CSCI 5654 - Linear Programming - Homework – 3

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P1

A. Given problem as

$$\min_{x_1,x_2,x_3\in\mathbb{R}^3} \max(2x_1+3x_2-5x_3,\ x_1,\ x_2,\ 2x_1-x_2+x_3).$$

Consider $\max(2x_1+3x_2-5x_3,\ x_1,\ x_2, 2x_1-x_2+x_3) \le Z$

Considering $x_1 = x_{1+} + x_{1-}$, $x_2 = x_{2+} + x_{2-}$, $x_3 = x_{3+} + x_{3-}$

Then, we can write the LP as follows

$$\begin{array}{lrrrrrrcl}
\min & & & & Z & & & & \\
\text{s.t.} & 2x_{1+} & -2x_{1-} & +3x_{2+} & -3x_{2-} & -5x_{3+} & +5x_{3-} & \le & Z \\
 & x_{1+} & -x_{1-} & & & & & \le & Z \\
 & & & +x_{2+} & -x_{2-} & & & \le & Z \\
 & 2x_{1+} & -2x_{1-} & -x_{2+} & +x_{2-} & +x_{3+} & -x_{3-} & \le & Z \\
 & \multicolumn{8}{c}{x_{1+}, x_{2+}, x_{3+}, x_{1-}, x_{2-}, x_{3-} \ge 0}
\end{array}$$

B. Given problem as

$$\min_{x_1,x_2,x_3\in\mathbb{R}^3} (|x_1+x_2|+|x_2-x_3|+|x_3-x_1|+|x_1+x_2+x_3|).$$

In order to handle the mod values, we can do the following:

Consider $y_1 \ge x_1 + x_2$ and $y_1 \ge -(x_1+x_2)$

Since $|x_1+x_2|$ is $+(x_1+x_2)$ or $-(x_1+x_2)$, we get $y_1 \ge |x_1+x_2|$. Similarly doing it for the other parts in the equation we get

$$y_1 \ge |x_1+x_2|$$
$$y_2 \ge |x_2-x_3|$$
$$y_3 \ge |x_3-x_1|$$
$$y_4 \ge |x_1+x_2+x_3|$$

To add the $\ge$ constraint we do:

$$x_1 = x_{1+} + x_{1-}$$
$$x_2 = x_{2+} + x_{2-}$$
$$x_3 = x_{3+} + x_{3-}$$

Then, the LP can be formulated as,

$$
\begin{array}{llllllllll}
\min & & & & y_1 + y_2 + y_3 + y_4 & & & & \\
\text{s.t.} & x_{1+} & -x_{1-} & +x_{2+} & -x_{2-} & & & \leq & y_1 \\
& -x_{1+} & +x_{1-} & -x_{2+} & +x_{2-} & & & \leq & y_1 \\
& & & x_{2+} & -x_{2-} & -x_{3+} & +x_{3-} & \leq & y_2 \\
& & & -x_{2+} & +x_{2-} & +x_{3+} & -x_{3-} & \leq & y_2 \\
& -x_{1+} & +x_{1-} & & & +x_{3+} & -x_{3-} & \leq & y_3 \\
& x_{1+} & -x_{1-} & & & -x_{3+} & +x_{3-} & \leq & y_3 \\
& x_{1+} & -x_{1-} & +x_{2+} & -x_{2-} & +x_{3+} & -x_{3-} & \leq & y_4 \\
& -x_{1+} & +x_{1-} & -x_{2+} & +x_{2-} & -x_{3+} & +x_{3-} & \leq & y_4 \\
& & \multicolumn{6}{l}{x_{1+}, x_{2+}, x_{3+}, x_{1-}, x_{2-}, x_{3-} \geq 0} & \\
\end{array}
$$

C. Given the problem

$$
\begin{array}{rl}
\min & \max(|x_1|,\ |x_2|,\ |x_3|,\ |x_1 + x_2|) \\
\textsf{subj.to} & x_1 - x_2 \leq 5 \\
& x_2 \leq 3
\end{array}
$$

