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Closure Properties of Regular Languages

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Closure Properties of Regular Languages

Closure properties on regular languages are defined as certain operations on regular language which are guaranteed to produce regular language.

Closure refers to some operation on a language, resulting in a new language that is of same “type” as originally operated on i.e., regular.

Closure Properties of Regular Language

Theorem: If L1 and L2 are regular language, then

- $L1 \cup L2$, $L1 . L2$ and $L1^*$ are also regular.

For a regular language, we can assume that there is a NFA accepting it with a unique start and accept state.

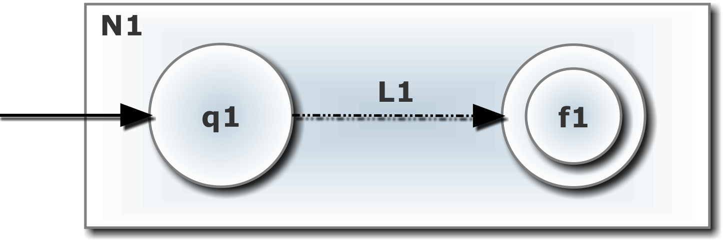
Constructive Proof

Assume L1 and L2 are the regular languages for NFA N1 and N2 respectively.

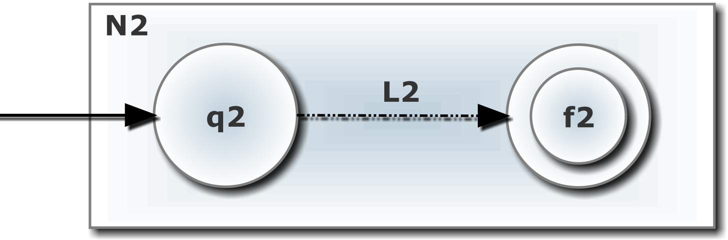
- Similarly, consider q1 and q2 are the initial states whereas, f1 and f2 are the accept states of N1 and N2 respectively.

Hence, L1 = L (N1) and L2 = L (N2)

