 144°, 216°, and 288°.
However, the reflections about opposite sides or opposite corners do not exist in the pentagon.
Instead, we reflect about an axis of symmetry running from one corner to the middle of an opposite side.
There are five such reflections, for a total of 10 symmetries.

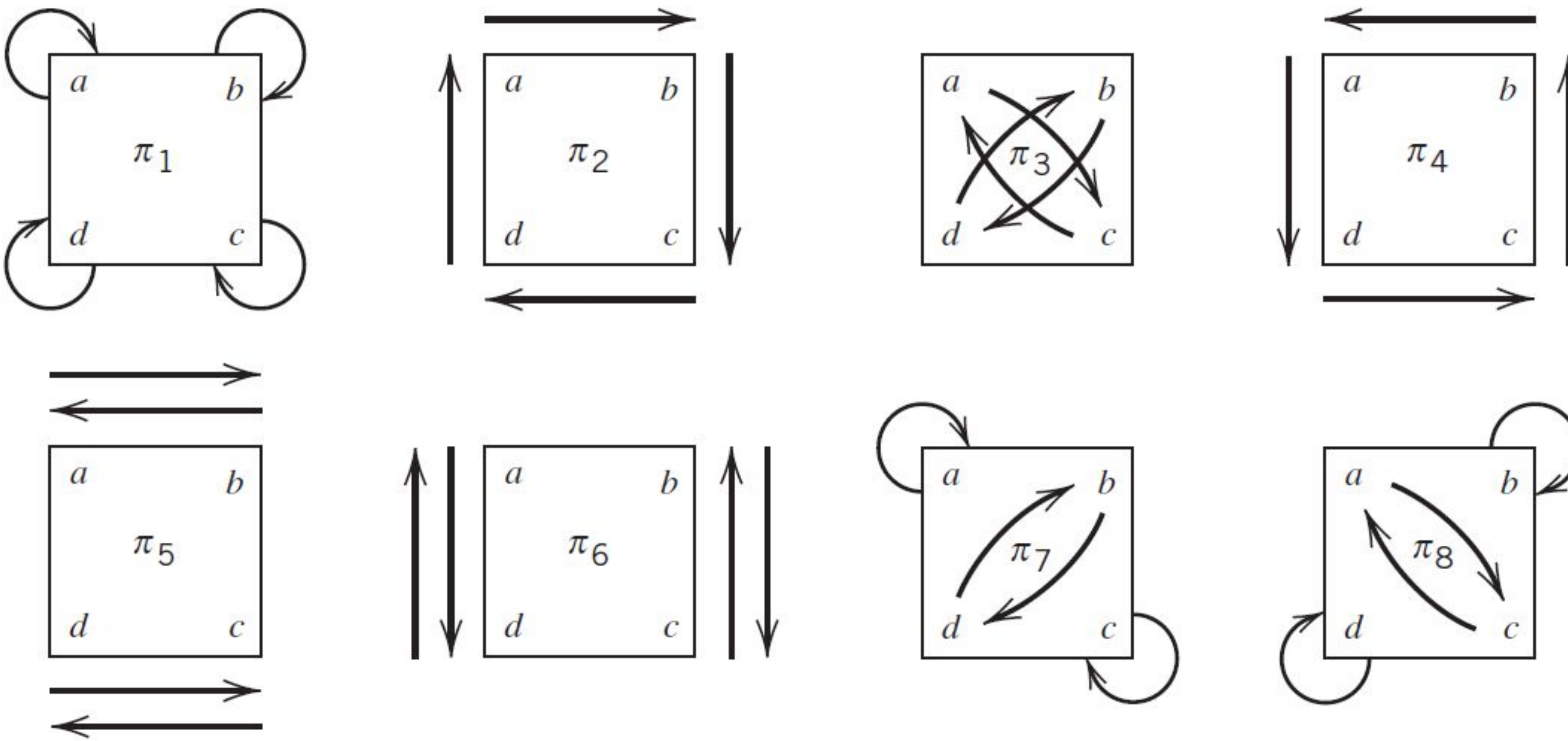
In a regular odd n-gon, there are n such flips, along with n rotations. Summing rotations and reflections, we find that a regular odd n-gon also has **2n symmetries**.

