5.4 DISTRIBUTIONS

How many ways are there to assign 3 different diplomats to 2 different continents?

How many ways are there to assign 9 diplomats to 3 continents if 3 diplomats must be assigned to each continent?

How many ways are there to distribute 3 red balloons among 2 children.

Basic Models for Distributions

Distinct Objects The process of distributing *r* distinct objects into *n* different boxes is equivalent to putting the distinct objects in a row and stamping one of the *n* different box names on each object. The resulting sequence of box names is an arrangement of length *r* formed from *n* items (box names) with repetition.

Thus there are $n \times n \times \ldots \times n$ (r ns) = n^r distributions of the r distinct objects.

If r_i objects must go in box i, $1 \le i \le n$, then there are $P(r; r_1, r_2, \ldots, r_n)$ distributions.

Theorem 1

If there are n objects, with r_1 of type 1, r_2 of type 2, ..., and r_m of type m, where $r_1 + r_2 + \cdots + r_m = n$, then the number of arrangements of these n objects, denoted $P(n; r_1, r_2, \ldots, r_m)$, is

$$P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2\cdots-r_{m-1}}{r_m}$$

$$= \frac{n!}{r_1!r_2!\ldots r_m!} \qquad (*)$$

How many ways are there to assign 100 different diplomats to five different continents?

How many ways if 20 diplomats must be assigned to each continent?

How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?

Identical Objects The process of distributing r identical objects into n different boxes is equivalent to choosing an (unordered) subset of r box names with repetition from among the n choices of boxes.
Thus there are $C(r + n - 1, r) = (r + n - 1)!/ r!(n - 1)!$ distributions of the r identical objects.

Theorem 2

The number of selections with repetition of r objects chosen from n types of objects is $C(r + n - 1, r)$.

How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?

Remark.

Distributions of distinct objects are equivalent to arrangements

and

Distributions of identical objects are equivalent to selections

Ways to Arrange, Select, or Distribute r Objects from n Items or into n Boxes

	Arrangement (Ordered Outcome) or Distribution of Distinct Objects	*Combination (Unordered Outcome) or Distribution of Identical Objects*
No repetition	$P(n, r)$	$C(n, r)$
Unlimited repetition	n^r	$C(n+r-1, r)$
Restricted repetition	$P(n; r_1, r_2, \ldots r_m)$	—

How many ways are there to distribute four identical oranges and six distinct apples (each a different variety) into five distinct boxes? In what fraction of these distributions does each box get exactly two objects?

Compute the number of ways to distribute r identical balls into n distinct boxes with

- at least one ball in each box
- at least r_1 balls in the first box, at least r_2 balls in the second box, . . . , and at least r_n balls in the n^{th} box

How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$? How many solutions with $x_i \geq 1$? How many solutions with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

Remark. Equations with integer-valued variables are called *diophantine equations.* They are named after the Greek mathematician Diophantus, who studied them 2,250 years ago.

How many arrangements of the letters *a, e, i, o, u, x, x, x, x, x, x, x, x* (eight *x*s) are there if no two vowels can be consecutive?

Hw. What fraction of binary sequences of length 10 consists of a (positive) number of 1s, followed by a number of 0s, followed by a number of 1s, followed by a number of 0s? An example of such a sequence is 1110111000.

84/1024

Equivalent Forms for Selection with Repetition

1. The number of ways to select r objects with repetition from n different types of objects.
2. The number of ways to distribute r identical objects into n distinct boxes.
3. The number of nonnegative integer solutions to $x_1 + x_2 + \cdots + x_n = r$.

Ways to Arrange, Select, or Distribute r Objects from n Items or into n Boxes

	Arrangement (Ordered Outcome) or Distribution of Distinct Objects	*Combination (Unordered Outcome) or Distribution of Identical Objects*
No repetition	$P(n, r)$	$C(n, r)$
Unlimited repetition	n^r	$C(n+r-1, r)$
Restricted repetition	$P(n; r_1, r_2, \ldots r_m)$	—