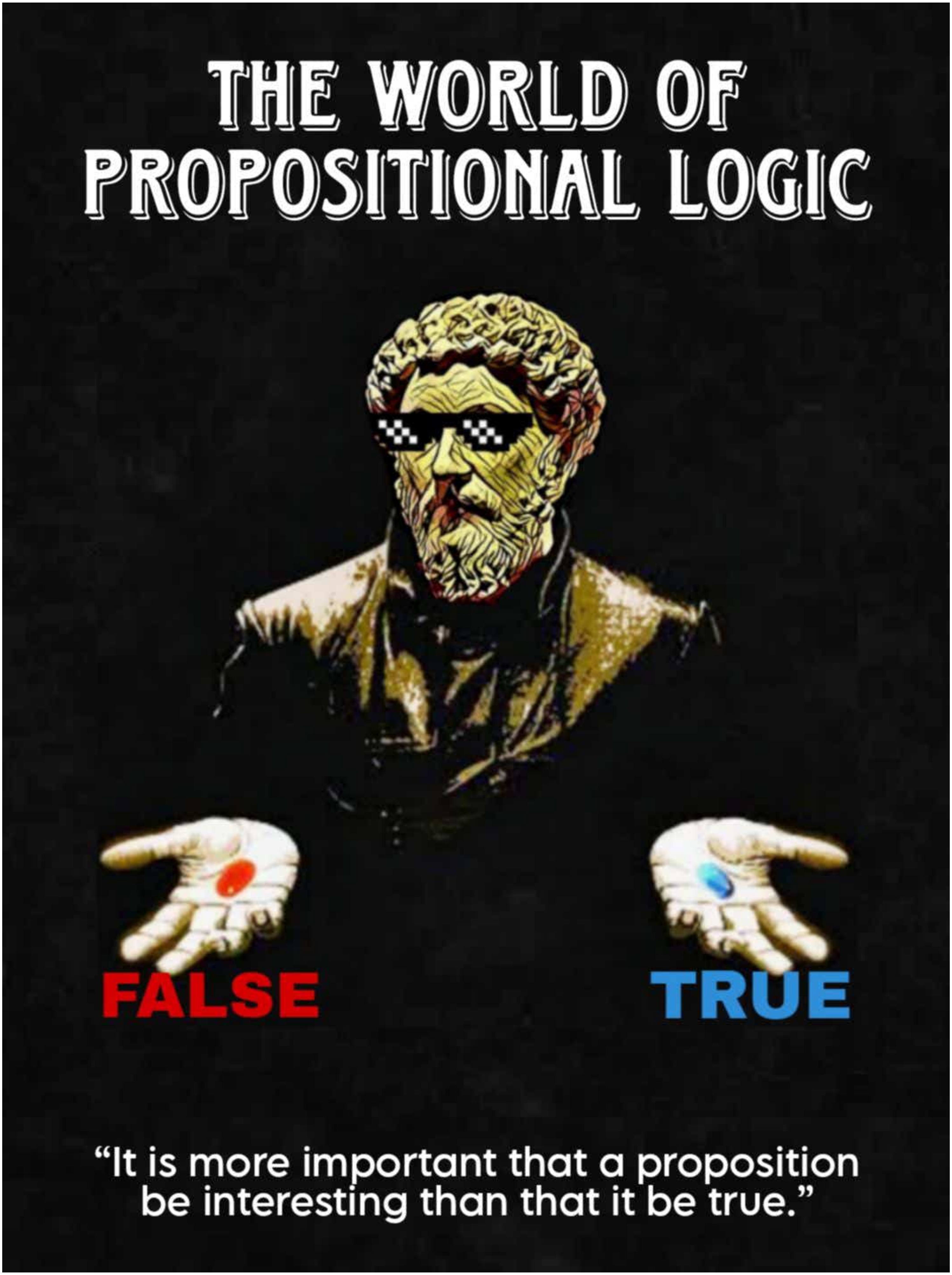

THE WORLD OF
PROPOSITIONAL LOGIC
FALSE
TRUE
“It is more important that a proposition be interesting than that it be true.”

What is Discrete Mathematics?

DM is the part of mathematics devoted to the study of discrete objects.

However, there is no exact definition of the term "discrete mathematics".

But roughly we'll get the idea of what comes under DM & what

→ DM : field of mathematics
↳ we study Discrete objects

Ram, shyam, a, b, 1, 2, ...

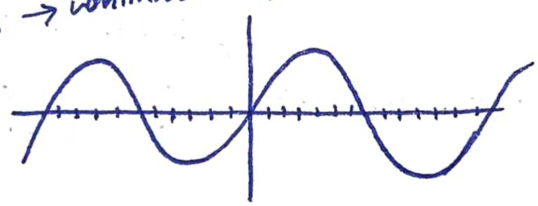
1, 2, 3, 4, 5, ...

Graph : 

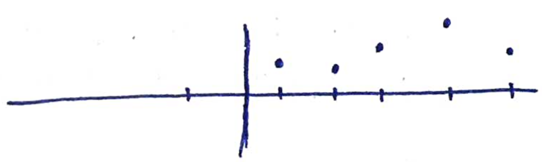
In DM → we study : set, Graphs, Functions, statements, ...

→ calculus → continuous objects

sin x



DM



→ Discrete Maths v/s ~~Calcu~~ continuous Maths

The objects studied in DM —such as integers, graphs, & statements in logic— do not vary smoothly.
Topics include sets theory, probability etc.

The objects studied in continuous math —such as real nos— vary smoothly.
Topics include, calculus, etc.

Integers : $0, -1, 1, -2, 2, -3, 3, \ldots$ → these are discrete objects.

Real No.s :
↳ continuous objects.

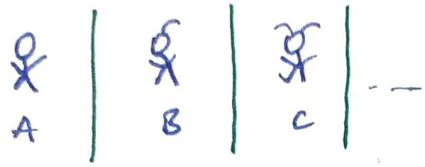
$-\infty \longleftarrow -2 \quad -1 \quad 0 \quad 1 \quad 2 \longrightarrow \infty$ → Real line

0 , 0.01
↓
0.0001

The more we zoom b/w 2 no.s the more no.s we get

not.

People :
↓
Discrete objects



o→ In Theory of computation (TOC) subject, we'll study

- → countable set → Discrete set
- → uncountable set → continuous set

o→ like we cannot define Mathematics precisely. likewise, we cannot define DM precisely. It is all a rough idea.

Why do we need DM ?
(Mathematics of computer Science) → name of DM course in MIT

DM is the backbone of CS. It has become popular in recent decades bcoz of its application to CS. DM improves our problem solving skills & computer science is all about solving problems by means of step by step procedure or automated tools.
concepts & Notations from DM are useful in studying & describing objects & problems in all branches of CS, such as computer Algo, Programming Lang, Cryptography, Automata Theory etc

MATHEMATICAL LOGIC

- lang of Mathematicians
- lang of computers
- 1st chapter of AI books

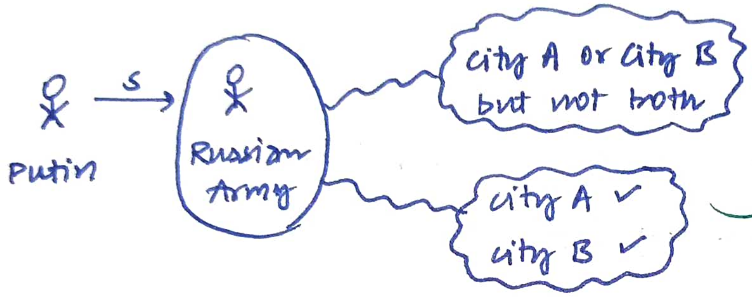
The rules of logic specify the meaning of mathematical statements.

Logic is the basis of all mathematical reasoning, & of all automated reasoning

Natural Languages are Ambiguous, but computer should not have ambiguity

ex: missile system

ex S : Attack City A or City B.



Ambiguity of English → Taken care by logic

ex Mary is a singer or poet.

- Exactly one of them.
  OR
- Atleast one of them.

Ambiguity of Natural Lang.

ex All men are Mortal, Socrates is a man. Hence Socrates is Mortal. — Argument 1 Right ✓

ex All cats like fish. Silvy isn't a cat. Hence, Silvy doesn't like fish. — Argument 2 Wrong ✗

↳ if you are a cat ⇒ you like fish

but if silvy isn't a cat, its not necessary that silvy like fish.

Maybe, silvy is Bengali :)

From Current knowledge $\xrightarrow[\text{derive}]{\text{correctly}}$ New knowledge

$\underbrace{\hspace{12em}}$

Mathematical Reasoning → will help to find out whether what we derived is correct or not

current knowledge ↓

| Cat → like fish
silvy Not cat |

→ (Not correct derivation) →

New knowledge ↓

| silvy not like fish |

→ MATHEMATICAL LOGIC :

When most people say 'logic', they mean either propositional logic or first-order predicate logic. However, the precise definition is quite broad, & literally hundreds of logic have been studied by philosophers, computer scientists, & Mathematicians.

Mathematical logic

- propositional logic (0^{th} order logic) → simplest type of logic } This is what we gonna study
- 1^{st} order logic
- 2^{nd} order logic
- 3^{rd} order logic
- ⋮ - - - - -
- fuzzy logic
- - - - - - -

There are 100s of type of ~~the~~ logics studied under Mathematical logic

PROPOSITIONAL LOGIC (0th order logic)

A world of True or False
It is the simplest type of logic

'Not True' = False
'Not False' = True

Earth (World of Humans)

not everything is True or False

Propositional logic world

True or False (Not both)

- **Proposition**: It is a declarative sentence (i.e, a sentence that declares a fact) that is either True or False; but not both.

  ↓
  It is an Assertive sentence

Propositional logic → world of propositions → T or F

Assertive sentences → propositions
Questions "
Commands "
Exclamations "
} Not propositions

→ Examples of Propositions: each statement is either T or F.

(i) "Drilling for oil caused dinosaurs to become extinct" is a proposition. ✓

(ii) All cows are brown. ✓

(iii) The Earth is farther from the Sun than Venus. ✓

(iv) $2 \times 2 = 5$ ✓

(v) There is a 1,00,000 digit prime no. ✓

(vi) DM is useful for Gate CSE exam. ✓

→ Things that are not propositions:

(i) Look Out! → exclamation sentence
(ii) sit down
(iii) Fly you Fools

} commands can't be propositions.

(iv) what is the Time?
(v) How far is it to the next town?

} Questions can't be propositions.

(vi) He is tall → not a proposition.
He → pronoun.

(vii) $x+1=2$ → T if $x=1$
F if $x\neq 1$

} Not a proposition → Truth value depends on 'x' (variables)

↓ same

(viii) $2x = 2+x$ → This is a declerative sentence,
but unless x is assigned a value or is otherwise prescribed, the sentences neither T or F, hence, not a proposition.

→ How to determine a sentence is proposition or not?

	Author 1	Author 2	Author 3
(i) Today is Friday	✓	✗	✗
(ii) He is a good boy	✓	✗	✗
(iii) John is a good boy	✓	✓	✗

Author 1 ↓ Keneth Rosen, Stanford CSE course

Author 3 ↳ pure mathematicians

→ These types depends on the author

A proposition is a claim; something that is T or F.

All these 3 are sentences that can be used to express propositions, athough the context will have to make clear exactly what is being claimed. For ex, the 1st claim's Truth- ~~table~~ value depends on what day you make the claim, the 2nd depends on what 'he' is referring to, & even when you use 'John' in the 3rd sentences, the context will have to make clear which of the many 'John's you are taking about. Moreover, what exactly constitutes a 'good boy'?

Some texts will say that bcoz of these ambiguities or unresolved indexicals, none of these are propositions. Other texts may say that 'He is a good boy' is not, but 'John is a good boy' is.

So, there is a lot of confusion & disagreement on exactly when a sentence is, or expresses, a proposition...

→ I suggest not to waste time on 'whether a given sentence is proposition or not'.
Any Good Exam will Never ask you "which sentence is prop?"

ques. Which is a proposition?
(i) Today is Monday.
(ii) He is 56 kg.
(iii) Ram is a good boy

Answers:
Dont waste time bro,
go text with your gf

↳ The good news is that once this introductory stuff is out of the way, it wont actually matter whether 'Tomorrow is Monday' is a prep or not, bcoz thats not the kind of statement you'll be reasoning about. Mathematics is not the study of what day it is ;) We're on much safer ground with "0 is an integer" or "f is a continuous func".
Then I think it is an unfair ques. you shouldn't be asked to guess for yourself whether "Tomorrow is Monday" will or not be an acceptable prop, without knowing what you going to do with it.
At this introductory level the imp point should be that a prop can be considered either T or F for the purposes of the logic you're about to do.

→ The questions will give you the prep; They will not ask you.

→ Some standard conclusions:
what is not a prep → 1) Questions
2) command, Exclamation sentences.

↳ Mathematical logic deals with Propositional Variables → either T or F

Mathematical logic ↓ doesn't deal with exact sentences of proposition
eg. Tommorow is Sunday ✗

Propositional Variables ↓ a variable which represents a prop.

→ Propositional Variables : → variables that represents propositions : p, q, r, s

Each ~~variable~~ prop will be represented by a prop. variables.
We use letters to denote prop variables (or statement variables), that is, variables that represent variables.
Each variable represents some proposition.

eg. Proposition p ↦ "Today is Friday".
→ Truth Values → T or F

Truth values : The truth value of a prop is true, denoted by T, if it is a true proposition, & the truth value of a proposition is false, denoted by F, if it is a false proposition.

Each (prop) variable can take one of 2 values : True or False

↳ 'p' is a prop variable

p → True or False } but not both
↓
representing some proposition → T or F

propositional variable ⟵ will be used interchangably ⟶ proposition

p is a prop variable ✓
p is a proposition. ✓

e.g. S : "S is False" → not a proposition.

case 1 : Assume S = True
↳ S = False

case 2 : Assume S = False
↳ S = True

} S is neither True, nor False

so, S is self contradictory
⇒ S is Paradox. → liar Paradox in logic.

Atomic Prop : Truth values of these prop doesn't depend on Anyone else.

Compound Prop :
we can produce new propositions from those we already have.
Many mathematical statements are constructed by combining one or more propositions.
New prop, called compound propositions, are formed ~~by~~ from existing propositions ~~called~~ using logical operators.

ex P : This book is interesting
R : I am staying at home
} Atomic propositions

This book is interesting, and I am staying at home
This book is not interesting
This book is interesting or I am staying at home.
This book is interesting or I am staying at home but not both
If this book is interesting then I am staying at home.

compound prop → created by Atomic prop.

↳ P, R → Atomic prop
with help of connectives
↓
New prop { P and R ; P or R }
(compound prop)

↳ logical connectives
↳ 8 standard logical connectives

1. Negation ¬
2. AND ∧
3. OR ∨
4. XOR ⊕
5. Implication →
6. Bi-Implication ↔
7. NAND ↑
8. NOR ↓

Proposition : $p \to$ True, $p \to$ False

Connectives : $\neg$, $\wedge$, $\vee$, $\oplus$, $\to$, $\leftrightarrow$, $\uparrow$, $\downarrow$

propositional logic : world of propositions , a mathematical system for reasoning about propositions & how they relate to one another.

T or F $\leftarrow$ $p, q, r, s, \ldots$

$\neg p$, $p \wedge q$, $p \vee q$, $p \oplus q$ $\to$ T or F

$p \to q$, $p \leftrightarrow q$, $p \uparrow q$, ... $\to$ T or F

LOGICAL OPERATORS on propositions

(i) **Negation** : unary operator $\to$ operation on 1 proposition

when applied to a prop 'p', changes the truth value of 'p'.

It is denoted by $\neg p$, $\tilde{p}$, p' , $\bar{p}$, $\sim p$

read as "not p".

T $\to$ $\boxed{\text{Negation}}$ $\to$ F

F $\to$ $\boxed{\text{Negation}}$ $\to$ T

Truth Table :

p	$\neg p$
T	F
F	T

ex p : This book is interesting

$\neg p$ can be read as :

(i) This book is not interesting

(ii) This book is uninteresting

(iii) It is not the case that this book is interesting

(iv) It is not True that this book is interesting

ex q : Micheal's PC runs Linux.

then $\neg q$: Micheal's PC doesnot runs Linux

or, $\neg q$: It is not the case that $\underbrace{\text{Micheal's PC runs Linux.}}_{q}$

ques. p : Today is Monday. what $\neg p$?

It is not the case that Today is Monday.

(i) Today is Wednesday ✗

(ii) Today is Thursday ✗

(iii) Today is not Monday ✓

(ex) p : Vandana's smartphone has atleast 32 GB of Memory

$\neg p$: It is not the case that Vandana's smartphone has atleast 32 GB of Memory

It is not the case that 'p'.

$\neg p$: Vandana's smartphone doesnot have atleast 32 GB of Memory

$\neg p$: Vandana's smartphone has less than 32 GB of memory.

(ex) p : 2 is even no. : T

$\neg p$: 2 is not even no : F

p	$\neg p$
T	F

p : 3 is even no. : F

$\neg p$: 3 is not even no. : T

p	$\neg p$
F	T

→ **Truth Table** is used to determine when a compound statement is T or F.

p	$\neg p$
F	T
T	F

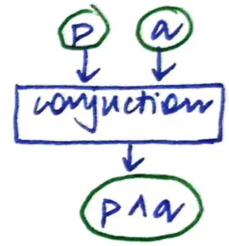
Truth Table tells you the definition/behaviour of an operation.

CONJUCTION (AND)

{ $\wedge$ is logical AND. In C prog, logical AND is represented as && }

let p & q be the propositions. The conjuction of p and q, denoted by $p \wedge q$, is the proposition "p and q".

The conjuction $p \wedge q$ is T when both p & q is T, and is F otherwise.



↳ conjuction Operator, "and", has symbol '$\wedge$'.

$p \wedge q$, $p \cdot q$, pq ← binary connective ↳ applies on 2 things

ex p : 9 is divisible by 3.

q : 5 is an odd no.

s : $\underbrace{\text{9 is divisible by 3}}_{p}$ and $\underbrace{\text{5 is an odd no.}}_{q}$

s : p and q — English statement

s : $p \wedge q$ } Mathematical statement

ex p : This book is interesting

q : I am staying at home.

p and q $\equiv$ $p \wedge q$: This book is interesting and I am staying at home.

ex P : 10 is divisible by 3 → F

Q : 4 is an odd no. → F

$P \wedge Q$: 10 is divisible by 3 and 4 is an odd no. ↳ F

↓ AND/conjuction

P	Q	$P \wedge Q$
F	F	F

(ex) P : 9 is divisible by 3. → T

Q : 4 is an odd no. → F

P ∧ Q : [9 is divisible by 3 <u>and</u> 4 is an odd no.] → F (overall)

P	Q	P ∧ Q
T	F	F

(ex) P : 10 is divisible by 3. → F

Q : 4 is an even no. → T

P ∧ Q : [10 is divisible by 3 <u>and</u> 4 is an even no.] → F (overall)

P	Q	P ∧ Q
F	T	F

(ex) P : 9 is divisible by 3. → T

Q : 4 is an even no. → T

P ∧ Q : [9 is divisible by 3 <u>and</u> 4 is an even no.] → T

P	Q	P ∧ Q
T	T	T

→ conjuction is a binary operⁿ as it operates on 2 prop.

P	Q	P ∧ Q
F	F	F
F	T	F
T	F	F
T	T	T

P ∧ Q → conjucⁿ of P, Q

P and Q is T only when both P & Q are T.

→ $\underbrace{\text{P and Q}}_{\text{english statement}} \equiv \underbrace{P \wedge Q}_{\text{prop. logic statement/expression}}$

o→ In logic, we have

AND = But = Though = Even Though = However = yet
= Nonetheless = Nevertheless = Moreover } → $\wedge$

english statements such as "even though", 'however', 'but', 'yet' all have usages that contrast facts, but they are all translated to 'AND' in propositional logic.

P but Q $\approx$ P AND Q

P however Q $\approx$ P AND Q

(ex) P : Joe has received below a C in a class.
Q : Joe has maintained a B average.

All these are $P \wedge Q$

↳ Joe has maintained a B average even though he did receive a grade below a C in a class.

Joe has maintained a B average and he did receive a grade below a C in a class.

Joe has maintained a B avg though he did receive a grade below a C in a class.

Joe did receive a grade below a C in a class, however, Joe has maintained a B average.

Joe did receive a grade below a C in a class, yet, Joe has maintained a B average.

Joe has maintained a B avg moreover he ~~did~~ received a grade below a C in a class.

Joe received a grade below a C in a class, nevertheless, Joe has maintained a B avg.

Joe has received a grade below C but Joe maintained B avg.