The symmetries of a square are naturally characterized by the way they permute the corners of the square. Thus, the 180◦ rotation π_3 (see Figure below) can be described as the corner permutation:

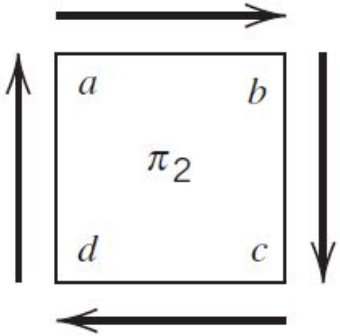
$a \longrightarrow c$, $b \longrightarrow d$, $c \longrightarrow a$, $d \longrightarrow b$;

in tabular form, we write

$$\begin{bmatrix} a & b & c & d \\ c & d & a & b \end{bmatrix}$$



Similarly, the 90◦ rotation π_2 can be described as
a ⟶ b, b ⟶ c, c ⟶ d, d ⟶ a, or a ⟶b ⟶ c ⟶ d ⟶ a.

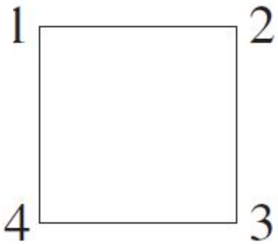


A permutation of the form $x_1 \to x_2 \to x_3 \cdots \to x_n \to x_1$ is called a cyclic permutation or **cycle.** Thus, π_2 is a cycle of length 4. Cycles are usually written in the form $(x_1 x_2 x_3 \ldots x_n x_1)$. So $\pi_2 = (abcd)$. Any permutation can be expressed as a product of disjoint cycles (proof of this claim is an exercise). For example, $\pi_3 = (ac)(bd)$, $\pi_4 = (adcb)$, and $\pi_7 = (a)(bd)(c)$.

Find the group of permutations that describe the symmetries of the square shown below.

$$\rho = (1234); \quad \rho^2 = (13)(24); \quad \rho^3 = (1432); \quad \rho^4 = e = (1)(2)(3)(4)$$

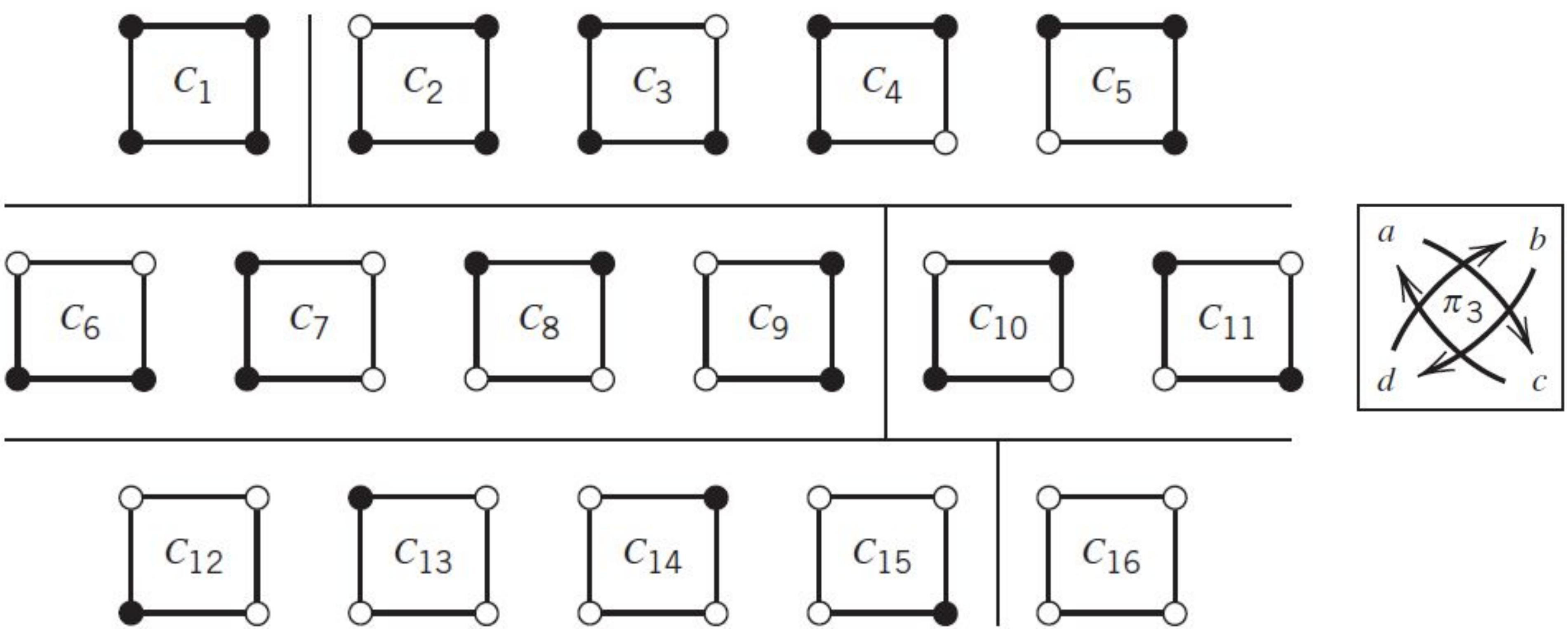
We will use α, β, γ, and δ to represent the listed reflections.



