CSCI 5654 Spring 2020: Assignment #4
Reading: Games (Vanderbei Chapter 11), Slides on ILP (Vanderbei Chapter 23).
Due Date: Friday, March 20, 2020 (midnight)
Your Name:

P1. (30 points) Consider a two player game with the following payoff matrix:

$$\begin{bmatrix} -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -2 & -2 & 3 \end{bmatrix}$$

Note that a vector is *stochastic* if all its entries are non-negative and sum up to 1.

(a) Let us fix the row player strategy as the vector $\mathbf{x}_0 : (0.2, 0.3, 0.2, 0.2, 0.1)^T$. Pose an solve an LP for the best column player strategy corresponding to this strategy.

Write down the LP (you should express your LP in terms of $\mathbf{x}_0$ and A).

Solve the LP for the given $\mathbf{x}_0$ and write down its solution in terms of the strategy $\mathbf{y}$ of the column player. Note that $\mathbf{y}$ just needs to have a single non-zero entry in it.

What does this mean in terms of the player B's strategy? (just a single short sentence interpretation).

(b) Show that for $\mathbf{c} : (c_1, \ldots, c_n)^T$, the optimal solution for the LP

$$\min \mathbf{c}^T \mathbf{y} \text{ s.t. } \mathbf{y} \text{ is a stochastic vector}\,.$$

is equal to

$$\min\{c_1, c_2, \ldots, c_n\}\,.$$

(c) Use part (b) to show that

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}^T A \mathbf{y}\,,$$

for stochastic vectors $\mathbf{x}, \mathbf{y}$ is the solution to the LP

$$\max_{\mathbf{x}} \min \left(\mathbf{x}^T A_{(*,1)}, \mathbf{x}^T A_{(*,2)}, \ldots, \mathbf{x}^T A_{(*,n)}\right)\,.$$

Note: $A_{(*,j)}$ is the j^{th} column of matrix A.
Hint: Writing $\mathbf{x}^T A = \mathbf{c}$, what would c_j be? Use the result from part (b).

(d) Convert the problem

$$\begin{array}{ll} \max_{\mathbf{x}} & \min(\mathbf{c}_1^T \mathbf{x},\ \mathbf{c}_2^T \mathbf{x}, \ldots, \mathbf{c}_n^T \mathbf{x}) \\ \text{s.t.} & \mathbf{1}^T \mathbf{x} = 1 \\ & \mathbf{x} \geq 0 \end{array}$$

into a linear program.

(e) Show that the dual of the LP:

$$\begin{array}{ll} \max & t \\ & -A^T\mathbf{x} + \mathbf{1}t \ \leq 0 \\ & \mathbf{1}^T\mathbf{x} \qquad\quad\ = 1 \\ & \mathbf{x} \qquad\qquad\ \geq 0 \end{array}$$

is the LP:

$$\begin{array}{ll} \min & z \\ & -A\mathbf{y} + z \ \geq 0 \\ & \mathbf{1}^T\mathbf{y} \qquad\ = 1 \\ & \mathbf{y} \qquad\quad\ \geq 0 \end{array}$$

Note: $A_{(i,*)}$ is the i^{th} row of matrix A, it is a row vector.

(e) Compute the optimal strategy for the row and column players for the game in part (a). Write down the value of the game and the "equilibrium" strategies for each player.

P2. (10 points) Solve each of the problems below using a branch-and-bound method. Write down the solution obtained and the enumeration tree obtained. You may use any LP solver of your choice to solve the subproblems. Please do not use a ILP solver directly.

(A)

$$\begin{array}{llllll} \max & 2x_1 & -3x_2 & -2x_3 & -x_4 & \\ \text{s.t.} & x_1 & -x_2 & +x_3 & & \leq 5 \\ & x_1 & -2x_2 & -x_3 & +x_4 & \leq 3 \\ & x_1 & -x_2 & -x_3 & -x_4 & \leq -1 \\ & x_1, & x_2, & x_3, & x_4 & \in [-5, 5] \\ & x_1, & x_2, & x_3, & x_4 & \in \mathbb{Z} \end{array}$$

Hint: Use x_4 as the variable to branch on.

(B)

$$\begin{array}{llllll} \max & 2x_1 & & +3x_3 & +x_4 & \\ \text{s.t.} & x_1 & -x_2 & & +x_4 & \leq 1 \\ & & 2x_2 & & -x_4 & \leq 2 \\ & x_1 & & -x_3 & -2x_4 & \leq -1 \\ & -x_1 & & & +x_4 & \leq 1 \\ & x_1 & & & & \in \{-2, -1, 0, 1, 2\} \\ & & x_2 & & & \in \{-1, 0, 1\} \\ & & & x_3 & & \in \{0, 1\} \\ & & & & x_4 & \in \{-1, 0, 1\} \end{array}$$

P3. (10 points) Consider the final dictionaries for the LP relaxation of a few ILPs. Assuming all variables are integers, write down all the cutting planes:

Dictionary # 1:

x_1	0.666666666667	$-0.666667x_5$	$+0.333333x_4$
x_2	1	$-1x_5$	
x_3	2	$+4x_5$	$-1x_4$
z	1	$-1x_5$	

Dictionary # 2:

x_4	4.33333333333	$+0.333333x_8$	$+0.666667x_9$	$-0.333333x_3$
x_5	8.66666666667	$-0.333333x_8$	$+0.333333x_9$	$-2.666667x_3$
x_6	10			$-1x_3$
x_7	3	$-3x_8$	$+1x_9$	$-18x_3$
x_1	5.66666666667	$-0.333333x_8$	$-0.666667x_9$	$+0.333333x_3$
x_2	1.33333333333	$+0.333333x_8$	$-0.333333x_9$	$+2.666667x_3$
z	7		$-1x_9$	$-2x_3$

P4 (15 points) We are given sets of numbers $\langle S_1, \ldots, S_k \rangle$ such that each $S_i \subseteq \{1, \ldots, n\}$. For example, $n = 10$ and the sets are

$$S_1 : \{1, 3, 6\},\ S_2 :\ \{2, 7, 8\},\ S_3 :\ \{1, 8, 9\},\ S_4 :\ \{1, 6, 5, 3\}\,.$$

Our goal is to select a subset $S \subseteq \{1, 2, \ldots, n\}$ such that $S \cap S_i \neq \emptyset$ for $i = 1, \ldots, k$ and the sum of elements in the chosen set S is minimized.

Formulate a $0-1$ ILP for the problem for given $n, k, \langle S_1, \ldots, S_k \rangle$. Also, solve it for the example above.