CSCI 5654 - Linear Programming Homework - 1

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1 P1

Step 1

$$\begin{aligned} \text{maximize} \quad & -3x_1 + 5x_2 \\ \text{subject to} \quad & -4x_1 - x_2 \le 4 \\ & -2x_1 + x_2 \le 8 \\ & x_1 + 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned}$$

Step 2

$$\begin{aligned} \text{maximize} \quad & -3x_1 + 5x_2 \\ \text{subject to} \quad & -4x_1 - x_2 \le 4 \\ & -2x_1 + x_2 \le 8 \\ & x_1 + 2x_2 \le 4 \\ & x_2 = x_+ + x_- \\ & x_1 \ge 0,\ \ x_+ \ge 0,\ \ x_- \ge 0 \end{aligned}$$

Step 3 - Standard Form

$$\begin{aligned} \text{maximize} \quad & -3x_1 + 5x_+ - 5x_- \\ \text{subject to} \quad & -4x_1 - x_+ + x_- \le 4 \\ & -2x_1 + x_+ - x_- \le 8 \\ & x_1 + 2x_+ - 2x_- \le 4 \\ & x_2 = x_+ + x_- \\ & x_1 \ge 0,\ \ x_+ \ge 0,\ \ x_- \ge 0 \end{aligned}$$

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2 P2

2.a

Step 1 - Adding Slack variables

$$\begin{array}{rl} \text{maximize} & z = -3x_1 + 5x_+ - 5x_- \\ \text{subject to} & -4x_1 - x_+ + x_- + w_1 = 4 \\ & -2x_1 + x_+ - x_- + w_2 = 8 \\ & x_1 + 2x_+ - 2x_- + w_3 = 4 \\ & x_2 = x_+ + x_- \\ & x_1 \geq 0, \;\; x_+ \geq 0, \;\; x_- \geq 0 \end{array}$$

Step 2 - Initial Dictionary

$$\begin{array}{l} w_1 = 4 + 4x_1 + x_+ - x_- \\ w_2 = 8 + 2x_1 - x_+ + x_- \\ w_3 = 4 - x_1 - 2x_+ + 2x_- \\ \hline \;\; z = 0 - 3x_1 + 5x_+ - 5x_- \end{array}$$

2.b

Step 1 - Feasible Test

Setting the non-basic variables x_1, x_+, x_- to 0

By doing this we get the solution: $w_1 = 4, w_2 = 8, w_3 = 4, z = 0$

Since the solution satisfies the constraints, the dictionary is feasible.

Step 2 - Simplex - Pivot

The variable $+5x_+$ maximizes the objective function

Hence it can be chosen as the incoming variable

Choosing x_+ as entering we see that it the variable x_+ has no constraint on w_1

For w_2, $x_+ \leq 8$ and for w_3, $x_+ \leq 2$The variable w_3 imposes lowest constraint.

Hence it can be chosen as the leaving variable

Rearranging and using x_+ we get

$$\begin{aligned} x_+ &= 2 - 0.5x_1 + x_- - 0.5w_3 \\ w_1 &= 6 + 3.5x_1 + 0x_- - 0.5w_3 \\ w_2 &= 6 + 2.5x_1 + 0x_- + 0.5w_3 \\ z &= 10 - 5.5x_1 + 0x_- - 2.5w_3 \end{aligned}$$

2.c

The x_+ variable is the incoming variable and w_3 is the outgoing variable

2.d

Resulting Dictionary

$$\begin{array}{l} w_1 = 6 + 3.5x_1 + 0x_- - 0.5w_3 \\ w_2 = 6 + 2.5x_1 + 0x_- + 0.5w_3 \\ x_+ = 2 - 0.5x_1 + x_- - 0.5w_3 \\ \hline z = 10 - 5.5x_1 + 0x_- - 2.5w_3 \end{array}$$

3 P3

Based on the conditions given in the problem, the following linear program is set up. I have used the python pulp package to solve this problem.

maximize $1.451x_1 + 2.683x_2 + 5.898x_3 + 2.102x_4 + 5.709x_5 + 4.519x_6 + 7.176x_7 + 6.075x_8 + 5.718x_9 + 7.442x_10 + 1.234x_11 + 4.680x_12 + 7.229x_13 + 9.589x_14 + 6.497x_15$

Constraints in next page.

Solving the Linear problem in python an optimal solution is obtained.

Solution is - Profit - 8513.474870720001

x10 = 147.6378, x12 = 53.763441, x14 = 459.68558, x3 = 467.14419 and 0 for all other variables. The given problem is feasible since we have obtained a solution. And it is the optimal solution.

It is bounded since, if unbounded we would obtain infinity as an answer.