$$\alpha = (24); \quad \beta = (13); \quad \gamma = (12)(34); \quad \delta = (14)(23)$$

In permuting the corners, the symmetries create permutations of the colorings of the corners. For example, if C_i is the ith square in Figure 9.1, then π_3 is the following permutation of colorings:

$$\pi_3 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_4 & C_5 & C_2 & C_3 & C_8 & C_9 & C_6 & C_7 & C_{10} & C_{11} & C_{14} & C_{15} & C_{12} & C_{13} & C_{16} \end{pmatrix} \qquad (1)$$



The point is that while a symmetry π_i is easily visualized by how it moves the corners of the square, what we are really interested in is the way π_i takes one coloring into another (making them equivalent). Thus, we formally define our coloring equivalence as follows:

$$\text{Colorings C and } C' \text{ are } \textit{equivalent, } C \sim C', \text{ if there exists a symmetry } \pi_i \text{ such that } \pi_i(C) = C' \qquad (2)$$

The properties of the set G of symmetries that interest us are the ones that make the relation $C \sim C'$ in Eq. (2) an equivalence relation. These properties of G are (here $\pi_i \cdot \pi_j$ means *applying motion* π_i *followed by motion* π_j):

1. *Closure:* If $\pi_i, \pi_j \in G$, then $\pi_i \cdot \pi_j \in G$;
2. *Identity:* G contains an identity motion π_I such that $\pi_\mathrm{I} \cdot \pi_i = \pi_i$ and $\pi_i \cdot \pi_\mathrm{I} = \pi_i$;
3. *Inverses:* For each $\pi_i \in G$, there exists an inverse in G, denoted π_i^{-1}, such that $\pi_i^{-1} \cdot \pi_i = \pi_\mathrm{I}$ and $\pi_i \cdot \pi_i^{-1} = \pi_\mathrm{I}$;

1. *Closure:* If $\pi_i, \pi_j \in G$, then $\pi_i \cdot \pi_j \in G$; $\pi_2 \cdot \pi_5 = ?$

