Stirling Numbers

Stirling numbers of the second kind S(n, k)

The Stirling number of the second kind $\left\{ n \atop k \right\}$, read "$n$ subset k", is the number of ways to partition a set with n elements into k non-empty subsets.

Compute $\left\{ 4 \atop 2 \right\}$

$\{1,2,3\} \cup \{4\}$ $\quad$ $\{1,2,4\} \cup \{3\}$ $\quad$ $\{1,3,4\} \cup \{2\}$ $\quad$ $\{2,3,4\} \cup \{1\}$

$\{1,2\} \cup \{3,4\}$ $\quad$ $\{1,3\} \cup \{2,4\}$ $\quad$ $\{1,4\} \cup \{2,3\}$

Hence $\left\{ 4 \atop 2 \right\} = 7$

Compute S(n, k) for k = 0, 1, n , n – 1, 2

$k = 0$ We can partition a set into no nonempty parts if and only if the set is empty.

That is: $\left\{ {n \atop 0} \right\} = [n = 0]$.

$k = 1$ We can partition a set into one nonempty part if and only if the set is nonempty.

$k = n$ If $n > 0$, the only way to partition a set with n elements into n nonempty parts, is to put every element by itself.

That is: $\left\{ {n \atop n} \right\} = 1$. (This also matches the case $n = 0$.)

$k = n - 1$ Choosing a partition of a set with n elements into $n - 1$ nonempty subsets, is the same as choosing the two elements that go together.

That is: $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

$k=2$ Let X be a set with two or more elements.

- Each partition of X into two subsets is identified by two ordered pairs $(A, X \setminus A)$ for $A \subseteq X$.
- There are 2^n such pairs, but $(\emptyset, X)$ and $(X, \emptyset)$ do not satisfy the nonemptiness condition.
- Then $\left\{ {n \atop 2} \right\} = \dfrac{2^n - 2}{2} = 2^{n-1} - 1$ for $n \geqslant 2$.

In general, $\left\{ {n \atop 2} \right\} = (2^{n-1} - 1)[n \geqslant 1]$

Compute $\left\{ {n \atop k} \right\}$

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty groups, then decide which group to add the nth element to.

2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty groups out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}, \qquad \text{for } n > 0,$$

that yields the following triangle:

Calculate the value $\left\{ {4 \atop 2} \right\}$ by applying the recurrence above.

$$\left\{ {4 \atop 2} \right\} = \left\{ {3 \atop 1} \right\} + 2\left\{ {3 \atop 2} \right\} = \left\{ {2 \atop 0} \right\} + 1\left\{ {2 \atop 1} \right\} + 2\left(\left\{ {2 \atop 1} \right\} + 2\left\{ {2 \atop 2} \right\}\right)$$

We use $S(2,0) = 0$ and $S(2,1) = S(2,2) = 1$ to simplify further.

$$\left\{ {4 \atop 2} \right\} = 0 + \left\{ {2 \atop 1} \right\} + 2\left\{ {2 \atop 1} \right\} + 4(1) = 3\left\{ {2 \atop 1} \right\} + 4$$

So, $\left\{ {4 \atop 2} \right\} = 7$.

To extend the table or verify any entry in it, choose an element. Multiply that element by the number at the top of its column, and add to it the element on its left (or 0 if there is no element to the left). The result should be the number below the chosen element.

$n\backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	3	1	0	0	0
4	0	1	7	6	1	0	0
5	0	1	15	25	10	1	0
6	0	1	31	90	65	15	1

How many ways are there to factor the number $2 \cdot 3 \cdot 5 \cdot 11 \cdot 23 = 7590$ into a product of two factors each greater than 1? How many for three such factors?

Consider the set {2, 3, 5, 11, 23}. Now we need to break this set into 2 sets which can be done in S{5, 2} = 15 ways.
S{5, 3} = 25.

Angered by all the candies being given to children at the beginning of this section, a group of indignant parents take five distinct pamphlets on the dangers of childhood obesity to give to two textbook authors. In how many ways may this be done if the two authors are considered distinct? In how many ways if the authors are considered identical? In either case, no author escapes being given at least one pamphlet, and there is only one copy of each of the five pamphlets.