Hence, the problem is feasible, bounded and has an optimal solution.

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SUBJECT TO
_C1: 2.563 x1 + 10.16 x10 + 1.961 x11 + 9.3 x12 + 11.672 x13 + 10.877 x14
 + 12.137 x15 + 4.307 x2 + 6.422 x3 + 3.488 x4 + 6.581 x5 + 8.993 x6
 + 11.481 x7 + 11.73 x8 + 9.27 x9 <= 10000

_C2: 2.563 x1 + 10.16 x10 + 1.961 x11 + 9.3 x12 + 11.672 x13 + 10.877 x14
 + 12.137 x15 + 4.307 x2 + 6.422 x3 + 3.488 x4 + 6.581 x5 + 8.993 x6
 + 11.481 x7 + 11.73 x8 + 9.27 x9 >= 0

_C3: 2.563 x1 + 10.16 x10 + 11.672 x13 + 3.488 x4 <= 3500

_C4: 2.563 x1 + 10.16 x10 + 11.672 x13 + 3.488 x4 >= 1500

_C5: 10.877 x14 + 4.307 x2 + 6.581 x5 + 11.73 x8 + 9.27 x9 <= 6500

_C6: 10.877 x14 + 4.307 x2 + 6.581 x5 + 11.73 x8 + 9.27 x9 >= 4500

_C7: 1.961 x11 + 12.137 x15 + 6.422 x3 + 7.176 x7 <= 3000

_C8: 1.961 x11 + 12.137 x15 + 6.422 x3 + 7.176 x7 >= 1000

_C9: 9.3 x12 + 8.993 x6 <= 2500

_C10: 9.3 x12 + 8.993 x6 >= 500

_C11: 2.563 x1 + 1.961 x11 + 11.73 x8 <= 3000

_C12: 2.563 x1 + 1.961 x11 + 11.73 x8 >= 0

_C13: 12.137 x15 + 4.307 x2 + 6.422 x3 + 6.581 x5 + 8.993 x6 + 11.481 x7
 <= 4000

_C14: 12.137 x15 + 4.307 x2 + 6.422 x3 + 6.581 x5 + 8.993 x6 + 11.481 x7 >= 0

_C15: 11.672 x13 + 3.488 x4 + 9.27 x9 <= 5000

_C16: 11.672 x13 + 3.488 x4 + 9.27 x9 >= 0

_C17: 10.16 x10 + 9.3 x12 + 10.877 x14 <= 7000

_C18: 10.16 x10 + 9.3 x12 + 10.877 x14 >= 0

_C19: 2.563 x1 + 10.16 x10 + 1.961 x11 + 11.672 x13 + 4.307 x2 + 6.422 x3
 + 8.993 x6 + 11.481 x7 + 11.73 x8 + 9.27 x9 <= 10000

_C20: 2.563 x1 + 10.16 x10 + 1.961 x11 + 11.672 x13 + 4.307 x2 + 6.422 x3
 + 8.993 x6 + 11.481 x7 + 11.73 x8 + 9.27 x9 >= 2000
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Figure 1: Constraints for the problem P3

4 P4

4.a

It is given that a set S is considered convex if any two points $x_1, x_2 \in$ S and the point

$\lambda\, x_1$ + (1 - λ)$x_2 \in$ S for all $\lambda \in$ [0, 1]

Let Set F be a set of feasible solutions F : {x | Ax ≤ b}. Now consider 2 points $x_1, x_2 \in$ F.

Using the above conditions, F can be set to be convex if the point $\lambda\, x_1$ + (1 - λ)$x_2 \in$ F.

Which means $\lambda\, x_1$ + (1 - λ)x_2 must be one of the feasible solutions. Hence it should satisfy Ax ≤ b.

Since x_1 and x_2 are feasible solutions, we have Ax_1 ≤ b and Ax_2 ≤ b

$$\lambda A x_1 \leq \lambda b$$
$$(1-\lambda) A x_2 \leq (1-\lambda) b$$

Adding the above equations we get

$$\lambda A x_1 + (1-\lambda) A x_2 \leq \lambda b + (1-\lambda) b$$
$$A(\lambda x_1 + (1-\lambda) x_2) \leq \lambda b + b - \lambda b$$
$$A(\lambda x_1 + (1-\lambda) x_2) \leq b$$

From above equation we see that even ($\lambda\, x_1$ + (1 - λ)x_2) is a feasible solution and so it $\in$ F.

Hence F is a Convex set.

