6.3 PARTITIONS

A **partition** of a group of r identical objects divides the group into a collection of (unordered) subsets of various sizes.
Analogously, we define a partition of the integer r to be a collection of positive integers whose sum is r.

Compute the number of partitions of 5?

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

Construct a generating function for a_r , the number of partitions of the integer 5. First model it as integer-solution-to-an-equation problem.

Find the generating function for a_r , the number of ways to express r as a sum of distinct integers.

Find a generating function for a_r, the number of ways that we can choose 2¢, 3¢, and 5¢ stamps adding to a net value of r¢.

Using generating function, show that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

Find the generating function for a_r , the number of ways to express r as a sum of distinct powers of 2.

How many ways are there to express n as a sum of integers no greater than k?

How many ways are there to express n as sum of integers no greater than k, one of which is k itself?

The number of ways to partition n into nonzero parts of which the largest is k is equal to the number of ways to partition n into k nonzero parts

as in this example for n = 8 and k = 3:

$$
\begin{aligned}
3+1+1+1+1+1 &\leftrightarrow 6+1+1\\
3+2+1+1+1 &\leftrightarrow 5+2+1\\
3+2+2+1 &\leftrightarrow 4+3+1\\
3+3+1+1 &\leftrightarrow 4+2+2\\
3+3+2 &\leftrightarrow 3+3+2
\end{aligned}
$$

What do you observe

To exhibit this relationship, we have recourse to a visual technique for presenting partitions:

Definition 1. The *Ferrers diagram* for the partition $a_1 + a_2 + a_3 + \cdots + a_k$ for $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k > 0$ consists of k left-justified rows of equally-spaced dots with a_i dots in the ith row, for each i.

For instance, here we have a Ferrers diagram for $3 + 2 + 2 + 1$:

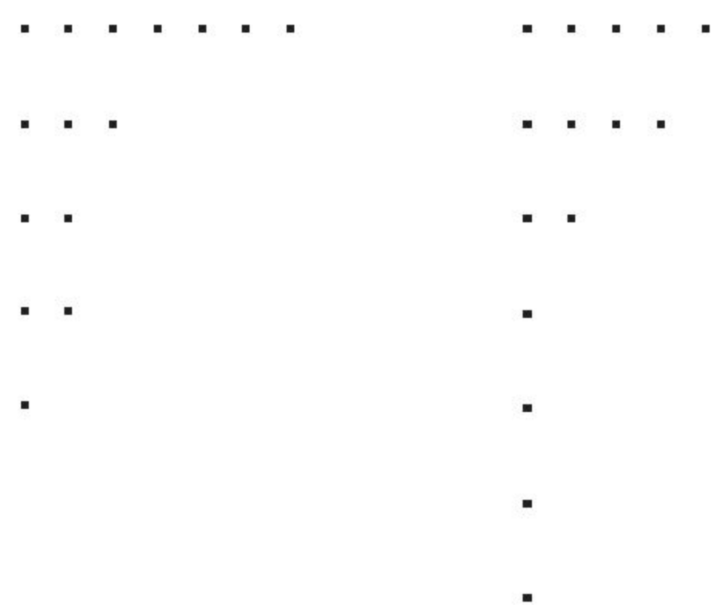
$$
\begin{array}{lll}
\bullet & \bullet & \bullet\\
\bullet & \bullet & \\
\bullet & \bullet & \\
\bullet & &
\end{array}
$$

The partition 1+2+2+3+7 of 15 is shown in the Ferrers diagram in Figure 6.1a.

If we **transpose** the rows and columns of a Ferrers diagram of a partition of r, we get a Ferrers diagram of another partition of r.

This diagram is called the **conjugate** of the original Ferrers diagram.

For example, Figure 6.1b shows the conjugate of the Ferrers diagram in Figure 6.1a. Here the partition of 15 is 1+1+1+1+2+4+5.



6.1 *(a)* *(b)*

Show that the number of partitions of an integer r as a sum of m positive integers is equal to the number of partitions of r as a sum of positive integers, the largest of which is m.

If we draw a Ferrers diagram of a partition of r into m parts, then the Ferrers diagram will have m rows.
The transposition of such a diagram will have m columns, that is, the largest row will have m dots.

Thus there is a one-to-one correspondence between these two classes of partitions.

A partition is **self-conjugate** if it is equal to its conjugate, or in other words, if its Ferrers diagram is symmetric about the diagonal.
For example, the Ferrers diagram for the partition 10 = 4+3+3+1 is self-conjugate (see Figure below) .

```
*                                        *
* * *                                    * * *
* * *                                    * * *
* * * *      with conjugate              * * * *
```

Prove that the number of partitions of n into parts that are both odd and distinct is equal to the number of self-conjugate partitions of n.

A generating function for the first object is

$$\prod_{j=0}^{\infty}(1+x^{2j+1}),$$

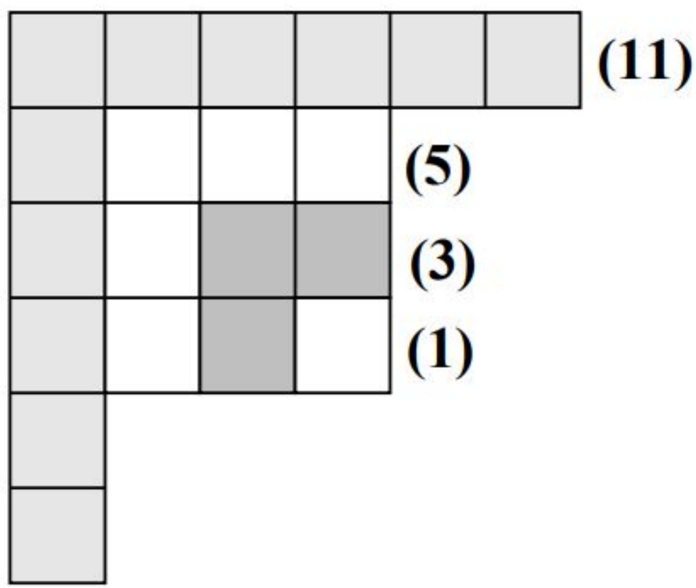
But a generating function for the latter object is not obvious.

However, using Ferrers diagrams, a bijective proof is straightforward.

The general idea is to 'bend' each odd, distinct part at the middle cell and then join the bent pieces together.

This yields a self-conjugate partition, a process that is clearly reversible.

As an example, the partition of 20 into the odd, distinct parts 11+5+3+1 is illustrated in Figure below.



Converting the partition $20 = 11 + 5 + 3 + 1$ into one that is self-conjugate