Counting the number of Non-Isomorphic Graphs

Count the number of graphs with n vertices.

There are **C(n, 2)** possible edges, each of which can be included or excluded. Thus, there are $2^{C(n, 2)}$ labeled graphs on n vertices.

A more difficult problem arises when we want to start counting **non-isomorphic graphs** on n vertices.
Since any vertex can be mapped to any other vertex, the symmetric group S_4 acts on the vertices.

To account for edges, we move from the symmetric group S_4 to its **pair group** S_4^2 (we need non-isomorphic graphs when n = 4 which depends on the edges between the 4 vertices). The objects that S_4^2 permutes are the 2-element subsets of {1, 2, 3, 4}.

Consider n = 4 and S_4. First we will review the permutations in S_4 and their cycle structures as follows.

1. **The permutation (1)(2)(3)(4) has four cycles of length one**, giving the term s_1^4.
2. **The permutation (1)(2)(34) has structure $s_1^2 s_2$.**
   There are six of these permutations, for each of the six possible pairs of vertices: (12),(13),(14),(23),(24), and (34).
3. **The permutation (1)(234) has structure $s_1 s_3$.**
   There are eight possible differing cycles of length three, and thus eight permutations with this structure. The other seven cycles of length three are (123),(132),(134),(143),(124),(142), and (243).
4. **The permutations (12)(34),(13)(24), and (14)(23)** have structure s_2^2.
5. **The permutation (1234) has structure s_4.**
   The other five with this structure include (1243), (1423), (1342),(1324), (1432).

Thus

$$Z(S_4) = \frac{1}{24}(s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 + 6s_4).$$

To evaluate permutations in $S_4^{(2)}$ we will switch from permuting vertices to permuting *pairs* of vertices, since we are trying to eventually count edges. For notation, we will let $\overline{ij}$ be the pair of vertices i and j, and $\overline{ij} = \overline{ji}$.

1. If our term in S_4 is s_1^4, the corresponding term in $S_4^{(2)}$ is s_1^6. The new permutation is $(\overline{12})(\overline{13})(\overline{14})(\overline{23})(\overline{24})(\overline{34})$.

2. If our term in S_4 is $s_1^2 s_2$, the corresponding term in $S_4^{(2)}$ is $s_1^2 s_2^2$. The permutation $(1)(2)(34)$ would become $(\overline{12})\ (\overline{34})\ (\overline{14}\ \overline{13})\ (\overline{23}\ \overline{24})$.

3. If our term in S_4 is $s_1 s_3$, the corresponding term in $S_4^{(2)}$ is s_3^2. The permutation (1)(243) would become $(\overline{12}\ \overline{14}\ \overline{13})(\overline{23}\ \overline{24}\ \overline{34})$.

4. If our term in S_4 is s_2^2, the corresponding term in $S_4^{(2)}$ is $s_1^2 s_2^2$. The permutation (12)(34) would become $(\overline{12})(\overline{34})(\overline{24}\ \overline{13})(\overline{23}\ \overline{14})$.

5. If our term in S_4 is s_4, the corresponding term in $S_4^{(2)}$ is $s_2 s_4$. The permutation (1432) would become $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{41})$.

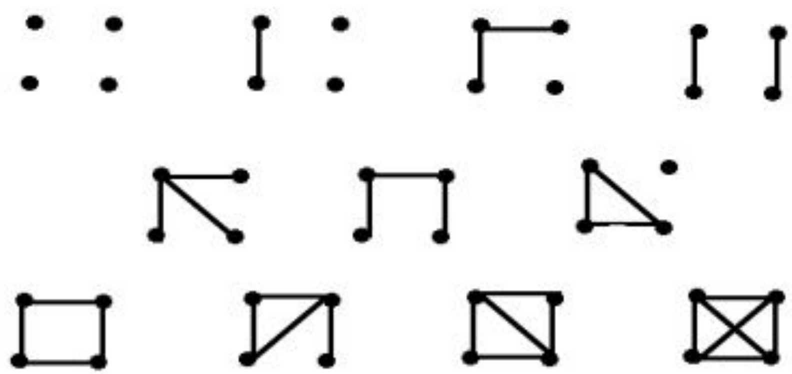
Thus

$$\begin{aligned} Z(S_4^{(2)}) &= \frac{1}{24}(s_1^6 + 6s_1^2 s_2^2 + 8s_3^2 + 3s_1^2 s_2^2 + 6s_2 s_4) \\ &= \frac{1}{24}(s_1^6 + 9s_1^2 s_2^2 + 8s_3^2 + 6s_2 s_4). \end{aligned}$$

To apply Polya’s Theorem and count the number of different graphs, we need a weight structure. If the edge exists, it will have weight 1, and it if does not, it will have weight 0, giving f(x) = 1 + x as the **generating function** for the set of colors.

$$
\begin{aligned}
g_4(x) &= Z(S_4^{(2)}, 1+x) \\
&= \frac{1}{24}((x+1)^6 + 9(x+1)^2(x^2+1)^2 + 8(x^3+1)^2 + 6(x^2+1)(x^4+1)) \\
&= x^6 + x^5 + 2x^4 + 3x^3 + 2x^2 + x + 1.
\end{aligned}
$$

There are **eleven** distinct graphs on four vertices including one graph each with six edges, five edges, one edge, or no edges, two graphs each with four edges or two edges, and three graphs with three edges.



