INCLUSION–EXCLUSION FORMULA

In this section we generalize the inclusion–exclusion formula for counting $N(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3)$ to n sets $A_1, A_2, \ldots, A_n$. To simplify notation, we will omit the intersection symbol "$\cap$" in expressions and write intersected sets as a product. For example, $A_1 \cap A_2 \cap A_3$ would be written $A_1 A_2 A_3$. Using this new notation, the number of elements in none of the sets $A_1, A_2, \ldots, A_n$ will be written $N(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n)$.

Theorem 1 **Inclusion–Exclusion Formula**

Let $A_1, A_2, \cdots A_n$, be n sets in a universe $\mathcal{U}$ of N elements. Let S_k denote the sum of the sizes of all k-tuple intersections of the A_is. Then

$$N(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n \qquad (1)$$

To clarify the definition of the S_ks, $S_1 = \sum_i N(A_i)$, $S_2 = \sum_{ij} N(A_i A_j)$, S_k is the sum of the $N(A_{j_1} A_{j_2} \cdots A_{j_k})$s for all sets of k A_j's, and finally $S_n = N(A_1 A_2 \cdots A_n)$. We

$$N(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3) = N - \sum_i N(A_i) + \sum_{ij} N(A_i \cap A_j) - N(A_1 \cap A_2 \cap A_3)$$

Corollary

Let $A_1, A_2 \cdots A_n$ be sets in the universe $\mathcal{U}$. Then

$$N(A_1 \cup A_2 \cup \cdots \cup A_n) = S_1 - S_2 + S_3 - \cdots + (-1)^{k-1} S_k + \cdots + (-1)^{n-1} S_n \tag{4}$$

How many ways are there to select a 6-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

For n = 4

$$N(\overline{A}_1\overline{A}_2\cdots\overline{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n$$

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4) = \binom{52}{6} - 4\binom{39}{6} + 6\binom{26}{6} - 4\binom{13}{6} + 0$$

How many ways are there to distribute r distinct objects into five (distinct) boxes with at least one empty box?

We do not need to determine $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$ in this problem, because it does not concern outcomes where some property does not hold for all boxes. Rather, this is a union problem, using the corollary's formula.

It is easy to mistake union problems, which use phrases such as "with at least one empty box," with standard inclusion–exclusion problems, which use phrases such as "at least one object in every box."

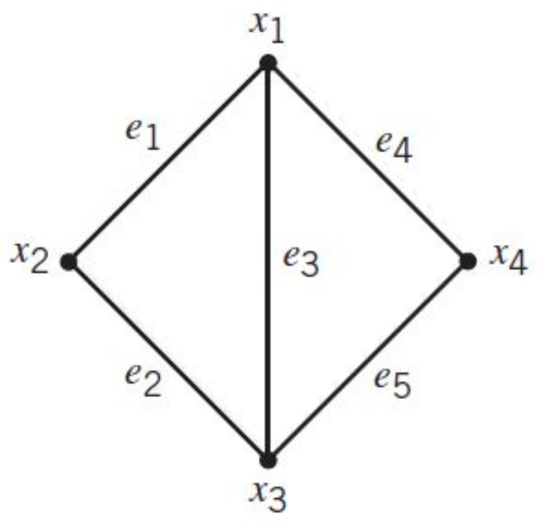
Let U be all distributions of r distinct objects into five boxes.
Let A_i be the set of distributions with a void in box i. Then the required number of distributions with at least one void is $N(A_1 \cup A_2 \cup \cdots A_5)$.