Here $n = 5$ refers to the pamphlets, and $k = 2$ refers to the authors. From the table, $\left\{ {5 \atop 2} \right\} = 15$, so if the authors are distinct the answer is $2! \times 15$ or 30, and if the authors are identical we get 15. $\square$

Q. Properly chastened, the authors meet with three indignant parents and offer them a collection of six distinct snacks. Assuming the three parents are distinct, in how many ways can this be done if each parent is to receive at least one snack? In how many ways if a parent may refuse a snack (all snacks are distributed)? In how many ways if we do not require that all snacks be distributed, and any parent may refuse a snack?

The first problem is just a matter of looking at the table; $n = 6$, $k = 3$, so we get $3! \times \left\{ {6 \atop 3} \right\} = 6 \times 90 = 540$.

The second problem must be done in pieces. If all three parents accept a snack, 540 is still the answer; if one parent refuses, there are three ways to choose a parent to refuse and $2! \times \left\{ {6 \atop 2} \right\} = 62$ ways to distribute the snacks to the other two parents, so we get another 186 possibilities. Then if two parents refuse, we get another three possibilities (three choices of the parent to accept all six snacks, or if you like $\binom{3}{1} \times \left\{ {6 \atop 1} \right\} = 3$). Adding these, we find a total of 729 possible distributions of snacks.

The third problem can be solved by considering the trash can as a fourth potential snack recipient.

If all four possible recipients get a snack, we have $4! \times \left\{ {6 \atop 4} \right\} = 1560$. If one recipient (parent or trash can) gets no snack, we count $\binom{4}{1} \times 3! \times \left\{ {6 \atop 3} \right\} = 2160$. Two snackless recipients yields $\binom{4}{2} \times 2! \times \left\{ {6 \atop 2} \right\} = 372$. Finally, if all the snacks go to one recipient, there are four possibilities. Adding these numbers yields 4096 ways. Realizing that this answer is just 4^6, we see that we could have saved ourselves much trouble by saying: Associate with each snack a number from $\{0, 1, 2, 3\}$ according to which recipient receives it. Placing these numbers in order gives us a six-digit base-4 number, of which there are 4^6. $\square$

We use the notation $[r]_k = r!/(r-k)! = r(r-1)\ldots(r-k+1)$, the *falling factorial* notation.

(A Stirling Number Formula)

$$r^n = \sum_{k=0}^{r} \left\{ {n \atop k} \right\} [r]_k$$

Proof. Suppose we have r recipients of n objects, and, although all the objects must be distributed, some recipients may get no object. If all r recipients get an object, we have $r! \times \left\{ {n \atop r} \right\}$, which is the $k = r$ summand of the formula. For the case in which one recipient gets no object, we count $\binom{r}{1} \times (r-1)! \times \left\{ {n \atop r-1} \right\}$, which is equivalent to the $k = r-1$ summand. In general, if i recipients get no objects, we have $\binom{r}{i} \times (r-i)! \times \left\{ {n \atop r-i} \right\} = \frac{r!}{i!} \left\{ {n \atop r-i} \right\}$

which is the $r-i$th summand. On the other hand, the quantity we are counting is the number of n-digit base-r integers, of which there are r^n. $\square$

Write the polynomial x^3 as a linear combination of the polynomials 1, x, $x(x-1)$, and $x(x-1)(x-2)$.

In the preceding theorem, substitute 3 for n and x for r; this gives us

$$x^3 = \sum_{k=0}^{3} \left\{ {3 \atop k} \right\} \frac{x!}{(x-k)!}$$

and we simplify this to get $x^3 = 0 \cdot 1 + 1 \cdot x + 3 \cdot x(x-1) + 1 \cdot x(x-1)(x-2)$, or $x^3 = x + 3x(x-1) + x(x-1)(x-2)$. Even though the recurrence requires that x be an integer, the identity holds for any value of x. $\square$

We have three distinct snacks and wish to put them into bags. How many ways can this be done?

There is exactly one way to do this with three bags (each snack into its own bag); three ways to do this with two bags (depending on which snack goes into a bag by itself); and one way with one bag. Thus $B_3 = 5$. $\square$

We define the **Bell number** B_n to be the number of ways to partition a set of n distinct objects.

$$B_n = \sum_{i=0}^{n} \left\{ {n \atop i} \right\}.$$

How many ways are there of writing $2 \times 3 \times 5 \times 7 = 210$ as a product of distinct integers?

$B_4 = 15.$

(Bell Number Recurrence)

$$B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$$

Note that B_0 is 1.