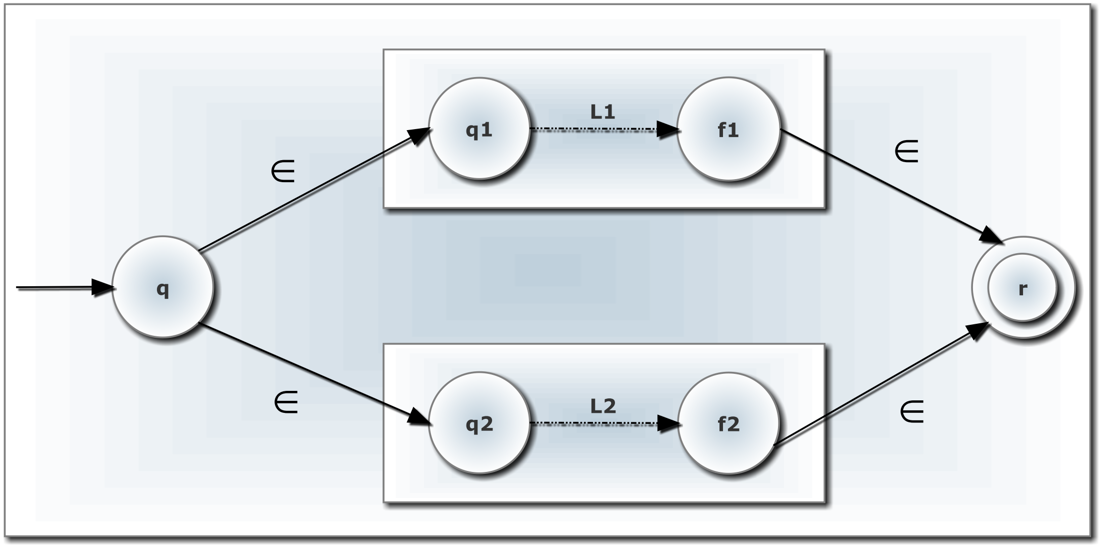
Closure Properties: Union



NFA for L1

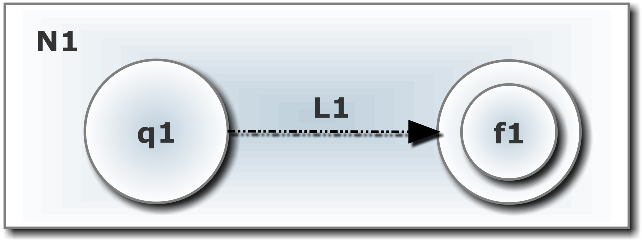


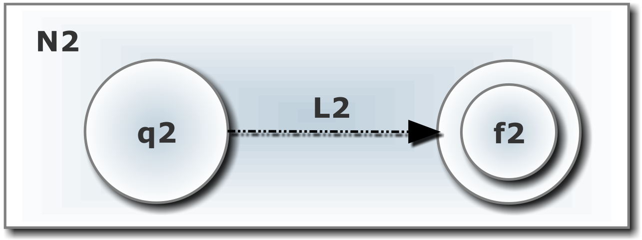
NFA for L2



NFA for L1 + L2

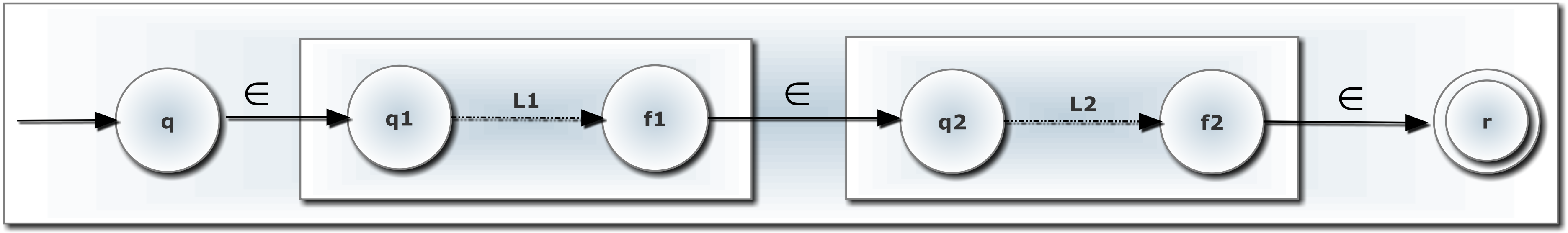
Closure Properties: Concatenation





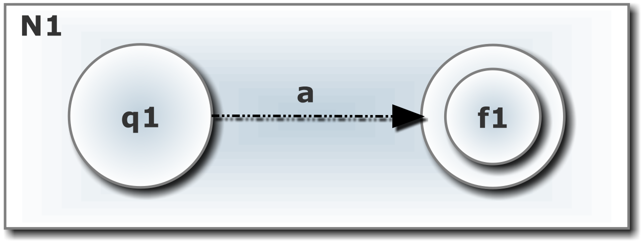
NFA for L1

NFA for L2

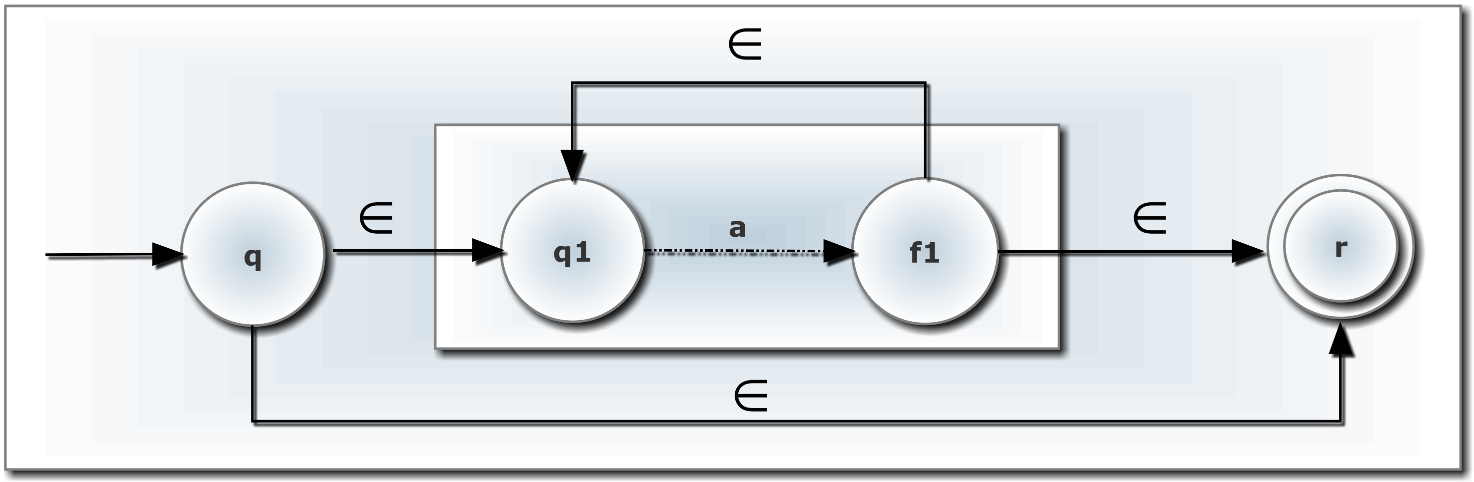


NFA for L1 . L2

Closure Properties: Kleene



NFA for L1 = a



NFA for L1 = a*

Closure Properties: Complement

Complement of L is

- $L` = \Sigma^* - L$

If L is regular then L` is also regular.

Closure Properties: Complement

Let D = (Q, Σ, δ, q0, F) be some DFA for L

We can also construct a DFA D\` for L\` where

- D\` = (Q, Σ, δ, q0, **Q - F**)

Closure Properties: Intersection

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B\}$$