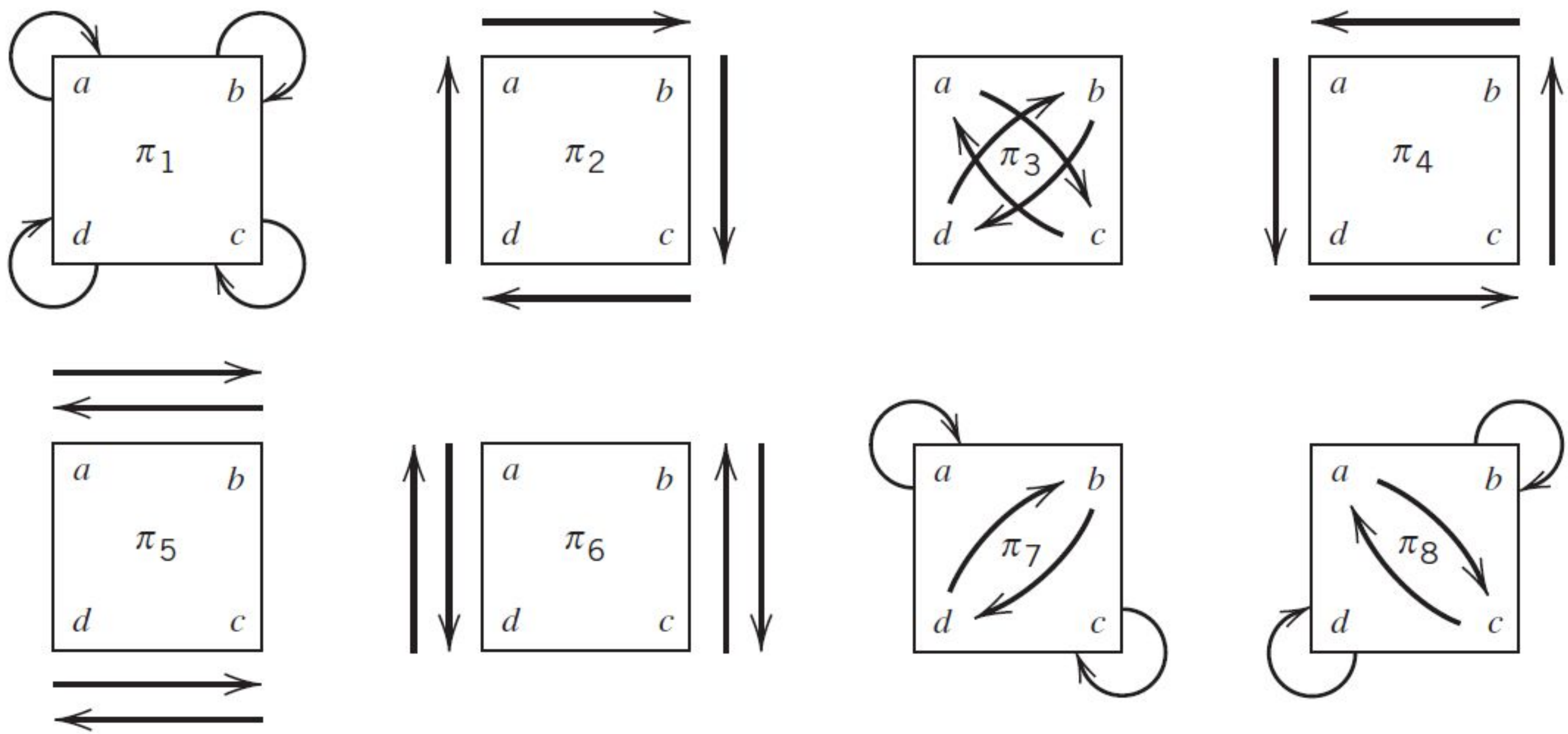
π_2 (a)(b)(c)(d) = (d)(a)(b)(c) and π_5 (d)(a)(b)(c) = (a)(d)(c)(b) = π_7

2. *Identity:* G contains an identity motion π_I such that $\pi_I \cdot \pi_i = \pi_i$ and $\pi_i \cdot \pi_I = \pi_i$;

π_I is π_1

3. *Inverses:* $\pi_2^{-1} =$ π_2 (a)(b)(c)(d) = (d)(a)(b)(c), π_4 (d)(a)(b)(c) = (a)(b)(c)(d) implies