Using the same formulations from the problems P1.A and P1.B we can do the following:

Consider $\max(|x_1|, |x_2|, |x_3|, |x_1 + x_2|) \leq Z$

Considering $x_1 = x_{1+} + x_{1-}$, $x_2 = x_{2+} + x_{2-}$, $x_3 = x_{3+} + x_{3-}$

Then, the LP can be formulated as

$$
\begin{array}{llllllllll}
\min & & & & Z & & & & \\
\text{s.t.} & x_{1+} & -x_{1-} & & & & & \leq & Z \\
& -x_{1+} & +x_{1-} & & & & & \leq & Z \\
& & & x_{2+} & -x_{2-} & & & \leq & Z \\
& & & -x_{2+} & +x_{2-} & & & \leq & Z \\
& & & & & +x_{3+} & -x_{3-} & \leq & Z \\
& & & & & -x_{3+} & +x_{3-} & \leq & Z \\
& x_{1+} & -x_{1-} & +x_{2+} & -x_{2-} & & & \leq & Z \\
& -x_{1+} & +x_{1-} & -x_{2+} & +x_{2-} & & & \leq & Z \\
& x_{1+} & -x_{1-} & -x_{2+} & +x_{2-} & & & \leq & 5 \\
& & & x_{2+} & -x_{2-} & & & \leq & 3 \\
& & \multicolumn{6}{l}{x_{1+}, x_{2+}, x_{3+}, x_{1-}, x_{2-}, x_{3-} \geq 0} & \\
\end{array}
$$

P2

Given that $x_1 < x_2 < \cdots < x_{2k+1}$ are 2k+1 distinct number.

A. We need to show that the number x that minimizes the 2-norm $f_2(x):\sqrt{\sum_{j=1}^{2k+1}\left(x_j - x\right)^2}$ is the mean.

The number x which minimizes $\sqrt{\sum_{j=1}^{2k+1}\left(x_j - x\right)^2}$ is the same as the number which minimizes $\sum_{j=1}^{2k+1}\left(x_j - x\right)^2$ since we are squaring numbers and taking root.

So, to find x which minimizes $f_2(x) = \sum_{j=1}^{2k+1}\left(x_j - x\right)^2$ we take its derivative with respect to x and then equate it to 0.

$$\frac{d}{dx}\left(f_2(x)\right) = \frac{d}{dx}\left(\sum_{j=1}^{2k+1}\left(x_j - x\right)^2\right) = \frac{d}{dx}\left((x_1 - x)^2 + \cdots + (x_{2k+1} - x)^2\right)$$
$$= 2(x_1 - x) + 2(x_2 - x) + \cdots + 2(x_{2k+1} - x)$$

Equating it to zero, so $\frac{d}{dx}\left(f_2(x)\right) = 0$

$$2(x_1 - x) + 2(x_2 - x) + \cdots + 2(x_{2k+1} - x) = 0$$
$$2(x_1 + x_2 + x_3 + \cdots + x_{2k+1}) - 2k + 1(2x) = 0$$
$$(x_1 + x_2 + x_3 + \cdots + x_{2k+1}) - 2k + 1(x) = 0$$
$$(x_1 + x_2 + x_3 + \cdots + x_{2k+1}) = 2k + 1(x)$$
$$x = \frac{(x_1 + x_2 + x_3 + \cdots + x_{2k+1})}{2k + 1}$$

So, the x which minimizes the function $\sqrt{\sum_{j=1}^{2k+1}\left(x_j - x\right)^2}$ is the mean of the numbers $x_1 < x_2 < \cdots < x_{2k+1}$

B. We need to show the number x that minimizes the 1-norm $f_1(x):\sum_{j=1}^{2k+1}|x_j - x|$ is their median x_{k+1}.

Consider the value of $f(x_{k+1})$. We get $f_1(x_{k+1}):\sum_{j=1}^{2k+1}|x_j - x_{k+1}|$