If L1 and L2 are regular, then L1 $\cap$ L2 is also regular.

Closure Properties: Intersection

$$L1 \cap L2 = \overline{\bar{L1} \cup \bar{L2}}$$

$$\bar{L1}\, and\, \bar{L2}\ are\ regular under complement$$

$$\bar{L1} \cup \bar{L2}\ are\ also regular under union$$

$$Hence,\ \overline{\bar{L1} \cup \bar{L2}}\ \ is\, also\ regular.$$

Hence, regular languages are closed under intersection operation.

Closure Properties: Set Difference

A - B = { x | x ∈ A and x ∉ B}

$$A - B = A \cap \overline{B}$$

If L1 and L2 are regular, then L1 - L2 is also regular.

$$L1 - L2 = L1 \cap \overline{L2}$$

Closure Properties: Reversal

Let $w = a_1\ a_2$ ---- a_n be a string then rev $(w) = a_n\ a_{n-1}$---- a_1

For a language $L \subseteq \Sigma^*$,

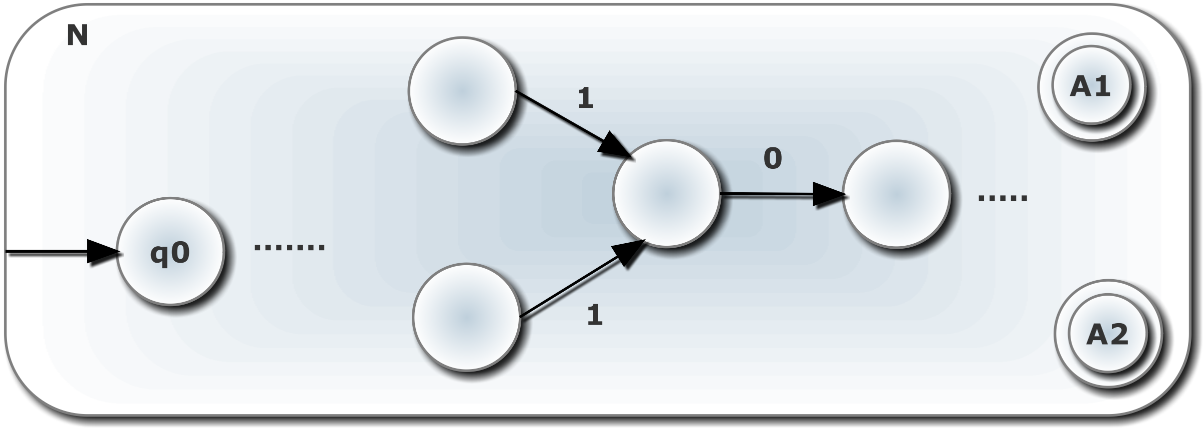
- $rev(L) = \{w \in \Sigma^* \mid rev(w) \in L\}$

