HOMOGENEOUS LINEAR RECURRENCES

Linear Recurrences

If $A = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ is a sequence, we say that A satisfies a linear recurrence relation with constant coefficients if for all $n \geq m$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m} + g(n)$, where $c_1, c_2, \ldots, c_m$ is some set of constants and $g(n)$ is a function depending on n. If $c_m \neq 0$, we say that the recurrence relation has order m, and in case $g(n) = 0$, we say the recurrence is homogeneous.

In this section we study methods for solving homogeneous recurrence relations, which usually means that we want to find a solution $a_n = f(n)$ of the recurrence that satisfies the initial conditions $a_0 = b_0$, $a_1 = b_1, \ldots, a_{m-1} = b_{m-1}$, for a given set of constants $b_0, b_1, b_2, \ldots, b_{m-1}$.

Solve $a_n = ra_{n-1}$ with $a_0 = k$.

$\mathbf{a_n = kr^n}$

In this section we show how to solve recurrence relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} \tag{1}$$

The general solution to (1) will involve a sum of individual solutions of the form $a_n = \alpha^n$.

Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ with $a_0 = a_1 = 1$.

Find a formula for the number of ways for the elf in Example 2 of Section 7.1 to climb the n stairs.

The recurrence relation obtained in Example 2 of Section 7.1 was the Fibonacci relation $a_n = a_{n-1} + a_{n-2}$, with the initial conditions $a_1 = 1$, $a_2 = 2$, or equivalently, $a_0 = a_1 = 1$. The associated characteristic equation is obtained by setting $a_n = \alpha^n$:

The Case of Repeated Roots

In the first part of this section, we saw that when the characteristic equation of a homogeneous linear recurrence relation had distinct roots, a general solution could be obtained by looking at linear combinations of m fundamental solutions, one for each characteristic root.

In this case, for each characteristic root, there was an associated fundamental solution $\mathbf{a_n = r_i^n}$.

However, when the characteristic equation has repeated roots, two or more fundamental solutions coincide. Therefore we have fewer than m fundamental solutions that are exponential functions. Thus the linear combinations of exponential functions cannot form a general solution because there are not enough fundamental solutions.

It is still possible, though, to find a general solution for a homogeneous linear recurrence even when the characteristic equation has repeated roots by looking at a slightly larger class of fundamental solutions.

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

Find a formula for a_n satisfying the relation $a_n = -2a_{n-2} - a_{n-4}$ with $a_0 = 0, a_1 = 1, a_2 = 2$, and $a_3 = 3$.