This can also be written as:

$$f_1(x_{k+1}) = \sum_{j=1}^{k}\left|x_j - x_{k+1}\right| + |x_{k+1} - x_{k+1}| + \sum_{j=k+2}^{2k+1}|x_j - x_{k+1}|$$

Since $x_1 < x_2 < \cdots < x_{2k+1}$ we get

$$f_1(x_{k+1}) = \sum_{j=1}^{k}\left|x_j - x_{k+1}\right| + \sum_{j=k+2}^{2k+1}|x_j - x_{k+1}|$$
$$= \sum_{j=1}^{k}\left|x_j - x_{k+1}\right| + \sum_{j=k+2}^{2k+1}|x_j - x_{k+1}|$$

Since $x_1 < x_2 < \cdots < x_{2k+1}$ we get to know that $x_j - x_{k+1}$ for all j from 1 to k will be a negative value and $x_j - x_{k+1}$ for all j from k+2 to 2k+1 will be a positive value.

$$f_1(x_{k+1}) = (x_{k+1} - x_1) + \cdots + (x_{k+1} - x_k) + (x_{k+2} - x_{k+1}) + \cdots + (x_{2k+1} - x_{k+1})$$

Now consider the value of $f(x_{k+1} + \varepsilon)$. Where $\varepsilon \in [0, x_{k+2} - x_{k+1})\ or\ (x_k - x_{k+1}, 0]$

We get $f_1(x_{k+1} + \varepsilon) :\ \sum_{j=1}^{2k+1} |x_j - (x_{k+1} + \varepsilon)|$

This can also be written as:

$$f_1(x_{k+1} + \varepsilon) = \sum_{j=1}^{k} |x_j - (x_{k+1} + \varepsilon)| + |x_{k+1} - (x_{k+1} + \varepsilon)| + \sum_{j=k+2}^{2k+1} |x_j - (x_{k+1} + \varepsilon)|$$

Consider above equation as Equation [**1**]

Since $x_1 < x_2 < \cdots < x_{2k+1}$ we get to know that $x_j - x_{k+1}$ for all j from 1 to k will be a negative value and $x_j - x_{k+1}$ for all j from k+2 to 2k+1 will be a positive value. So $x_j - x_{k+1} - \varepsilon$ for all j from 1 to k will also be a negative value and $x_j - x_{k+1} - \varepsilon$ for all j from k+2 to 2k+1 will also be a positive value.

Hence,

$$\sum_{j=1}^{k} |x_j - (x_{k+1} + \varepsilon)| = \sum_{j=1}^{k} x_{k+1} - x_j + \varepsilon = (x_{k+1} - x_1 + \varepsilon) + \cdots + (x_{k+1} - x_k + \varepsilon)$$
$$= (x_{k+1} - x_1) + \cdots + (x_{k+1} - x_k) + k\varepsilon \text{ ----- Equation } [\mathbf{2}]$$

$$\sum_{j=k+2}^{2k+1} |x_j - (x_{k+1} + \varepsilon)| = \sum_{j=k+2}^{2k+1} x_j - x_{k+1} - \varepsilon$$
$$= (x_{k+2} - x_{k+1} - \varepsilon) + \cdots + (x_{2k+1} - x_{k+1} - \varepsilon)$$
$$= (x_{k+2} - x_{k+1}) + \cdots + (x_{2k+1} - x_{k+1}) - k\varepsilon \text{ ----- Equation } [\mathbf{3}]$$

Substituting equations [2] and [3] in [1] we get,

$$f_1(x_{k+1} + \varepsilon) = (x_{k+1} - x_1) + \cdots + (x_{k+1} - x_k) + k\varepsilon + |x_{k+1} - (x_{k+1} + \varepsilon)| + (x_{k+2} - x_{k+1}) + \cdots + (x_{2k+1} - x_{k+1}) - k\varepsilon$$