DISJUNCTION (OR)

{ In C prog → || ; In DLC → OR GATE }

Let p & q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the prop "p or q". The disjunction $p \vee q$ is F when both p and q are F & is T otherwise.

↳ Disjunction Operator, inclusive "or", has symbol '∨'.

$p \vee q$, $p+q$ ✓, p or q ✓

In logic:
Or = unless = Inclusive Or

ex
p : This book is interesting.
q : I am staying at home

$\underbrace{p \vee q}$: This book is interesting, or I am staying at home

↓ p or q

→ Disjunction of p, q

ex
p : 9 is divisible by 3.
q : 5 is an odd no.

s : $\underbrace{\text{9 is divisible by 3}}_{p}$ OR $\underbrace{\text{5 is an odd no.}}_{q}$

s : p OR q English statement

s : $p \vee q$ } prop logic statement

ex p : 9 is divisible by 3. → T
q : 5 is an odd no. → T

$p \vee q$: [9 is divisible by 3 OR 5 is an odd no.] → T

p	q	$p \vee q$
T	T	T

ex p : 9 is divisible by 3. → T
q : 4 is an odd no. → F

$p \vee q$: [9 is divisible by 3 OR 4 is an odd no.] → T

p	q	$p \vee q$
T	F	T

ex p : 10 is divisible by 3. → F
q : 5 is an odd no. → T

p∨q : [10 is divisible by 3 OR 5 is an odd no.] → overall T

p	q	p∨q
F	T	T

ex p : 10 is divisible by 3. → F
q : 4 is an odd no. → F

p∨q : [10 is divisible by 3 OR 4 is an odd no.] → overall F

p	q	p∨q
F	F	F

→ Truth Table of Disjunction

P	Q	P ∨ Q
F	F	F
F	T	T
T	F	T
T	T	T

P ∨ Q → disjunction of P, Q

"P or Q" is T, even if P and Q both are T.
Remember there are 3 ways for "P OR Q" to be T.

Now, P : You can come on Sunday.
Q : You can come on Monday.

Me → Receptionist

S : You can come on Sunday OR on Monday.
OR → inclusive OR

Monday	Sunday	S
✓	✗	✓
✗	✓	✓
✓	✓	✓

→ We went on both the days.
Receptionist : You fool. ek din ana tha bas
↓
The real fool who doesn't know maths logic

She must have said,

S : You can come on Sunday OR on Monday But not both
→ Exclusive OR

Natural Languages → Ambiguous

Prop logic → Not Ambiguous

(ex) Mary is Poet or singer → Ambiguous English Statement

$P \vee Q$ → Not Ambiguity

prop logic expression

$\underbrace{P \oplus Q}$ → Not Ambiguity

Mary cannot be both singer, Poet.

→ Mary is Poet <u>or</u> singer → $P \vee Q$ ✓

Mary is Poet <u>or</u> singer <u>but not both</u> → $P \oplus Q$ ✓

→ P or Q ≡ $P \vee Q$ inclusive OR (OR)

P	Q	$P \vee Q$
T	T	T

P or Q $\boxed{\text{but not both}}$ ≡ $P \oplus Q$ Exclusive OR

P	Q	$P \oplus Q$
T	T	F

EXCLUSIVE OR $\oplus$, XOR {In DLC → XOR Gate}

let p & q be propositions. The 'exclusive or' of p & q, denoted by $p \oplus q$, is the prop that is T when exactly one of p & q is T and is F otherwise.

↳ Exclusive Or Operator, "XOR", has symbol '$\oplus$'.

(ex) p : This book is interesting.

q : I am staying at home.

$p \oplus q$: Either this book is interesting, or I am staying at home, but not both.

↳ p or q but not both

→ Exclusive OR

Truth Table :

p	q	p⊕q
F	F	F
F	T	T
T	F	T
T	T	F

(p EXOR q)

same as inclusive OR

→ when both p, q are T, p⊕q is F.

ex Exclusive (News):

P ⊕ Q
P EXOR Q

will be True when Exclusively P, Q is True

Aajtak
Zee News
(Godi Media)

same News → Not Exclusive

k if Aajtak, Zee News dont have that particular News → it is also Not Exclusive

A	Z	A ⊕ Z	
F	F	F	→ Not Exclusive
F	T	T	Exclusive
T	F	T	Exclusive
T	T	F	→ Not Exclusive

$\boxed{\text{NOTE}}$ → $p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$

$= (p \vee q) \wedge (\neg p \vee \neg q)$

proof :

p	q	p⊕q	(p∧¬q)	(¬p∧q)	(p∧¬q)∨(¬p∧q)
T	T	F	T∧F = F	F∧T = F	F
T	F	T	T∧T = T	F∧F = F	T
F	T	T	F∧F = F	T∧T = T	T
F	F	F	F∧T = F	T∧F = F	F

≅

NAND, NOR

NAND : Negation of AND NAND : $\uparrow$

NOR : Negation of OR NOR : $\downarrow$

$$P \text{ NAND } Q = P \uparrow Q = \overline{(P \wedge Q)}$$

$$P \text{ NOR } Q = P \downarrow Q = \overline{(P \vee Q)}$$

P	Q	$P \wedge Q$	$P \uparrow Q$	$P \vee Q$	$P \downarrow Q$
F	F	F	T	F	T
F	T	F	T	T	F
T	F	F	T	T	F
T	T	T	F	T	F

CONDITIONAL OPERATION / IMPLICATION

Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q". The conditional statement $p \to q$ is F when p is T and q is F, & T otherwise.

LHS of implication ← $p \to q$ → RHS of implication

p: hypothesis, premise, antecedent, condition

q: conclusion, consequence

↳ Implication operator, "if ... then ...", has symbol '$\to$'.

(ex) "If $\underbrace{\text{score is 2}}_{p}$ then $\underbrace{\text{score is even.}}_{q}$"

If p, then q] English

$p \to q$] prop. logic

If (P) then (Q) ✓

condition $\xrightarrow{\text{implies}}$ conclusion

→ P implies Q ✓ $\boxed{P \rightarrow Q}$ ✓

implication symbol

ex If $\underbrace{\text{you dont study}}_{\text{condition/ Hypothesis}}$, $\underbrace{\text{you'll not crack GATE}}_{\text{consequence/ conclusion}}$.

$\boxed{\text{Not study}} \xrightarrow{\text{implies}} \boxed{\text{Not crack GATE}}$

→ Now, lets study the behaviour of implications with example:

ex ~~Claim1~~: "If our party ABC wins, you'll get free macbook air".

When will you say that this claim is False?

↳ Only when ABC wins AND you dont get free macbook air.

ABC wins	Free MacAir	Claim1
T	F	F ✓✓
T	T	T
F	F	T
F	T	T

} claim 1 is not False, as they didn't win.

Claim1: (conditional claim)

$\boxed{\text{ABC wins}} \rightarrow \boxed{\text{Free MacAir}}$

ex Claim 2 : " If you ~~reach~~ take train, you'll reach Delhi on time".

Take Train	Reach Delhi	Claim 2
T	F	F
T	T	T
F	F	T
F	T	T

} claim 2 is not False as you didn't take train (ex) Claim 3: "If he eats this poison, he'll die".

poison	Die	claim3
✓	×	F
✓	✓	T
×	×	T
×	✓	T

claim cannot be False

(ex) claim 4: "If you work, I'll give you salary".
(of a company)

work	salary	claim 4 (work → salary)
✓	×	F
✓	✓	T
×	×	T
×	✓	T

claim cannot be False

NOTE →

$T \to T = T$

$\boxed{T \to F = F}$ → only situation to make implication F.

$\underbrace{F \to \bigcirc} = T$ $\left\{ \begin{array}{l} F \to F = T \\ F \to T = T \end{array} \right\}$ If the condition is not T, Then the implication cannot be F

i.e., $P \to Q$

P: condition, Q: conclusion

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

NOTE →

$P \to Q$
$Q \leftarrow P$
same ⇒ $\underbrace{\text{If P then Q}}_{\text{P implies Q}}$

Q: RHS of implication
P: LHS of implication

ques. If $\underbrace{n \text{ is } 2}_{P}$ then $\underbrace{n \text{ is even}}_{Q}$: $P \to Q$

To make $\underset{n \text{ is even}}{Q}$ true, Is $\underset{n \text{ is } 2}{P}$ necessary or sufficient?

↳ P is sufficient but not necessary.

(ex) if you want $\underbrace{\text{even Amount}}_{Q}$

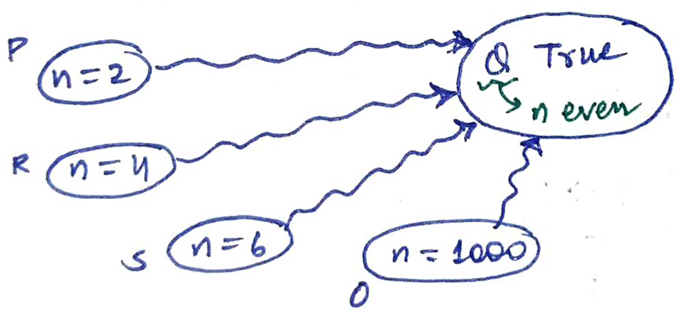
- Is it "necessary" that I give ₹2 ?

↳ NO! Not necessary! why? → bcoz we can give ₹4, ₹6, ₹1000, ...

- But is it "sufficient" (enough) that I give ₹2 ?

↳ yes!

So, there are many ways for Q to be T, P is not necessary.



P is sufficient for Q ✓

R is sufficient for Q ✓

S is sufficient for Q ✓

P is necessary for Q ✗

$P \to Q$

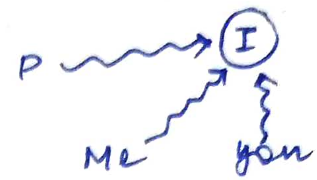
P is sufficient for Q.

"n being 2" is sufficient for "n to be even".

(ex) S: If $\underbrace{\text{you are India's PM}}_{P}$ then $\underbrace{\text{you are Indian}}_{I}$. } This statement is True

S : $P \rightarrow I$

P is sufficient for I. ✓
(enough)



P is necessary for I. ✗
P: being India's PM
I: being Indian

NOTE → Implication ($\rightarrow$)

- $p \rightarrow q$ means 'if p is true, q is true aswell'
  i.e. if condition (p) is satisfied (True) then conclusion (q) must be satisfied (True).
- $p \rightarrow q$ says nothing about what happens if p is false
  i.e. if condition (p) is NOT satisfied (False) then conclusion (q) can be anything.

$p \rightarrow q$ says nothing about casuality; it just says that if p is true, q will be true as well.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

- F, F and F, T: in both these, p is False, so statement ($p \rightarrow q$) 'if p then q' is vacuosly True
- T, F: $p \rightarrow q$ should mean when p is T, q is T as well. But here p is T & q is F
  The only way for $p \rightarrow q$ to be F.
- T, T: $p \rightarrow q$ means if we ever find that p is T, we'll find that q is T as well.

The implication $p \to q$ is the prop that is often read "if p, then q"
'if p then q' is false precisely when P is T but q is F.

Only way to think of the meaning of $p \to q$ is to consider it a contract that says if the 1st condition is satisfied, then the 2nd will also be satisfied. If the 1st condition, P, is not satisfied, then the condition of the contract is null & void.
In this case, it doesnot matter if the 2nd condition is satisfied or not, the contract is still upheld.

Now,

P : If $\underline{x \text{ is natural no}}$ then $\underline{x \text{ is integer}}$ ✓ True

S : If $\underbrace{x \text{ is integer}}_{A}$ then $\underbrace{x \text{ is natural no.}}_{B}$ ✗ False

S : $A \to B$

if $x = -5 \rightsquigarrow \{\boxed{A = T ; B = F}\}$ makes $A \to B$ False

↗ we found atleast one 'x' for which 'x' is integer but not natural no.

P : if $\underbrace{x \text{ is natural}}_{N}$ no then $\underbrace{x \text{ is integer.}}_{I}$ } True

$N \to I$

if N then I

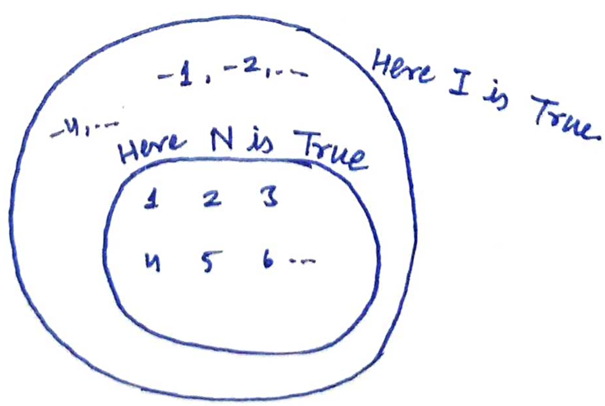
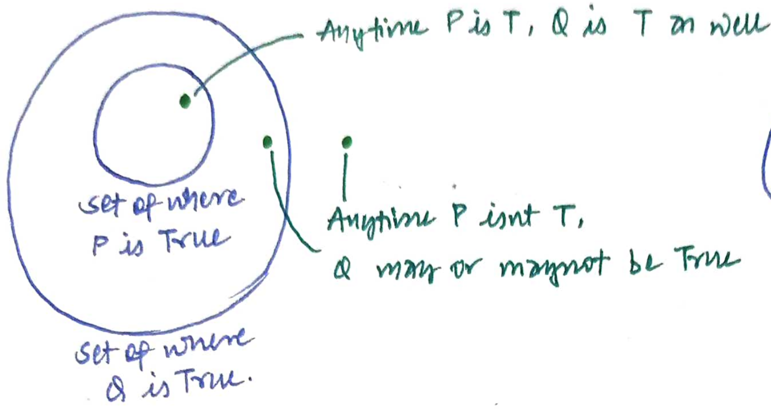


Diagram Representation of Implication

when $\boxed{P \to Q \text{ is True}}$: whenever P is True then Q is True



P→Q is True
T → T
F → T/F

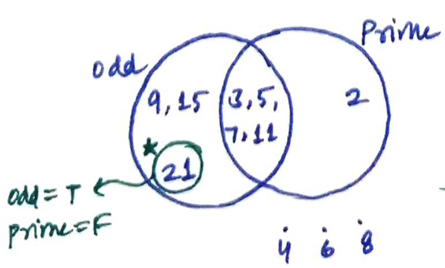
When P doesnot Imply Q :

- P → Q means 'if P is T, q is T as well'.
- The only way for P→Q to be F is if we know that P is T but Q is F.
- Rationale:
  - ↳ if P is F, P→Q doesnt guarantee anything. Its true, but it's not meaningful.
  - ↳ if P is T & Q is T, then the statement "if P is T, then Q is also T" is itself T.
  - ↳ if P is T & Q is F, then the statement "if P is T, then Q is also T" is itself F.

$\boxed{\text{when } P \to Q \text{ is F}}$ means there is atleast 1 situation where P is T & Q is F.

ex) if x is odd then x is Prime } False

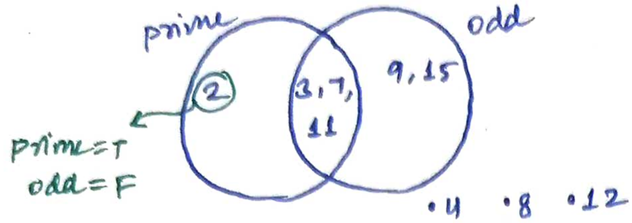
R : odd → prime



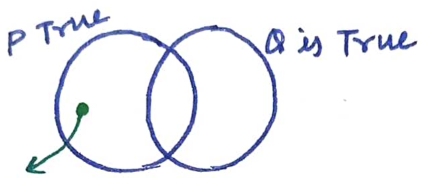
so, odd → Prime is F.

ex) if x is prime, then x is odd. } False

R: prime $\rightarrow$ odd



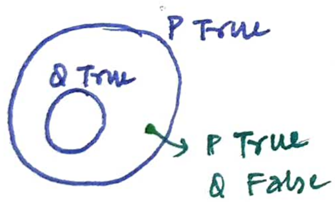
NOTE → General Diagram when $P \rightarrow Q$ is False : there will be atleast one situation where P is T & Q is F.



$\left.\begin{matrix} P = T \\ Q = F \end{matrix}\right\} \Rightarrow P \rightarrow Q = F$

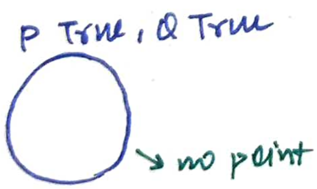
ques. It is known that $P \rightarrow Q$ is False, then which is a possible diagram Representation

(A) ✓✓



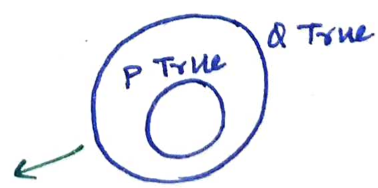
(B) ✓✓

(C) ✗



no point where P is True & Q is False

(D) ✗



↳ $p \to q$ means

if p, then q.

if p is T, then q is T.

$p \to q$

$\neg p \to q$ or $\neg p \to \neg q$

$p \to q$ is F only when p True AND q False.

$p \to q$ is True when p False OR q True.

↳ { p = T, q = T ; p = F, q = T ; p = F, q = F } 3 situations

False implying Anything

→ F → F/T is True

F/T → T is True

Anything implying True

T → F is False.

→ Conditional statement : If P then Q ≡ $P \to Q$

when P True → Q True

But when P False → Q True or → Q False

$\boxed{\text{So, conditional statements are one way}}$ $P \to Q$

$p \to q = q \to p$?? NO

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Different Truth Tables.

- $p \to q$ is F only when $p = T$, $q = F$. (p: condition)
- $q \to p$ is F only when $q = F$, $p = T$. (q: condition)

$p \to q \neq q \to p$

+ $p \to q$: if p then q, $\underset{\text{condition}}{p} \longrightarrow \underset{\text{conclusion}}{q}$

+ $q \to p$: if q then p. $\underset{\text{condition}}{q} \longrightarrow \underset{\text{conclusion}}{p}$

ex P : you are PM of India.

Q : you are Indian

↳ $P \to Q$ ✓
$Q \to P$ ✗ } $P \to Q \neq Q \to P$

$Q \to P$ says "If you are Indian then you are PM".

ex A : x is a natural no.

B : x is integer

↳ $A \to B$ ✓
$B \to A$ ✗ } $A \to B \neq B \to A$

Sufficiency and Necessity

↳ "Filling the Gate Form" (P) is Necessary for "cracking GATE Top-10 Rank" (Q).

$P \to Q$ ✗
$Q \to P$ ✓ } P is Necessary for Q ✓

↳ "Being Natural" (P) is sufficient (enough) for "being integer" (Q).

$P \to Q$ ✓
$Q \to P$ ✗ } P is sufficient for Q

↳ "Being Natural" is Necessary for "being integer". ✗ ex -9 Not natural But Integer

Sufficient condition : A condition 'P' is said to be sufficient for condition 'Q', if (and only if) the truth (/existence /occurence) of 'P' guarantees (or bring about) the truth (/existence /occurence) of 'Q'

ex while air is necessary condition of human life, it is by no means a sufficient condition, i.e, it doesnot, by itself, i.e. alone, suffice for human life. While someone may have air to brethe, that person will still die if s/he lacks water, has taken poison, etc.

$\boxed{\text{P is sufficient for Q}}$

happening of P is enough for happening of Q.

If P is True then Q is True ≡ If Q is False then P is False.

$P \rightarrow Q$

If Q is F then P cannot be T.
bcoz if P is T, Q must be T.

ex for whole no > 2

being odd (P) is Necessary for being prime (Q).

$P \rightarrow Q$ ✗ Odd > 2 $\not\rightarrow$ Prime

$Q \rightarrow P$ ✓ Prime > 2 $\rightarrow$ odd ✓

ex for whole no > 2

being odd is sufficient for being prime. ✗

ex: 9 odd ✓
prime ✗

Necessary condition: A condition 'P' is said to be necessary for a condition 'Q', if (and only if) the falsity (/nonexistence / non-occurence) [as the case may be] of 'P' guarantees (or brings about) the falsity (/nonexistence/non occurrence) of 'Q'.

ex Air (P) is necessary for Human life (Q) ≡ without Air, NO Human life

Air → Human life ✗

Human life → Air ✓

ques why not sufficient ?
bcoz we need water, sleep etc

$\boxed{\text{P is Necessary for Q}}$

$Q \rightarrow P$ ✓

$P \rightarrow Q$ ✗

(ex) "Age > 18" is sufficient for "Voting Right" } ✗

$\underbrace{\text{Age} > 18}_{P}$ ... $\underbrace{\text{Voting Right}}_{Q}$

Age 18 → Voting Right ✗ You also need to be indian, you must not be prisoner.

