Pumping Lemma for Regular Languages

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Weak Points of Finite Automaton (FA)

Every language is not regular.

- There exists some non-regular languages as well.
- Non-regular languages cannot be recognized by FA

Intuitively, languages that have some memory element are generally not regular.

Pumping Lemma for Regular Languages

Pumping Lemma

If L is a regular language, then there exists a constant p such that for every string $w \in L$ s.t. $|w| \geq p$ there exists a partition of w in strings x, y, and z s.t. $w = xyz$ such that

- $|y| > 0$,
- $|xy| \leq p$, and
- for all $i \geq 0$ we have that $xy^iz \in L$.

Contrapositive form of Pumping Lemma

If L is a regular language, then there exists a constant p such that for every string $w \in L$ s.t. $|w| \geq p$ there exists a partition of w in strings x, y, and z s.t. $w = xyz$ such that
$|y| > 0$,
$|xy| \leq p$, and
for all $i \geq 0$ we have that $xy^iz \in L$.

- Let $L \subseteq \Sigma^*$ if
- $\forall\, p \geq 0$,
- $\exists\, w \in L$ s.t. $|w| \geq p$
- s.t. for all partitions
- $w = xyz$ where
  - $|xy| \leq p$ and $|y| > 0$
- $\exists\, i \geq 0$ s.t. $xy^iz \notin L$
- Then L is not regular.

Contrapositive form of Pumping Lemma (Continued….)

- Let $L \subseteq \Sigma^*$ if
- $\forall$ $p \geq 0$,
- $\exists$ $w \in L$ s.t. $|w| \geq p$
- s.t. for all partitions
- $w = xyz$ where
  - $|xy| \leq p$ and $|y| > 0$
- $\exists$ $i \geq 0$ s.t. $xy^iz \notin L$
- Then L is not regular.

Example

Prove that the language $L = \{0^n1^n | n \geq 0\}$ is not regular.

Proof: Given $p \geq 0$

Choose $w = 0^p1^p$

Now, given a partition

$w=xyz$ where $|xy| \leq p$ and $|y| > 0$

Note that x and y consist only of 0's.

Choose $i = 0$

$xy^0z = xz = 0^r1^p$ where $r = p - |y| \neq p$

Hence, $xy^0z \notin L$, Therefore, L is not regular.

Example

Prove that the language $L = \{a^l b^m c^n \mid l+m \leq n\}$ is not regular.

Proof: Given p

Choose $w = a^p b^p c^{2p}$

Now, given a partition

$w=xyz$ where $|xy| \leq p$ and $|y| > 0$

We have $y = a^t$ for some $t > 0$

Choose $i = 2$

$xy^2z = a^{p+t}b^p c^{2p}$ which implies $|a|+|b|>|c|$

Hence, $xy^2z \notin L$, Therefore, L is not regular.

Revisited Pumping Lemma

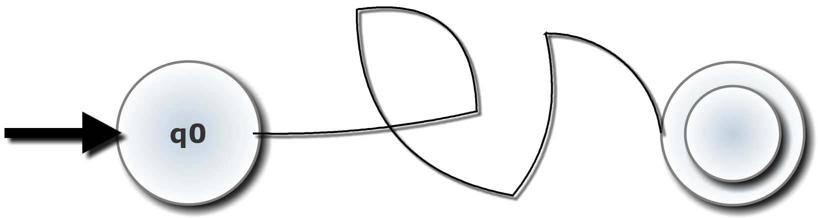
If L is a regular language, then there exists a constant p such that for every string $w \in L$ s.t. $|w| \geq p$ there exists a partition of w in strings x, y, and z s.t. $w = xyz$ such that

- $|y| > 0$,
- $|xy| \leq p$, and
- for all $i \geq 0$ we have that $xy^iz \in L$.

Proof of Pumping Lemma

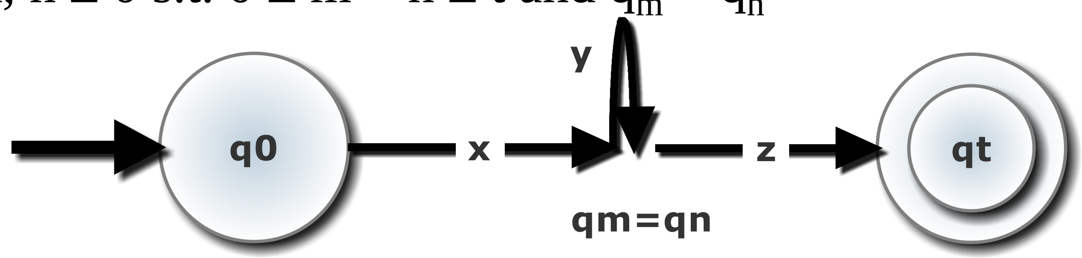
Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA for L.

Set $p = |Q|$



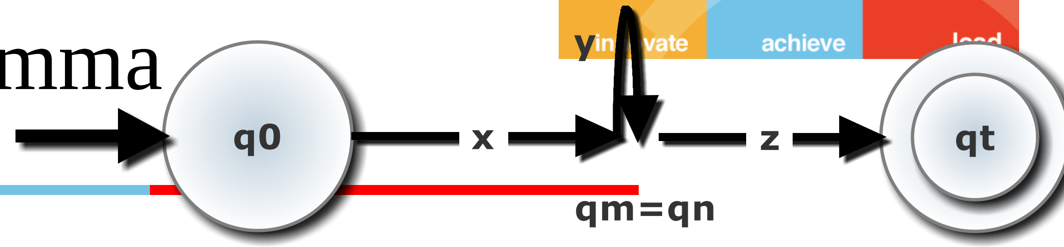
Given $w \in L$ s.t. $|w| \geq p$, $\exists$ a sequence of states q0, q1, q2, ------ q_t s.t. $t \geq p$ and $q_t \in F$

By pigeonhole principle, $\exists$ m, n ≥ 0 s.t. $0 \leq m < n \leq t$ and $q_m = q_n$



Set x = String from q0 to q_m
y = String on the loop
z = Remaining String
Hence, $\forall i$, $xy_iz \in L$

Proof of Pumping Lemma



In other words, the DFA traverses from q0 to qm on x, traverses from qm to qn (qn is the same as qm) on y and then proceeds to qt .

- Since $m < n$ therefore $|y| > 0$.

Now for all $i \geq 0$, $\delta(q0, xy^i) = qm$, as the automaton loops on the state qm on the string y.

- Therefore $\delta(q0, xy^iz) = qt$ and hence, $xy^i z \in L$.

Important Observation

We are now "overloading" the definition of δ to accommodate strings as well instead of input alphabet symbols only.

- We can formalize this by defining δ recursively as follows:
- $\delta : Q \times \Sigma^* \rightarrow Q$ such that

$$\delta(q, \in) = q,$$

$$\delta(q, xa) = \delta(\delta(q, x), a).$$

Home Assignment

Prove that the language $L = \{0^p \mid p \text{ is a prime}\}$ is not regular.