4.b

It is given that optimal objective value $z^* = c^t x_1 = c^t x_2$ and both $x_1 x_2$ satisfy Ax ≤ b

$$A x_1 \leq b$$
$$A x_2 \leq b$$

Adding the above equations we get

$$A x_1 + A x_2 \leq 2b$$
$$A \frac{(x_1 + x_2)}{2} \leq b$$

So, $\frac{(x_1+x_2)}{2}$ is another point which satisfies the linear program and is equal to z^*

In other words, the mid point satisfies the program and results in the same optimal value.

We can now choose another new point x_i which is between x_1 and $\frac{(x_1+x_2)}{2}$.

This new point x_i will also provide us with the same z^*.

Hence, by continuing this method we can obtain infinite distinct points which provide the same optimal solutions for the problem.

5 P5

5.a

Given x_5 must be between x_1 and x_3. So, $x_1 \leq x_5 \leq x_3$ or $x_3 \leq x_5 \leq x_1$

If $x_1 \leq x_5 \leq x_3$ Then the linear program can be expressed as

$$\begin{aligned} &x_1 - x_5 \leq 0 \\ &x_5 - x_3 \leq 0 \\ &x_1, x_3, x_5 \geq 0 \end{aligned}$$

If $x_3 \leq x_5 \leq x_1$ Then the linear program can be expressed as

$$\begin{aligned} &x_3 - x_5 \leq 0 \\ &x_5 - x_1 \leq 0 \\ &x_1, x_3, x_5 \geq 0 \end{aligned}$$

5.b

x1 must be between 40% and 60% of the $\sum_{i=1}^{n} x_i$

$$0.4 \sum_{i=1}^{n} x_i \leq x_1 \leq 0.6 \sum_{i=1}^{n} x_i$$

Then the linear program can be expressed as

$$\begin{aligned} &-0.6x_1 + 0.4 \sum_{i=2}^{n} x_i \leq 0 \\ &0.4x_1 - 0.6 \sum_{i=2}^{n} x_i \leq 0 \\ &x_1, x_2, \ldots .x_n \geq 0 \end{aligned}$$

5.c

Given at least one of $x_1, \ldots, x_n$ must be 0

Which can be expressed as $\prod_{i=1}^{n} x_i = 0$. Which cannot be expressed as a linear program.

Let Set F be a set of feasible solutions F : {x | Ax ≤ b}

Consider that all points which satisfy $\prod_{i=1}^{n} x_i = 0$ form a convex set of feasible solutions F.

Now consider 2 points x_1 and x_2 where the points are of dimension $n = 2$.

The points are $x_1 = (x_a, x_b)$ and $x_2 = (x_c, x_d)$

In order for F to be a convex set it must satisfy $\lambda\, x_1$ + (1 - λ)$x_2 \in$ F.

According to the constraint, in the point x_1 either $x_a = 0$ or $x_b = 0$ and in point x_2 either $x_c = 0$ or $x_d = 0$, we can see for any λ = (0,1) we get a new point which does not satisfy $\prod_{i=1}^{n} x_i = 0$

For example when λ = 0.5, we get the point as $(0.5x_a, 0.5x_d)$

Thus, for any λ != 0,1 we get points not in the set F.

So, proof by contradiction, we can say F is not convex.

Also, since we got points for n=2 which doesn't belong to set F, by proof of induction we can show that for n=n also we can find similar points which does not make F convex.

Hence proved that, since F is not a convex set, the initial equation, $\prod_{i=1}^{n} x_i = 0$ is not a convex constraint and the problem cannot be expressed as a linear program.

6 P6

Given $0 \le x_1 \le 10$ and $0 \le x_2 \le 15$. We also have indicator variables $w_1, w_2 \in 0, 1$. Goal is to express $x_1 x_2 = 0$ as linear inequalities

Using the hint provided in question, We can force x_1 to be 0 when $w_1 = 1$ and $w_1 = 0$ places no constraint on x_1.

This can be expressed by the equations,

$$x_1 + 10w_1 \le 10$$
$$x_2 + 15w_2 \le 15$$

By doing this when $w_1 = 1$,

$$x_1 + 10 \le 10$$
$$x_1 \le 0$$

which forces $x_1 = 0$ since x_1 must be ≥ 0.

Similarly when $w_2 = 1$,

$$x_2 + 15 \le 15$$
$$x_2 \le 0$$

which forces $x_2 = 0$ since x_2 must be ≥ 0.

This can be applied to both equations forces the conditions provided in the hint.

By adding the condition, $w_1 + w_2 = 1$ we can force either w_1 or w_2 to be 1 at a particular time.

Hence, $x_1 x_2 = 0$ as linear inequalities as below,

$$\begin{aligned}
&x_1 + 10w_1 \leq 10 \\
&x_2 + 15w_2 \leq 15 \\
&w_1 + w_2 = 1 \\
&x_1, x_2, w_1, w_2 \geq 0
\end{aligned}$$