"Age > 18" is Necessary for "Voting Right"

$\underbrace{\text{Age} > 18}_{P}$ ... $\underbrace{\text{Voting Right}}_{Q}$

without Age > 18, you can't have voting right

If you have voting right then Age > 18

$Q \rightarrow P$ ✓

(ex) Reading chemistry is Neither sufficient Nor necessary to crack GATE CSE exam

$\underbrace{\text{Reading chemistry}}_{P}$ ... $\underbrace{\text{crack GATE CSE exam}}_{Q}$

$\boxed{\text{NOTE}}$ → P is sufficient for Q ≡ $P \rightarrow Q$

≡ Q is necessary for P ≡ $P \rightarrow Q$

≡ If P then Q ≡ P implies Q

(ex) If x is Natural No then x is integer.

$\underbrace{x \text{ is Natural No}}_{N}$ ... $\underbrace{x \text{ is integer}}_{I}$

≡ $N \rightarrow I$

≡ N is sufficient for I ≡ I is necessary for N

$\boxed{\text{NOTE}}$ → P is sufficient for Q ≡ $P \rightarrow Q$ ✓ $Q \rightarrow P$ ✗

P is necessary for Q ≡ $Q \rightarrow P$ ✓ $P \rightarrow Q$ ✗

Equivalent forms of "if p then q"

$p \rightarrow q$

p: hypothesis, premise, antecedent, condition

q: conclusion, consequence

- if p then q
- if p is True, then q is True
- p implies q
- p only if q
- q if p
- q when p
- q follows from p
- q whenever p
- A sufficient condition for q is p
- A necessary condition for p is q
- q is necessary for p
- if p, q
- q unless $\neg p$

In logic,

if = when = whenever

= provided that = given that

↳ makes a condition

if p then Q ≡
when p then Q ≡
whenever p then Q ≡
provided that p then Q ≡
given that p, Q ≡
if p, Q ≡

$\Big\} \; P \rightarrow Q$

Trick : $p \rightarrow q$: p is sufficient for q ≡ $p \rightarrow q$
q is necessary for p ≡ $q \rightarrow p$
$\Big\} \equiv p \rightarrow q$

↳ P only if Q means without Q, P can't happen
= P will happen only if Q happen

✓ P is sufficient for Q
✗ P is necessary for Q

Q is necessary for P
P is sufficient for Q
$P \rightarrow Q$ ✓

↳ P $\boxed{\text{only if}}$ Q ≡ $P \to Q$

↳ P $\boxed{\text{if}}$ Q ≡ $\underbrace{\text{If } Q, P}_{Q \to P}$

$P \leftarrow Q$

TRICK :

α $\boxed{\text{only if}}$ β ≡ $\alpha \to \beta$

α $\boxed{\text{if}}$ β ≡ $\alpha \leftarrow \beta$

ques. (i) only if Q, P ≡ $P \to Q$ → Q is necessary for P

(ii) if Q, P ≡ $Q \to P$

(iii) P only if Q ≡ $P \to Q$

(iv) Q if P ≡ $P \to Q$

↳ P is sufficient for Q

{ if P, Q ≡ Q if P
P only if Q ≡ Only if Q, P }

ex P : If $\underbrace{x \text{ is natural no}}_{N}$ then $\underbrace{x \text{ is integer}}_{I}$.

↳ $N \to I$ ≡ if N, I ≡ N only if I ≡ I if N

$\boxed{\text{NOTE}}$ → $p \to q \equiv \neg p \vee q$

↳ $p \to q$

↳ converse : $q \to p$

↳ contrapositive : $\neg q \to \neg p$

↳ Inverse : $\neg p \to \neg q$

ex $\underbrace{\text{I will stay}}_{p}$ only if $\underbrace{\text{you go}}_{q}$

I will stay → you go.

converse: $q \to p$

If you go then I will stay.

Using "$\rightarrow$" only one way statements can be represented.

ex A : You are from UP B : You are Indian

↳ If you are from UP then you are Indian. $A \rightarrow B$ ✓ $B \rightarrow A$ ✗

ex P : x is even Q : 2x is even

↳ $P \rightarrow Q$ ✓ $Q \rightarrow P$ ✗ ↳ says if 2x is even then x is even
ex 2x3 is even but 3 not even.

↳ But sometimes, we have, Two way statements

ex $\underbrace{x \text{ is } 2}_{P} \rightleftarrows \underbrace{x \text{ is even prime}}_{Q}$

$P \rightarrow Q$ ✓ $Q \rightarrow P$ ✓

ex $\underbrace{x \text{ is even}}_{P} \rightleftarrows \underbrace{x+2 \text{ is even}}_{Q}$

$P \rightarrow Q$ ✓ if x is even then x+2 is even ✓

$Q \rightarrow P$ ✓ if x+2 is even then x is even

↳ Implication is only one way. To represent 2 way statements, we need a new connective : bi-Implication ($\leftrightarrow$)

$P \rightarrow Q \equiv$ P True $\rightarrow$ Q True ✓

Q True $\rightarrow$ P True ✗

$P \leftrightarrow Q \equiv P \rightleftarrows Q$

$\left.\begin{array}{l} P \rightarrow Q \\ \text{and } Q \rightarrow P \end{array}\right\}$ P True $\rightarrow$ Q True ✓
Q True $\rightarrow$ P True ✓

BICONDITIONAL (iff)

Let p & q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is True when p and q have the same truth values, & is false otherwise. Biconditional statements are also called bi-implications.

ex p : This book is interesting q : I am staying at home.

$p \leftrightarrow q$: This book is interesting if and only if I am staying at home.

o→ $P \leftrightarrow Q$ means $P \to Q$ (P implies Q)
and
$Q \to P$ (Q implies P)

So, $\boxed{P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)}$

ex P : x is even
Q : 2x is even
$P \to Q$ ✓
$Q \to P$ ✗
⇒ $P \to Q$ AND $Q \to P$ ⇒ F ⇒ $\boxed{P \leftrightarrow Q}$ ✗

ex P : x is even
Q : x+2 is even
$P \to Q$ ✓
$Q \to P$ ✓
$\boxed{P \leftrightarrow Q}$
P double implies Q

P	Q	$P \to Q$	$Q \to P$	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

for $P \leftrightarrow Q$ to be True either both P & Q must be True or both must be False.

Truth Table of Biimplication

o→ $P \leftrightarrow Q \equiv \boxed{P \to Q}$ and $\boxed{Q \to P}$

P is sufficient for Q and P is necessary for Q

⇒ $P \leftrightarrow Q \equiv$ P is sufficient and necessary for Q ✓

o→ $P \leftrightarrow Q \equiv \boxed{P \to Q}$ and $\boxed{Q \to P}$

P only if Q and P if Q

⇒ $P \leftrightarrow Q \equiv$ P if and only if Q ✓

$P \leftrightarrow Q$:
- P double implies Q
- P if and only if Q
- P iff Q
- P is necessary and sufficient for Q
- if p then Q, and conversely
- if p then Q and if Q then P
- Q unless $\neg P$ and P unless $\neg Q$

$\boxed{\text{NOTE}}$ → EXOR, Bimplication are negation of each other.

i.e. $P \leftrightarrow Q \equiv \overline{(P \oplus Q)}$ or $P \oplus Q \equiv \overline{P \leftrightarrow Q}$

↳ negation of XOR
↳ exclusive OR

P	Q	$P \leftrightarrow Q$	$P \oplus Q$
F	F	T	F
F	T	F	T
T	F	F	T
T	T	T	F

↳. Definition : Definitions are always "if & only if statements" (even if they are written as 'if')

ex Definition of even no : A no. is even no. if it is integer multiple of 2
(even; m2)

m2 → even ✓
even → m2 ?? Yes

so correct statement is : "even if and only if m2".

↳ Implication tells you about 'PROPERTY'.

↳ BiImplication tells you about 'DEFINITION'.

⇛ Definitions of a even number :

D1 : integer multiple of 2

D2 : integer divisible by 2

D3 : n mod 2 = 0

$D1 \leftrightarrow D2$

$D2 \leftrightarrow D3$

$D1 \leftrightarrow D3$

⇛ Prime > 2 → odd ✓

is "being odd" definition of "prime > 2" ? NO

But "being odd" is property of "prime > 2". ✓

⇛ Boy → Human : "being Human" is PROPERTY of "being boy".

Define a Boy : $\underbrace{\text{Boy is Human}}_{\text{not correct definition}}$ → bcoz girl is also Human.

⇛ P → Q :

P is property of Q ✗ Q is property of P ✓

↳ Every P has property Q.

⇛ Properties of Natural No:

1) Integer 3) Rational 5) Odd or even

2) > 0 4) Real

N → I ✓ N → Real ✓ N → Rational ✓ N → odd/even ✓ N → '> 0' ✓

⇛ PM of India → Indian

So, PM has property that he/she is Indian.

But being Indian is not the definition of PM of India.

⇛ $x=2 \Leftrightarrow x=$ even prime $\equiv$ $x=2 \leftrightarrow x=$ even prime

so "being 2" is definition of "even prime".

"being even prime" is definition of "being 2".

$\boxed{\text{NOTE}}$ →

$P \leftrightarrow Q$ $\equiv$ P is definition of Q $\equiv$ Q is definition of P

$\equiv$ P, Q are definition of each other.

$P \rightarrow Q$ $\equiv$ Q is property of P (P satisfies property Q)

Definitions of Even Prime :

D1 : is 2 $\longleftrightarrow$ D2 : prime < 3 $\longleftrightarrow$ D3 : is prime & divisible by 2.

$\updownarrow$ even prime (for D1), $\updownarrow$ even prime (for D2), $\updownarrow$ even prime (for D3)

⇛ TOC subject has countability topic.

↳ we will study 3 diff definitions of "being countable".

D1 ✓ D2 ✓ D3 ✓ $\quad$ D1 $\leftrightarrow$ D2 $\quad$ D2 $\leftrightarrow$ D3 $\quad$ D1 $\leftrightarrow$ D3

D3 ↓ most easy

$\boxed{\text{NOTE}}$ → Logical Equivalence : $\Leftrightarrow$, $\equiv$

$p \Leftrightarrow q$ means that $p \leftrightarrow q$ is a tautology.

Logical Implication : $\Rightarrow$

$p \Rightarrow q$ mean that $p \rightarrow q$ is a tautology

↳ we'll see the definition of tautology soon.

The Operator Precedence in Logic :

Lec - 8

Highest - $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ - Lowest

For remaining connectives, proper parenthesis will (must) be given

✗ $p \oplus q \uparrow r$ ⇝ poor question

✓ $(p \oplus q) \uparrow r$ or $p \oplus (q \uparrow r)$ must be given ✓

eg.
- $\neg p \vee q \equiv (\neg p) \vee q$
- $\neg p \wedge q \vee p \equiv ((\neg p) \wedge q) \vee p$
- $\neg p \vee q \rightarrow s \equiv ((\neg p) \vee q) \rightarrow s$
- $p \vee q \wedge r$ → $p \vee (q \wedge r)$ ✓
  ✗ $(p \vee q) \wedge r$

qus. Let p represents a True statement, while q & r represent False statements. Find the Truth value of the compound statements. $p = T \quad q = r = F$

(i) $\sim[(\sim p \wedge \sim q) \vee \sim q]$
$= \sim[(\neg T \wedge \neg F) \vee \neg F]$
$= \sim[(F \wedge T) \vee T]$
$= \neg(F \vee T) = \neg T = F$ ✓

(ii) $\sim(p \wedge q) \wedge (r \vee \sim q)$
$\neg(T \wedge F) \wedge (F \vee \neg F)$
$= \neg F \wedge (F \vee T)$
$= T \wedge T = T$ ✓

(iii) $\sim(\sim p \wedge \sim q) \vee (\sim r \vee \sim p)$
$\neg(\neg T \wedge \neg F) \vee (\neg F \vee \neg T)$
$= \neg(F \wedge T) \vee (T \vee F)$
$= \neg F \vee T = T \vee T = T$ ✓

qus. Suppose the statement $((P \wedge Q) \vee R) \rightarrow (R \vee S)$ is False. Find the Truth values of P, Q, R, S.

Solⁿ $((P \wedge Q) \vee R) \rightarrow (R \vee S)$ is False, it means:

$(P \wedge Q) \vee R = \text{True}$
$\Rightarrow (PQ) \vee F = \text{True}$
↳ $P = T$ and $Q = T$

$R \vee S = \text{False}$
↳ $R = F$ and $S = F$

$\boxed{\text{NOTE}}$ ⇒ $\alpha \rightarrow \beta$ is False means $\alpha = T,\ \beta = F$

qus. If $P = Q = T$ and $R = S = F$. Then Truth values of : Precedence :→ $\neg > \wedge > \vee > \rightarrow > \leftrightarrow$

(a) $P \rightarrow Q \wedge \neg R \vee \neg S$
$P \rightarrow (\underset{T}{Q} \wedge \underset{T}{\neg R}) \vee \underset{T}{\neg S}$
$P \rightarrow T \vee T$
$T \rightarrow T \equiv T$ ✓

(b) $\neg P \rightarrow \neg Q \vee \neg R \wedge \neg S$
$F \rightarrow F \vee (T \wedge T)$
$F \rightarrow F \vee T$
$F \rightarrow T \equiv T$ ✓

TRUTH TABLE FOR STATEMENTS

$S : P \vee Q$

P < T, F
Q < T, F

P	Q	S
F	F	F
F	T	T
T	F	T
T	T	T

↳ $\alpha : p \wedge (p \rightarrow q)$ $\quad \beta : p \vee (p \rightarrow q)$

Truth Table of α & β,

p	q	α	β
F	F	F	T
F	T	F	T
T	F	F	T
T	T	F	T

or you can go in a sequential manner.

p	q	$p \to q$	$p \land (p \to q)$	$p \lor (p \to q)$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

$p \boxed{\land} (p \to q)$ ← we cant evaluate this until we have a value for $p \to q$.

↳ $(p \to q)$ is evaluated first.

↳ The we evaluate $p \land (p \to q)$ & it gives the final Truth value for the expression.

ques. with 1 prop. variable, then how many rows in the Truth Table? How many Truth-value combinations possible?

Ans. 2 P < T, F

P
T
F

ques. with 2 prop. variable.—

Ans. 4 2 choices ← P   Q → 2 choices

P < T, F   Q < T, F

P — T < Q = T, Q = F; P — F < Q = T, Q = F } 4 comb. of Truth value; 4 rows in Truth Table

ques with 3 prop. variables.—

Ans. 8 Rows in Truth ~~Values~~. Table

P (2 choices)   q (2 choices)   r → 2 choices. $= 2 \times 2 \times 2 = 2^3 = 8$ rows.

↳ combination of Truth values

ques. with n prop. variables...

Ans. a_1 a_2 a_3 ...... a_n → 2, 2, 2, 2

2^n rows { $2 \times 2 \times 2 \times \ldots \times 2 = 2^n$ combinations

∴ The Truth Table approach may not be always feasible, as when $n \uparrow$, no. of rows ↑ exponentially. It is better to use properties to prove the propositions.

ques. Does order of combination matters in Truth Table?

Ans. NO

p	q	$\alpha \equiv q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

=

q	p	$\alpha \equiv q \to p$
F	T	T
T	F	F
T	T	T
F	F	T

ques. Truth table for $(P \lor Q) \land \neg(P \land Q)$.

Ans.

P	Q	$P \lor Q$	$P \land Q$	$\neg(P \land Q)$	$(P \lor Q) \land \neg(P \land Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

these 3 colums are just HELPER COLUMNS & are not necessary part of the table. In writing truth tables, you may choose to omit such cols if you are confident about your work.

ques. $p \vee q \rightarrow r$, In Truth table, no. of rows & columns?

Sol^n $p \vee q \rightarrow r$ ⇝ will be evaluated as $(p \vee q) \rightarrow r$ $\vee > \rightarrow$ ✓

In Truth table, #Rows $= 2^3$

#columns $= 4$

p	q	r	$(p \vee q) \rightarrow r$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

{ $Y : \alpha \rightarrow \beta$
If β = True then Y = True
If α = False then Y = True }

In Truth Table, order of Truth values combination doesn't matter but anyways generally, we have a standard order. (in order of binary number)
(To avoid any confusion in combinations)

True ≡ T ≡ 1   False ≡ F ≡ 0

	p
0 →	0
1 →	1

	p	q
0 →	0	0
1 →	0	1
2 →	1	0
3 →	1	1

≡

p	q
F	F
F	T
T	F
T	T

} generally standard order of Truth Table

ques. construct the Truth Table of the compound proposition
$Y : (p \vee \neg q) \rightarrow (p \wedge q)$ ⇝ $\{\alpha \rightarrow \beta\}$

Sol^n

	p	q	$(p \vee \neg q) \rightarrow (p \wedge q)$
0 →	0	0	0
1 →	0	1	1
2 →	1	0	0
3 →	1	1	1

⇝ $Y \equiv q$ ✓

{ here we used the tricks
$\boxed{\text{Anything} \rightarrow \text{True}}$ is True
$\boxed{\text{False} \rightarrow \text{Anything}}$ is True }

E.g. $\neg x \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$
$\equiv \underset{P}{(\neg x)} \rightarrow \underset{Q}{(y \vee z)} \rightarrow \underset{R}{[x \vee (y \wedge z)]}$ $\equiv$ $\underbrace{P \rightarrow Q \rightarrow R}_{\text{Proper parenthesis must be given}}$ — $(P \rightarrow Q) \rightarrow R$ / $P \rightarrow (Q \rightarrow R)$

{ Proper parenthesis should be given
It is author dependent. }

E.g.

p	q	$p \vee \neg p$	$p \rightarrow q$	$p \wedge \neg p$
F	F	T	T	F
F	T	T	T	F
T	F	T	F	F
T	T	T	T	F
		Always True	sometimes True; sometimes False	Always False

Prop. Expression
- Always True → Tautology
- Always False → contradiction
- Atleast one time True; Atleast 1 time False → contingency

contingency means neither Tautology nor contradiction
→ not always True
→ not always False

①

Ques. Build a Truth Table to verify that the proposition $\underbrace{(p \leftrightarrow q) \wedge (\neg p \wedge q)}_{\alpha}$ is a contradiction.

Sol^n

p	q	α
F	F	F
F	T	F
T	F	F
T	T	F

} α is always False.
↳ is contradiction.

Ques. Which of the following propositional logic expressions are Tautology?

Sol^n
1) p : contingency
2) $\neg p$: contingency
3) $T \to p$: contingency
4) $p \to p$: Tautology
5) $p \to \neg p$: contingency
6) $\neg p \to p$: contingency
7) T : Tautology
8) F : contradiction
9) $p \to F$: contingency
10) $p \to T$: Tautology
11) $F \to p$: Tautology

LOGICAL EQUIVALENCE

The compound propositions p and q are called 'logically equivalent' if $p \leftrightarrow q$ is a Tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.
$\equiv$ → Equivalence symbol

i.e. Two expressions, $\alpha \equiv \beta$, iff they have same Truth Table.

↳ if $\underbrace{\alpha \equiv \beta}_{\alpha, \beta \text{ have same T.T}}$ } $\alpha \leftrightarrow \beta$ will be tautology.

ex $\alpha : p \to q$; $\beta : \bar{p} \vee q$ } $\alpha \equiv \beta$??

p	q	$\overset{\alpha}{p \to q}$	$\overset{\beta}{\bar{p} \vee q}$	$\alpha \leftrightarrow \beta$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	T	T	T

} Tautology

2 prop. expressions α, β are equivalent iff {i.e $\alpha \equiv \beta$}
↳ they have same Truth Table.
OR
↳ $\alpha \leftrightarrow \beta$ is Tautology.
OR
↳ $\alpha \equiv \beta$ iff {whenever α True, β True.
whenever β True, α True

↳ $\alpha \equiv \beta$ means, both have same Truth values in every situation.
{either both are True or both are False}

↳ $\alpha \equiv \beta$ & $\alpha \Leftrightarrow \beta$ are same thing just different representation.

