CSCI 5654 - Linear Programming - Homework – 2

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P1

$$
\begin{array}{r|rrrrrrr}
w_1 & 8 & -5x_1 & -4x_2 & +x_5 & -2x_6 & +2w_3 & -w_4 \\
w_2 & 2 & -x_1 & & -x_5 & -x_6 & -w_3 & \\
x_3 & 2 & & +x_2 & -x_5 & -x_6 & -w_3 & -2w_4 \\
x_4 & 3 & -x_1 & -x_2 & -x_5 & -x_6 & +2w_3 & +w_4 \\
w_5 & 4 & -x_1 & -2x_2 & -4x_5 & & & +w_4 \\
\hline
z & 11 & -x_1 & +2x_2 & +x_5 & +x_6 & +3w_3 & -2w_4
\end{array}
$$

Entering variable – we choose the entering variable based on the "+ve" sign in the objective function of the dictionary since we need to maximize the objective function. Hence x_2, x_5, x_6, w_3 can be the entering variables.

Leaving variable – For leaving variable we choose the "-ve" sign elements in our constraints and calculate δ which will place the least constraint or goes down to 0 the first. Example – for x_2 the leaving variables can be w_1, x_4, w_5. For these leaving variables δ can be calculated as follows:

$$\delta\_w_1 = \frac{8}{-(-4)} = 2 \qquad \delta\_x_4 = \frac{3}{-(-1)} = 3 \qquad \delta\_w_5 = \frac{4}{-(-2)} = 2$$

By comparing the values, we can choose w_1, w_5 as the leaving variables as they have lesser δ. Similarly, we can calculate the leaving variables for the others.

Objective function value – Objective function value in the next dictionary will be $Z^* = C_0 + \delta C_i$. So, for the first case, the value will be 11 + (2 * 2) = 15. Similarly, for the other three.

Degeneracy – The next dictionary will be degenerate if the objective function remains the same or we have at least one 0 in the constant column of our next dictionary. We can calculate this by δ *coeff of entering variable in each constraint (except for the constraint variable leaving since it rearranges). For the first case, $\delta = 2$ and for w_5 we will have 0 in the constant column in next dictionary. Hence it is degenerate. Same way for other cases.

Entering	Leaving	Objective Fn. value in next dictionary	Next Dictionary Degenerate
x_2	w_1, w_5	15	Yes
x_5	w_5	12	No
x_6	w_2, x_3	13	Yes
w_3	w_2, x_3	17	Yes

P2

$$
\begin{array}{r|llllll}
x_{B,1} & b_1 & +a_{11}x_{N,1} & +\cdots & +a_{1j}x_{N,j} & \cdots & +a_{1n}x_{N,n} \\
\vdots & \vdots & & \ddots & \vdots & \ddots & \\
x_{B,i} & b_i & +a_{i1}x_{N,1} & +\cdots & +a_{ij}x_{N,j} & \cdots & +a_{in}x_{N,n} \\
\vdots & & \ddots & & & & \\
x_{B,m} & b_m & +a_{m1}x_{N,1} & +\cdots & +a_{mj}x_{N,j} & \cdots & +a_{mn}x_{N,n} \\
\hline
z & c_0 & +c_1x_{N,1} & +\cdots & +c_jx_{N,j} & \cdots & +c_nx_{N,n}
\end{array}
$$

A. Objective co-efficient in the next dictionary:

Given that in a dictionary

$$x_{B,i} = b_i + a_{i1}x_{n,1} + \cdots + a_{i,j}x_{N,j} + \cdots + a_{i,n}x_{N,n}$$

Since $x_{n,j}$ is chose as the entering variable, we can re-write the above equation as

$$-a_{i,j}x_{N,j} = b_i + a_{i1}x_{n,1} + \cdots + a_{i,n}x_{N,n} - x_{B,i}$$

$$x_{N,j} = \frac{b_i + a_{i1}x_{n,1} + \cdots + a_{i,n}x_{N,n} - x_{B,i}}{-a_{i,j}}$$

Given,

$$Z = C_o + c_1x_{n,1} + \cdots + c_jx_{N,j} + \cdots + c_nx_{N,n}$$

Substituting $x_{N,j}$ we get,

$$Z = C_o + c_1x_{n,1} + \cdots + c_j\left(\frac{b_i + a_{i1}x_{n,1} + \cdots + a_{i,n}x_{N,n} - x_{B,i}}{-a_{i,j}}\right) + \cdots + c_nx_{N,n}$$

Hence, co-eff of $x_{B,i} = \boldsymbol{\frac{c_j}{a_{i,j}}}$

B. We choose $x_{n,j}$ to be entering since we know that c_j is a positive number.

So, $C_j > 0$

We also choose $x_{B,i}$ as the leaving variable since we know that $a_{i,j}$ is a negative number.