Find B_6.

The recurrence gives $B_6 = B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 + B_5$. We may compute these by summing rows in the triangle, or by recursively applying Theorem 7.6 to get $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, and $B_5 = 52$. This gives us $B_6 = 1 + 5 + 20 + 50 + 75 + 52 = 203$.

Stirling numbers of the first kind s(n, k) $\left[n \atop k \right]$

The *falling factorial polynomial* of degree n is

$$(x)_n = x(x-1)(x-2)(x-3)\cdots(x-n+1),$$

a polynomial of degree n in one indeterminate x. If we evaluate the polynomial at m, we get the number of n-permutations chosen from a set of size m:

$$(m)_n = m(m-1)(m-2)(m-3)\cdots(m-n+1) = \frac{m!}{(m-n)!} = P(m,n).$$

Here are the first few of these polynomials.

$$\begin{aligned}
&(x)_0 = 1 \text{ (the empty product!)}\\
&(x)_1 = x\\
&(x)_2 = x(x-1) = x^2 - x\\
&(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x\\
&(x)_4 = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x\\
&(x)_5 = x(x-1)(x-2)(x-3)(x-4) = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x
\end{aligned}$$

The coefficients appearing in $(x)_n$ are called *Stirling numbers of the first kind*. The coefficient of x^k in $(x)_n$ is denoted $s(n,k)$, thus

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k.$$

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} - (m-1)\begin{bmatrix} m-1 \\ n \end{bmatrix}$$

Proof. We begin by using the definition and applying a bit of algebra.

$$\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} x^j = x(x-1)\ldots(x-m+1) = [x]_m = (x-m+1)[x]_{m-1}$$

Replacing the falling factorial on the right with the equivalent sum gives us:

$$\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} x^j = (x-m+1)\sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} x^j$$

and with the distributive law we obtain

$$\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} x^j = x\sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} x^j - (m-1)\sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} x^j.$$

What is the coefficient of x^n on each side of the equation above? We get $\begin{bmatrix} m \\ n \end{bmatrix}$ on the left and $\begin{bmatrix} m-1 \\ n-1 \end{bmatrix} - (m-1)\begin{bmatrix} m-1 \\ n \end{bmatrix}$ on the right. Since the two polynomials are equal, the coefficients are equal for each n, establishing the theorem. $\square$

As with Stirling numbers of the second kind, we may present them in the form of a table and refer to it, shown below. To extend the table or verify any entry in it, choose an element. Multiply that element by the number n at the left of its row, and subtract it from the element on its left (or 0 if there is no element to the left). The result should be the number below the chosen element.

$n \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	−1	1	0	0	0	0
3	0	2	−3	1	0	0	0
4	0	−6	11	−6	1	0	0
5	0	24	−50	35	−10	1	0
6	0	−120	274	−225	85	−15	1

(Unsigned Stirling Number Recurrence)

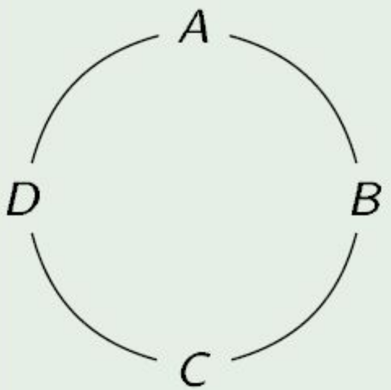
$$\left|\begin{bmatrix} m \\ n \end{bmatrix}\right| = \left|\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right| + (m-1)\left|\begin{bmatrix} m-1 \\ n \end{bmatrix}\right|$$

The absolute value of $s(n,k)$ is denoted $|s(n,k)|$ and is called an *unsigned Stirling number of the first kind.* The signs alternate, so $s(n,k) = (-1)^{n-k}|s(n,k)|$.

Unsigned Stirling numbers of the first kind

The Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$, read "$n$ cycle k", is the number of ways to partition of a set with n elements into k non-empty circles.

Circle is a cyclic arrangement



- Circle can be written as $[A, B, C, D]$;
- It means that $[A, B, C, D] = [B, C, D, A] = [C, D, A, B] = [D, A, B, C]$;
- It is not same as $[A, B, D, C]$ or $[D, C, B, A]$.