Remark : The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound prop, but rather is the statement that $p \leftrightarrow q$ is a Tautology.

ques. $p \to q$; $\neg q \to \neg p$ } are they equivalent?

sol^n • method 1 : Truth Table → stupid & inefficient & boring

p	q	$p \to q$	$\neg q \to \neg p$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

} $p \to q \equiv \neg q \to \neg p$

• method 2 $\alpha : p \to q$ $\beta : \neg q \to \neg p$

case I : $p = T$

$\left.\begin{array}{l}\alpha : q \\ \beta : q\end{array}\right\}$ same

$T \to q \equiv q$ (F/T) (F/T)

$q' \to F \equiv q'$ (F/T) (T/F) } $q' \to F \equiv q$ ✓

$\Rightarrow \alpha \equiv \beta$ in case I

case II : $p = F$

$\left.\begin{array}{l}\alpha : T \\ \beta : T\end{array}\right\}$ same

{ $F \to \_ \equiv T$ }

{ $\_ \to T \equiv T$ }

$\Rightarrow \alpha \equiv \beta$ in case II

↳ $\alpha \equiv \beta$ ✓

ques Show that $(p \to q) \wedge (q \to p)$ is logically equivalent to $p \leftrightarrow q$.

sol^n • Method 1 : make Truth table of both. It will be equivalent

bcoz $(p \to q) \wedge (q \to p) \overset{\text{definition}}{\equiv} p \leftrightarrow q$

• Method 2 : $\alpha : p \leftrightarrow q$ $\beta : (p \to q) \wedge (q \to p)$

case I : $p = T$

$\left.\begin{array}{l}\alpha : q \\ \beta : q\end{array}\right\}$ same

$T \leftrightarrow q \equiv q$ (F/T) (F/T)

$(T \to q) \wedge (q \to T)$

$\equiv (T \to q) \wedge T \equiv q$

↳ $\alpha \equiv \beta$ ✓

case II : $p = F$

$\left.\begin{array}{l}\alpha : q' \\ \beta : q'\end{array}\right\}$ same

$F \leftrightarrow q \equiv q'$ (F/T) (T/F)

$(F \to q) \wedge (q \to F)$

$\equiv T \wedge q' \equiv q'$

ques. check if $\underbrace{\neg(p \to q)}_{\alpha} \equiv \underbrace{p \wedge \bar{q}}_{\beta}$

sol^n Case I : $p = True$

$\left.\begin{array}{l}\alpha = q' \\ \beta : q'\end{array}\right\}$ same

$\neg(T \to q) \equiv \neg q$

$T \wedge \bar{q} \equiv \bar{q}$

case II : $p = False$

$\left.\begin{array}{l}\alpha : F \\ \beta : F\end{array}\right\}$ same

$\neg(F \to q) \equiv \bar{T} \equiv F$

$F \wedge \bar{q} \equiv F$

↳ $\alpha \equiv \beta$ ✓

→ <u>with 3 variables</u> : p, q, r

w.r.t p → 2 cases
p=T or p=F

w.r.t r → 2 cases
r=T or r=F

w.r.t p, q, r → 8 cases } this will be the complete Truth Table

wrt q, r → 4 cases
q, r = F, F
q, r = F, T
q, r = T, F
q, r = T, T

<u>ques.</u> $\underbrace{p \vee (q \wedge r)}_{\alpha} \equiv \underbrace{(p \vee q) \wedge (p \vee r)}_{\beta}$??

<u>soln</u> wrt p, q → 4 cases possible :→

<u>case I</u> : p=F, q=F	<u>case II</u> : p=F, q=T	<u>case III</u> : p=T, q=F	<u>case IV</u> : p=T, q=T
α : F, β : F } same	α : r, β : r } same	α : T, β : T } same	α : T, β : T } same

→ $\alpha \equiv \beta$ ✓

wrt p → 2 cases possible :→ [FASTER Method as cases are less]

<u>case I</u> : p = T

α : T
β : T } same

<u>case II</u> : p = F

α : $(q \wedge r)$
β : $(q \wedge r)$ } same

→ $\alpha \equiv \beta$ ✓

<u>ques</u> $p \wedge (q \rightarrow (r \vee s)) \equiv (p \wedge q) \rightarrow (r \vee s)$??

<u>soln</u> <u>case I</u> : p = T

no need to check

<u>case II</u> : p = F

α : F
β : T } So, $\alpha \neq \beta$
we dont need to check case I now.

NOTE →

Tautology : Any proposition (simple or complex) is a Tautology if it is always True.
i.e the Truth value is True for all possible matches.

Contradiction / Fallacy / Invalid : If a prop 'P' is always False, then we call the prop as contradiction.

In other words, if the Truth table results in False for all combinations of propositional variables, then it is called contradiction.

Contingency : If a proposition is sometimes True and sometimes False.

NOTE → If an expression (propositional statement) is a tautology or a contingency, it is said to be satisfiable.

If an expression is a contradiction, it is said to be non-satisfiable as no combination of Truth values (T/F) assignments to the propositional variables, will result in a True for the whole expression.

SOME STANDARD LOGICAL EQUIVALENCES (Logical Identities)

1) IDENTITY LAWS :

$$p \wedge T \equiv p$$
$$p \vee F \equiv p$$

proof 1 : $p \wedge T \equiv p$ (T → Fixed → True) $p \vee F \equiv p$

p: T, F p: T, F p: T, F p: T, F

proof 2 : $\underbrace{p \wedge T}_{\alpha} \equiv \underbrace{p}_{\beta}$

Case 1 : $p = T$	Case 2 : $p = F$
$\alpha = T$, $\beta = T$ } same	$\alpha = F$, $\beta = F$ } same

$\amalg^{ly}$ these can be proven :

(i) $p \oplus T \equiv p'$
(ii) $p \oplus F \equiv p$
(iii) $p \leftrightarrow T \equiv p$
(iv) $p \leftrightarrow F \equiv p'$

} Not identity laws but similar to them → we have proved them before

2) DOMINATION LAWS :

$$p \vee T \equiv T$$
$$p \wedge F \equiv F$$

Anything $\vee\ T \equiv T$

Anything $\wedge\ F \equiv F$

3) IDEMPOTENT LAWS :

$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$

General Definition :→ operation # is Idempotent iff $\boxed{p \# p = p}$ $\forall\ p$

$p \xleftarrow{} \#$

ex In Number Theory,

↳ Multiplication : not idempotent	↳ Addition : not idempotent
$2 \times 2 \neq 2$ $1 \times 1 = 1$	$2 + 2 \neq 2$
$3 \times 3 \neq 3$	$3 + 3 \neq 3$

qus. for natural numbers n, $n^2 = n$?

↳ FALSE

$2^2 \neq 2$
$3^2 \neq 3$ ✓
$1^2 = 1$

qus. In prop logic, which connective is Idempotent ?

(i) $\wedge$ ✓ (ii) $\vee$ ✓ (iii) $\rightarrow$ ✗ (iv) $\leftrightarrow$ ✗ (v) $\oplus$ ✗ (vi) $\uparrow$ ✗ (vii) $\downarrow$ ✗

$P \wedge P \equiv P$

$P \vee P \equiv P$

$P \rightarrow P \not\equiv P$

$P \rightarrow P \equiv T$

$P \leftrightarrow P \not\equiv P$

↳ $P \leftrightarrow P \equiv T$

$P \oplus P \equiv F$

$P \uparrow P = \overline{P \cdot P} = \overline{P}$

$P \downarrow P = \overline{P \vee P} = \overline{P}$

↳ can you prove that $P \leftrightarrow P \equiv T$?

Proof: $\underbrace{P \leftrightarrow P}_{LHS} \equiv \underbrace{T}_{RHS}$

Case 1 : P = True

LHS : T
RHS : T } same

Case 2 : P = False

LHS : T
RHS : T } same

↳ $P \leftrightarrow P \equiv T$
$F \leftrightarrow F$ — T
$T \leftrightarrow T$ — T

↳ $P \oplus P \equiv F$
$T \oplus T$ F
$F \oplus F$ F

$P \leftrightarrow P \equiv \underbrace{(P \rightarrow P)}_{T} \wedge \underbrace{(P \rightarrow P)}_{T} \equiv T$ ✓ $\{\alpha \leftrightarrow \beta \equiv (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)\}$

4) DOUBLE NEGATION LAW :

$\neg(\neg P) = P$

5) COMMUTATIVE LAW :

$x \vee y = y \vee x$

$x \wedge y = y \wedge x$

General Definition :→

operation # is commutative iff $\boxed{a \# b = b \# a}$ $\forall a, b$

ex In Number Theory,

↳ commutative operⁿ : $+$, $\times$

$2+3 = 3+2$ $\qquad$ $2 \times 3 = 3 \times 2$

$a+b = b+a$ $\qquad$ $a \times b = b \times a$

↳ Not commutative operⁿ : subtraction, division

$4/2 \neq 2/4$ $\qquad$ $4-2 \neq 2-4$

ques. In prop logic, which connective is commutative?

(i) $\wedge$ ✓ (ii) $\vee$ ✓ (iii) $\rightarrow$ ✗ (iv) $\leftrightarrow$ ✓ (v) $\oplus$ ✓ (vi) $\uparrow$ ✓ (vii) $\downarrow$ ✓

$p \vee q \equiv q \vee p$ ✓ $\qquad$ $p \leftrightarrow q \equiv q \leftrightarrow p$ ✓ $\qquad$ $p \downarrow q \equiv q \downarrow p$ ✓

$p \wedge q \equiv q \wedge p$ ✓ $\qquad$ $p \oplus q \equiv q \oplus p$ ✓

$p \rightarrow q \not\equiv q \rightarrow p$ ✗ $\qquad$ $p \uparrow q \equiv q \uparrow p$ ✓

6) ASSOCIATIVE LAW :

General Definition :→ operation # is Associative iff

$\boxed{(a \# b) \# c = a \# (b \# c)}$ $\forall a, b, c.$

ex In Number Theory,

↳ Associative

$+$, $\times$

$(a+b)+c = a+(b+c)$

↳ Non Associative

$-$, $\div$

$(5-2)-1 \neq 5-(2-1)$

Qun. In prop logic, which connective is Associative?

(i) $\wedge$ ✓ (ii) $\vee$ ✓ (iii) $\rightarrow$ ✗ (iv) $\leftrightarrow$ ✓ (v) $\oplus$ ✓ (vi) $\uparrow$ ✗ (vii) $\downarrow$ ✗

↳ '$\rightarrow$' is not Associative.

i.e, $(P \rightarrow Q) \rightarrow R \not\equiv P \rightarrow (Q \rightarrow R)$

Proof 1 : Truth Table Method

P	Q	R	$P\rightarrow Q$	$Q \rightarrow R$	$(P\rightarrow Q)\rightarrow R$	$P\rightarrow(Q\rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Not equivalent as diff Truth tables

Proof 2 : By case Method

$\underbrace{(P \rightarrow Q) \rightarrow R}_{\alpha} \not\equiv \underbrace{P \rightarrow (Q \rightarrow R)}_{\beta}$

case 1 : $P = \text{True}$

$\left.\begin{aligned}\alpha &= Q \rightarrow R \\ \beta &= Q \rightarrow R\end{aligned}\right\}$ same

case 2 : $P = \text{False}$

$\left.\begin{aligned}\alpha &= R \\ \beta &= T\end{aligned}\right\}$ not equivalent

$\alpha \equiv \beta$ means, in every case/situation, α, β must be equivalent.

$\alpha \not\equiv \beta$ means, in atleast one case/situation, α, β must be different.

↳ $\oplus$ is Associative

i.e. $\underbrace{(p \oplus q) \oplus r}_{LHS} \equiv \underbrace{p \oplus (q \oplus r)}_{RHS}$

proof : By case Method :

case 1 : $p = \text{True}$

$\left.\begin{aligned}LHS &: q' \oplus r \\ RHS &: \overline{q \oplus r}\end{aligned}\right\}$ equivalent

Case 2 : $p = \text{False}$

$\left.\begin{aligned}LHS &: q \oplus r \\ RHS &: q \oplus r\end{aligned}\right\}$ same

$\underbrace{\bar{q} \oplus r}_{\alpha} \equiv \underbrace{\overline{q \oplus r}}_{\beta}$

↳ $q = T$

$\left.\begin{aligned}\alpha &= r \\ \beta &= r\end{aligned}\right\}$ same

$q = F$

$\left.\begin{aligned}\alpha &= \bar{r} \\ \beta &= \bar{r}\end{aligned}\right\}$ same

$T \oplus y \equiv \bar{y}$

$F \oplus y = y$

Kashti Leharon se takrayegi toh hi kinare naseeb honge.

7) DISTRIBUTIVE PROPERTY :

operation # is distributive over operation * iff

$$\boxed{a \# (b * c) = (a \# b) * (a \# c)} \quad \forall\, a, b, c .$$

ex) In Number Theory,

✓ $a \times (b + c) = (a \times b) + (a \times c)$

✗ $a + (b \times c) = (a + b) \times (a + c)$

So, multiplication is distributive over addition,
But Addition is not distributive over Multiplication.

↳ In Propositional logic,

→ $\wedge$ is distributive over $\vee$. i.e, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ ✓

$\vee$ is distributive over $\wedge$. i.e, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ ✓

↳ Prove that : $\vee$ is distributive over $\wedge$.

$$\underbrace{P \vee (Q \wedge R)}_{LHS} \equiv \underbrace{(P \vee Q) \wedge (P \vee R)}_{RHS}$$

Proof : P = True

LHS : T
RHS : T } equivalent

P = False

LHS : $Q \wedge R$
RHS : $Q \wedge R$ } equivalent

ques. Prove or Disprove :

(i) $\rightarrow$ is dist. over $\wedge$.

(ii) $\wedge$ is " " $\rightarrow$.

(iii) $\rightarrow$ is " " $\vee$.

(iv) $\vee$ is " " $\rightarrow$.

(v) $\oplus$ is " " $\rightarrow$.

(vi) $\rightarrow$ is " " $\oplus$.

Sol^n (i) '$\to$' is dist over '$\wedge$'.

To Prove: $\underbrace{P \to (Q \wedge R)}_{LHS} \equiv \underbrace{(P \to Q) \wedge (P \to R)}_{RHS}$

Proof:

P = True	P = False
LHS : $Q \wedge R$ } same	LHS : True } same
RHS : $Q \wedge R$	RHS : True

$\Rightarrow$ '$\to$' is dist. over '$\wedge$' $\quad P \to (Q \wedge R) \equiv (P \to Q) \wedge (P \to R)$

(ii) '$\wedge$' is dist over '$\to$'.

To Prove: $\underbrace{P \wedge (Q \to R)}_{LHS} \equiv \underbrace{(P \wedge Q) \to (P \wedge R)}_{RHS}$

Proof:

P = True	P = False
LHS : $Q \to R$	LHS : F } Not same
RHS : $Q \to R$	RHS : T

$\Rightarrow$ '$\wedge$' is not dist. over '$\to$'. $\quad P \wedge (Q \to R) \not\equiv (P \wedge Q) \to (P \wedge R)$

(iii) '$\to$' is dist. over '$\vee$'.

To Prove: $\underbrace{P \to (Q \vee R)}_{LHS} \equiv \underbrace{(P \to Q) \vee (P \to R)}_{RHS}$

Proof:

P = True	P = False
LHS : $(Q \vee R)$ } same	LHS : True } same
RHS : $(Q \vee R)$	RHS : True

$\Rightarrow$ '$\to$' is dist. over '$\vee$'. $\quad P \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$

(iv) '$\vee$' is dist. over '$\to$'

To Prove: $P \vee (Q \to R) \equiv (P \vee Q) \to (P \vee R)$

Proof:

P = True	P = False
LHS : $\cancel{(Q \to R)}$ True } same	LHS : $(Q \to R)$ } same
RHS : True	RHS : $(Q \to R)$

$\Rightarrow$ '$\vee$' is dist. over '$\to$'. $\quad P \vee (Q \to R) \equiv (P \vee Q) \to (P \vee R)$

(v) '$\oplus$' is dist. over '$\rightarrow$'.

To prove: $\underbrace{P \oplus (Q \rightarrow R)}_{LHS} \equiv \underbrace{(P \oplus Q) \rightarrow (P \oplus R)}_{RHS}$

Proof:

P = True

LHS : $\overline{(Q \rightarrow R)}$
RHS : $\bar{Q} \rightarrow \bar{R}$ } not same

↳ $\underbrace{\overline{Q \rightarrow R}}_{\alpha} \equiv \underbrace{\bar{Q} \rightarrow \bar{R}}_{\beta}$

Q = True

α : $\bar{R}$
β = T } not same

Q = False

α : F
β : $\bar{R}$ } not same

$\Rightarrow \overline{Q \rightarrow R} \not\equiv \bar{Q} \rightarrow \bar{R}$

P = False

LHS : $(Q \rightarrow R)$
RHS : $(Q \rightarrow R)$ } same

$T \oplus y \equiv \bar{y}$ (T, F → F, T)

$F \oplus y \equiv y$ (T, F → T, F)

$\Rightarrow$ '$\oplus$' is not dist. over '$\rightarrow$' ✓ $P \oplus (Q \rightarrow R) \not\equiv (P \oplus Q) \rightarrow (P \oplus R)$

(vi) '$\rightarrow$' is dist over '$\oplus$'.

To Prove : $\underbrace{P \rightarrow (Q \oplus R)}_{LHS} \equiv \underbrace{(P \rightarrow Q) \oplus (P \rightarrow R)}_{RHS}$

Proof:

P = True

LHS : $(Q \oplus R)$
RHS : $(Q \oplus R)$ } same

P = False

LHS : True
RHS : False } not same

$\Rightarrow$ '$\rightarrow$' is not dist over '$\oplus$' ✓ $P \rightarrow (Q \oplus R) \not\equiv (P \rightarrow Q) \oplus (P \rightarrow R)$

8) DE-MORGAN'S LAW

- Negation of (P and Q) = $\bar{P} \vee \bar{Q}$ i.e., $\overline{(P \wedge Q)} \equiv \bar{P} \vee \bar{Q}$
- Negation of $(P \vee Q)$ = $\bar{P} \wedge \bar{Q}$ i.e., $\overline{(P \vee Q)} \equiv \bar{P} \wedge \bar{Q}$

↳ Prove : $\underbrace{\overline{(P \wedge Q)}}_{\alpha} \equiv \underbrace{\bar{P} \vee \bar{Q}}_{\beta}$ ✓

P = True	P = False
$\left.\begin{array}{l}\alpha : \bar{Q} \\ \beta : \bar{Q}\end{array}\right\}$ equivalent	$\left.\begin{array}{l}\alpha : T \\ \beta : T\end{array}\right\}$ equivalent

NOTE

- let's suppose we have a propositional eqn / Formula 'G' of $\vee, \wedge, \neg$ only. then to find '$\bar{G}$' i.e to negate 'G', do these following changes in the expression of 'G' $\Rightarrow$ $\left\{\begin{array}{l} \vee \rightarrow \wedge \\ \wedge \rightarrow \vee \\ P \rightarrow \bar{P} \;;\; \bar{P} \rightarrow P \end{array}\right\}$

ex)

$G : P \wedge Q$
$\bar{G} : \bar{P} \vee \bar{Q}$
$\overline{(P \wedge Q)} = \bar{P} \vee \bar{Q}$

$G : P \vee Q$
$\bar{G} : \bar{P} \wedge \bar{Q}$
$\overline{(P \vee Q)} = \bar{P} \wedge \bar{Q}$

qus. $G : (\bar{P} \wedge Q) \vee R$. $\bar{G} = ?$

↳ $\overline{(\bar{P} \wedge Q) \vee R} = (P \vee \bar{Q}) \wedge \bar{R}$ ✓

qus. $G : P \vee Q \wedge R$. $\bar{G} = ?$

(i) $(\bar{P} \wedge \bar{Q}) \vee \bar{R}$ (ii) $\bar{P} \wedge (\bar{Q} \vee \bar{R})$

↳ $G : P \vee Q \wedge R$

$G : P \vee (Q \wedge R)$

$\Rightarrow \bar{G} : \bar{P} \wedge (\bar{Q} \vee \bar{R})$

Priority : $\neg > \wedge > \vee > \rightarrow > \leftrightarrow$

9) ABSORPTION LAW :

$$\left\{ \begin{aligned} P \wedge (P \vee Q) &\equiv P \\ P \vee (P \wedge Q) &\equiv P \end{aligned} \right\}$$

Rat $P \wedge ((P+Q)) \equiv P$

$P + (PQ) \equiv P$

Proof : $P + PQ$

$= P(1+Q) = (P \cdot 1) + (PQ)$

$P + PQ$

$= P(1)$

$= P$ ✓

$\left.\begin{aligned} 1 &\equiv T \\ 0 &\equiv F \end{aligned}\right\}$ ✓

Proof : $P(P+Q) = P \cdot P + P \cdot Q = P + PQ = P$ ✓

ques. $P + PQ + PQ\bar{R}\bar{S}$

↳ $P + PQ + PQ\bar{R}\bar{S} \equiv P$ ✓ {using Absorption Law}

$P + P\alpha = P$ ✓ $\quad P(P+\alpha) = P$ ✓

10)
$$\left\{ \begin{aligned} P \wedge \bar{P} &\equiv F \\ P \vee \bar{P} &\equiv T \\ P \wedge P &\equiv P \\ P \vee P &\equiv P \end{aligned} \right\}$$

11) $\boxed{P \rightarrow Q \equiv \bar{P} \vee Q}$

i.e. $\alpha \rightarrow \beta \equiv \bar{\alpha} \vee \beta$

$\square \rightarrow \Delta \equiv \square' + \Delta$

12) Important Property :→

$\alpha + \alpha'\beta \equiv \alpha + \beta$

$\boxed{P + \bar{P}Q \equiv P + Q}$

$\boxed{\text{NOTE}}$ → $\underbrace{\alpha \equiv \beta}$ means $\alpha \leftrightarrow \beta$ is Tautology.

