CSCI 5654 Spring 2020: Assignment #3
Due Date: Thursday, Feb 27, Midnight

P1. (10 points) Express the problems below as linear programs by adding suitable additional variables.

(A) $\min_{x_1,x_2,x_3 \in \mathbb{R}^3} \ \max(2x_1 + 3x_2 - 5x_3,\ x_1,\ x_2,\ 2x_1 - x_2 + x_3)$.

(B) $\min_{x_1,x_2,x_3 \in \mathbb{R}^3} \ (|x_1 + x_2| + |x_2 - x_3| + |x_3 - x_1| + |x_1 + x_2 + x_3|)$.

(C)

$$\begin{array}{rl} \min & \max\left(|x_1|,\ |x_2|,\ |x_3|,\ |x_1 + x_2|\right) \\ \textsf{subj.to} & x_1 - x_2 \leq 5 \\ & x_2 \leq 3 \end{array}$$

P2. (20 points) Let $x_1 < x_2 < \ldots < x_{2k+1}$ be $2k+1$ distinct numbers. (a) Show that the number x that minimizes the 2-norm $f_2(x)$: $\sqrt{\sum_{j=1}^{2k+1}(x_j - x)^2}$ is the mean and (b) the number x that minimizes the 1-norm $f_1(x)$: $\sum_{j=1}^{2k+1} |x_j - x|$ is their median x_k.
Hints: (A) just use calculus to minimize the derivatives; (B) Compare the value of $f_1(x_k)$ against $f_1(x_k + \epsilon)$ for any ϵ. Drawing a picture helps.

P3. (20 points) The attached spreadsheet `euro-usd-data.csv` has data over the past 5 years of the exchange rate of USD vs. Euros. We wish to model the rate $r(t)$ over time t where t is number of months elapsed since February 2015. For your convenience, we have provided this under the column "t" in the sheet.
(A) Use regression to fit a model :

$$r(t) \simeq c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \sum_{j=1}^{60} a_j \cos(\frac{2\pi t}{j}) + \sum_{j=1}^{60} b_j \sin(\frac{2\pi t}{j}) \,.$$

Use L_2 norm least squares problems and write down the coefficients of the model you obtain. Also plot the histogram of the residuals. We can provide help for solving this problem using Matlab or Python.
(B) Solve problem (A) but use the L_1 norm least squares. Also add a penalty term that penalizes the L_1 norm of the coefficients of the model. Compare the resulting model and the residuals.

For both parts, submit the code separately. To your submission, attach the plots of the fit for each model against original data and the residuals. Make sure that all plots are titled and axes labeled clearly.

P4 (20 points). Consider the linear programming problem below:

$$
\begin{array}{llllllll}
\max & -2x_1 & -3x_2 & -x_3 & -x_4 & & +x_6 & \\
\text{s.t.} & x_1 & -x_2 & & & & -x_6 & \leq 3 \\
 & x_1 & & & -x_4 & -x_5 & & \leq -1 \\
 & & & -x_3 & & & -x_6 & \leq -2 \\
 & & -x_2 & -x_3 & +x_4 & & +x_6 & \leq 4 \\
 & x_1 & & +x_3 & & +x_5 & +x_6 & \leq 6 \\
 & -x_1 & & & -x_4 & +x_5 & -x_6 & \leq -2 \\
 & & & & & x_1, & \ldots, \; x_6 & \geq 0
\end{array}
$$

We add slack variables $w_1, w_2, \ldots, w_6$ for the 6 constraints. **(A)** Compute the solution associated with the dictionary and the objective rows of the dictionaries for the following set of basic variables. Do not forget to write down the value of the objective functions.

1. $\{x_1, x_2, x_3, w_1, w_2, w_3\}$.
2. $\{x_1, x_2, x_5, w_3, w_5, w_6\}$.
3. $\{x_1, x_2, x_6, w_4, w_5, w_6\}$.

(B) Perform one step of the revised simplex method for the dictionary with the basis:

$$\{x_3, x_4, x_5, w_1, w_2, w_6\}$$

1. Compute the constant column and objective rows of this dictionary.
2. Choose the entering variable with the largest coefficient.
3. Set up the equations to construct the column for the entering variable.
4. Choose the leaving variable.
5. Write down the basic variables in the next dictionary.

You may use MATLAB or Python to perform the matrix calculations required. You are encouraged to code up this process as this will be important for your programming assignment.
(C) Write down the final dictionary for the problem corrsponding ot the optimal solution

$$x_1 = 0, \; x_2 = 0, \; x_3 = 0, \; x_4 = 0, \; x_5 = 1, \; x_6 = 4\,.$$

You should use matlab or numpy to perform the matrix calculations.
P5 (10 points). Prove that the dual of the problem (not in standard form):

$$
\begin{array}{lll}
\max & \mathbf{c}^t\mathbf{x} & \\
\text{s.t.} & A\mathbf{x} & \leq \mathbf{b}
\end{array}
$$

is the LP

$$
\begin{array}{lll}
\min & \mathbf{b}^t\mathbf{y} & \\
\text{s.t.} & A^t\mathbf{y} & = \mathbf{c} \\
 & \mathbf{y} & \geq 0
\end{array}
$$