CSCI 5654 Spring 2020: Assignment #5
Reading: Interior Point Method (Vanderbei Chapters 17, 18).
Due Date: Friday, April 24, 2020
Your Name:

P1. (15 points) (A, 10 points) Consider the equality constrained minimization problem:

$$\min\ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}\,.$$

wherein Q is a $n \times n$ positive definite matrix, $\mathbf{c}$ is a $n \times 1$ vector, A is a $m \times n$ matrix where $m < n$ and $\mathbf{b}$ is a $m \times 1$ vector. Show that any local minimum $\mathbf{x}$ satisfies the constraints:

$$\begin{aligned} Q\mathbf{x} + A^T\mathbf{y} &= -\mathbf{c} \\ A\mathbf{x} &= \mathbf{b} \end{aligned}$$

wherein $\mathbf{y}$ is a $m \times 1$ vector.
(B, 5 points) Using the fact that Q is invertible and assuming that $(AQ^{-1}A^T)$ is also invertible, show that the system of equations above has a unique solution for $(\mathbf{x}, \mathbf{y})$.

P2. (35 points) Consider the optimization problem:

$$\begin{array}{lll} \min & \sum_{i=1}^n q_i x_i^2 & \\ \text{s.t} & A\mathbf{x} + \mathbf{w} & = \mathbf{b} \\ & \mathbf{x}, \mathbf{w} & \geq 0 \end{array}$$

For simplicity, the objective may be written as $\mathbf{x}^t Q\mathbf{x}$ where $Q = \mathsf{diag}(\mathbf{q})$.

(A, 5 points) Formulate a log barrier problem with the log terms weighted by the factor $\mu \geq 0$.

(B, 5 points) Write down the Langrangian for the log barrier problem involving the primal variables $(\mathbf{x}, \mathbf{w})$ and dual variables $\mathbf{y}$ corresponding to the equality constraints $A\mathbf{x} + \mathbf{w} = \mathbf{b}$.

(C, 15 points) Write down the equations for the minimizers of the Langrangian. For convenience, you may use $X = \mathsf{diag}(\mathbf{x})$, $W = \mathsf{diag}(\mathbf{y})$, and $Y = \mathsf{diag}(\mathbf{y})$. Also you can use the variables $\mathbf{z}$ to denote the expression $A^t\mathbf{y} - 2Q\mathbf{x}$, and $Z = \mathsf{diag}(\mathbf{z})$.

(D, 10 points) Write down the KKT equations for a step $(\Delta x, \Delta y, \Delta w, \Delta z)$ given the current point $(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z})$. You may assume that $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z} > 0$.

P3. (20 point) Let $f(\mathbf{c})$ be defined as the function that maps input $\mathbf{c}$ to the optimal solution of the LP max $\mathbf{c}^T$ $\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}$. For simplicity, we assume that $A\mathbf{x} \leq \mathbf{b}$ describes a non empty and bounded polyhedron (i.e, we assume $f(\mathbf{c})$ is defined for all $\mathbf{c}$).

(A, 7 points) Show that for all $\mathbf{c}_1, \mathbf{c}_2$, we have $f(\mathbf{c}_1 + \mathbf{c}_2) \leq f(\mathbf{c}_1) + f(\mathbf{c}_2)$.

(B, 8 points) Show that $f(\mathbf{c})$ is a convex function.

(C, 5 points) Show that $f(\mathbf{y})$ can be written as a piecewise linear function

$$\max(\mathbf{a}_1^t\mathbf{y}, \cdots, \mathbf{a}_N^t\mathbf{y})$$

for some $\mathbf{a}_1, \ldots, \mathbf{a}_N$. (Hint: If a LP has an optimal solution then one of the vertices of its feasible region is also optimal).

(P4. extra credit Consider ellipsoids $E_1 : \ E(\mathbf{0}, A_1)$ and $E_2 : \ E(\mathbf{0}, A_2)$.

(A) Show that $E_1 \subseteq E_2$ iff $\mathbf{x}^t A_2^{-1} \mathbf{x} \leq \mathbf{x}^t A_1^{-1} \mathbf{x}$ for all $\mathbf{x}$.

(B) Show that $A_1 - A_2$ is positive semidefinite iff $(A_2^{-1} - A_1^{-1})$ is positive semidefinite.

(C) Show that $E_1 \subseteq E_2$ iff $A_1 - A_2$ is positive semidefinite.