5.3 ARRANGEMENTS AND SELECTIONS WITH REPETITIONS

How many arrangements are there of the six letters b, a, n, a, n, a?

There are C(6, 3) × C(3, 2) × C(1, 1) = 20 × 3 × 1 = 60 arrangements.

This is equivalent to

$$P(6; 3, 2, 1) = \binom{6}{3}\binom{3}{2}\binom{1}{1} = \frac{6!}{3!3!} \times \frac{3!}{2!1!} \times \frac{1!}{1!} = \frac{6!}{3!2!1!}$$

Theorem 1

If there are n objects, with r_1 of type 1, r_2 of type 2, ..., and r_m of type m, where $r_1+r_2+\cdots+r_m=n$, then the number of arrangements of these n objects, denoted $P(n; r_1, r_2, \ldots, r_m)$, is

$$P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2\cdots-r_{m-1}}{r_m}$$

$$= \frac{n!}{r_1!r_2!\ldots r_m!} \qquad (*)$$

Proof. Then there are *n!* arrangements of the n distinct objects.

$$n! = P(n; r_1, r_2, \ldots, r_m)r_1!r_2!\ldots r_m!$$

or

$$P(n; r_1, r_2, \ldots, r_m) = \frac{n!}{r_1!r_2!\ldots r_m!} \quad \blacklozenge$$

How many different ways are there to select six hot dogs from three varieties of hot dog?

Can we apply Theorem 1 directly, why or why not?
It can not be applied because the number of items of type i are not specified.

Model it as arrangement-with-repetition problem

Theorem 2

The number of selections with repetition of r objects chosen from n types of objects is $C(r + n - 1, r)$.

Proof

We make an "order form" for a selection just as in Example 2, with an x for each object selected. As before, the xs before the first | count the number of the first type of object, the xs between the first and second |s count the number of the second type, $\ldots$, and the xs after the $(n - 1)$-st | count the number of the nth type ($n - 1$ slashes are needed to separate n types). The number of sequences with r xs and $n - 1$ |s is $C(r + (n - 1), r)$. ◆

How many ways are there to form a sequence of 10 letters from four as, four bs, four cs, and four ds if each letter must appear at least twice?

How many ways are there to pick a collection of exactly 10 balls from a pile of red balls, blue balls, and purple balls if there must be at least five red balls? If at most five red balls?

How many arrangements are there of the letters b, a, n, a, n, a such that:

a. The b is followed (immediately) by an a

b. The pattern bnn never occurs

Ans. 30, 56