$\pi_2^{-1} = \pi_4.$



Observe that closure makes our coloring relation $\sim$ satisfy transitivity

For suppose $C \sim C'$ and $C' \sim C''$. Since $C \sim C'$, there must exist $\pi_i \in G$ such that $\pi_i(C) = C'$. Similarly, there is a $\pi_j \in G$ such that $\pi_j(C') = C''$. Then by closure, there exists $\pi_k = \pi_i \cdot \pi_j \in G$ with $\pi_k(C) = (\pi_i \cdot \pi_j)(C) = C''$. Thus $C \sim C''$. Similarly, properties (2) and (3) of the symmetries imply that our coloring relation satisfies properties (ii) and (iii) of an equivalence relation, respectively

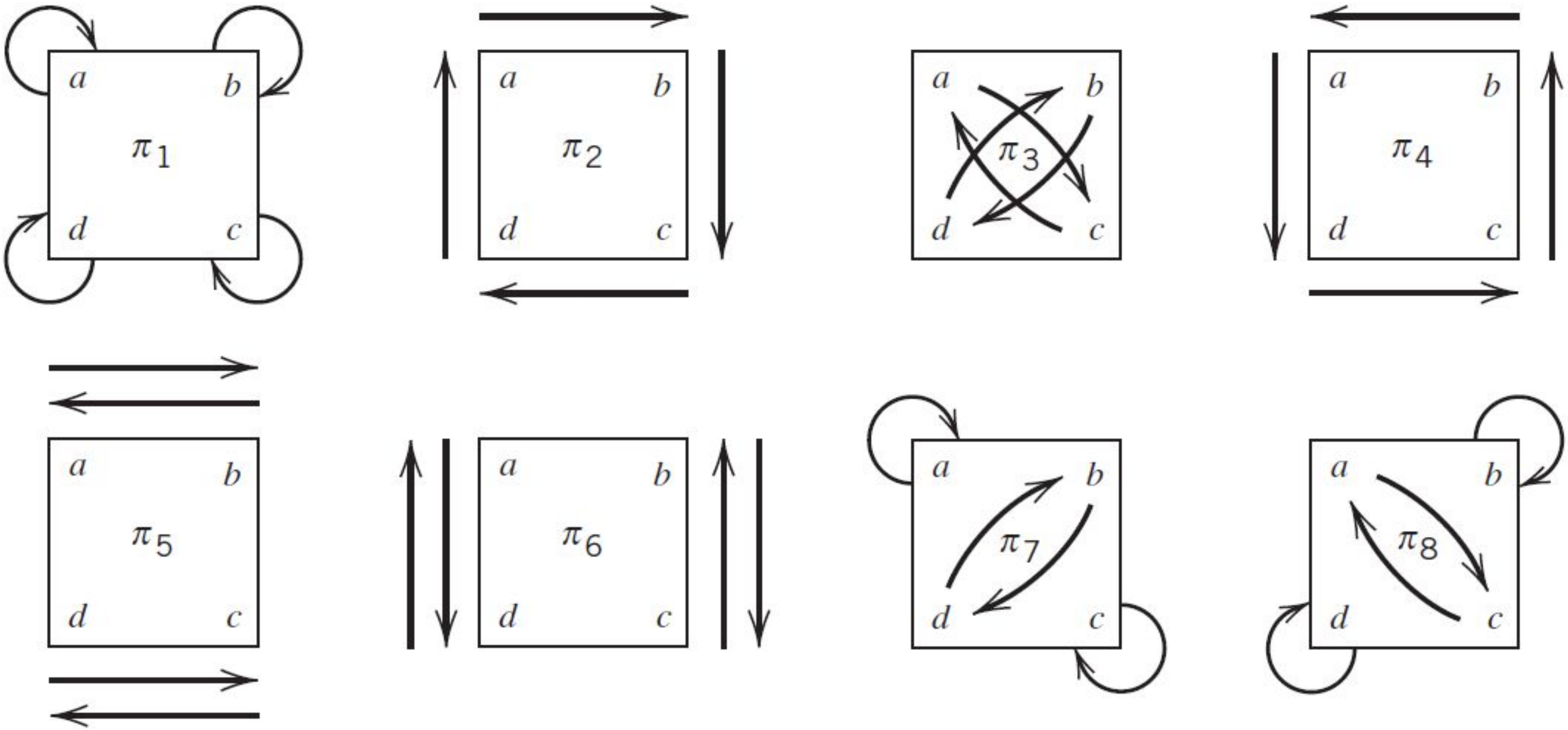
A collection G of mathematical objects with a binary operation is called a **group** if it satisfies properties 1, 2, and 3 along with the associativity property—$(\pi_i \cdot \pi_j) \cdot \pi_k = \pi_i \cdot (\pi_j \cdot \pi_k)$. Thus we have the following theorem.