T T
F F

i.e,

$\left\{ \begin{array}{c} \alpha \to \beta \text{ is Tautology} \\ \text{AND} \\ \beta \to \alpha \text{ is Tautology} \end{array} \right\}$

same ✓
$\alpha \leftrightarrow \beta \equiv (\alpha \to \beta) \wedge (\beta \to \alpha)$

ques. check Tautology or not

$$(P \wedge Q) \to (P \vee Q)$$

↳ $\boxed{\text{Method 1}}$: Truth Table

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \to (P \vee Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

all T's, so it is Tautology.

↳ $\boxed{\text{Method 2}}$: by case, let $\alpha : (P \wedge Q) \to (P \vee Q)$

case1 : P = True

$\alpha : Q \to T$

α : True

case2 : P = False

$\alpha : F \to Q$

α : True

so, α is Tautology

↳ $\boxed{\text{Method 3}}$: Simplification

use all the properties that we studied

$(P \wedge Q) \to (P \vee Q)$

$\equiv (P \wedge Q)' + P + Q$ $\quad \{\alpha \to \beta \equiv \alpha' + \beta\}$

$P + Q \equiv P \vee Q$
↳ OR

$\equiv \bar{P} + \bar{Q} + P + Q$

$\equiv \underbrace{\bar{P} + P}_{T} + \underbrace{\bar{Q} + Q}_{T}$

$\equiv$ True ✓ ⇒ Tautology ✓ ⇝ $(P \wedge Q) \to (P \vee Q) \equiv$ True ✓

So, Tautology

Method 4 :

$\boxed{\text{NOTE}}$

If we want to know whether $\alpha \to \beta$ is Tautology or not ?

Try your best to make $\alpha \to \beta$ False i.e. $\alpha = T$ & $\beta = F$, simultaneously

If you can $\Rightarrow$ $\alpha \to \beta$ is not Tautology

If you can't $\Rightarrow$ $\alpha \to \beta$ is Tautology.

for $P \to Q$,

- 3 ways to make implication T,

$P = T$, $Q = T$

$P = F$, $Q = T$

$P = F$, $Q = F$

- Only 1 way to make implication F,

$\underbrace{P = T, Q = F}_{\text{simultaneously}}$

Now, $\alpha \to \beta$

Try to make $\alpha = T$, $\beta = F$.

Approach 1

1st make $\alpha = T$, then try to make $\beta = F$

Approach 2

1st make β = False, then Try to make α = True

Now, $y : \underbrace{(P \wedge Q)}_{LHS} \to \underbrace{(P \vee Q)}_{RHS}$

Approach 1 → make LHS True,

LHS : $P \wedge Q = T$

$P = T$, $Q = T$

Now Try to make RHS False

RHS : $P \vee Q$, $P \to T$, $Q \to T$ } cannot make RHS False. so, we can never make y False

So, y is Always Tautology.

make RHS False ← **Approach 2**

RHS : $P \vee Q = F$

$P = F$, $Q = F$

Now Try to make LHS True,

LHS : $P \wedge Q$, $P \to F$, $Q \to F$ } we cannot make LHS True. So, we cannot make y False.

So, y is Always Tautology.

$\Rightarrow$ $(P \wedge Q) \to (P \vee Q)$ is Tautology

Qus. $[(P\to Q)\wedge(Q\to R)]\to(P\to R)$ check Tautology or not?

↳ **Method 1** : Using Truth Table
↳ 8 Rows → Very Boring, let's have some fun :)

↳ **Method 2** : Simplification Method :

$[(P\to Q)\wedge(Q\to R)]\to(P\to R)$

$[(\bar P+Q)(\bar Q+R)]\to(\bar P+R)$ $\{\alpha\to\beta\equiv\bar\alpha+\beta\}$

$\overline{(\bar P+Q)(\bar Q+R)}+\bar P+R$ $\{\alpha\to\beta\equiv\bar\alpha+\beta\}$

$(P\bar Q)+(Q\bar R)+\bar P+R$ {Using DeMorgan's Law}

$\bar P+P\bar Q+R+\bar R Q$ $\{\alpha+\bar\alpha\beta\equiv\alpha+\beta\ ,\ \bar\alpha+\alpha\beta\equiv\bar\alpha+\beta\}$

$\bar P+\bar Q+R+Q\equiv\bar P+R+Q+\bar Q\equiv\bar P+R+T\equiv T$ ✓

So, True, i.e. Tautology.

↳ **Method 3** : By case ⇒ $y:[(P\to Q)\wedge(Q\to R)]\to(P\to R)$

Case I : P = True

$y:[(T\to Q)\wedge(Q\to R)]\to(T\to R)$

$y:[Q\wedge(Q\to R)]\to R$

$y:[Q(\bar Q+R)]\to R$

$y:[\underbrace{Q\bar Q}_{F}+QR]\to R\equiv QR\to R$

$y:\overline{(QR)}+R\equiv\bar Q+\underbrace{\bar R+R}_{T}\equiv T$ ✓

Case II : P = False

$y:[(F\to Q)\wedge(Q\to R)]\to(F\to R)$

$y:(Q\to R)\to T$

$y:$ True ✓

So, y is Tautology ✓

↳ **Method 4** : $\underbrace{[(P\to Q)\wedge(Q\to R)]}_{LHS}\to\underbrace{(P\to R)}_{RHS}$

make RHS False : P = T, R = F ✓

Now, Try to make LHS True,

LHS : $(\underset{T}{P}\to Q)\wedge(Q\to\underset{F}{R})\equiv Q\wedge\bar Q\equiv$ False } After making RHS False, we cannot make LHS True, So LHS → RHS is Tautology ✓

$\left.\begin{array}{l}T\to Q\equiv Q\\ Q\to F\equiv\bar Q\end{array}\right\}$ ✓

L-10

Tautology : A statement is called Tautology if the final column in its truth table contains only 1's.

↓ Always 1 (Always T)

Contradiction : A statement is called a contradiction if the final column in its truth table contains only 0's.

↓ Always 0 (Always F)

Contingency : A statement is called a contingency or contingent if the final column in its truth table contains both 0's and 1's.

↓ Neither Tautology, nor contradiction

$\{0 \equiv \text{False} \equiv F,\ 1 \equiv \text{True} \equiv T\}$

ex

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F
contingency		Tautology	contradiction

★ PROPOSITIONAL SATISFIABILITY

A compound prop is satisfiable if there is an assignment of Truth values to its variables that makes it true. When no such assignment exists, i.e, when the compound prop is false for all assignments of Truth values to its variables, the compound prop is unsatisfiable. Note that the compound prop is unsatisfiable iff its negation is True for all assignment of Truth values to the variables, that is, iff its negation is a Tautology.

SATISFIABLE : α is satisfiable if it is possible to make α True in atleast one case.

→ satisfiable ≡ either Tautology or contingency

satisfiable ≡ atleast one time True

UNSATISFIABLE: α is unsatisfiable if it is not possible to make α True in all cases.

→ unsatisfiable ≡ contradiction ≡ Always False ex) y : $P \vee (P \to Q)$ can we satisfy y ?

↳ In prop logic, making True means satisfy.

↳ $P = T$ will satisfy y ↳ satisfiable.

ex) $\underbrace{P \wedge \neg P}$

Not satisfiable ≡ unsatisfiable

$\begin{cases} P = T \,;\; P \wedge \neg P = F \\ P = F \,;\; P \wedge \neg P = F \end{cases}$

ex) P → satisfiable (contingency) $P = T$ ✓, $P = F$

$P \wedge Q$ → satisfiable (contingency) $P = T, Q = T$ ✓

$P \wedge (Q \wedge \bar{P})$ → unsatisfiable $(P \wedge \bar{P}) \wedge Q \equiv F \wedge Q \equiv F$ (contradiction)

$P \vee \bar{P}$ → satisfiable (Tautology)

$P \to P$ → satisfiable (Tautology)

ques. Assume G is contradiction. Then what is $\bar{G}$?

(A) Tautology (B) contingency (C) satisfiable

↳ G is contradiction means G is always False

⇒ $\bar{G}$ must be always True,

↳ $\bar{G}$ is Tautology as well as satisfiable.

G	$\bar{G}$
F	T
F	T
F	T
F	T
⋮	⋮

$\boxed{\text{NOTE}}$ → Tautology $\underset{\not\leftarrow}{\longrightarrow}$ satisfiable

satisfiable → (Tautology OR contingency)

In propositional logic,

- Valid ≡ Tautology
- Not Valid ≡ Not Tautology
  ≡ contingency OR contradiction

$\boxed{\text{NOTE}}$ → In prop logic,

- Tautology ≡ Valid ≡ Always True ≡ Equivalent to True.
- Invalid ≡ Not Tautology ≡ contradiction or contingency
- contradiction ≡ unsatisfiable ≡ Always False ≡ Equivalent to False
- satisfiable ≡ Atleast one time True ≡ Tautology or contingency
- Atleast one time False ≡ contradiction or contingency ≡ Not Tautology
- contingency ≡ Neither Tautology Nor contradiction

Qus. Assume G is Falsifiable. Then $\overline{G}$?

↳ G can be made False in atleast one case.

(i) Tautology
(ii) contingency
(iii) Not valid
(iv) contradiction
(v) satisfiable

↳ ex: $G_1 : P \wedge \overline{P}$ — Falsifiable, contradiction

$G_2 : P \rightarrow Q$ — Falsifiable, contingency

$G_3 : P \vee \overline{P}$ — Not falsifiable

↳ Falsifiable ≡ Not Valid ≡ Not Tautology
↳ contradiction OR contingency

when we say G is Falsifiable it means, we can make it false.

P Q R S ---	G	$\overline{G}$
	⋮	⋮
	F	T
	⋮	⋮

$\overline{G}$ → Tautology OR contingency

→ satisfiable → in atleast on situation $\overline{G}$ is T.
⇓
Tauto OR contingency

if G is Falsifiable, we cannot say with surety that $\overline{G}$ is Tautology or $\overline{G}$ is contingency or $\overline{G}$ is Not valid.

woz G is Falsifiable ⇒ G is F in atleast one situation
↳ $\overline{G}$ is T in atleast one situation.

$\boxed{\text{NOTE}}$ → $\overline{\text{Tautology}}$ = contradiction

$\overline{\text{contradiction}}$ = Tautology

$\overline{\text{contingency}}$ = contingency

G	$\overline{G}$
T	F
T	F
T	F
T	F

Tautology — contradiction

y	$\overline{y}$
T	F
F	T
F	T
T	y

↳ contingency

Ques. Check satisfiability of Propositional logic Expressions:

(a) $(p \vee \neg p)$

↳ It is satisfiable & Tautology.

Method: Try to make them True

To check if formulae G is satisfiable or not,
Try to make G True.

(b) $(p \to p)$

↳ It is satisfiable & Tautology

(c) $(p \to (q \to p))$

if we put $p = F$; $F \to (q \to F) \equiv T$

$p = T$; $T \to (q \to T) \equiv T \to T \equiv T$

↳ It is satisfiable & Tautology

$\alpha \to \beta$

To make it True $\left\{ \begin{matrix} \alpha \text{ False} \\ \text{OR} \\ \beta \text{ True} \end{matrix} \right\}$

(d) $\neg(p \wedge \neg p)$

↳ $p \wedge \neg p \equiv F$

$\neg(p \wedge \neg p) \equiv T$ ✓

↳ It is satisfiable & Tautology.

(e) $\neg(p \vee \neg p)$

↳ $\neg(T) \equiv F$ ⇒ It is not satisfiable & contradiction

(f) $(p \wedge \neg p)$

↳ $p \wedge \neg p \equiv F$ ⇒ Not satisfiable & is contradiction

(g) p

↳ $p = T$ or $p = F$ ⇒ Satisfiable & contingency

(h) $((p \to q) \to p)$

if we put $p = T$; $((T \to q) \to T) \equiv q \to T \equiv T$

$p = F$; $((F \to q) \to F) \equiv T \to F \equiv F$

↳ satisfiable & contingency

$\boxed{\text{NOTE}}$ → **Checking Tautology for conditional statements :**

Checking if given implication statement "$x \to y$" is Tautology or not.

Logical Method :

We know that implication statement is ALWAYS TRUE except when $x=T, y=F$

so, try to make $x \to y$ False

if you can, then $x \to y$ is not Tautology

if you canNOT, then $x \to y$ is Tautology

{ P → Q
Antecedent (LHS) — Consequent (RHS) }

Approach 1 :

Make x True.

Now, Try to make y False.

If you can,
then $x \to y$ is not Tautology

If you canNOT,
then $x \to y$ is Tautology

Approach 2 :

Make y False

Now, Try to make x True.

If you can,
then $x \to y$ is not Tautology.

If you canNOT,
then $x \to y$ is Tautology.

↳ Approach 2 is VERY IMP method when we study Arguments.

ques. $(p \oplus q) \to (p \vee q)$. check Tautology or not?

↳ Approach 2 : make RHS false → consequent of Implication

$\left.\begin{array}{l} p=F \\ q=F \end{array}\right\}$ only way

Try to make LHS True → Antecedent

LHS : $p \oplus q$

$(F \oplus F) \to$ False

so, we cannot make LHS True, RHS False. So, Tautology ✓.

↳ Approach 1 : make Antecedent (LHS) True

Two ways $\left\{\begin{array}{l} p=T, q=F \\ p=F, q=T \end{array}\right.$

Now Try to make RHS False

way 1

$p=T, q=F$

$RHS = p \vee q \equiv T$

way 2

$p=F, q=T$

$RHS = p \vee q \equiv T$

so, after making LHS True, we cannot make RHS False. so, Tautology ✓.

<u>Qus.</u> Check Tautology or not?

$$(P\wedge Q) \to (P \uparrow Q)$$

↳ $(P\wedge Q) \to (P\uparrow Q)$

$\equiv (P\wedge Q) \to \overline{(P\wedge Q)}$

$\alpha \to \bar{\alpha}$ $\{A\to B \equiv \bar{A}+B\}$

$\equiv \bar{\alpha}+\bar{\alpha} \equiv \bar{\alpha}$

$\equiv \overline{(P\wedge Q)} \equiv \underbrace{\bar{P}\vee\bar{Q}}_{\text{contingency}}$ } → we can make this T or F

⇒ Not Tautology.

↳ $\boxed{\text{Method 2}}$ Using <u>Approach 1</u>:

make (LHS = $P\wedge Q$) True.

$P = T$, $Q = T$ } only way

Now Try to make RHS False

$P\uparrow Q \equiv T\uparrow T \equiv$ False

⇒ LHS = True, RHS = False } succeeded ✓   so, Not Tautology ✓

<u>Qus.</u> $(P\leftrightarrow Q) \to (P\downarrow Q)$. Tautology or not?

↳ <u>Approach 1</u>: make LHS True,

2 ways { $P=T$, $Q=T$; $P=F$, $Q=F$

Now Try to make RHS False

<u>way 1</u>

$P=T$, $Q=T$

RHS = $\overline{P\vee Q} \equiv$ F

<u>way 2</u>

$P=F$, $Q=F$

RHS = $\overline{P\vee Q}$ = T

We made RHS False ✓ success ✓

LHS = True, RHS = False } succeeded

so, not Tautology ✓

↳ <u>Approach 2</u>: make RHS False, i.e. $\overline{(P\vee Q)} = F$

3 ways { $P=T$, $Q=T$; $P=T$, $Q=F$; $P=F$, $Q=T$

Now Try to make LHS True

<u>way 1</u>

$P=T$, $Q=T$

RHS = $P\leftrightarrow Q \equiv$ T

<u>way 2</u>

$P=T$, $Q=F$

RHS = $P\leftrightarrow Q \equiv F$

<u>way 3</u>

$P=F$, $Q=T$

RHS = $P\leftrightarrow Q \equiv F$

We made LHS True ✓

LHS = True, RHS = False } succeeded

so, not Tautology ✓

Que. $[(P \to Q) \wedge (R \to S) \wedge (P \vee R)] \to (Q \vee S)$. Check Tautology.

↳ $\boxed{\text{Method 1}}$: Truth Table ⇝ 16 Rows ⇝ Too much work for Lazy & Smart guys

↳ $\boxed{\text{Method 2}}$: Simplification

$$[(P \to Q) \wedge (R \to S) \wedge (P \vee R)] \to (Q \vee S)$$

$$\equiv [(\bar{P}+Q)(\bar{R}+S)(P+R)] \to (Q+S) \qquad \left\{ \begin{array}{l} \alpha \to \beta \equiv \bar{\alpha}+\beta \\ \vee \equiv + \ , \ \wedge = \cdot \end{array} \right\}$$

$$\equiv \overline{[(\bar{P}+Q)(\bar{R}+S)(P+R)]} + (Q+S)$$

$$\equiv (P\bar{Q}) + (R\bar{S}) + (\bar{P}\bar{R}) + Q + S \qquad \{\text{using De-Morgan's Law}\}$$

$$\equiv Q + P\bar{Q} + S + R\bar{S} + \bar{P}\bar{R} \qquad \left\{ \begin{array}{l} \alpha + \bar{\alpha}\beta \equiv \alpha+\beta \\ \overline{(\alpha+\beta)} = \bar{\alpha}\bar{\beta} \end{array} \right\}$$

$$\equiv Q + P + S + R + \overline{PR}$$

$$\equiv Q + P + S + R + \bar{P} \equiv Q + S + R + \overbrace{P + \bar{P}}^{T} \equiv T$$ so, Tautology ✓

↳ $\boxed{\text{Method 3}}$: Approach 2 :

$$\underbrace{[(P \to Q) \wedge (R \to S) \wedge (P \vee R)]}_{LHS} \to \underbrace{(Q \vee S)}_{RHS}$$

Make RHS = $Q \vee S$ = False

$$\left. \begin{array}{l} Q = F \\ S = F \end{array} \right\} \text{only way}$$

Now, Try to make LHS = True.

$$LHS = (P \to \underset{\downarrow}{Q}) \wedge (R \to \underset{\downarrow}{S}) \wedge (P \vee R) \qquad \left\{ \begin{array}{l} y \to F \equiv \bar{y} \\ y \to F \equiv \bar{y} + F \equiv \bar{y} \end{array} \right\}$$

$$\qquad\quad F \qquad\qquad F$$

$$\bar{P} \wedge \bar{R} \wedge (P \vee R)$$

i.e $(\bar{P})(\bar{R})(P+R) \equiv \bar{P}\bar{R}P + \bar{P}\bar{R}R \equiv F + F \equiv F$

After making RHS False, we cannot make LHS True.

$\Rightarrow [(P \to Q) \wedge (R \to S) \wedge (P \vee R)] \to (Q \vee S)$ is Tautology. ✓

NOTE → Checking contradiction for conditional statements :

checking if given implication statement "$x \to y$" is contradiction or not:

Logical Method :

we know that Implication statement is ALWAYS TRUE except when $x=T, y=F$.

so, try to make $x \to y$ True.

If you can, then $x \to y$ is NOT contradiction.

If you cannot, then $x \to y$ is contradiction.

ques. $y : P \to (P \vee Q)$. Is it contradiction or not?

↳ Just Try to make y True.

if $P = F \rightsquigarrow \underbrace{y = \text{True}}_{\text{Not contradiction.}}$ $\{F \to (F \vee Q) \equiv T\}$

$F \to Q \equiv T$ ✓

$\boxed{LHS \to RHS}$ contradiction or not?

↳ Method : Just make LHS False

OR

Just make RHS True.

ques. $[P \leftrightarrow (Q \wedge R)] \to (P \oplus Q)$. contradiction or not?

↳ Just make RHS True,

$\{P = F, Q = T\}$

We would make it True
so not contradiction

Anything → True is True

ques. $(P \vee \bar{P}) \to (Q \wedge \bar{Q})$. contradiction or not?

↳ we cannot make LHS False
we cannot make RHS True.

we cannot make it true.

so, LHS = True, RHS = False, so contradiction.

ques. Prove the following:

(i) $(p \vee q) \vee (p \vee \neg q)$ is a Tautology

Simplification is the best method for this question

$(p+q) + (p+\bar{q})$

$= p+q+p+\bar{q} = p+p+\underbrace{q+\bar{q}}_{\text{True}} =$ Anything + True $=$ True

So, Tautology ✓

(ii) $[(p \to r) \wedge (q \to r) \wedge (p \vee q)] \to r$ is a Tautology.