$|x_{k+1} - (x_{k+1} + \varepsilon)| = |\varepsilon|$ for any value of ε

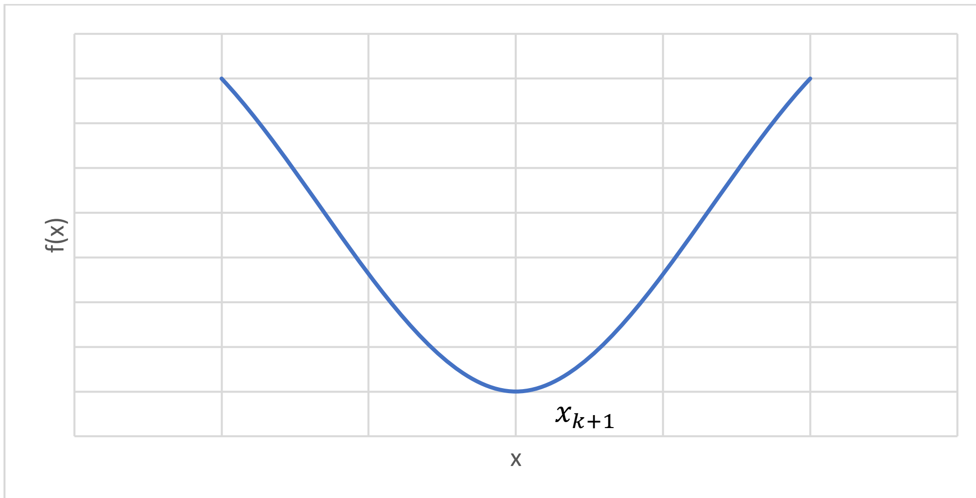
So, we get:

$$f_1(x_{k+1} + \varepsilon) = (x_{k+1} - x_1) + \cdots + (x_{k+1} - x_k) + |\varepsilon| + (x_{k+2} - x_{k+1}) + \cdots + (x_{2k+1} - x_{k+1})$$

Using the equation for: $f_1(x_{k+1})$ in the above equation we get

$$f_1(x_{k+1} + \varepsilon) = f_1(x_{k+1}) + |\varepsilon|$$

We see that as x increases the value of $f(x)$ tends to decrease and then again increase as shown in the diagram below. The point at which the function has the minimum value is at the point when $x = x_{k+1}$.



Also, since $f_1(x)$ is a convex function, the local minimum obtained is also a global minimum of the function.