Now we will find the number of graphs on five vertices.

First consider the cycle index for S_5. We will review the permutations in S_5 and their cycle structures. Also, $|S_5| = 5! = 120$

1. The permutation (1)(2)(3)(4)(5) has five cycles of length one, giving the term s_1^5.
2. The permutation (12)(3)(4)(5) has structure $s_1^3 s_2$. There are ten of these permutations, which can be found by all the combinations of five objects into pairs, i.e. $\binom{5}{2}$.
3. The permutation (123)(4)(5) has structure $s_1^2 s_3$. There are twenty, which can be counted by first recognizing the ten combinations for the cycle of length two, and then for each remaining cycle of length three there are two possibilities for their arrangement.
4. The permutation (1234)(5) has structure $s_1 s_4$. There are six different ways that four vertices can be permuted, and there are five choices for our cycle of length one. Thus there are thirty of these permutations.
5. The permutation (12)(34)(5) has structure $s_1 s_2^2$. From the previous example with four vertices, we saw that there were three possible permutations with the same structure as (12)(34). Now we have five possibilities for which number we choose to be the cycle of length one, so we have a total of fifteen permutations with this structure.
6. The permutation (123)(45) has structure $s_2 s_3$. There are twenty, which can be counted in the same way as permutation 3.
7. Finally, the permutation (12345) has structure s_5 and there are $4! = 24$ of them.

Thus

$$Z(S_5) = \frac{1}{120}(s_1^5 + 10s_1^3 s_2 + 20s_1^2 s_3 + 30s_1 s_4 + 15s_1 s_2^2 + 20s_2 s_3 + 24s_5). \tag{22}$$

Now we will examine each structure in $S_5^{(2)}$.

1. The permutation (1)(2)(3)(4)(5) becomes $(\overline{12})(\overline{13})(\overline{14})(\overline{15})(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})$, giving the term s_1^{10}. This again makes intuitive sense, because there are a total of ten edges on a graph of five vertices.

2. The permutation (12)(3)(4)(5) becomes $(\overline{12})(\overline{34})(\overline{35})(\overline{45})(\overline{15}\ \overline{25})(\overline{14}\ \overline{24})(\overline{13}\ \overline{23})$, giving the term $s_1^4 s_2^3$. The cycles of length two are depicted by the red, blue, and purple edges in Figure 27.

3. The permutation (123)(4)(5) becomes $(\overline{45})(\overline{12}\ \overline{23}\ \overline{31})(\overline{14}\ \overline{24}\ \overline{34})(\overline{15}\ \overline{25}\ \overline{35})$, giving the term $s_1 s_3^3$, as in Figure 28.

4. The permutation (1234)(5) becomes $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{14})(\overline{15}\ \overline{25}\ \overline{35}\ \overline{45})$, giving the term $s_2 s_4^2$, as in Figure 29.

5. The permutation (12)(34)(5) becomes $(\overline{12})(\overline{34})(\overline{14}\ \overline{23})(\overline{13}\ \overline{24})(\overline{15}\ \overline{25})(\overline{35}\ \overline{45})$, giving structure $s_1^2 s_2^4$. These cycles are shown in Figure 30, where the gray edges are fixed.

6. The permutation (123)(45) becomes $(\overline{45})(\overline{13})(\overline{12}\ \overline{23})(\overline{14}\ \overline{35}\ \overline{24}\ \overline{15}\ \overline{3425})$. The cycle structure is then $s_1 s_3 s_6$, as reflected in Figure 31.

7. Finally, the permutation (12345) would become $(\overline{12}\ \overline{23}\ \overline{34}\ \overline{45}\ \overline{51})(\overline{13}\ \overline{24}\ \overline{35}\ \overline{41}\ \overline{52})$, giving the term s_5^2. The cycles are shown in Figure 32.

Thus

$$Z(S_5^{(2)}) = \frac{1}{120}(s_1^{10} + 10s_1^4s_2^3 + 20s_1s_3^3 + 30s_2s_4^2 + 15s_1^2s_2^4 + 20s_1s_3s_6 + 24s_5^2). \tag{23}$$

Finally, we can substitute $x + 1$ in this cycle index and use WolframAlpha to give

$$\begin{aligned} g_5(x) &= Z(S_5^{(2)}, x+1) \\ &= \frac{1}{120}((x+1)^{10} + 10(x+1)^4(x^2+1)^3 + 20(x+1)(x^3+1)^3 + 30(x^2+1)(x^4+1)^2 \\ &\quad + 15(x+1)^2(x^2+1)^4 + 20(x+1)(x^3+1)(x^6+1) + 24(x^5+1)^2) \\ &= x^{10} + x^9 + 2x^8 + 4x^7 + 6x^6 + 6x^5 + 6x^4 43x^3 + 2x^2 + x + 1, \end{aligned} \tag{24}$$

which is, unsurprisingly, the same result we acheived using Theorem 8. There is 1 graph with each of zero, one, nine, or ten edges, 2 graphs with each of two or eight edges, 4 graphs with each of three or seven edges, and 6 graphs with each of five, six, or seven edges. This gives a total of 34 graphs on five vertices.