Best method → Approach 2.

Put RHS = ~~True~~ False

$r = F$ } Now try to make LHS = True

$(p \to F)(q \to F)(p+q)$

$\equiv \bar{p}\,\bar{q}\,(p+q) \equiv \bar{p}\,\bar{q}\,p + \bar{p}\,\bar{q}\,q \equiv F+F =$ False

couldn't make LHS True. So, Tautology ✓

(iii) $(p \vee q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

↳ let y : $(p+q)(\bar{p}\,\bar{q})$

Try to make y True

$p+q = T$ }→ this cannot happen

$\bar{p}\,\bar{q} = T \to p=F,\ Q=F$ } only way

So, it is contradiction ✓.

{ $AB =$ True means $A=T$, $B=T$ }

Method 2 : simplification.

$(p+q)(\bar{p}\,\bar{q})$

$\equiv p\bar{p}\bar{q} + q\bar{p}\bar{q}$

$\equiv \underbrace{p\bar{p}}_{F}\bar{q} + \bar{p}\underbrace{q\bar{q}}_{F} \equiv F+F \equiv F$

So contradiction ✓

(iv) $\neg(q \to p) \wedge (p \wedge q \wedge s \to r) \wedge p$ is a contradiction

↳ $\overline{(q \to p)}\,(pqs \to r)\,p$

$q \to p \equiv \bar{q} + p$

$\overline{(q \to p)} \equiv \overline{(\bar{q} + p)} \equiv q\bar{p}$

$(q\underbrace{\bar{p})\,p}_{\text{False}}\,(pqs \to r)$

$q \cdot \text{False} \cdot \text{Anything} = \text{False}$ ✓

So, contradiction ✓

(v) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$\left\{\begin{array}{l} \text{For } \underbrace{\alpha \equiv \beta} \\ \quad \alpha \to \beta \text{ must be Tautology} \\ \quad \beta \to \alpha \text{ must be Tautology} \end{array}\right\}$

Best Method : By Case

$\underbrace{(p \to q) \wedge (p \to r)}_{\alpha} \equiv \underbrace{p \to (q \wedge r)}_{\beta}$

Case 1 : $p = \text{True}$	Case 2: $p = \text{False}$
$\left.\begin{array}{l}\alpha : (q \wedge r) \\ \beta : (q \wedge r)\end{array}\right\}$ same	$\left.\begin{array}{l}\alpha : T \\ \beta : T\end{array}\right\}$ equivalent

$\Rightarrow \alpha \equiv \beta$ ✓ ; $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ ✓

We can also use cases on r.

Case 1 : $r = \text{True}$	Case 2 : $r = \text{False}$
$\left.\begin{array}{l}\alpha : p \to q \\ \beta : p \to q\end{array}\right\}$ equivalent	$\left.\begin{array}{l}\alpha : (p \to q)\bar{p} \equiv (\bar{p} + q)\bar{p} \equiv \bar{p} \\ \beta : \bar{p}\end{array}\right\}$ equivalent

$\Rightarrow \alpha \equiv \beta$ ✓ ; $(p \to q)(p \to r) \equiv p \to qr$ ✓

qus. Is it contingency?

$(p \to q)(\neg p + q)$

{ To check contingency ↳ somehow make True AND somehow make False }

↳ $\boxed{\text{Method 1}}$: Truth Table

p	q	$p \to q$	$\sim p + q$	$(p \to q)(\neg p + q)$
T	T	T	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

↳ $(p \to q)(\neg p + q)$ is contingency. ✓

$\boxed{\text{Method 2}}$: simplification

$(p \to q)(\neg p + q)$

$\equiv (p \to q)(p \to q) \equiv (p \to q)$ it is implication, & it is contingency ✓

$\boxed{\text{Method 3}}$: Try to make ~~the~~ it True and False.

$(p \to q)(\neg p + q) \equiv (p \to q)$

make True : $p = T, q = T$; $p = F, q = F$; $p = F, q = T$ } we can make it True so it is not contradiction

make False : $p = T, q = F$ } it is not Tautology.

$\Rightarrow (p \to q)(\neg p + q)$ is contingency ✓

qus. $\bar{p} \wedge \bar{q} \wedge (p \vee q)$. contradiction or not ?

↳ $\bar{p}\,\bar{q}\,(p+q)$

simplification :

$\underbrace{\bar{p}\,\bar{q}\,p}_{F} + \underbrace{\bar{p}\,\bar{q}\,q}_{F}$

$= F$

so, contradiction ✓

Try to make it True

for $\bar{p}\,\bar{q}\,(p+q)$ to be True

$\bar{p} = T \rightsquigarrow p = F$

$\bar{q} = T \rightsquigarrow q = F$

$p + q = T \rightsquigarrow$ cannot be ~~the~~ done bcoz $p = F$ & $q = F$

so, contradiction ✓

{ $A \cdot B \cdot C = T$ means $A = T$, $B = T$, $C = T$ } ✓

ques. Find if the given expressions are Tautology or contradiction or contingency?

(i) $(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$

$\boxed{\text{Method 1}}$: simplification

$AB \leftrightarrow \overline{(\bar{A}+\bar{B})}$

$\equiv \underbrace{AB \leftrightarrow AB}_{\text{is Tautology}} \equiv T$ {using De-Morgan's Law}

{$P \leftrightarrow P$ is Tautology}

$\boxed{\text{Method 2}}$: By case $\quad y: AB \leftrightarrow \overline{(\bar{A}+\bar{B})}$

case 1 : $A = T$; $y = B \leftrightarrow B \equiv$ True

case 2 : $A = F$; $y = F \leftrightarrow F \equiv$ True

} so, Tautology

(ii) $(A \rightarrow B) \leftrightarrow \neg(\neg A \vee B)$

using simplification

$(A \rightarrow B) \leftrightarrow \overline{(\bar{A}+B)}$

$(A \rightarrow B) \leftrightarrow \overline{(A \rightarrow B)} \equiv F$ $\quad$ {$P \leftrightarrow \bar{P}$ is contradiction}

so, contradiction.

$\boxed{\text{Method 2}}$: By case $\quad y: (A \rightarrow B) \leftrightarrow \overline{(\bar{A}+B)}$

case 1 : $A = T$; $y = B \leftrightarrow \bar{B} \equiv$ False

case 2 : $A = F$; $y = T \leftrightarrow F \equiv$ False

} so, contradiction

(iii) $(A \wedge B) \rightarrow \neg(\neg C \vee \neg D) \leftrightarrow \neg(A \wedge B) \vee (C \wedge D)$

$\left[AB \rightarrow \overline{(\bar{C}+\bar{D})}\right] \leftrightarrow \left[\overline{(AB)} + (CD)\right]$

$\equiv (AB \rightarrow CD) \leftrightarrow \left[\overline{(AB)} + (CD)\right]$

$\equiv \underbrace{(AB \rightarrow CD)}_{\alpha} \leftrightarrow \underbrace{(AB \rightarrow CD)}_{\alpha}$ $\quad$ {$\alpha \leftrightarrow \alpha$ is tautology}

so, Tautology.

(iv) $(p \vee q) \wedge (\neg p \wedge \neg q)$

$\hookrightarrow$ $(p+q)(\bar{p}\,\bar{q})$

$\equiv \underbrace{p\bar{p}}_{F}\bar{q} + \bar{p}\underbrace{q\bar{q}}_{F} \equiv F$ so, contradiction

(v) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow \neg q)$

$\hookrightarrow$ $(p \leftrightarrow q)(\bar{p} \leftrightarrow \bar{q})$: y

By case

case 1 : $p = T$

$y = (T \leftrightarrow q)(F \leftrightarrow \bar{q})$

$\equiv q(\bar{\bar{q}}) \equiv q \begin{smallmatrix} \nearrow T \\ \searrow F \end{smallmatrix}$

case 2 : $p = F$

$y = (F \leftrightarrow q)(T \leftrightarrow \bar{q})$

$\equiv \bar{q}\cdot\bar{q} \equiv \bar{q} \begin{smallmatrix} \nearrow T \\ \searrow F \end{smallmatrix}$

so, contingency.

Using Truth Table

p	q	$p \leftrightarrow q$	$\bar{p}$	$\bar{q}$	$\bar{p} \leftrightarrow \bar{q}$	$(p \leftrightarrow q) \wedge (\bar{p} \leftrightarrow \bar{q})$
T	T	T	F	F	T	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T

$\rightarrow$ contingency

(vi) $(p \vee q) \wedge (\neg p \vee r) \rightarrow (p \vee r)$

$\hookrightarrow$ $(p+q)(\bar{p}+r) \rightarrow (p+r)$

$\equiv \overline{[(p+q)(\bar{p}+r)]} + (p+r)$ $\{\alpha \rightarrow \beta \equiv \bar{\alpha} + \beta\}$

$\equiv \bar{p}\bar{q} + \underbrace{p\bar{r} + p}_{} + r$ {Demorgan's Law}

$\equiv \bar{p}\bar{q} + p + r$ {Absorption law : $\alpha + \alpha\beta \equiv \alpha$}

$\equiv p + \bar{q} + r$ $\{\alpha + \bar{\alpha}\beta \equiv \alpha + \beta\}$

T F

$\Rightarrow$ contingency

$\boxed{\text{Method 2}}$: using case Method

$y : (p+q)(\bar{p}+r) \to (p+r)$

case 1 : $p = \text{True}$

$y = (T+q)(F+r) \to (T+r)$

$\equiv r \to T \equiv T$

case 2 : $p = \text{False}$

$y = (F+q)(T+r) \to (F+r)$

$\equiv q \to r$

T F

when, p is F , y can be T or F $\Rightarrow$ contingency

(vii) $((p\to q)\to r) \leftrightarrow (p \to (q \to r))$

$y : ((p\to q)\to r) \leftrightarrow (p\to (q\to r))$

$\boxed{\text{Method 1}}$: Using case Method

case 1 : $p = \text{True}$

$y = ((T\to q)\to r) \leftrightarrow (T \to (q\to r))$

$y = \underbrace{(q\to r)}_{\alpha} \leftrightarrow \underbrace{(q\to r)}_{\alpha}$

$y = T$

case 2 : $p = \text{False}$

$y = ((F\to q)\to r) \leftrightarrow (F \to (q\to r))$

$y = (T\to r) \leftrightarrow T$

$y = r \leftrightarrow T \equiv r$

T F

~~so Tautology~~

when p is F,

$y = r$, can be F or T $\Rightarrow$ contingency

so, contingency

$\boxed{\text{Method 2}}$: simplification

$((p\to q)\to r) \leftrightarrow (p\to (q\to r))$

$\equiv ((\bar{p}+q)\to r) \leftrightarrow (p \to (\bar{q}+r))$

$\equiv \overline{(\bar{p}+q)} + r \leftrightarrow \bar{p} + (\bar{q}+r)$

$\equiv p\bar{q} + r \leftrightarrow \bar{p}+\bar{q}+r$

$\equiv [(p\bar{q}+r) \to (\bar{p}+\bar{q}+r)][(\bar{p}+\bar{q}+r) \to (p\bar{q}+r)]$

$\equiv [\overline{(p\bar{q}+r)} + \bar{p}+\bar{q}+r][\overline{(\bar{p}+\bar{q}+r)} + p\bar{q}+r]$

$\equiv [(\bar{p}+q)\bar{r} + \bar{p}+\bar{q}+r][pq\bar{r} + p\bar{q}+r]$

{use Absorption Law: $\alpha + \alpha\beta \equiv \alpha$ }

$\equiv (\bar{p}+\bar{q}+r)(pq+p\bar{q}+r) \equiv (\bar{p}+\bar{q}+r)(p+r)$

$$\equiv (\bar{p} + \bar{q} + r)(p + r)$$

$$\equiv \underbrace{p\bar{p}}_{F} + \bar{p}r + \bar{q}p + \bar{q}r + rp + \underbrace{rr}_{r}$$

$$\equiv p\bar{q} + r$$

{Absorption law: $\alpha + \alpha\beta \equiv \alpha$}

$$\equiv \left[(\bar{p}+q)\bar{r} + \bar{p} + \bar{q} + r\right]\left[pq\bar{r} + p\bar{q} + r\right] \qquad \{\alpha + \bar{\alpha}\beta \equiv \alpha + \beta\}$$

$$\equiv \left[\bar{p} + q + r + \bar{p} + \bar{q}\right]\left[pq + r + p\bar{q}\right]$$

$$\equiv (\bar{p} + \bar{r} + q + \bar{q})(p(q + \bar{q}) + r)$$

$$\equiv T \cdot (p + r) \equiv p + r$$

$p + r \to T$, $p + r \to F$

} It can be both True or False.

So, contingency

CONVERSE, CONTRAPOSITTIVE, INVERSE

L-11

(of conditional statements)

Consider the conditional statement :

"If $\underbrace{\text{the weather is nice}}_{w}$, then $\underbrace{\text{I'll wash the car}}_{c}$"

w = the weather is nice c = I'll wash the car

Now the statement is : if w, then c, which can also be written as $w \to c$.

We can also make the negations, or "nots" of w and c.

$\neg w$ = the weather is not nice $\neg c$ = I won't wash the car

Using these "nots" & switching the order of w and c, we can create 3 new statements.

Converse : $c \to w$

↳ If $\underbrace{\text{I wash the car}}_{c}$, then $\underbrace{\text{the weather is nice}}_{w}$.

Inverse : $\neg w \to \neg c$

↳ If $\underbrace{\text{the weather is not nice}}_{\neg w}$, then $\underbrace{\text{I wont wash the car}}_{\neg c}$.

Contrapositive : $\neg c \to \neg w$

↳ If $\underbrace{\text{I dont wash the car}}_{\neg c}$, then $\underbrace{\text{the weather is not nice}}_{\neg w}$.

→

y : If p then q	y : $p \to q$
• Converse of y : If q then p	$q \to p$
• contrapositive of y : If not q then not p	$\neg q \to \neg p$
• Inverse of y : If not p, then not q	$\neg p \to \neg q$

→ We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are 3 related conditional statements that occur so often that they have special names.

The proposition $q \to p$ is called **converse** of $p \to q$.

$\neg q \to \neg p$ is called **contrapositive** of $p \to q$.

$\neg p \to \neg q$ is called **inverse** of $p \to q$.

We'll see that of these 3 conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

ex) converse of $q \to p$: $p \to q$

" " $\underbrace{\neg q}_{x} \to \underbrace{\neg p}_{y}$: $\underbrace{\neg p}_{y} \to \underbrace{\neg q}_{x}$

converse of $\alpha \to \beta$: $\beta \to \alpha$

Inverse of $\alpha \to \beta$: $\neg\alpha \to \neg\beta$

ques. What are the contrapositive, the converse, and the inverse of the conditional statement
"$\underbrace{\text{The home team wins}}_{W}$ whenever $\underbrace{\text{it is raining}}_{R}$."

↳ W whenever R $\equiv$ W if R $\equiv$ $R \to W$

- Converse : $W \to R$; If the home team wins then it is raining
- Contrapositive : $\neg W \to \neg R$; If the home team doesnot win, then it is not raining
- Inverse : $\neg R \to \neg W$; If it is not raining, then the home team doesnot win.

↳ only the contrapositive is equivalent to the original statement.

ques. "If this book is interesting, then I am staying at home." $B \to H$

- converse : $H \to B$; If I am staying at home, then this book is interesting.
- contrapositive : $\neg H \to \neg B$; If I am not staying at home, then the book is not interesting.
- Inverse : $\neg B \to \neg H$; If the book is not interesting, then I am not staying at home.

→ Truth Table of all the statements.

p	q	original statement $p \to q$	contrapositive $\neg q \to \neg p$	converse $q \to p$	Inverse $\neg p \to \neg q$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

<u>Observations</u> : (i) conditional statement ≡ its contrapositive

$$p \to q \equiv \neg q \to \neg p$$

(ii) converse of statement ≡ Inverse of statement

$$q \to p \equiv \neg p \to \neg q$$

<u>Proof</u>:

$p \to q$ false only when

$p = T$
$q = F$

$\neg q \to \neg p$ false only when

$\neg q = T$; $q = F$
$\neg p = F$; $p = T$

$$\{p \to q \equiv \neg q \to \neg p\}$$

↳ We use them in Proof Techniques.

<u>Direct Proof</u>

To prove

If P then Q

P
≡
≡
≡
Q

correct knowledge

start from P, then apply some knowledge and you get Q

<u>Proof by contraposition (contrapositive)</u>

To prove

If P then Q

¬Q
≡
≡
≡
¬P

correct knowledge

Assume Q is False, then you should get P is False.

↳ Both are same {if P then Q ≡ if ¬Q then ¬P}

but we use whichever is easier to apply

→ ex What is the converse, contrapositive & inverse of $\neg q \to \neg p$

↳ converse : $\neg p \to \neg q$

contrapositive : $\neg(\neg p) \to \neg(\neg q) \equiv p \to q$

Inverse : $\neg(\neg q) \to \neg(\neg p) \equiv q \to p$

→ ✓ <u>Propositional Variable</u> $\equiv$ $\underbrace{\text{Atomic proposition}}_{P, Q, R, \ldots}$

v/s

✓ <u>Propositional Formula / Expression / Form</u> :

$P, Q, \neg P, P \to Q, P \wedge \bar{P}, \ldots$

↳ If I say P, Q, R etc are prop variable then

P → <u>Atomic proposition</u> → "by itself" True or False

cannot be broken

P Q are <u>prop variable</u>.

$P \to Q$ - proposition

$P \wedge \bar{P}$ - proposition

P - proposition

Q - proposition

P	Q	$P \to Q$	$P \wedge \bar{P}$
F	F	T	F
F	T	T	F
T	F	F	F
T	T	T	F

values are assigned to prop variables

These values are calculated.

Proposition $\equiv$ propositional expression

Proposition → Atomic (Prop variables), compound

ex <u>In Algebra,</u>

$x + y^2 = xy$ → is NOT a variable

x, y: variable

xy: Expression

We'll assign the value to x & y

We will NOT assign the value to xy

but it is calculated.

$x \to$ integer

$y \to$ integer

$xy \to$ integer

$y^2 \to$ integer

values Assigned

x	y	xy
2	3	--

we dont assign value of xy

↓

it is calculated

<u>In Prop logic,</u>

variable → can be assigned T or F.

$\equiv$

prop variable $\equiv$ Atomic proposition

- P, Q are prop. variable then all Truth values combinations for P, Q are possible.

- α, β are prop formula then some truth value combinations for α, β might not be possible.

(ex) $\alpha : P \wedge \bar{P}$ $\quad\quad$ $\beta : P \vee \bar{P}$

α	β	
T	T	
T	F	
F	T	
F	F	

→ WRONG
Dont make this type of Truth Table

P	α	β
T	F	T
F	F	T

✓ The truth value will have 2 rows.

(ex) $\alpha : P \to (Q \vee R)$ $\quad\quad$ $\beta : P \wedge \bar{Q}$

α	β	
T	T	
T	F	
F	T	
F	F	

→ WRONG

P	Q	R	α ✓	β ✓

8 Rows

This truth table is correct

$$\underbrace{\alpha}_{\text{prop formula}} : \underbrace{P}_{} \to (\underbrace{Q}_{} \vee \underbrace{R}_{})$$

prop variable → Atomic prop

qun. Assume P, Q are $\boxed{\text{prop variable}}$ then which is correct?

(i) $P \to Q$ is NEVER equivalent to $Q \to P$.

(ii) $P \to Q$ is ALWAYS equivalent to $\neg Q \to \neg P$.

P	Q	$P \to Q$	$Q \to P$	$\neg Q \to \neg P$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	T	T

$\neg Q \to \neg P \equiv P \to Q$

$P \to Q \not\equiv Q \to P$

qus. Assume α, β are $\boxed{\text{prop Expression}}$ then which is correct.

(i) $\alpha \to \beta$ is NEVER equivalent to $\beta \to \alpha$. ✗

(ii) $\alpha \to \beta$ is ALWAYS equivalent to $\neg\beta \to \neg\alpha$. ✓

↳ $\alpha \to \beta \equiv \neg\beta \to \neg\alpha$ is always True.

$\boxed{\text{Any}}$ conditional statement ≡ It's contrapositive ✓

ex) lets take any α, β randomly

$\alpha : P \to Q$ $\beta : P \wedge Q$

P	Q	$\alpha \to \beta$	$\neg\beta \to \neg\alpha$
F	F	F	F
F	T	F	F
T	F	T	T
T	T	T	T

$\alpha \to \beta \equiv \neg\beta \to \neg\alpha$

In any case if $\alpha \to \beta$ is False

it means $\alpha = T$, $\beta = F$,

it makes $\neg\beta \to \neg\alpha$ also False.

