7.1 RECURRENCE RELATION MODELS

Recurrence Relation

A recurrence relation for a sequence $A = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ is a formula that relates a_n to one or more of the preceding terms $a_0, a_1, \ldots, a_{n-1}$ in a uniform way, for any integer n greater than or equal to some initial integer k. The values of the first terms needed to start computing with a recurrence relation are called the initial conditions.

Find a recurrence relation for the number of ways to arrange n distinct objects in a row. Find the number of arrangements of eight objects.

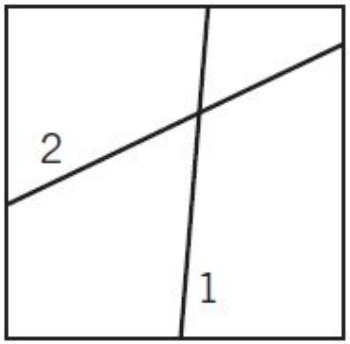
An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation for a_n, the number of different ways for the elf to ascend the n-stair staircase.

Find a recurrence relation for the number of different ways to hand out a piece of chewing gum (worth 1\$) or a candy bar (worth 10\$) or a doughnut (worth 20\$) on successive days until n\$ worth of food has been given away.

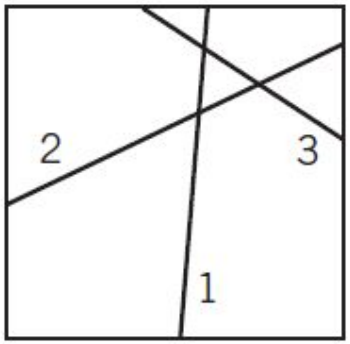
Suppose we drawn straight lines on a piece of paper so that every pair of lines intersect (but no three lines intersect at a common point). Into how many regions do these n lines divide the plane?

With one line, the paper is divided into two regions.

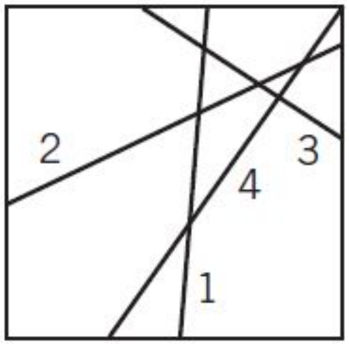
With two lines, we get four regions—that is, $a_2 = 4$. $\mathbf{a_3 = ?}$ $\mathbf{a_3 = a_2 + ?}$



(a)



(b)



(c)

Now $a_3 = 7$ (why).

3rd line cross each of the other two lines (at different points) and cuts through three of the regions formed by the first two lines, which form three new regions. Thus

Thus $a_3 = a_2 + 3 = 4 + 3 = 7$.

Let a_n be the number of regions into which the plane is divided by n lines. It means n – 1 lines are dividing the plane into a_{n-1} regions.
When n^{th} line is drawn, it intersects each of the previous lines and divide n of the previous regions into half, creating n additional regions. Thus,
$\mathbf{a_n = a_{n-1} + n}$, $n > 1$.

Find a recurrence relation for the number of binary sequences of length n with no consecutive 0's.

Find a recurrence relation for the number of binary sequences of length n with no block of three consecutive 0's.

Let a_n be the number of binary sequences of length n that contain no block of three consecutive 0's. A sequence of length n with no three consecutive 0's may start with (look for all possibilities):

: 1 and be followed by any sequence of length n – 1 that contains no three consecutive 0's,

: or it may start with a 01 and be followed by any sequence of length n – 2 that contains no three consecutive 0's,

:or it may start with a 001 and be followed by any sequence of length n – 3 that contains no three consecutive 0's.

In this case, we obtain the recurrence relation

$\mathbf{a_n = a_{n-1} + a_{n-2} + a_{n-3}}$

satisfying the initial conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.

A single coin is flipped n times. Each outcome is represented by a sequence of n digits; each digit is an H or a T. Find a recurrence relation for the number of outcomes with at least two consecutive heads.

Find a recurrence relation for a_n, the number of n-digit ternary sequences without any occurrence of the subsequence “012.”

A ternary sequence is a sequence composed of 0s, 1s, and 2s.

The recurrence relations we discussed involved only one variable and a_n can be easily computed using **recursive backward substitution** or **Mathematical Induction.**

Using recursive backward substitution, the relation $a_n = a_{n-1} + n$ becomes $a_n = (a_{n-2} + n - 1) + n = \cdots = 1+2+3 + \cdot \cdot \cdot + n - 1 + n$.

Using, Mathematical Induction solve the recurrence relation

$a_n = 2a_{n-1} + 1$, $a_1 = 1$.