CSCI 5654 - Linear Programming - Homework – 4

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P1

A. Given the payoff matrix

$$\begin{bmatrix} -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -2 & -2 & 3 \end{bmatrix}$$

And the row player strategy as $x_0 = [0.2, 0.3, 0.2, 0.2, 0.2\,]^T$. The LP for choosing the best column player strategy can be formulated as below

$$\begin{aligned} &\min \ x_0^T A y \\ &\text{s.t } 1^T y = 1 \\ &y \geq 0 \end{aligned}$$

y representing the column player's strategy. Using x_0 the LP can be written as

$$\begin{aligned} &\min -0.2y_1 + 0y_2 + 0.4y_3 - 0.5y_4 + 0.3y_5 \\ &\text{s.t } y_1 + y_2 + y_3 + y_4 + y_5 = 1 \\ &y_1, y_{2,}, y_3, y_4, y_5 \geq 0 \end{aligned}$$

Solving the LP using the code we get the best column player strategy:

$$y = [0, 0, 0, 1, 0]$$

Which means the column player chooses column 4 that is $[0, -1, 1, -1, -2\,]^T$ as his best strategy to penalize the row player.

B. Given that the vector c = $(c_1, \ldots, c_n)^T$ and the LP $min\ c^T\ y$ s.t. y is a stochastic vector, the LP can we written as

$$\min c^T y$$
$$\text{s.t } 1^T y = 1$$
$$y \geq 0$$

So,

$$c^T y = (c_1, \ldots, c_n)^T (y_1 \ldots y_n)$$
$$= c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

The LP can be written in standard form as below which is the primal.

$$\max -c^T y$$
$$\text{s.t } \sum_{i=1}^{n} y_i \leq 1$$
$$\sum_{i=1}^{n} -y_i \leq -1$$
$$y \geq 0$$

The dual of the above LP using w_1, w_2 as the dual variables will be:

$$\min\ w_1 - w_2$$
$$\text{s.t } \sum_{i=1}^{n} w_1 - w_2 \geq -c_i$$
$$w_1, w_2 \geq 0$$

The dual constraint is same as $\sum_{i=1}^{n} w_2 - w_1 \leq c_i$.

For all the constraints to hold true, the difference between $w_2\ and\ w_1$ must be less than the minimum of all c_i, which will be $\min(c_1, c_2 \ldots c_n)$.

Based on the objective function the values of w should be such that $w_1 - w_2$ be the minimum, that will be possible when we have optimal solution equal to $-\min(c_1, c_2 \ldots c_n)$.

So, optimal solution of the dual is therefore $-\min(c_1, c_2 \ldots c_n)$

Using the strong duality theorem, the primal should also have the same solution which will be $-\min(c_1, c_2 \ldots c_n)$.

Hence the solution for the original primal LP is $-\min(c_1, c_2 \ldots c_n)$.

But the problem is in standard form, so we are $\max -c^T y$. going back to the original problem where we have objective to $\min c^T y$. The optimal solution to the original problem will be $\min(c_1, c_2 \ldots c_n)$.

C. We are given that the vectors x and y are stochastic. When the row player has a fixed strategy x, then we know the LP can be formulated as

$$\min x^T A y$$
$$\text{s.t } 1^T y = 1$$
$$y \geq 0$$

Let us consider that $x^T A = c$. Then, based on the results from Part B we know that the optimal solution for the above LP is $\min(c_1, c_2 \dots c_n)$.

We know that based on A the c will be a $1 \times n$ matrix

$$c_i = x^T A_{(*,i)}$$

We have to show that the LP

$$\max_{\mathbf{x}} \min\left(\mathbf{x}^T A_{(*,1)}, \mathbf{x}^T A_{(*,2)}, \dots, \mathbf{x}^T A_{(*,n)}\right)$$

is equal to $\max_x \min_y x^T A y$

Substituting the value of c_i calculated we get

$$\max_x \min_y x^T A y = \max_x \min(x^T A_{(*,1)}, x^T A_{(*,2)}, \dots, x^T A_{(*,n)})$$
$$\text{s.t } 1^T x = 1$$
$$x \geq 0$$