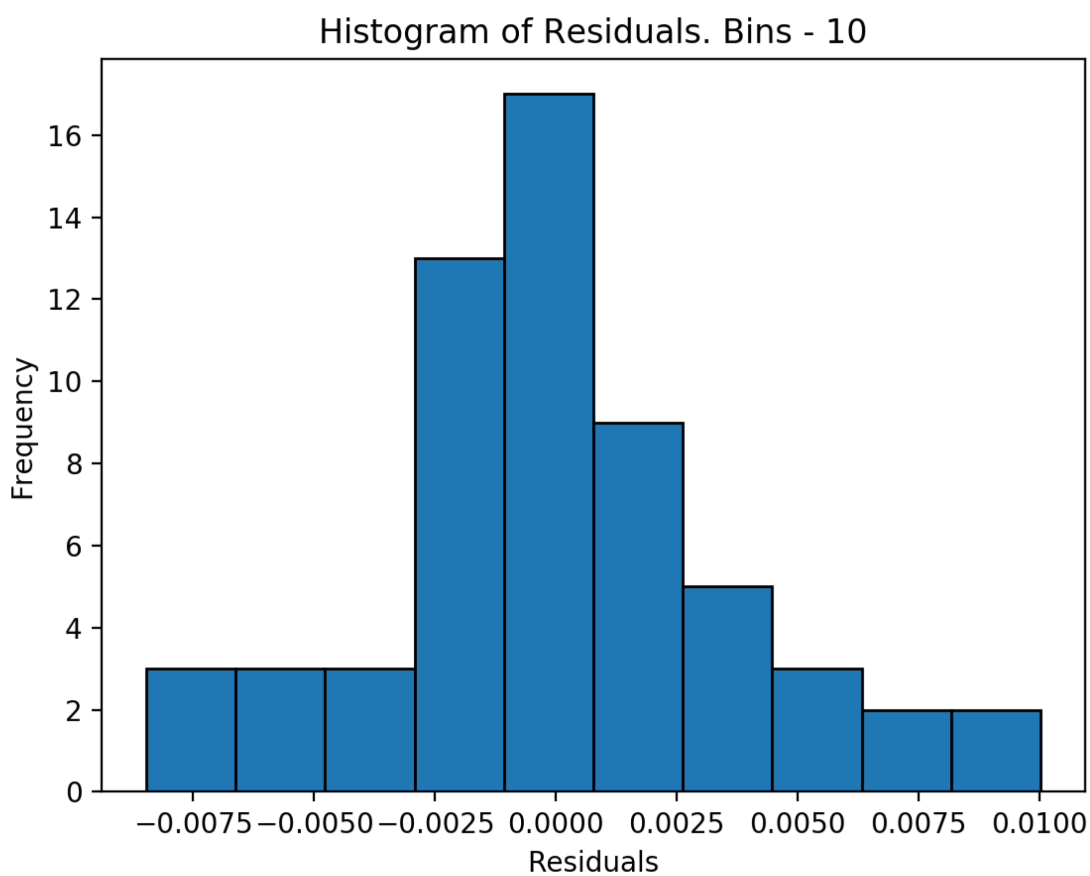
P3

A. Using regression to fit the model below for the provided data the residuals were obtained.

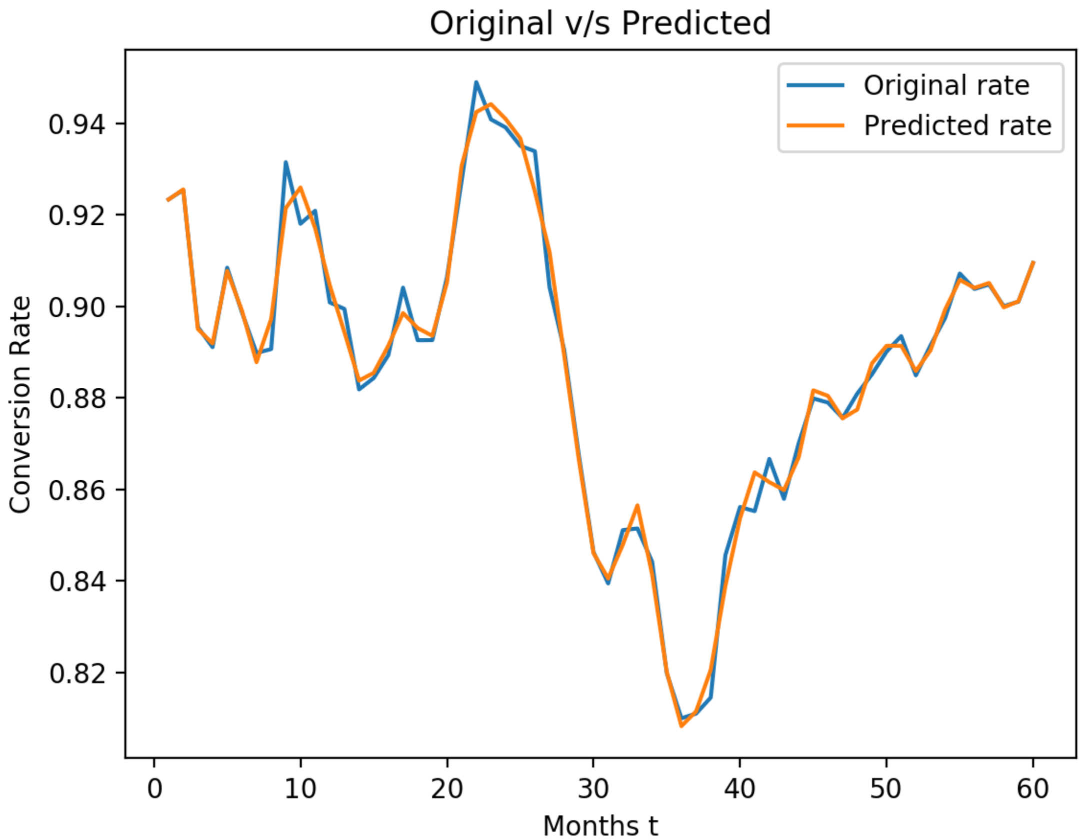
$$r(t) \simeq c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \sum_{j=1}^{60} a_j \cos(\frac{2\pi t}{j}) + \sum_{j=1}^{60} b_j \sin(\frac{2\pi t}{j}) .$$