Closure Properties: Reversal

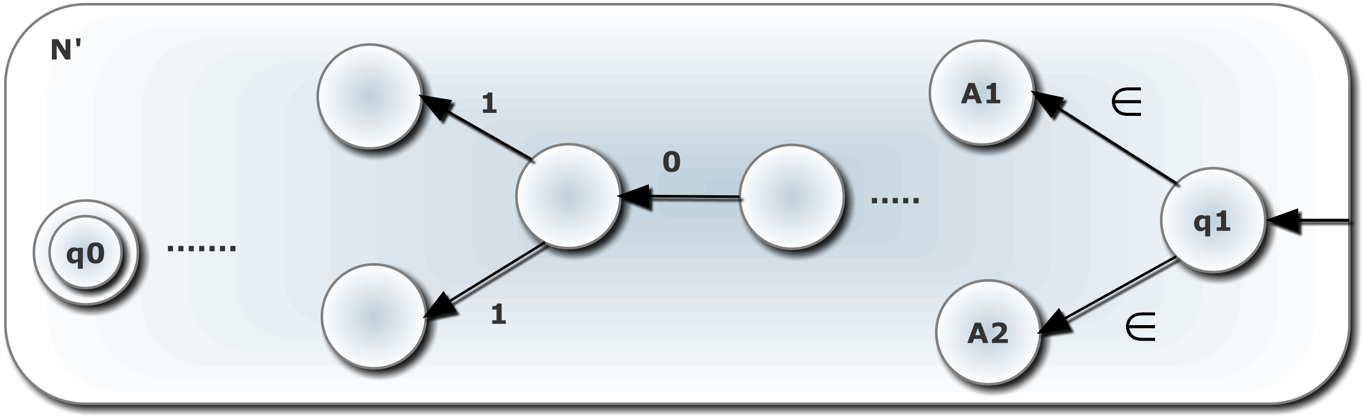
If L is regular then, rev(L) is also regular.

Let D = (Q, Σ, δ, q0, F) be some DFA for L

Closure Properties: Reversal



N\` = (Q\`, Σ, δ\`, q0\`, F\`)
where
Q\` = Q ∪ {q1}
δ\`(q,a) = {r | δ(r,a) = q }
δ\`(q1,∈) = {A | A ∈ F}
F\` = {q0}
Then L(N\`) = rev (L)



Closure Properties: Homomorphism

Let Σ, γ be two alphabets

A homomorphism is a map

- h: $\Sigma \rightarrow \gamma^*$
- Let $w = a_1\ a_2\ a_3$---a_n be a string in Σ^*.Then, $h(w) = h(a_1)h(a_2)$----$h(a_n)$

Closure Properties: Homomorphism

A homomorphism on an alphabet is a function that gives a string for each symbol in that alphabet.

- Example: $h(0) = ab$; $h(1) = \varepsilon$.

Extend to strings by $h(a_1 \ldots a_n) = h(a_1) \ldots h(a_n)$.

- Example: $h(01010) = ababab$.

Closure under Homomorphism

If L is a regular language, and h is a homomorphism on its alphabet, then h(L) = {h(w) | w is in L} is also a regular language.

Proof: Let E be a regular expression for L.

- Apply h to each symbol in E. Language of resulting RE is h(L).

Closure under Homomorphism

Let $h(0) = ab$; $h(1) = \in$.

Let L be the language of regular expression $01^* + 10^*$.

- Then, h(L) is the language of regular expression $ab\in^* + \in(ab)^*$.

Closure under Homomorphism

- $ab\in^* + \in(ab)^*$ can be simplified.
- $\in^* = \in$, so $ab\in^* = ab\in$
- $\in$ is the identity under concatenation.
- That is, $\in E = E\in = E$ for any RE E.
- Thus, $ab\in^* + \in(ab)^* = ab\in + \in$
- $(ab)^* = ab + (ab)^*$.
- Finally, L(ab) is contained in L((ab)*), so a RE for h(L) is $(ab)^*$.

Closure Properties: Subset Operation

Regular languages are not closed under subset operation

The subset of a language may or may not be regular.

- $a^*b^* \subseteq (a + b)^*$
- $a^nb^n \mid n \geq 1$ is not regular