So, converting the $\max_x \min_y x^T A y$ into an LP we get,

$$\max_x \min_y x^T A y = \max_x \min(x^T A_{(*,1)}, x^T A_{(*,2)}, \dots, x^T A_{(*,n)})$$
$$\text{s.t } 1^T x = 1$$
$$x \geq 0$$

which is the question, and hence solution to this LP will be the same solution for the original problem $\max_x \min_y x^T A y$

D. Given problem as

$$\begin{aligned} \max_{\mathbf{x}} \quad & \min(\mathbf{c}_1^T \mathbf{x},\ \mathbf{c}_2^T \mathbf{x}, \dots, \mathbf{c}_n^T \mathbf{x}) \\ \text{s.t.} \quad & \mathbf{1}^T \mathbf{x} = 1 \\ & \mathbf{x} \geq 0 \end{aligned}$$

Consider a variable v such that $v \geq \min(c_1^T x, c_2^T x, \dots, c_n^T x)$

Then we can formulate the LP as

$$\max v$$
$$\text{s.t } v \leq c_1^T x$$
$$\dots$$
$$\dots$$
$$v \leq c_n^T x$$
$$1^T x = 1$$
$$x \geq 0$$

E. Given the primal LP

$$\begin{aligned} \max \quad & t \\ & -A^T\mathbf{x} + \mathbf{1}t && \leq 0 \\ & \mathbf{1}^T\mathbf{x} && = 1 \\ & \mathbf{x} && \geq 0 \end{aligned}$$

In order to take the dual let us use y representing the dual variable corresponding to the 1st constraint and z be the dual variable corresponding to the 2nd constraint. Based on the constant term in the second condition, the objective of the dual will be to minimize z.

In order to formulate the constraints, we take the values of x row wise, so they will be $-A^T$ for 1st constraint and 1^T for the 2nd constraint. The coeff of x in the objective is 0. This gives the following constraint

$$-Ay + 1z \geq 0$$

We get the next constraint using the dual variable y which corresponds to $1t$ in the primal. The coeff of t in the objective is 1. This gives the constraint

$$1^T y = 1$$

Based on these factors on the dual variables, the LP for the dual can be formulated as,

$$\begin{gathered} \min z \\ s.t - Ay + 1z \geq 0 \\ 1^T y = 1 \\ y \geq 0 \end{gathered}$$

F. The LP for best strategy using the calculations from the above sections can be formulated as

$$\begin{gathered} \max\ t \\ s.t - A^T x + 1t \leq 0 \\ 1^T x = 1 \\ x \geq 0 \end{gathered}$$

Using the payoff matrix given and substituting it in the above LP we get

$$\begin{gathered} \max\ t \\ s.t.\ \ t + x_1 + x_3 - x_4 \leq 0 \\ t - 2x_1 + x_2 + x_5 \leq 0 \\ t - 2x_2 + 2x_5 \leq 0 \\ t + x_2 - x_3 + x_4 + 2x_5 \leq 0 \\ t + x_1 - x_2 - 3x_5 \leq 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

Using the above LP using code, we get the best strategy for the row player as

$$x = (0, 0, 0.5, 0.5, 0)$$

Solving the dual using Part E, will give us the best strategy for the column player as

$$y = (0.5, 0, 0, 0.5, 0)$$

The value of the game is = 0.0

P2

A. Given the ILP

$$\begin{array}{rlllll}
\max & 2x_1 & -3x_2 & -2x_3 & -x_4 & \\
\text{s.t.} & x_1 & -x_2 & +x_3 & & \leq 5 \\
& x_1 & -2x_2 & -x_3 & +x_4 & \leq 3 \\
& x_1 & -x_2 & -x_3 & -x_4 & \leq -1 \\
& x_1, & x_2, & x_3, & x_4 & \in [-5, 5] \\
& x_1, & x_2, & x_3, & x_4 & \in \mathbb{Z}
\end{array}$$

To solve the LP using branch and bound, we first remove the integrality constraint and solve the problem as a regular LP. Then we choose a variable to branch on and continue solving. (Code attached with submission)

Solving the LP removing the integrality constraint we get the solution:

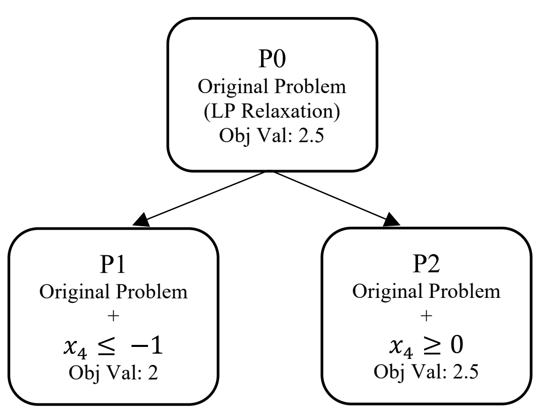
$$x_1 = -5.0 \quad x_2 = -5.0$$
$$x_3 = +1.5 \quad x_4 = -0.5$$
$$\text{Objective value: } 2.5$$

Now we can branch on variable x_3 or x_4. Using hint provided, choosing x_4. This adds constraints $x_4 \leq -1$ and $x_4 \geq 0$ as new constraints in 2 different branches.

New added constraint	$x_4 \leq -1$	$x_4 \geq 0$
Solution	$x_1 = -5.0$ $x_2 = -5.0$ $x_3 = 2.0$ $x_4 = -1.0$ $Objective\ value$: 2.0	$x_1 = -5.0$ $x_2 = -4.0$ $x_3 = 0.0$ $x_4 = 0.0$ $Objective\ value$: 2.0

Since we have reached a solution with all integers, we can consider them as solutions for our LP with the integrality constraint. Also, since the objective value of both solutions are maximum upper limit integer values compared to just LP solution, we can conclude **both the solutions are optimal**.

Enumeration tree:



B. Given the ILP as

$$
\begin{array}{llllll}
\max & 2x_1 & & +3x_3 & +x_4 & \\
\text{s.t.} & x_1 & -x_2 & & +x_4 & \leq 1 \\
& & 2x_2 & & -x_4 & \leq 2 \\
& x_1 & & -x_3 & -2x_4 & \leq -1 \\
& -x_1 & & & +x_4 & \leq 1 \\
& x_1 & & & & \in \{-2,-1,0,1,2\} \\
& & x_2 & & & \in \{-1,0,1\} \\
& & & x_3 & & \in \{0,1\} \\
& & & & x_4 & \in \{-1,0,1\}
\end{array}
$$

We again first remove the integrality constraint and solve the problem as a regular LP. Solving the LP removing the integrality constraint we get the solution:

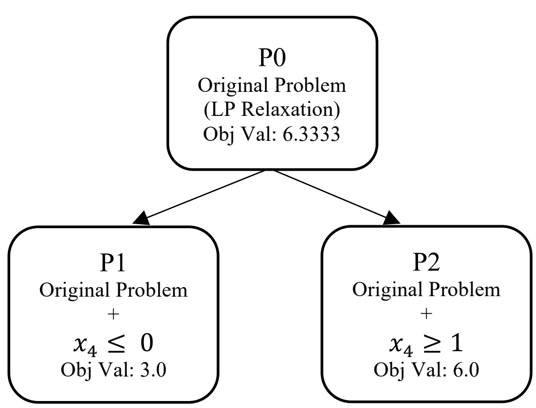
$$x1 = 1.3333333 \quad x2 = 1.0$$
$$x4 = 0.6666667 \quad x3 = 1.0$$
Objective value: 6.33333327

Now we can branch on variable x_1 or x_4. We can choose to branch on x_4 since it has lesser possible values. Branching on x_4 will help fixing value for it and it better to branch on variables with lesser constraints. Doing this adds two more constraints $x_4 \leq 0$ and $x_4 \geq 1$. These are the new constraints in 2 different branches.

New added constraint	$x_4 \leq 0$	$x_4 \geq 1$
Solution	$x_1 = 0$ $x_2 = -1$ $x_3 = 1$ $x_4 = 0$ $\mathit{Objective\ value}$: 3.0	$\boldsymbol{x_1 = 1.0}$ $\boldsymbol{x_2 = 1.0}$ $\boldsymbol{x_3 = 1.0}$ $\boldsymbol{x_4 = 1.0}$ ***Objective value*: 6.0**

Since we have reached a solution with all integers, we can consider them as solutions for our LP with the integrality constraint. Also, since the objective value of one of the solutions is a maximum upper limit integer value compared to just LP solution, we can conclude **the solutions is optimal for branch $x_4 \geq 1$**.