The coefficients obtained using L_2 norm least squares and the residuals is attached as a file: *HW3_ PE3_ A_ Output.txt*

The histogram of the residuals:



Comparison of the fit and the actual values is given below:



B. The given problem after adding a penalty term that penalizes the L_1 norm of the coefficients of the model can be formulated as follows:

$$\min ||Ax-b|| - ||x||$$

Consider,

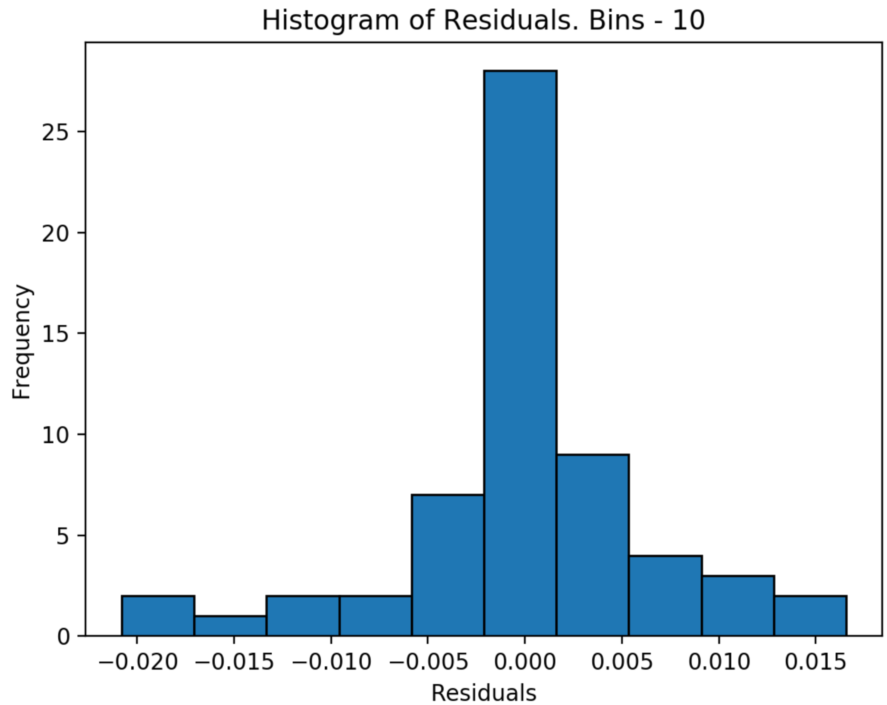
$$|Ax-b| \leq u \text{ and } |x| \leq v$$