& In any case if $\alpha \to \beta$ is True

it means $\alpha \neq T$, $\beta \neq F$

it makes $\neg\beta \to \neg\alpha$ also True

and vice-versa.

so $\alpha \to \beta$ is always equivalent to $\neg\beta \to \neg\alpha$ ✓

$\alpha, \beta \Rightarrow$ propositional expression

$\alpha : P \wedge \bar{P}$ $\beta : Q \wedge \bar{Q}$

P	Q	α	β	$\alpha \to \beta$	$\beta \to \alpha$
F	T	F	F	T	T
F	T	F	F	T	T
T	F	F	F	T	T
T	T	F	F	T	T

$\alpha \to \beta \equiv \beta \to \alpha$

But it is not the case also always.

it is just one example when

$\alpha \to \beta \equiv \beta \to \alpha$

so it is not the case that,

$\alpha \to \beta$ is never equivalent to $\beta \to \alpha$ ✓

if α & β were prop variables then only we can say that $\alpha \to \beta$ is never equivalent to $\beta \to \alpha$.

qus. If α, β are prop formula such that $\alpha \to \beta \equiv \beta \to \alpha$.

then what can we say about α, β ?

↳ then $\alpha \equiv \beta$ ✓

$\alpha \to \beta \equiv \beta \to \alpha$ is Tautology

ex)

P Q R ...	α	β	$\alpha \to \beta$	$\beta \to \alpha$
— — — — —	~~T~~	~~F~~	F	T
— — —	~~F~~	T	T	F
— — —	F	F	T	T
— — —	T	T	T	T

for any combination this cannot happen can NEVER happen

only this can happen both α, β must have same Truth values → α, β can be Tauto or contradiction, but must be same

LOGIC - ENGLISH TRANSLATION

Translating between English and Propositional logic

Negation is easy to recognize b/oz it almost includes the word 'not', as "it is not the case that E" or "its not true that E". other instances include declarative expressions containing and embedded 'not'. Example:

(a) Aditya went to the BYD cafe.

(b) Aditya didnot go to the BYD cafe.

If (a) is translated as 'ϕ', then (b) is translated as '$\neg\phi$'.

ex) The sun is out. : P

The sun is not out. — $\neg P$
It is Not the case that the sun is out. — $\neg P$
It is Not true that the sun is out. — $\neg P$
It is false that the sun is out. — $\neg P$
The sun is out, is false. — $\neg P$

} Negation of P

ex) P : This book is interesting.

This book is not interesting. $\neg P$
It is not the case that this book is interesting. $\neg P$
It is not true that this book is interesting. $\neg P$
It is false that this book is interesting. $\neg P$
The book is interesting, is false $\neg P$

} Negation of P

ex) P

It is Not the case that P. ($\neg P$)
It is Not true that P. ($\neg P$)
It is false that P. ($\neg P$)
P is false. ($\neg P$)

} Negations of P

NOTE → $\boxed{P \text{ is false}} \equiv \neg P$ ✓
$\boxed{P \text{ is True}} \equiv P$

↳ $\neg\alpha$ is false $\equiv \alpha$
$\neg\alpha$ is True $\equiv \alpha$

↳ P
P is true ✓

↳ $\neg P$
P is false ✓

ex) The sun is out. : P
The sun is out, is true. : P
} same

ex) We are students. : Q
We are students, is true : Q
} same

ex) Delhi is capital of India - True
Delhi is capital of India, is true. - True
} same

ex) Kolkata is capital of India. - False
Kolkata is capital of India, is true - False
} same

↳ $\underbrace{\text{P is true}}_{P}$ and $\underbrace{\text{Q is true}}_{Q} \equiv P \wedge Q$
P and Q
} same

↳ $\underbrace{\text{P is false}}_{\neg P}$ $\underbrace{\text{but}}_{\wedge}$ $\underbrace{\text{Q is true}}_{Q} \equiv \neg P \wedge Q$
Not P, But Q $\equiv \neg P \wedge Q$
} same

(But ≡ AND)

↳ $\underbrace{\text{P is false}}_{\neg P}$ and $\underbrace{\text{Q is false}}_{\neg Q}$. $\equiv \neg P \wedge \neg Q$
Not P, and Not Q. $\equiv \neg P \wedge \neg Q$
} same

↳ $P \rightarrow Q$
If $\underbrace{\text{P is true}}_{P}$ then $\underbrace{\text{Q is True}}_{Q}$. $\equiv P \rightarrow Q$
If P then Q. $\equiv P \rightarrow Q$
} same

↳ If $\underbrace{\text{P is false}}_{\neg P}$ then $\underbrace{\text{Q is true}}_{Q}$. $\equiv \neg P \to Q$

If Not P then Q. $\equiv \neg P \to Q$ } same

↳ If $\underbrace{\text{P is false}}_{\neg P}$ then $\underbrace{\text{Q is false}}_{\neg Q}$ $\equiv \neg P \to \neg Q$

If Not P then Not Q. $\equiv \neg P \to \neg Q$ } same

How to write formula/expressions :

- $(P \to Q)$ is True $\equiv$ P is false OR Q is True

$$\boxed{P \to Q \equiv \neg P \vee Q \equiv \bar{P} + Q}$$ ✓

- $P \oplus Q$ is True $\equiv$ $P = T, Q = F$ OR $P = F, Q = T$

$$\boxed{P \oplus Q \equiv P\bar{Q} + \bar{P}Q}$$ ✓

- $P \leftrightarrow Q \equiv P = T, Q = T$ OR $P = F, Q = F$

$$\boxed{P \leftrightarrow Q \equiv PQ + \bar{P}\bar{Q}}$$ ✓

- $\underbrace{\bar{P}}_{\alpha} \oplus Q \equiv \underbrace{P}_{\bar{\alpha}} Q + \underbrace{\bar{P}}_{\alpha} \bar{Q}$

$\bar{P} \oplus Q \equiv PQ + \bar{P}\bar{Q} \equiv P \leftrightarrow Q$

$$\boxed{\bar{P} \oplus Q \equiv P \leftrightarrow Q}$$ ✓

- $\underbrace{\bar{P}}_{\alpha} \leftrightarrow \underbrace{\bar{Q}}_{\beta} \equiv \alpha\beta + \bar{\alpha}\bar{\beta}$

$\equiv \bar{P}\bar{Q} + PQ$

$$\boxed{\bar{P} \leftrightarrow \bar{Q} \equiv P \leftrightarrow Q}$$

P	Q	R	α	β
F	F	F	F	F
F	F	T	F	F
F	T	F	F	T
F	T	T	T	T
T	F	F	T	T
T	F	T	F	F
T	T	F	T	T
T	T	T	F	T

Just focus on when α is T

$$\alpha \equiv \bar{P}QR + P\bar{Q}\bar{R} + PQ\bar{R}$$

Just focus on when β is T

$$\beta \equiv \bar{P}Q\bar{R} + \bar{P}QR + P\bar{Q}\bar{R} + PQ\bar{R} + PQR$$

- $P \oplus \bar{Q} \equiv PQ + \bar{P}\bar{Q} \equiv P \leftrightarrow Q$
- $\bar{P} \oplus \bar{Q} \equiv P\bar{Q} + \bar{P}Q \equiv P \oplus Q$

Conjunction sometimes involves the word 'and', but not always. If E and F are English declaratives, then E and F, E but F, E nonetheless F, E however F, E nevertheless F, & E moreover F all translated as Prop logic conjuctions.

In logic, AND ≡ But ≡ Although ≡ Though ≡ Even though ≡ However = yet ≡ still ≡ Moreover ≡ Nevertheless ≡ Nonetheless ≡ comma

P and Q ≡ $P \wedge Q$

P is True and Q is True. ≡ $P \wedge Q$

P but Q. ≡ $P \wedge Q$

P yet Q. ≡ $P \wedge Q$

P still Q. ≡ $P \wedge Q$

P However Q. ≡ $P \wedge Q$

P, Q ≡ $P \wedge Q$

Although P, Q ≡ $P \wedge Q$

P though Q ≡ $P \wedge Q$

P nonetheless Q ≡ $P \wedge Q$

Disjunction: usually uses the word 'or' (but is inclusive in PL). sentences like E or F and either E or F are translated as $P \vee Q$.

Implication If P then Q
$P \rightarrow Q$
} Has many forms as studied before.

ex) consider the statement :

$\underbrace{\text{Being 18+}}_{P}$ is necessary to $\underbrace{\text{join army}}_{Q}$ $\equiv Q \to P$

What does it mean?

- If $\underbrace{\text{you are in army}}_{Q}$ then $\underbrace{\text{you are 18+}}_{P}$. ✓ $Q \to P$
- If $\underbrace{\text{you are 18+}}_{P}$ then $\underbrace{\text{you are in army}}_{Q}$. ✗ $P \to Q$

$\hookrightarrow$ $\boxed{P \text{ is Necessary for } Q \equiv Q \text{ is sufficient for } P \equiv Q \to P}$

ex) consider the statement :

- $\underbrace{\text{Being Rajasthani}}_{P}$ is sufficient $\underbrace{\text{to be Indian}}_{Q}$ $\equiv P \to Q$

what does it mean?

- If $\underbrace{\text{you are Rajasthani}}_{P}$ then $\underbrace{\text{you are Indian}}_{Q}$. ✓ $P \to Q$
- If $\underbrace{\text{you are Indian}}_{Q}$ then $\underbrace{\text{you are Rajasthani}}_{P}$. ✗ $Q \to P$

ex) consider the statement :

$\underbrace{\text{you can join army}}_{A}$ only if $\underbrace{\text{you are 18+}}_{18+}$. $\equiv A \to 18+$

what does it mean?

- If $\underbrace{\text{you are in army}}_{A}$ then $\underbrace{\text{you are 18+}}_{18+}$. ✓ $A \to 18+$
- If you are 18+ then you are in army. ✗ $18+ \to A$
- $\underbrace{\text{Being 18+}}_{18+}$ is necessary condition for $\underbrace{\text{being in army}}_{A}$ ✓ $\equiv A \to 18+$
- without being 18+, you can't be in army. ✓

$\underbrace{\text{If you are not 18+}}_{\neg 18+}$, $\underbrace{\text{you cannot join army}}_{\neg P}$ ✓ $\equiv \neg 18+ \to \neg A$ $\equiv A \to 18+$ $\Big\}$ contra positive

$\boxed{\text{NOTE}} \to$ we have seen in previous lectures that :

P only if Q $\equiv$ without Q, P cannot happen $\equiv$ Q is necessary for P.

$\equiv P \to Q \equiv$ If not Q then not P $\equiv \underbrace{\neg Q \to \neg P}_{\text{contra positive}} \equiv \underbrace{P \to Q}$

o→ $\underset{\text{If}}{\underline{\text{provided that}}}$ $P, Q \equiv P \to Q$ | Q $\underset{\text{if}}{\underline{\text{provided that}}}$ $P \equiv P \to Q$

If $P, Q \equiv P \to Q$ | Q if P

o→ P if $Q \equiv$ If $Q, P \equiv Q \to P$

$\boxed{P}$ if $\boxed{Q}$

$P \leftarrow Q$

o→ P only if $Q \equiv P \to Q$

o→ Only if $P, Q \equiv Q \to P$ → P is necessary for $Q \equiv Q$ is sufficient for P

$\equiv Q$ $\boxed{\text{only if}}$ P $\equiv Q \to P$

Bi-Implication makes a stronger claim than the conditional. It's used to translate English sentences of the form 'P if and only if Q'.

There are some other common ways to express $P \leftrightarrow Q$

- P is necessary and sufficient for Q. $\equiv P \leftrightarrow Q$
- if P then Q, and conversely. $\equiv P \leftrightarrow Q$
  if Q then P
- P iff Q $\equiv P \leftrightarrow Q$

English
o→ Sentences like neither... nor... are essentially a negated disjunction, a -ve version of either... or...

ex Aditya neither studied nor went to gym.

with Aditya studied as 'P' & Aditya went to gym as 'Q'

we translated it as $\neg(P \vee Q)$.

↳ Neither P nor Q $\equiv$ P is F and Q is F $\equiv \bar{P} \wedge \bar{Q} \equiv \overline{P \vee Q} \equiv P \downarrow Q$

o→ sometimes we can also negate conjunctions in English. This kind of sentences usually take the form "it's not true that both P and Q" or "not P and Q"

ex it is not true that Aditya went to gym and studied both.

P : Aditya went to gym.
Q : Aditya studied.

} The translation is $\neg(P \wedge Q) = \bar{P} \vee \bar{Q}$
$\equiv P \uparrow Q$

ex A : There is a lion outside my room.

B : Lions can open doors.

C : I am in my room right now.

D: My room has doors.

E : I am going to be eaten by a lion.

convert the statement to prop formulae

(i) $\underbrace{\text{I wont be eaten by a lion}}_{\neg E}$ if $\underbrace{\text{there isn't a lion outside my room}}_{\neg A}$

↳ $\neg E$ if $\neg A$ $\equiv$ $\neg A \rightarrow \neg E$ $\equiv$ $E \rightarrow A$ ✓

(ii) If $\underbrace{\text{there is a lion outside my room}}_{A}$, but $\underbrace{\text{it cant open doors}}_{\neg B}$, $\underbrace{\text{I am not going to be eaten by a lion}}_{\neg E}$

↳ If (A but ¬B) then ¬E ✓ $\equiv$ $A\bar{B} \rightarrow \bar{E}$ ✓

If A then ¬B , ¬E ✗

(iii) I am only in my room when there is no lion outside.

$\equiv$ $\underbrace{\text{I am in my room}}_{C}$ only when $\underbrace{\text{there is no lion outside}}_{\neg A}$

↳ C only when ¬A $\equiv$ $C \rightarrow \neg A$ $\equiv$ $A \rightarrow \neg C$
(contrapositive)

¬A is necessary for C $\equiv$ $C \rightarrow \neg A$

C is sufficient for ¬A $\equiv$ $C \rightarrow \neg A$

→ "P only when Q" translates to "$P \rightarrow Q$".

ques. Which of the following statements are equivalent to "If $\underbrace{x \text{ is even}}_{xe}$, then $\underbrace{y \text{ is odd}}_{yo}$" ? $xe \rightarrow yo$

(i) $\underbrace{y \text{ is odd}}_{yo}$ $\underbrace{\text{only if}}_{\rightarrow}$ $\underbrace{x \text{ is even}}_{xe}$.

$yo \rightarrow xe$ Not equivalent.

(ii) $\underbrace{x \text{ is even}}_{x_e}$ is sufficient for $\underbrace{y \text{ to be odd}}_{y_o}$.

↳ $x_e \rightarrow y_o$ Equivalent ✓

(iii) $\underbrace{x \text{ is even}}_{x_e}$ is necessary for $\underbrace{y \text{ to be odd}}_{y_o}$.

↳ $x_e \leftarrow y_o \equiv y_o \rightarrow x_e$ Not equivalent.

(iv) If $\underbrace{x \text{ is odd}}_{\neg x_e}$, then $\underbrace{y \text{ is even}}_{\neg y_o}$.

↳ $\overline{x_e} \rightarrow \overline{y_o} \equiv y_o \rightarrow x_e$ Not equivalent.

(v) $\underbrace{x \text{ is even}}_{x_e}$ and $\underbrace{y \text{ is even}}_{\neg y_o}$.

↳ $x_e \wedge \overline{y_o}$ Not equivalent

(vi) $\underbrace{x \text{ is odd}}_{\neg x_e}$ or $\underbrace{y \text{ is odd}}_{y_o}$.

↳ $\overline{x_e} + y_o \equiv x_e \rightarrow y_o$ Equivalent ✓

ques. p : "I will prove this by case".

q : "There are more than 500 cases"

r : "I can find another way".

(i) state $(\neg r \vee \neg q) \rightarrow p$ in simple english

↳ If I cannot find another way or There are less than 500 cases then I will prove this case.

(ii) converse of (i)

↳ $p \rightarrow (\neg r \vee \neg q) \equiv p \rightarrow \overline{(rq)}$

If I will prove this by case then it not true that I can find another way and there are more than 500 cases.

(iii) Inverse

↳ $\overline{(\neg r \vee \neg q)} \rightarrow \overline{p} \equiv rq \rightarrow \overline{p}$

If I can find another way and there are more than 500 cases then I will not prove this by case.

(iv) contrapositive

↳ $\overline{p} \rightarrow \overline{(\overline{r} \vee \overline{q})} \equiv \overline{p} \rightarrow rq$

'UNLESS' in English

There are numerous ways to express conditionals in English.
We have seen many already several condition-forming expressions, including 'if', 'provided', 'only if'.
Now we'll consider a further conditional-forming expression – "unless".

$P \Rightarrow Q$
- If P, then Q
- Q if P
- P only if Q
- Whenever P, then also Q
- ⋮

There is a lot of words for saying "if P then Q" in English
We can even say it without using "if... then..."

eg (imagine a stubborn child "Jen")
$\underbrace{\text{Jen wont go to the party}}_{P}$
UNLESS $\underbrace{\text{Mary goes to the party}}_{Q}$.
≡ If Not Q, P
If Q doesnt happen the P will happen.

- $\underbrace{\text{Jen wont go the party}}_{P}$ UNLESS $\underbrace{\text{Mary goes to the party}}_{Q}$

means

IF $\underbrace{\text{mary doesnt go to the party}}_{\neg Q}$ THEN $\underbrace{\text{Jen wont go to the party}}_{P}$

$\equiv \quad \neg Q \to P$

means

IF not –(Mary goes to the party) THEN Jen wont go to the party

≡ If not – Q then P ≡ $\neg Q \to P$

- $\underbrace{\text{I wont study}}_{P}$ UNLESS $\underbrace{\text{you complete my demand}}_{Q}$.

≡ If $\neg Q$ then P ≡ $\neg Q \to P$

o→ I'll eat cake, unless I'm full.
$\underbrace{\text{I'll eat cake}}_{P}$, unless $\underbrace{\text{I'm full}}_{Q}$

$\equiv$ If $\neg Q$ then P $\equiv$ $\neg Q \rightarrow P$

Here the idea is that you fully intend to eat cake & you'll do so if you are not full
i.e., if you are not full, then you'll eat cake,
or "$\neg$Full → Eat cake".

o→ P UNLESS Q
P: intention; UNLESS $\equiv$ if not; Q: Just a chance

If not Q, P $\equiv$ $\boxed{\neg Q \rightarrow P}$

Ex) When Russia says,
we won't stop war (intention) unless you dont join NATO (Just a chance).
does it mean that
If ukraine doesnot joint NATO, Russia will stop war
No, coz even after ukraine aggreing to not joining NATO, war is there going
But it is guaranteed that
If they join Nato then it is war (war wont stop)

⇒ $\underbrace{\text{Russion wont stop war}}_{P \text{ (intensions)}}$ unless $\underbrace{\text{Ukraine doesnt join NATO}}_{Q}$ (Just a chance) not a guarantee

$\equiv$ If $\neg Q$ then P

$\equiv$ $\neg Q \rightarrow P$

Ex) $\underbrace{\text{you wont crak GATE}}_{P}$ unless $\underbrace{\text{you appear in GATE}}_{Q}$.

$Q \rightarrow \neg P$ ✗ $\qquad \neg Q \rightarrow P$ ✓

$\boxed{\text{NOTE}}$ → UNLESS $\equiv$ If Not ✓ | provided that $\equiv$ If ✓

P unless Q $\equiv$ P if not Q $\equiv$ $\neg Q \rightarrow P$

unless P, Q $\equiv$ If not P then Q $\equiv$ $\neg P \rightarrow Q$

Here, "if not" is short for "if it is not true ~~that~~". Notice this principle applies when "unless" appears at the beginning of the statement, as well as when it appears in the middle of the statement.

→ $\boxed{\text{P unless Q}}$

is equivalent to

P if not Q

$\equiv \boxed{\neg Q \rightarrow P}$

$\boxed{\text{unless P, Q}}$

is equivalent to

if not P, then Q

$\equiv \boxed{\neg P \rightarrow Q}$

→ P unless Q $\equiv$ P if not Q

$\equiv \neg Q \rightarrow P$

$\equiv Q + P \equiv$ P OR Q

$\boxed{\text{UNLESS} = \text{OR}}$ ✓

→ UNLESS $\equiv$ OR

UNLESS $\equiv$ If not

P unless Q $\equiv$ P or Q

$\equiv$ If not Q, P $\equiv P \vee Q \equiv \neg Q \rightarrow P$

NOTE → Do not Apply precedence in English statements.