Enumeration tree:



P3

A. Given the final dictionary

Dictionary # 1:

x_1	0.666666666667	$-0.666667x_5$	$+0.333333x_4$
x_2	1	$-1x_5$	
x_3	2	$+4x_5$	$-1x_4$
z	1	$-1x_5$	

Choosing cutting plane for the variable x_1 since it has a fractional coefficient. The constraint can be re-written as

$$x_1 + 0.666667x_5 - 0.33333x_4 = 0.666667$$

Converting the above equation into integral + fractional parts:

$$(1x_1 + 0x_5 - 1x_4) + (0.0x_1 + 0.666667x_5 + 0.666667x_4) = 0 + 0.666667$$

From the above separation, the cutting plane can be written as,

$$0.666667x_5 + 0.666667x_4 \geq 0.666667$$

$$x_5 + x_4 \geq 1$$

B. Given the final dictionary

Dictionary # 2:

x_4	4.33333333333	$+0.333333x_8$	$+0.666667x_9$	$-0.333333x_3$
x_5	8.66666666667	$-0.333333x_8$	$+0.333333x_9$	$-2.666667x_3$
x_6	10			$-1x_3$
x_7	3	$-3x_8$	$+1x_9$	$-18x_3$
x_1	5.66666666667	$-0.333333x_8$	$-0.666667x_9$	$+0.333333x_3$
x_2	1.33333333333	$+0.333333x_8$	$-0.333333x_9$	$+2.666667x_3$
z	7		$-1x_9$	$-2x_3$

Using the same methods from above, we can generate cutting planes for the variables x_4, x_5, x_1 and x_2.

$$x_4\text{: } 0.666667x_8 + 0.333333x_9 + 0.33333x_3 \geq 0.33333$$

$$x_5\text{: } 0.333333x_8 + 0.666667x_9 + 0.66667x_3 \geq 0.66667$$

$$x_1\text{: } 0.333333x_8 + 0.666667x_9 + 0.66667x_3 \geq 0.66667$$

$$x_2\text{: } 0.666667x_8 + 0.333333x_9 + 0.33333x_3 \geq 0.33333$$

There are 4 cutting planes as above. Among them 2 of them are unique.

P4

Given a set of numbers $S_1 \dots S_k$ such that $S_i \subseteq \{1 \dots n\}$ we need to select a subset S such that $S \cap S_i \neq \emptyset$ for $i = 1 \dots k$ and sum of elements in chosen set is minimized.
Consider each set is represented as a {0,1} vector where position i is set to 1 if i ⊆ that set. Example: if n = 3 then set $S_1 : \{1,3\}$ = [1 0 1].

We can represent the sets of numbers $\langle S_1, \dots, S_k \rangle$ in the form of a matrix A where row i in matrix is the {0,1} representation of the set S_i where $i = 1 \dots k$.
For the given example in question, when n = 10 and $S_1 : \{1,3,6\}, S_2 : \{2,7,8\}, S_3 : \{1,8,9\}, S_4 : \{1,6,5,3\}$ matrix A will be with dimensions $k \times n$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the ILP for this problem can be formulated as

$$\min \sum_{i=1}^{n} i * x_i$$
$$\text{s.t } Ax \geq 1$$
$$x \subseteq \{0,1\},\ x \subseteq Z$$

Here x will be column vector of dimensions $n \times 1$. This way we get a $k \times 1$ column vector which tells us all the numbers to include in set S and our objective ensures that we find a set S such that the sum of its elements is minimized.

Code to solve the ILP is attached. Solving the example in question we get the solution as follows:

X = [1 1 0 0 0 0 0 0 0 0]

So, the subset S = {1,2} and their sum is 3 which is the least possible sum.

Sources: Linear Programming – textbook by Vanderbei
Collaborated with - Ketan Ramesh and Siddartha Shankar for discussion on ideas.