Then the LP can be formulated as,

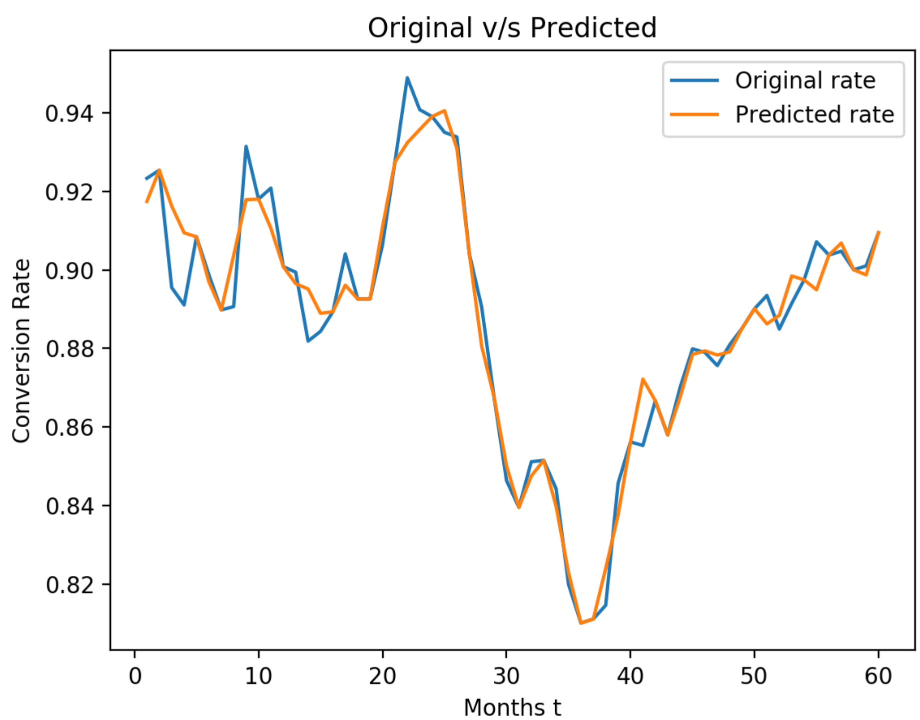
$$\begin{aligned} \min \sum_{i=1}^{60} u_i + \sum_{j=1}^{124} v_j \\ st. Ax - b \leq u \\ -Ax + b \leq u \\ x \leq v \\ -x \leq v \end{aligned}$$

The coefficients obtained using the L_1norm for the above problem is attached as the file *HW3_PE3_B_Output.txt*

The histogram of the residuals is given below:



Comparison of the fit and the actual values is given below:



P4

Given the LP as below

$$
\begin{array}{llllllll}
\max & -2x_1 & -3x_2 & -x_3 & -x_4 & & +x_6 & \\
\text{s.t.} & x_1 & -x_2 & & & & -x_6 & \leq 3 \\
 & x_1 & & & -x_4 & -x_5 & & \leq -1 \\
 & & & -x_3 & & & -x_6 & \leq -2 \\
 & & -x_2 & -x_3 & +x_4 & & +x_6 & \leq 4 \\
 & x_1 & & +x_3 & & +x_5 & +x_6 & \leq 6 \\
 & -x_1 & & & -x_4 & +x_5 & -x_6 & \leq -2 \\
 & & & & & x_1,\ \ldots,\ & x_6 & \geq 0
\end{array}
$$

A. Using Python numpy arrays and methods in it, a dictionary can be calculated by just knowing the basic and non-basic variables. By using the python program attached (*HW3_ P4.py*) and providing the matrix A, b and C of the LP the values were obtained as follows:

Equations used to obtain these solutions are:

$$b^* = A_b^{-1} b$$
$$Z = C_b A_b^{-1} + (-C_b A_b^{-1} A_n + C_n) x_n$$
$$-A_b^{-1} A_j = A_j^*$$