So, based on the rules of choosing leaving variable, $a_{i,j} < 0$

From answer in (P2.A) we know that co-eff of $x_{B,i} = \boldsymbol{\frac{c_j}{a_{i,j}}}$. From the above equations we can tell that $\boldsymbol{\frac{c_j}{a_{i,j}}} < 0$. So, co-eff of $x_{B,i}$ will be <u>negative</u> in the next dictionary.

Hence, it cannot be chosen as the entering variable in the next dictionary.

P3

A. A degenerate dictionary is one where the constant column has a 0. And an unbounded dictionary is one where the entering variable has no leaving variable. Based on these conditions, an example dictionary can be

$$\begin{array}{c|cccc} w_1 & 0 & +5x_1 & +5x_2 \\ w_2 & 0 & +10x_1 & +3x_2 \\ \hline z & 2 & +x_1 & -x_2 \end{array}$$

B. A degenerate dictionary, D, is one where the constant column has a 0. If on pivoting the next dictionary has to be also degenerate it should also have one of the constant columns as 0. One way this is possible is by choosing the entering and leaving variable such that it does not after the row or variable causing degeneracy. Based on these conditions, an example dictionary can be as below where choosing x_1 as entering has no change on w_2 and the next dictionary will also be degenerate with the objective increasing.

$$\begin{array}{c|cccc} w_1 & 10 & -5x_1 & +5x_2 \\ w_2 & 0 & & +3x_2 \\ \hline z & 2 & +x_1 & -x_2 \end{array}$$

C. It is <u>not possible</u> to have a non-degenerate dictionary D which upon pivoting yields another dictionary D′ but the value of the objective function stays the same.
This is because, the objective function value of next dictionary will be $Z^* = C_0 + \delta C_i$, and this can remain same only if $\delta = 0$ or $c_i = 0$. But, $c_i \neq 0$ since we always choose an entering variable with positive co-efficient. And we know $\delta = \frac{b_i}{-a_{i,j}}$ and $\delta = 0$ only if $b_i = 0$. But if b_i was 0 the dictionary had to be degenerate. Hence, it is not possible.

D. For any feasible dictionary such as below

$$\begin{array}{c|ccccccc} x_{B,1} & b_1 & +a_{11}x_{N,1} & +\cdots & +a_{1j}x_{N,j} & \cdots & +a_{1n}x_{N,n} \\ \vdots & \vdots & & \ddots & \vdots & \ddots & \\ x_{B,i} & b_i & +a_{i1}x_{N,1} & +\cdots & +a_{ij}x_{N,j} & \cdots & +a_{in}x_{N,n} \\ \vdots & & \ddots & & & & \\ x_{B,m} & b_m & +a_{m1}x_{N,1} & +\cdots & +a_{mj}x_{N,j} & \cdots & +a_{mn}x_{N,n} \\ \hline z & c_0 & +c_1x_{N,1} & +\cdots & +c_jx_{N,j} & \cdots & +c_nx_{N,n} \end{array}$$

We always choose a entering variable $x_{N,j}$ such that it's co-eff is positive. Also, we know that for leaving variable $a_{i,j} < 0$ and $\delta = \frac{b_i}{-a_{i,j}}$. In the next dictionary, all the elements in the constant column will increase as per $b_1' = b_1 + \delta a_{1i}$ and so on till $b_m' = b_m + \delta a_{mi}$. And since $a_{i,j} < 0$ we always have $\delta > 0$. So, $b_m' = b_m + \delta a_{mi}$ and all the elements in that column will be >= 0 in the next dictionary.

Hence, the next dictionary will always be feasible after pivot.

So, we <u>cannot have a dictionary that is feasible which upon pivoting yields an infeasible dictionary</u>.