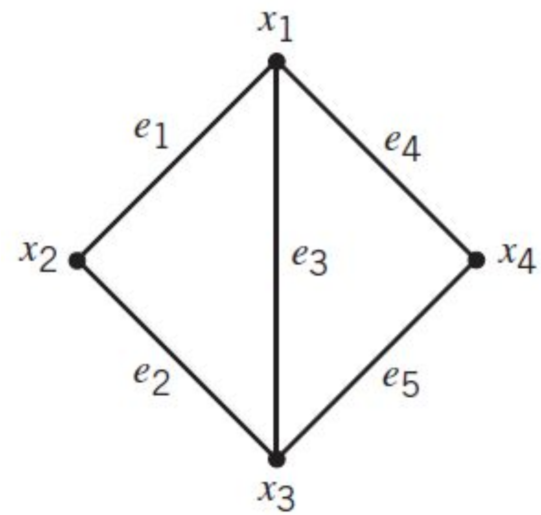
N = ? $N(A_i) = ?$ $N(A_i A_j) = ?$ $S_1 =?$ $S_2 =?$ $S_3 =?$ $S_4 = ?$ $S_5 =?$

How many different integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20 \quad 0 \leq x_i \leq 8$$

How many ways are there to color the four vertices in the graph shown in Figure below with n colors (such that vertices with a common edge must be different colors)?





x_1
e_1
e_4
x_2
e_3
x_4
e_2
e_5
x_3

$$
\begin{aligned}
N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= n^4 - \binom{5}{1}n^3 + \binom{5}{2}n^2 \\
&\quad - \left[2n^2 + \left(\binom{5}{3} - 2\right) \times n\right] + \binom{5}{4}n - n \\
&= n^4 - 5n^3 + 8n^2 - 4n \quad \blacksquare
\end{aligned}
$$

This expression is the chromatic polynomial of the given graph.

What is the probability that if n people randomly reach into a dark closet to retrieve their hats, no person will pick his own hat?

The problem treated in Example 4 is equivalent to asking for all permutations of the sequence $1, 2, \ldots, n$ such that no number is left fixed, that is, no number i is still in the ith position. Such rearrangements of a sequence are called **derangements.** The symbol D_n is used to denote the number of derangements of n integers. From Example 4, we have

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx n!e^{-1}$$

The number of arrangements of n different objects where no object is in its natural position is given by

$$\begin{aligned} D(n) &= n! - n(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \cdots + (-1)^n C(n,n)0! \\ &= n!(1 - 1/1! + 1/2! - 1/3! + \cdots + (-1)^n/n!) \end{aligned}$$

COUNTING RESTRICTED ARRANGEMENTS

As an application of the Principle of Inclusion-Exclusion, we consider some problems involving restricted arrangements.

Derangements

Consider the permutations of the set $\{1, 2, 3, 4, \ldots, n\}$. In some of these arrangements, which we call **derangements**, none of the n integers appear in their natural position, that is, 1 is not the first integer, 2 is not the second integer, 3 is not the third integer, and so on. We denote the number of derangements of n objects by $D(n)$. We will use the Principle of Inclusion-Exclusion to derive a formula for $D(n)$, which is valid for any positive integer n.

$D(2) = 1$	$D(3) = 2$	$D(4) = 9$		
21	231	2341	2143	4312
	312	3142	3412	2413
		4123	4321	3421

A math professor has typed five letters to five different persons and has also addressed five different envelopes for these letters. At the end of a particularly long day, he absentmindedly stuffs the five letters into the five envelopes at random. What is the probability that no letter is stuffed into the right envelope?

$$\begin{aligned} D(5) &= 5! - C(5,1)4! + C(5,2)3! - C(5,3)2! + C(5,4)1! - C(5,5)0! \\ &= 5!(1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5!) = 44 \end{aligned}$$

The probability that an arrangement of five letters is a derangement is thus $D(5)/5! = 44/120$. □

There are seven different pairs of gloves in a drawer. When seven children go out to play, each selects a left-hand glove and a right-hand glove at random.

a) In how many ways can the selection be made so that no child selects a matching pair of gloves?
b) In how many ways can the selection be made so that exactly one child selects a matching pair?
c) In how many ways can the selection be made so that at least two children select matching pairs?