CSCI 5654-Spring2020: Assignment #2 .
Due Date: Thu, Feb 13, 2020 (before class)
In-class: Assignment should be submitted on moodle.

Your Name: ____

P1. (10 points) Consider the following feasible dictionary:

$$\begin{array}{r|rrrrrrr}
w_1 & 8 & -5x_1 & -4x_2 & +x_5 & -2x_6 & +2w_3 & -w_4 \\
w_2 & 2 & -x_1 & & -x_5 & -x_6 & -w_3 & \\
x_3 & 2 & & +x_2 & -x_5 & -x_6 & -w_3 & -2w_4 \\
x_4 & 3 & -x_1 & -x_2 & -x_5 & -x_6 & +2w_3 & +w_4 \\
w_5 & 4 & -x_1 & -2x_2 & -4x_5 & & & +w_4 \\
\hline
z & 11 & -x_1 & +2x_2 & +x_5 & +x_6 & +3w_3 & -2w_4
\end{array}$$

Write down all possible entering variables. For each entering variable, write down all possible leaving variables corresponding to that entering variable, the value of the objective in the next dictionary and whether the next dictionary will be degenerate.. Express your answer as a table such as the one sample shown below.

Entering	Leaving	Objective Fn. value in next dictionary	Next Dictionary Degenerate (Y/N)?
x_2	w_1 or w_5		
$\vdots$	$\vdots$	$\vdots$	

P2. (10 points) Consider the following feasible dictionary:

$$\begin{array}{r|lllllll}
x_{B,1} & b_1 & +a_{11}x_{N,1} & +\cdots & +{\color{red}a_{1j}}x_{N,j} & \cdots & +a_{1n}x_{N,n} \\
\vdots & \vdots & & \ddots & \vdots & \ddots & \\
x_{B,i} & b_i & +a_{i1}x_{N,1} & +\cdots & +{\color{red}a_{ij}}x_{N,j} & \cdots & +a_{in}x_{N,n} \\
\vdots & & \ddots & & & & \\
x_{B,m} & b_m & +a_{m1}x_{N,1} & +\cdots & +{\color{red}a_{mj}}x_{N,j} & \cdots & +a_{mn}x_{N,n} \\
\hline
z & c_0 & +c_1x_{N,1} & +\cdots & +{\color{red}c_j}x_{N,j} & \cdots & +c_nx_{N,n}
\end{array}$$

(A) Suppose $x_{N,j}$ is chosen to enter and $x_{B,i}$ is the corresponding leaving variable, then write an expression for the objective row coefficient for $x_{B,i}$ in the next dictionary.
(B) Show that $x_{B,i}$ cannot be an entering variable in the next dictionary. (We can conclude that if simplex cycles, the cycle length has to be at least 3.)

P3 (15 points) Provide examples of dictionaries that satisfy the properties stated below. Try to construct examples that are as small as possible. If no such dictionaries can exist, briefly reason why.
(A) A degenerate dictionary that is also unbounded.

(B) A degenerate dictionary D which upon pivoting yields another degenerate dictionary D', but the objective value strictly increases.

(C) A non-degenerate dictionary D which upon pivoting yields another dictionary D' but the value of the objective function stays the same.

(D) A dictionary that is feasible but upon pivoting yields an infeasible dictionary.

(E) A dictionary that does not have leaving variable (is unbounded) for one choice of entering variable but has a leaving variable for a different choice of an entering variable.

P4 (10 points) This problem asks you to prove some facts about unbounded dictionaries.

(A) Show that an LP:

$$\begin{array}{rrcl} \max & \mathbf{c}^t\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

is unbounded if there exists a feasible point $\mathbf{x}_0 \geq 0$ such that $A\mathbf{x}_0 \leq \mathbf{b}$ and a ray $\mathbf{r} \geq 0$ such that (a) $\mathbf{c}^t\mathbf{r} > 0$ and (b) $A\mathbf{r} \leq 0$.

(**Hint:** Show that $\mathbf{x}_0 + \lambda\mathbf{r}$ is feasible. What value does it achieve for the objective?)

(B) Consider the following feasible dictionary:

$$\begin{array}{r|cccccc} x_{B,1} & b_1 & +a_{11}x_{N,1} & +\cdots & +{\color{red}a_{1j}}x_{N,j} & \cdots & +a_{1n}x_{N,n} \\ \vdots & \vdots & & \ddots & \vdots & \ddots & \\ x_{B,i} & b_i & +a_{i1}x_{N,1} & +\cdots & +{\color{red}a_{ij}}x_{N,j} & \cdots & +a_{in}x_{N,n} \\ \vdots & & \ddots & & & & \\ x_{B,m} & b_m & +a_{m1}x_{N,1} & +\cdots & +{\color{red}a_{mj}}x_{N,j} & \cdots & +a_{mn}x_{N,n} \\ \hline z & c_0 & +c_1x_{N,1} & +\cdots & +{\color{red}c_j}x_{N,j} & \cdots & +c_nx_{N,n} \end{array}$$

Let $X_{N,j}$ be an entering variable and let there be no leaving variable corresponding to it. Prove that the LP must be unbounded. (**Hint:** Show that $x_{N,j}$ can be set to any arbitrary positive number λ and the resulting solution is feasible. What happens to the objective function as λ is increased?)

P5 (10 points) Consider a standard form LP with two different non-degenerate optimal solutions $(\mathbf{x}_1, \mathbf{w}_1)$ and $(\mathbf{x}_2, \mathbf{w}_2)$ obtained from final dictionaries D_1 and D_2. Let the problem have n decision variables and m constraints.

(A) Write down the number of non-zero entries in they vectors: $(\mathbf{x}_1, \mathbf{w}_1)$, $(\mathbf{x}_2, \mathbf{w}_2)$ and $(\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2), \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}2_2)$.

(B) Write down the number of non-zero entries in any dual optimal solution $(\mathbf{y}, \mathbf{v})$ (use complementary slackness theorem).

(C) Show that every dual optimal dictionary is also degenerate.

P6 (10 points) Consider the LP below:

$$\begin{array}{rllllll} \max & 2x_1 & -3x_2 & +x_3 & -x_4 & +x_5 & \\ \text{s.t.} & x_1 & -x_2 & -x_3 & & & \leq 5 \\ & 2x_1 & & +x_3 & -x_4 & & \leq 6 \\ & -x_1 & +x_2 & & -x_4 & & \leq 4 \\ & -x_1 & & & & +x_5 & \leq 2 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0 \end{array}$$

(A) Write down the dual problem corresponding to the primal.

(B) Is the solution below optimal? If optimal show a dual feasible solution to certify optimality. If not, explain why.

$$x_1 = \frac{10}{3},\ x_2 = 8,\ x_3 = 0,\ x_4 = \frac{2}{3},\ x_5 = \frac{16}{3}\,.$$

Hint: First write down the values of the primal slack variables. Use complementary slackness theorem to formulate what constraints the dual must satify.

P7 (10 point) Consider the LP below:

$$\begin{array}{lll} \max & \mathbf{c}^t\mathbf{x} & \\ \text{s.t.} & A\mathbf{x} & \leq \mathbf{b} \\ & \sum_{i=1}^n x_i & \leq 1 \\ & \mathbf{x} & \geq 0 \end{array}$$

Prove that dual of this problem is always feasible. (**Hint:** Let $\mathbf{y}$ be the dual variables associated with the constraints $A\mathbf{x} \leq \mathbf{b}$. Associate a special dual variable y_0 with the last constraint $\sum_{i=1}^n x_i \leq 1$).