CSCI 5654 - Linear Programming - Homework – 5

Shreyas Gopalakrishna

May 2020

P1

A. Given the equality constrained minimization problem

$$\min\ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}.$$

We need to show that any local minimum x satisfies the below constraints

$$\begin{aligned} Q\mathbf{x} + A^T\mathbf{y} &= -\mathbf{c} \\ A\mathbf{x} &= \mathbf{b} \end{aligned}$$

In order to find the minimum, we can take the langrangian and then take partial derivative and equate it to 0

$$L(x, y) = \frac{1}{2}(x^T Q x) + c^T x + y^T(Ax - b)$$

Taking partial derivative with respect to x

$$\frac{\partial L}{\partial x} = \frac{1}{2}(Q + Q^T)x + c + A^T y = \frac{1}{2}2Qx + c + A^T y$$

$$0 = Qx + c + A^T y$$

$$Qx + A^T y = -c$$

Taking partial derivative with respect to y

$$\frac{\partial L}{\partial y} = Ax - b$$

$$0 = Ax - b$$

$$Ax = b$$

Therefore, any local minimum x must satisfy the below constraints

$$Qx + A^T y = -c$$

$$Ax = b$$

B. To find the solution of the given equations, do the following as we don't know if A is invertible.

$$Qx + A^T y = -c$$

Multiplying by AQ^{-1} (given Q is invertible)

$$Ax + AQ^{-1}A^T y = -AQ^{-1}c$$

$$b + AQ^{-1}A^T y = -AQ^{-1}c$$

And,

$$AQ^{-1}A^Ty = -AQ^{-1}c - b$$
$$y = -(AQ^{-1}A^T)^{-1}(AQ^{-1}c + b)$$
$$y = -({A^T}^{-1}QA^{-1})(AQ^{-1}c + b)$$
$$y = -({A^T}^{-1}QA^{-1}AQ^{-1}c + {A^T}^{-1}QA^{-1}b)$$
$$y = -({A^T}^{-1}c + {A^T}^{-1}QA^{-1}b)$$

And,

$$Qx + A^T(-({A^T}^{-1}c + {A^T}^{-1}QA^{-1}b)) = -c$$
$$Qx + (-(c + QA^{-1}b)) = -c$$
$$Qx + (-c - QA^{-1}b) = -c$$
$$Qx = (QA^{-1}b)$$
$$x = A^{-1}b$$

Since both x and y are fixed, we can say that the equations have a unique solution for any given (x, y).

P2

A. Given the problem

$$\begin{array}{lll} \min & \sum_{i=1}^{n} q_i x_i^2 & \\ \text{s.t} & A\mathbf{x} + \mathbf{w} & = \mathbf{b} \\ & \mathbf{x}, \mathbf{w} & \geq 0 \end{array}$$

Writing in standard form,

$$\max\ -x^T Q x$$
$$s.t\ Ax + w = b$$
$$x, w \geq 0$$

Formulating log barrier problem can be formulated as,

$$\max\ \ -x^T Q x + \mu \sum_{i=1}^{n} -\log(x_i) + \mu \sum_{j=1}^{m} -\log(w_j)$$
$$s.t\ Ax + w = b$$

B. Langrangian for the log barrier problem corresponding to the constrain $Ax + w = b$

$$L(x, w, y) = -x^T Q x + \mu \sum_{i=1}^{n} -\log(x_i) + \mu \sum_{j=1}^{m} -\log\left(w_j\right) + y^T(Ax + w - b)$$

C. Partial derivative of the langrangian will be

$$\frac{\partial L}{\partial x} = -2Qx - \mu \begin{pmatrix} \frac{1}{x_1} \\ \cdots \\ \frac{1}{x_n} \end{pmatrix} + y^T A$$

$$\frac{\partial L}{\partial w} = -\mu \begin{pmatrix} \frac{1}{w_1} \\ \cdots \\ \frac{1}{w_m} \end{pmatrix} - \ y^T$$

$$\frac{\partial L}{\partial y} = b - Ax - w$$

Equating it to 0 we get

For $\frac{\partial L}{\partial x}$,

$$2Qx + \mu \begin{pmatrix} \frac{1}{x_1} \\ \cdots \\ \frac{1}{x_n} \end{pmatrix} = y^T A$$

$$\mu \begin{pmatrix} \frac{1}{x_1} \\ \cdots \\ \frac{1}{x_n} \end{pmatrix} = A^T y - 2Qx$$

Considering for generic x_i and by using, $A^T y_j - 2Qx_i = z_i$ we get

$$\frac{\mu}{x_i} = z_i$$

$$\mu = x_i z_i$$

For $\frac{\partial L}{\partial w}$, for generic w_j we get

$$-\frac{\mu}{w_j} = y_j^T$$

$$\mu = -w_j y_j^T$$

For $\frac{\partial L}{\partial y}$,

$$Ax + w = b$$

We can use the x_i, w_j, y_j, z_i to extrapolate the complete matrix x,w,y and z.