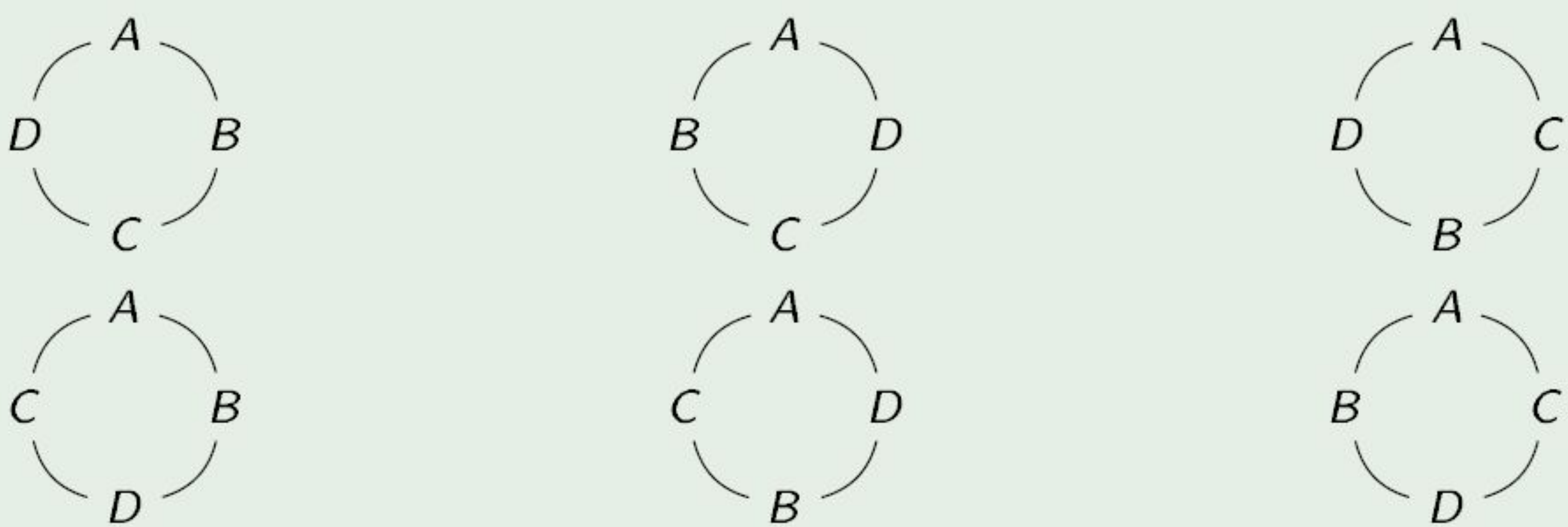
Compute $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$[1,2,3]\ [4]$ $[1,2,4]\ [3]$ $[1,3,4]\ [2]$ $[2,3,4]\ [1]$

$[1,3,2]\ [4]$ $[1,4,2]\ [3]$ $[1,4,3]\ [2]$ $[2,4,3]\ [1]$ Hence $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$

$[1,2]\ [3,4]$ $[1,3]\ [2,4]$ $[1,4]\ [2,3]$

Compute $\begin{bmatrix} n \\ k \end{bmatrix}$ for k = 1, n, n – 1



$k=1$ To arrange one circle of n objects: choose the order, and forget which element was the first. That is: $\begin{bmatrix} n \\ 1 \end{bmatrix} = n!/n = (n-1)!$.

$k = n$ Every circle is the singleton and there is just one partition into circles. That is, $\left[{n \atop n}\right] = 1$ for any n:

[1] [2] [3] [4]

$k = n-1$ The partition into circles consists of $n-2$ singletons and one pair. So $\left[{n \atop n-1}\right] = \binom{n}{2}$, the number of ways to choose a pair:

[1,2] [3] [4] [1,3] [2] [4] [1,4] [2] [3]

[2,3] [1] [4] [2,4] [1] [3] [3,4] [1] [2]

In the general case:

For $n \geqslant 1$, what are the options where to put the nth element?

1. Together with some other elements.
   To do so, we can first subdivide the other $n-1$ remaining objects into k nonempty cycles, then decide which element to put the nth one *after*.
2. By itself.
   Then we are only left to decide how to make the remaining $k-1$ nonempty cycles out of the remaining $n-1$ objects.

These two cases can be joined as the recurrent equation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \qquad \text{for } n > 0,$$

that yields the following triangle:

Compute the number of ways to seat 4 people around 2 circular tables
With no table left empty.