1. Basic variables: $\{x_1, x_2, x_3, w_1, w_2, w_3\}$

   Solution: $x_1 = 2, x_2 = -8, x_3 = 4, w_1 = -7, w_2 = -3, w_3 = 2$

   $x_4, x_5, x_6, w_4, w_5, w_6 = 0$

   Objective value: $+16$

   Objective row: $16 + 0x_4 - 6x_5 + 0x_6 - 3w_4 - 2w_5 - 4w_6$

2. Basic variables: $\{x_1, x_2, x_5, w_3, w_5, w_6\}$

   Solution: $x_1 = -1, x_2 = -4, x_5 = 0, w_3 = -2, w_5 = 7, w_6 = -3$

   $x_3, x_4, x_6, w_1, w_2, w_4 = 0$

   Objective value: $+14$

   Objective row: $14 + 4x_3 - 6x_4 - 6x_6 + 2w_1 + 0w_2 - 5w_4$

3. Basic variables: $\{x_1, x_2, x_6, w_4, w_5, w_6\}$

   Solution: $x_1 = -1, x_2 = -6, x_6 = 2, w_4 = -4, w_5 = 5, w_6 = -1$

   $x_3, x_4, x_5, w_1, w_2, w_3 = 0$

   Objective value: $+22$

   Objective row: $22 - 5x_3 - 6x_4 - 5x_5 - 3w_1 + 5w_2 + 4w_3$

B. Basic variables: $\{x_3, x_4, x_5, w_1, w_2, w_6\}$

Constant column will be [2.]
[6.]
[4.]
[3.]
[9.]
[0.]

Objective row: $-8 - 2x_1 - 4x_2 + 4x_6 - 2w_3 + 1w_4 + 0w_5$

Entering Variable: x_6

Using $-A_b^{-1}A_j = A_j^*$ the entering variable column for x_6 will be: [-1.]
[-2.]
[0.]
[1.]
[-2.]
[-1.]

Comparing the entering variable column and b^* the leaving variable will be w_6

Leaving Variable: w_6

Basic Variables in next Dictionary: $\{x_3, x_4, x_5, x_6, w_1, w_2\}$

C. Given the solution as

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = 1,\ x_6 = 4\,.$$

Finding the basic and non-basic variables by using the solution and plugging it to the original LP. We get $w_2, w_4 = 0$.

So, the non-basic variables will be $x_1, x_2, x_3, x_4, w_2, w_4$

And the basic variables will be $x_5, x_6, w_1, w_3, w_5, w_6$

Using these variables, the final dictionary will be

x_5	1	$+1x_1$			$-x_4$	$+w_2$	
x_6	4		$+x_2$	$+x_3$	$-x_4$		$-w_4$
w_1	7	$-1x_1$	$+2x_2$	$+x_3$	$-x_4$		$-w_4$
w_3	2		$+x_2$	$+2x_3$	$-x_4$		$-w_4$
w_5	1	$-2x_1$	$-x_2$	$-2x_3$	$+2x_4$	$-w_2$	$+w_4$
w_6	1		$+x_2$	$+x_3$	$+x_4$	$-w_2$	$-w_4$
z	4	$-2x_1$	$-2x_2$	$+0x_3$	$-2x_4$	$+0w_2$	$-w_4$

P5

Given the LP which is not in standard form as below

$$\max\ c^t x$$
$$\text{st. } Ax \leq b$$

We need to show that the dual of this problem is the LP below

$$\min\ b^t y$$
$$\text{st. } A^t y = c$$
$$y \geq 0$$

Converting the original LP into its standard form we need to add the constraint $x \geq 0$.
To do this let us consider $x = x_1 - x_2$ where $x_1, x_2 \geq 0$.
Now the problem becomes (in standard form)

$$\max\ c^t x_1 - c^t x_2$$
$$\text{st. } Ax_1 - Ax_2 \leq b$$
$$x_1, x_2 \geq 0.$$

The dual of the problem will be

$$\min\ b^T y$$
$$A^T y \geq c$$
$$-A^T y \geq -c$$
$$y \geq 0$$

Which is the same as

$$\min\ b^T y$$
$$A^T y = c$$
$$y \geq 0$$

This is the dual of the original LP problem we were trying to prove.

Collaborated with - Ketan Ramesh and Siddartha Shankar for discussion on ideas.