I have currently just mentioning the conditions as i and j

So, we get the equations for the minimizers of the Langrangian as,

$$\mu = x_i z_i, \quad \mu = -w_j y_j^T \text{ and } Ax + w = b \text{ where } A^T y - 2Qx = z$$

D. The KKT conditions for the above problem can be formulated as,

We found that,

$$\mu = x_i z_i,\ i = 1 \dots n$$

$$\mu = -w_j y_j^T,\ \ j = 1 \dots m$$

$$Ax + w = b$$

$$A^T y - z = 2Qx$$

The primal gap $\rho = b - Ax - w$

The dual gap $\rho' = 2Qx + z - A^T y$

For the step we write it as, (here $'$ is used to represent iteration)

$$A(x' + \Delta x') + (w' + \Delta w') = b$$

$$A^T(y' + \Delta y') - (z' + \Delta z') = 2Qx$$

$$(x' + \Delta x')(z' + \Delta z') = \mu$$

$$-(w_j' + \Delta w_j')(y_j'^T + \Delta y_j'^T) = \mu$$

Which can be equated individually as,

$$\mathrm{A}x' + A\Delta x' + w' + \Delta w' = b$$

$$A\Delta x' + \Delta w' = b - \mathrm{A}x' - w'$$

$$A\Delta x' + \Delta w' = \rho$$

$$A^T y' + A^T \Delta y' - z' - \Delta z' = 2Qx$$
$$A^T \Delta y' - \Delta z' = 2Qx + z' - A^T y'$$
$$A^T \Delta y' - \Delta z' = \rho'$$

Ignoring the multipliers of $\Delta' s$

$$(x'z' + \Delta x'z' + x'\Delta z') = \mu$$
$$-(w_j' y_j'^T + w_j' \Delta y_j'^T + \Delta w_j' y_j'^T) = \mu$$

Considering x_i, y_j, w_j, z_i all as diagonal matrices X, Y, W, Z where each element fall in the diagonal according to the formulation in the question, we get

$$Z\Delta x + X\Delta z = \mu - XZ1$$
$$-(W\Delta y^T + Y\Delta w) = \mu + YW1$$

So, the KKT equations for the step $(\Delta x, \Delta y, \Delta w, \Delta z)$ for a given point(x, y, w, z) can be written in a matrix as,

$$\begin{bmatrix} A & 0 & I & 0 \\ 0 & A^T & 0 & -I \\ Z & 0 & 0 & X \\ 0 & -W & -Y & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta w \\ \Delta z \end{bmatrix} = \begin{bmatrix} \rho \\ \rho' \\ \mu - XZ1 \\ \mu + YW1 \end{bmatrix}$$

P3

A. It is given that for LP given $Ax \leq b$ describes a non-empty and bounded polyhedron

$$\max c^T x$$
$$s.t\ Ax \leq b$$

$f(c)$ is a function which provides an optimal solution for a given c

In order to prove that for all c_1, c_2 we will have $f(c_1 + c_2) \leq f(c_1) + f(c_2)$

We can write $f(c_1 + c_2)$ as

$$\max(c_1 + c_2)^T x$$
$$s.t\ Ax \leq b$$

which has an optimal solution S_1

We can write $f(c_1)$ as

$$\max(c_1)^T x$$
$$s.t\ Ax \leq b$$

which has an optimal solution S_2

We can write $f(c_2)$ as

$$\max(c_2)^T x$$
$$s.t\ Ax \leq b$$

which has an optimal solution S_3

As we know that S_1 and S_2 are optimal solutions for $f(c_1 + c_2)$ and $f(c_1)$ respectively

The solution S_1must be a feasible (but not optimal) for $f(c_1)$. So we get

$$c_1 S_1 \leq c_1 S_2$$

Similarly, we get

$$c_1 S_1 \leq c_2 S_3$$

So we can write, $f(c_1 + c_2) = c_1 S_1 + c_2 S_1$ as S_1 is an optimal solution. Similarly, as S_2 *and* S_3 are also optimal solutions we can write

$$f(c_1) = c_1 S_2$$
$$f(c_2) = c_2 S_3$$

Using the above the equations and adding them we can say that,

$$c_1 S_1 + c_2 S_1 \leq\ c_1 S_2 + c_2 S_3$$

Hence, we get that

$$f(c_1 + c_2) \leq f(c_1) + f(c_2)$$

B. Consider a LP

$$\max c^T x$$
$$s.t\ Ax \leq b$$

Say that this LP has an optimal solution x'

The value of our objective will be $f(c)$

$$f(c) = c^T x'$$

Which can be expanded as,

$$f(c) = c_1x_1' + c_2x_2' + \cdots + c_nx_n'$$

Multiplying by a number k, $k \geq 0$ we get

$$kf(c) = kc_1x_1' + kc_2x_2' + \cdots + kc_nx_n'$$

Finding the value of $f(kc)$

For this the LP associated will be same with an objective: $\max\ (kc)^T x$

This value for the optimal solution for this LP will be

$$f(kc) = kc_1x_1' + kc_2x_2' + \cdots + kc_nx_n'$$

From the above equations we get

$$f(kc) = kf(c)$$

This proves that $f(c)$ is a convex function.

C. Consider the LP

$$\max c^T x$$
$$s.t\ Ax \leq b$$

Now, let us represent the co-eff of the decision variables as y and decision variables as a. Also, let D be the matrix representing the constraints.

Then, the LP is modified as,

$$\max y^T a$$
$$s.t\ Da \leq b$$

which is same as

$$\max a^T y$$
$$s.t\ Da \leq b$$

Since the solution for the LP forms a polyhedron with certain number of vertices say N which is bounded and non-empty. We can say the set $\{a_1, a_2 \ldots a_N\}$ be those vertices of the polyhedron which are the different feasible solutions for our LP.

So, the set of values for all the feasible solutions will be $\{a_1^T y, a_2^T y, \ldots a_N^T y\}$

In this set, the optimal value will be the maximum value since our LP is a maximization problem. And the corresponding vertex of the polyhedron is the optimal solution.

Hence, we can say that the value of optimal solution be

$$\max\left(a_1^T y, a_2^T y, \ldots a_N^T y\right)$$

The above equation is nothing but $f(y)$ since they are representing the same. Thus,

$f(y)$ can be written as a piecewise linear function as

$$\max\left(a_1^T y, a_2^T y, \ldots a_N^T y\right)$$

Sources: Linear Programming – textbook by Vanderbei and Posts on piazza
Collaborated with - Siddartha Shankar for discussion on ideas.