E. A dictionary that does not have leaving variable (is unbounded) for one choice of entering variable but has a leaving variable for a different choice of an entering variable. So, one column will have all positive co-eff for the leaving variable and another leaving variable will have at least one negative co-eff number.

w_1	3	$+x_1$	$-5x_2$
w_2	8	$+5x_1$	$-15x_2$
z	10	$-x_1$	$-x_2$

P4

A. Given that x_0 is a point which satisfies $Ax \leq b$. So, $Ax_0 \leq b$ and $x_0 \geq 0$.
Also given that $Ar \leq 0.$ Let us consider a point $x_0 + \lambda r$ (λ>0). To check if it is a feasible solution it should satisfy $Ax \leq b$.
So, $A(x_0 + \lambda r) \leq b$ must be true. Which gives,

$$Ax_0 + \lambda Ar \leq b$$

We know that $Ax_0 \leq b$ and we also know that $Ar \leq 0$. Considering $\lambda > 0$ we always have $\lambda Ar \leq 0.$ By this we know that $Ax_0 + \lambda Ar \textit{ is always} \leq b$ when $\lambda > 0$.
Hence, $x_0 + \lambda r$ is a feasible point.

The objective function value can be obtained by $C^t x.$ Which will be $C^t(x_0 + \lambda r)$.
Z = $C^t(x_0 + \lambda r) = C^t x_0 + \lambda C^t r.$

$$Z = C^t x_0 + \lambda C^t r \text{ will tend to } \infty$$

The value of Z tends to ∞ as λ keeps increasing to any value > 0.
Hence the LP is unbounded for a feasible point $x_0 \geq 0$ such that $Ax_0 \leq$ b and a ray r ≥ 0 such that $c^t r > 0$ and $Ar \leq 0$

B. Given that $x_{N,j}$ is entering variable and there is no leaving variable.
So, $a_{1,j}, a_{2,j} \dots \dots, a_{m,j} > 0$ in the column corresponding to $x_{N,j}$ and there is no other entering variable.
Now increasing $x_{N,j}$ by any value δ will not place any constraint on the constant column since all $a_{1,j}, a_{2,j} \dots \dots, a_{m,j} > 0$ for $x_{N,j}$. So, setting $x_{N,j}$ to any arbitrary λ>0 will still make the solution feasible as all b constant column > 0.
By setting $x_{N,j}$ to λ our objective function will be

$$Z = c_0 + c_1 x_{N,1} + \cdots + c_j \lambda \cdots + + c_n x_{N,n}, \lambda > 0$$

From above the value of the objective keeps increasing as λ → ∞. So Z also → ∞.
Hence, the LP will be unbounded.

P5

A. Given that the LP has two different non-degenerate optimal solutions (x1, w1) and (x2, w2) obtained from final dictionaries D1 and D2, and problem has n decision variables and m constraints.

Since the solution is optimal, and the dictionary will have n constraints and m decision variables, there will be n non-basic variables set to 0 at the end and m basic variables which are non-zero. Similarly, for (x_2, w_2)

$$\text{So, } nnz(x_1, w_1) = m = nnz(x_2, w_2)$$

For, $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(w_1 + w_2)\right)$, the final dictionaries will at minimum have $x_1 \neq x_2$ and both x will have non zeros which one value zero. So a total of m+1 non zero values. And the maximum value of non-zero values $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(w_1 + w_2)\right)$ can have is n+m since it belongs to $\mathbb{R}^{n+m}$

$$\text{So, m+1} \leq nnz\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(w_1 + w_2)\right) \leq m + n$$

B. In the dual optimal solution, the number of basic and non-basic variables interchange. So, there will be m dual basic variables and n slack variables.
From the calculation in P5.A, we have the same non zero counts for the primal and dual solutions $((\mathrm{x}, \mathrm{v}), (\mathrm{y}, \mathrm{w}))$.

$$\text{m+1 } nnz((x, v), (y, w)) < \text{m + n}$$
$$nnz((y, v) < n - 1$$

C.