Theorem

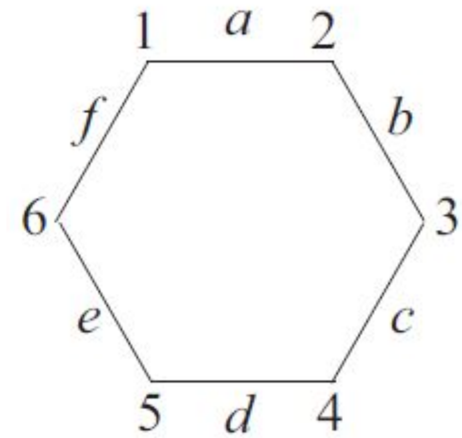
Let G be a group of permutations of the set S (corners of a square) and T be any collection of colorings of S (2-colorings of the corners). Then G induces a partition of T into equivalence classes with the relation $C \sim C' \Leftrightarrow$ some $\pi \in G$ takes C to C'.

Find two symmetries of the square π_i, π_j such that $\pi_i \cdot \pi_j \neq \pi_j \cdot \pi_i$ (this means the group of symmetries of a square is noncommutative).

$\pi_2 \cdot \pi_5$ is not equal to $\pi_5 \cdot \pi_2$



Find the group of permutations that describe the symmetries of the following figure



We have the rotation $\rho = (123456)$ representing a $60°$ clockwise rotation of the figure, and the powers $\rho^2 = (135)(246)$, $\rho^3 = (14)(25)(36)$, $\rho^4 = (153)(264)$, $\rho^4 = (165432)$, and $\rho^6 = e$. Now we list the six reflections; they are $\alpha = (12)(36)(45)$, $\beta = (13)(46)$, $\gamma = (23)(14)(56)$, $\delta = (15)(24)$, $\epsilon = (16)(25)(34)$, and $\phi = (26)(35)$.

Lemma

For any two permutations π_i, π_j in a group G, there exists a unique permutation $\pi_k = \pi_i^{-1} \cdot \pi_j$ in G such that $\pi_i \cdot \pi_k = \pi_j$.

Proof

First we show that $\pi_i \cdot \pi_k = \pi_j$. Since $\pi_k = \pi_i^{-1} \cdot \pi_j$,

$$\begin{aligned} \pi_i \cdot \pi_k = \pi_i \cdot \left(\pi_i^{-1} \cdot \pi_j\right) &= \left(\pi_i \cdot \pi_i^{-1}\right) \cdot \pi_j \quad \text{(by associativity)} \\ &= \pi_{\mathrm{I}} \cdot \pi_j = \pi_j \end{aligned}$$

as claimed. Next we show that π_k is unique. Suppose there also exists a permutation π_k' such that $\pi_i \cdot \pi_k' = \pi_j$. Then $\pi_i \cdot \pi_k = \pi_i \cdot \pi_k'$. Multiplying the equation by π_i^{-1}, we have

$$\begin{aligned} \pi_i^{-1} \cdot (\pi_i \cdot \pi_k) = \pi_i^{-1} \cdot (\pi_i \cdot \pi_k') &\Rightarrow (\pi_i^{-1} \cdot \pi_i) \cdot \pi_k = \left(\pi_i^{-1} \cdot \pi_i\right) \cdot \pi_k' \\ &\Rightarrow \pi_{\mathrm{I}} \cdot \pi_k = \pi_{\mathrm{I}} \cdot \pi_k' \Rightarrow \pi_k = \pi_k' \quad \blacklozenge \end{aligned}$$