P6

A. Given the primal LP

$$\begin{array}{llllllll}
\max & 2x_1 & -3x_2 & +x_3 & -x_4 & +x_5 & \\
\text{s.t.} & x_1 & -x_2 & -x_3 & & & \leq 5 \\
& 2x_1 & & +x_3 & -x_4 & & \leq 6 \\
& -x_1 & +x_2 & & -x_4 & & \leq 4 \\
& -x_1 & & & & +x_5 & \leq 2 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0
\end{array}$$

By using the co-eff of each variable, we can write the dual as follows:

$$\begin{array}{lllllll}
\min & 5y_1 & +6y_2 & +4y_3 & +2y_4 & & \\
\text{s.t.} & y_1 & +2y_2 & -y_3 & -y_4 & \geq & 2 \\
& -y_1 & & +y_3 & & \geq & -3 \\
& -y_1 & +y_2 & & & \geq & 1 \\
& & -y_2 & -y_3 & & \geq & -1 \\
& & & & y_4 & \geq & 1 \\
& & y_1, y_2, y_3, y_4 \geq 0 & & & &
\end{array}$$

B. Adding slack variables for the primal we get:

$$\begin{array}{c|cccccc}
w_1 & 5 & -x_1 & +x_2 & +x_3 & & \\
w_2 & 6 & -2x_1 & & -x_3 & +x_4 & \\
w_3 & 4 & +x_1 & -x_2 & & +x_4 & \\
w_4 & 2 & +x_1 & & & & -x_5 \\
\hline
z & 0 & +2x_1 & -3x_2 & +x_3 & -x_4 & +x_5
\end{array}$$

Given the values of x as

$$x_1 = \frac{10}{3},\ x_2 = 8,\ x_3 = 0,\ x_4 = \frac{2}{3},\ x_5 = \frac{16}{3}.$$

We get values of all slack variable as follows:

$$w_1 = \frac{29}{3},\ w_2, w_3, w_4 = 0$$

Using the complementary slackness theorem, we know that x.v = 0, w.y = 0 and the dual, primal objective are same in an optimal solution. Where x,w are primal decision and slack variables, y,v are dual decision and slack variables. Using the theorem, we get $v_1, v_2, v_4 = 0$ and $y_1 = 0$.

Since the solution to be optimal, the objective value obtained must be same on both the dual and primal. $c^t x = b^t y$

Value obtained for primal objective using x values = $c^t x$ = -$\frac{38}{3}$

And the dual objective $b^t y = 6y_2 + 4y_3 + 2y_4$ is always ≥ 0 since $y_1, y_2, y_3, y_4 \geq 0$.

As we have obtained $c^t x \neq b^t y$, the solution given is not optimal.

P7

Given the LP

$$\begin{array}{lll} \max & \mathbf{c}^t\mathbf{x} & \\ \text{s.t.} & A\mathbf{x} & \leq \mathbf{b} \\ & \sum_{i=1}^{n} x_i & \leq 1 \\ & \mathbf{x} & \geq 0 \end{array}$$

We need to show that the dual of this problem is always feasible. Let us consider y be the dual variables associated with the constraints Ax $\leq$ b and y_0 to be the dual variable associated with the constraint $\sum_{i=1}^{n} x_i \leq 1$.

Doing that, our new $A^* = [A\ 1]$ and $b^* = [b\ 1]$ $y^* = [y\ y_0]$ by the added constraint.

So now we will have m+1 constraints.

Using the above considerations, we will have the dual of as follows,

$$\min\ b^{*t}y^*$$
$$\text{st. } A^{*t}y^* \geq c$$
$$y \geq 0$$

In order to solve this dual we write the dictionary and then set $y_1, \dots y_n, y_0 = 0$. Doing this will not provide us with a feasible solution. Since we get $0 \ngeq c$. The conditions are not satisfied.

Expanding our conditions above we have $A^{*t}y + 1y_0 \geq C$.

Consider a condition where $y_1 \dots y_n = 0$. Then, the conditions which remains will be $y_0 \geq c$.

For the conditions to be satisfied at all times, y_0 values are greater, and they should not be less than any of the values in c. So, in other words, y_0 must be always greater than or equal to c for the dual to be feasible.

For this to hold true we must have $y_0 \geq \max(C_0,\ C_{1,} \dots, C_n)$. Only then we will have a feasible solution.

Collaborated with - Ketan Ramesh and Siddartha Shankar for discussion on ideas.