↳ we understand the feeling

$\{\neg > \wedge > \vee > \rightarrow > \leftrightarrow\}$ only for logical operators

eg If you dont leave early, (you'll miss the train unless train is late) ✗

↳ WRONG

If (you dont leave early, you'll miss the train) unless train is late.

main intention/statement — Just a chance

L : leave early

M : miss train

late : Train late

(if ¬L then M) unless late

$\equiv (\neg L \rightarrow M) + \text{late}$ ✓

OR

(if ¬L then M) unless late

$\equiv \neg \text{late} \rightarrow (\neg L \rightarrow M)$

$\equiv (\neg L \rightarrow M) \vee \text{late}$

equivalent

→ If p then q unless r.

↳ (If p then q) unless r

$\equiv (p \to q) \vee r$

$\equiv \neg r \to (p \to q)$

✓

qus. Determine which of the following statements are Tautologies. | L-12 |

(a) $p \to (\neg q \vee r \vee \neg r)$

↳ By simplification

$p \to (\neg q \vee \underbrace{r \vee \neg r})$

$p \to (\neg q + T)$

$p \to T \equiv$ True $\quad \therefore$ Tautology ✓

(b) $[\neg p \wedge (p \vee q)] \to q$

$A \to B$ is F only when $\left.\begin{array}{l} A = T \\ B = F \end{array}\right\}$ ✓

↳ using Approach 2

make RHS false i.e $q = $ False

now try to make LHS True

$LHS = \neg p \wedge (p \vee F)$

$\equiv \neg p \wedge p \equiv F$ } Failed to make LHS True $\Rightarrow$ it is Tautology ✓

(e) $\neg p \to (p \to q)$

↳ using Approach 1

make LHS True i.e $\neg p = T \Rightarrow p = F$

now try to make RHS False

$RHS = (p \to q)$

$\equiv F \to q \equiv$ True } Failed to make RHS False $\Rightarrow$ it is Tautology ✓

(f) $p \to (p \to q)$

↳ using Approach 1

make LHS True i.e $p = T$

now try to make RHS False

$RHS = p \to q$

$\equiv T \to q \equiv q$ ($q \to T$, $q \to F$) can be False $\Rightarrow$ it is not Tautology.

→ Implication is special.

We studied implication in detail.

Now we'll study implication in more details.

LOGICAL INFERENCE / IMPLICATION / CONSEQUENCE ARGUMENTS

$\boxed{\text{NOTE}}$ → Infer ≡ Deduce ≡ Entail ≡ Imply ≡ Deduction ≡ Derive

1) If $\underbrace{\text{it is raining}}_{R}$, $\underbrace{\text{He'll take umbrella}}_{U}$. It is raining. } knowledge Base (KB)

Can we infer $\underbrace{\text{"He'll take umbrella"}}_{\text{conclusion}}$? yes

↳ $KB = \{R \to U ; R\}$

conclusion : U

⇒ $\underbrace{R \to U, R}_{\text{from this knowledge we can}} \underbrace{\models}_{\text{inference}} U$ → this conclusion

infer

2) If it is raining, He'll take umbrella. It is not raining } knowledge Base (KB)

Can we $\underbrace{\text{infer}}_{\text{derive}}$, $\underbrace{\text{"He'll not take umbrella"}}_{\text{conclusion}}$? No,

He may take umbrella bcoz of sunlight

↳ $KB = \{R \to U ; \neg R\}$

conclusion $= \{\neg U\}$

⇒ $KB \not\models$ Conclusion } This inference we cannot make (deduction)

3) "If you have access to the N/W, then you can change your grade." } KB

"You have access to the N/W."

Can we infer, $\underbrace{\text{"You can change your grade"}}_{\text{conclusion/consequence}}$? yes

↳ $KB \underbrace{\models}_{\text{infers}}$ conclusion ✓

4) If $\underbrace{\text{I work all night on this H/W}}_{H}$, then $\underbrace{\text{I can answer all the exercises}}_{E}$.

If $\underbrace{\text{I answer all the excercises}}_{E}$, $\underbrace{\text{I will understand the material}}_{M}$.

$\underbrace{\text{I work all night on this H/W}}_{H}$.

Can we infer $\underbrace{\text{"I will understand the material"}}_{M \quad \text{(conclusion)}}$? yes ✓

↳ $KB = \{H \to E, E \to M, H\}$

conclusion $= \{M\}$

↳ $\underline{H \to E, E \to M, H} \underbrace{\models}_{\text{infers}} M$

ques. Which is correct?

(i) $\underbrace{P \wedge Q}_{\text{P is True and Q is True}} \models Q$ ✓ (ii) $P \wedge Q \not\models Q$ ✗

$\underbrace{\text{P is True and Q is True}}_{\text{knowledge}} \underbrace{\models}_{\text{infer}} \underbrace{\text{Q is True}}_{\text{conclusion}}$

$\Rightarrow$ $\left.\begin{array}{l} P \wedge Q \models Q \\ P \wedge Q \models P \end{array}\right\}$ ✓

ques. Which is correct?

(ii) $P \vee Q \models Q$ ✗ (ii) $P \vee Q \not\models Q$ ✓

$\underbrace{\text{P is True OR Q is True}}_{\text{from this knowledge}} \rightarrow$ we cannot make the judgement that Q is True.

↳ $P \vee Q, \neg Q \models P$ ✓ we can make this judgement.

(ex) To prove that "If n is even integer then n^2 is even integer".

$\underbrace{\text{n is even integer}}_{I}$

If $\underbrace{\text{n is even integer}}_{I}$, then $\underbrace{n = 2k, \text{ for some integer } k}_{K}$

If $\underbrace{n = 2k}_{K}$, then $\underbrace{n^2 = 4k^2}_{Q}$

If $\underbrace{n^2 = 4k^2}_{Q}$ then $\underbrace{n^2 \text{ is even.}}_{y}$

Can we infer $\underbrace{\text{"}n^2 \text{ is even"}}_{y}$?

↳ $I, I \rightarrow K, K \rightarrow Q, Q \rightarrow y \models y$ ✓ Knowledge Base $\models$ conclusion ✓

Inference ≡ Declaration ≡ Derivation ≡ Judgement

$\models$

Inference

KB $\models y$ → inference

If $\underline{KB = \text{True}}$ then $\underline{y \text{ should be True}}$

o→ KB is knowledge Base, which is a set of premises / hypothesis or assumptions.

$KB \models Q$ means "Q is logically inferred by KB" or "KB infers Q".

$KB \models Q$ iff there is no logically possible situation in which Q is False while all the premises in KB are True

Or, stated trely, Q is logically inferred by KB iff conclusion (Q) is T in every logically possible situation in which all the premise in KB are T.

o→ $\boxed{KB \models y}$ iff $\boxed{KB \rightarrow y \text{ is Tautology}}$.

$KB \models y$ means KB infers y

means if KB is true then y is True

means it is never possible that KB = True, But y = False

so, $KB \rightarrow y$ is Tautology

ques. which is correct?

(i) $\underbrace{P \rightarrow Q, Q}_{\text{from this info can we drive?}} \models P$ ✗ NO

$P \rightarrow Q$, Q is True

can we say that P must be True?

NO P need not be True

(ii) $P \oplus Q, Q \models \neg P$ ✓

(iii) $\underbrace{P \leftrightarrow Q, \neg Q}_{\text{from this knowledge we can}} \models \neg P$ ✓

infer

that P must be F.

(iv) $P \leftrightarrow Q \models P$ ✗ if $P \leftrightarrow Q$ is T then we cannot say that P must be T.

(v) $P \wedge Q \models P$ ✓

(vi) $P \rightarrow \bar{Q}, \bar{P} \models \bar{Q}$ ✗ if $P \rightarrow \bar{Q}$ is T and P is F, then it is not must that $\bar{Q}$ must be T, $\bar{Q}$ can be F also.

(vii) $P \rightarrow \bar{Q}, \bar{P} \models Q$ ✗ if P is F then $P \rightarrow \bar{Q}$ is T automatically. so Q can be F also.

$KB \models y$ iff $KB \to y$ is Tautology.

ex) $P \to Q, P \models Q$ ✓

↳ bcoz $(P \to Q) \wedge P \to Q$ is Tautology.

$A, B, C, D \models E$ iff $A \wedge B \wedge C \wedge D \to E$ is Tautology.

ex) $P \to \bar{Q}, \bar{P} \models \bar{Q}$ ✗

↳ bcoz $(P \to \bar{Q}) \wedge \bar{P} \to \bar{Q}$ is not Tautology

using Approach 2.

Make RHS = F $\Rightarrow$ $\bar{Q} = F$; $Q = T$

Try to make LHS = T ; $(P \to F) \wedge \bar{P} \equiv \bar{P} \wedge \bar{P} \equiv \bar{P}$

↳ can be True when $P = F$ } so not Tautology

ques. $KB = \{p \to q, p, s \to r\}$

Which of p, q, r, s are inferred by (entailed by) KB?

infer ≡ entail ≡ derive ≡ deduce

$p \to q, p, s \to r \models p$ ✓

$p \to q, p, s \to r \models q$ ✓

$p \to q, p, s \to r \models r$ ✗

$p \to q, p, s \to r \models s$ ✗

$p \to q, s \to r, p$

knowledge { p is T
if p is T then q must be T
bcoz $p \to q$ is T
$s \to r$ is T

We can say that { p must be T
q must be T

we cannot say that
{ r must be T
s must be T

• → Boy implies Human : True

Human implies Boy : False

Human doesnot imply boy.

Boy → Human is Tautology

bcoz

it can NEVER happen that

Boy = T , Human = F

- (P is T and Q is T) implies (P is T). True
  (P AND Q) implies P. True
  } both are same

  (P is T OR Q is T) implies (P is T) False
  (P OR Q) implies P False
  } both are same as seen before

- If I say "x implies y" means "$x \to y$" is Tautology

- If I say "x doesnot imply y" means $x \to y$ is NOT Tautology

  $x \to y$ is $\underbrace{\text{contradiction OR contingency}}_{\text{Falsifiable.}}$

- "x implies y" is false iff for some case $x \to y$ is False.

  "x implies y" is false iff $x \to y$ is Not Tautology

  - $x \to y$ is false
    ↳ $x \to y$ is falsifiable (contradiction OR contingency) eg : Human → Boy

- "x implies y" is equivalent to false iff for all cases $x \to y$ is False

  "x implies y" is equivalent to false iff $x \to y$ is contradiction.

  - $x \to y$ is equivalent to false
    ↳ $x \to y$ is contradiction.

LOGICAL IMPLICATION :

A formula "A logically implies B" iff $A \to B$ is Tautology.

'$\Rightarrow$' is symbol for Tautological Implication (implication which is Tautology), used by some authors.

So, $A \Rightarrow B$ means $A \to B$ is Tautology o→ $\boxed{\text{A (logically) implies B}}$ means $\boxed{A \to B \text{ is Tautology}}$

ex) P∧Q implies P True

bcoz $P \land Q \to P$ is Tautology

$P \land Q \Rightarrow P$ means $P \land Q \to P$ is Tautology

o→ logical consequence
logical Inference
logical Implication
logical Entailment
} All are same

logical consequence :
y is a (logical) consequence of x, iff "$x \to y$" is Tautology

logical Inference :
x (logically) infers y, iff "$x \to y$" is Tautology

logical Implication :
x (logically) implies y, iff "$x \to y$" is Tautology

ex)
Q is logical consequence of $P \land Q$.
$\equiv$
Q is logically inferred by $P \land Q$.
$\equiv$
$P \land Q \models Q$ = $P \land Q$ infers Q
$\equiv$
$P \land Q$ logically implies Q.

(LOGICAL) ARGUMENTS

(ex) "If you have a current password, then you can log onto the N/W."

"You have a ~~Ross~~ current password."

} set of premises

Therefore

"You can log onto the N/W." } conclusion

} Argument

↳ Valid Argument ✓

(ex) If it is raining, He'll take umbrella. } premises

It is not raining

Hence,

He'll not take umbrella } conclusion/consequence

} Argument

↳ Not Valid Argument (Invalid)

•→ P_1, P_2, P_3, P_4 } Premises

$\therefore C$ → conclusion/consequence

} Argument

$\therefore$ → So/Hence/Therefore

•→ P_1, P_2, P_3, $\therefore C$ } is VALID iff

If all premise are T then c must be T

P_1, P_2, P_3, $\therefore C$ } is VALID iff it is NEVER Possible that c is False but all premises are True.

i.e is VALID iff $(P_1 \land P_2 \land P_3) \to C$ is Tautology

•→ To check validity of Argument

P_1, P_2, P_3, $\therefore C$ }

Make conclusion False

Then try to make All premises True.

- If possible : Not Valid Argument ✓
- If NEVER possible : Valid Argument ✓

ex) Arg 1 :

$$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \\ \hline \end{array}$$

VALID ✓

make $Q = F$
now try to make all premises T.
$P = T$ ✓
Now $\underbrace{P}_{T} \to \underbrace{Q}_{F}$ } cannot be True

So couldnt make all premises True, so it is a VALID Argument.

(1) $\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \\ \hline \end{array}$ } is valid

(2) $P, P \to Q \models Q$

(3) $(P \wedge (P \to Q)) \to Q$ is Tautology

$\boxed{\text{Note : } 1 \equiv 2 \equiv 3}$

$\boxed{\text{NOTE}}$ →

(1) $\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \hline \therefore C \\ \hline \end{array}$ } is valid

(2) $P_1, P_2, P_3, P_4 \models C$

(3) $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \to C$ is Tautology

$\boxed{1 \equiv 2 \equiv 3}$

(1) $\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \hline \therefore C \\ \hline \end{array}$ } is Invalid

(2) $P_1, P_2, P_3 \not\models C$

(3) $P_1 \wedge P_2 \wedge P_3 \to C$ is Not Tautology

$\boxed{1 \equiv 2 \equiv 3}$

ex) Arg 2 :

$$\begin{array}{l} P \oplus Q \\ P \\ \hline \neg Q \\ \hline \end{array}$$

VALID ✓ Argument

make conclusion = F i.e $\neg Q = F \equiv Q = T$
now try to make all premises T.
$P = T$ ✓
Now $\underbrace{P}_{T} \oplus \underbrace{Q}_{T}$ } cannot be True
So all premises never T when $\neg Q$ (conclusion) is F.

or we can do it like this intutively.

$$\begin{array}{l} P \oplus Q \\ P \\ \hline \neg Q \\ \hline \end{array}$$

$\underbrace{P \oplus Q}_{T}, \underbrace{P}_{T} \models \overline{Q}$

must be F

if P is T & $P \oplus Q$ is T
then Q must be F
i.e $\overline{Q}$ must be T.
So VALID Argument ✓

$P \oplus Q, P \models \overline{Q}$ ✓

ex **Arg 3** :

$$\begin{array}{c} P \vee Q \\ R \vee S \\ \neg P \vee \neg R \\ \hline Q \vee S \end{array}$$

↳ VALID Arg ✓

make conclusion = F

i.e $Q \vee S = F \Rightarrow Q = F,\ S = F$

now try to make all premises T.

↳ $P \vee \underset{F}{Q} = T \Rightarrow P = T$

$R \vee \underset{F}{S} = T \Rightarrow R = T$

But $\neg P \vee \neg R \equiv F \vee F \equiv F$

} couldn't make all premise T ⇓ VALID ✓

or we can do it intutively -

$P \vee Q$ → Atleast one of P, Q is T

$R \vee S$ → Atleast one of R, S is T

$\neg P \vee \neg R$ → Atleast one of P, R is F

} if Atleast one of P, R is F, then atleast one of Q, S must be T to make $P \vee Q = T$ & $R \vee S = T$.

$Q \vee S$ } Atleast one of Q, S is True ✓

ex **Arg 4** :

$$\begin{array}{c} P \vee Q \\ R \vee S \\ \neg P \vee \neg R \\ \hline Q \oplus S \end{array}$$

$\neg P \vee \neg R$ → Atleast one of P, R is F

→ But when both are F, then for $P \vee Q = T$ & $R \vee S = T$ both $Q = T$ & $R = T$

$Q \oplus S$ } only one of Q, S is T ← so, Invalid Arg.

↳ INVALID ARG ✓

ex **Arg 5** :

$$\begin{array}{c} P \to Q \\ Q \to R \\ \hline P \to R \end{array}$$

↳ VALID ARG ✓

make conclusion = F

i.e $P \to R = F \Rightarrow P = T,\ R = F$

Now try to make all premises T.

↳ $\underset{T}{P} \to Q = T \Rightarrow Q = T$

$\underset{T}{Q} \to \underset{F}{R} \equiv F$ } cannot make True

} couldn't make all premises T ⇓ VALID ✓

$$\begin{array}{rl} \Rightarrow & P \to Q \\ & Q \to R \\ \hline \therefore & P \to R \end{array}$$

$$\begin{array}{ccc} P & \rightsquigarrow & Q \\ & \searrow\ \swarrow & \\ & R & \end{array}$$

$\boxed{\text{NOTE}}$: **Implication is Transitive.**

$$A \to B \to C$$

$(A \to B,\ B \to C)$ then $A \to C$.

ques Is this arg valid?

$$\left.\begin{array}{l} P \to Q \\ Q \to R \\ R \to S \\ S \to I \\ I \to U \\ \hline \therefore (P \to U) \wedge (Q \to I) \end{array}\right\} \text{VALID} \checkmark$$

There will be a Transitive Effect,

$$\overset{T}{P} \to \underset{T}{\overset{T}{Q}} \to \underset{T}{\overset{T}{R}} \to \underset{T}{\overset{T}{S}} \to \underset{T}{\overset{T}{I}} \to \overset{T}{U}$$

If P is T, then U is T. ✓

If Q is T, then I is T ✓

$\Rightarrow (P \to U) \wedge (Q \to I) \equiv T \wedge T \equiv T$ ✓

Arg:

$P \to Q$ → if P is T, then Q is T
$R \to S$ → if R is T, then S is T
$P \vee R$ → atleast one of P, R is T
$S \vee Q$ → atleast one of S, Q is T

} then we can say that atleast one of $Q \vee S$ must be T }

VALID ✓

or we can solve it by usual method.

$$\left.\begin{array}{l} P \to Q \\ R \to S \\ P \vee R \\ \hline S \vee Q \end{array}\right\} \text{VALID} \checkmark$$

make conclusion F, i.e $S \vee Q = F \Rightarrow S = F, Q = F$
now try to make all premises T,

$P \to \underset{F}{Q} = T \Rightarrow P = F$

$R \to \underset{F}{S} = T \Rightarrow R = F$

But, $\underset{F}{P} \vee \underset{F}{R} = F$

} cannot make all premises T. so it is a VALID Arg.

Arg:

$$\left.\begin{array}{l} P \to Q \\ R \to S \\ \bar{P} \vee \bar{R} \\ \hline \bar{Q} \vee \bar{S} \end{array}\right\} \text{INVALID}$$

$\bar{Q} \vee \bar{S} = F \Rightarrow Q = T$ and $S = T$

Now,

$P \to \underset{T}{Q} = T$, $P =$ Anything

$R \to \underset{T}{S} = T$, $R =$ Anything

$\bar{P} \vee \bar{R} = T$, when $P = F, R = T$
$P = T, R = F$

} We can make all premises T
⇓
INVALID Arg.

OR

$P \to Q$
$R \to S$
$\bar{P} \vee \bar{R}$ } Atleast one of P, R is F
$\bar{Q} \vee \bar{S}$ } This conclusion doesnt follow from Premises

} so we cannot say anything about Q, S } Q, S can be Anything

→ INVALID ARG ✓

Arg: $P \to Q$

$R \to S$

$\bar{Q} \vee \bar{S}$ } Atleast one of Q, S is F

$\bar{P} \vee \bar{R}$ } atleast one of P, R must be F

} if Q is F, P must be F
} if S is F, R must be F

} Atleast one of P, R must be F }

VALID Argument
as conclusion follows from premises

$\boxed{\text{NOTE}}$ → By an argument, we mean a sequence of statements that end with a conclusion.

By VALID, we mean that the conclusion, or final statement of the arguments, must follow from the truth of the preceding statements, or premises, of the argument.

That is, an argument is VALID iff if it is impossible for all the premises to be true and the conclusion to be false.

ex "If you have access to the N/W, then you can change your grade." (N, G)

"You have access to the N/W."

∴ "You can change your grade."

} Arg

↳ First of All change from English to logic.

logic →

$N \to G$

N

G

} Valid ✓

if N is T, then G is T

N is T

so G must be T ✓

ques. Is this Arg valid?

If it rains today, then we'll not have a Barbecue today.

If we donot have a barbecue today, then we'll have a barbecue Tomorrow.

Therefore, if it rains today, then we'll have a barbecue tomorrow.

↳

$R \to \bar{B}$

$\bar{B} \to BT$

$\therefore R \to BT$

} Valid ✓

$R \to \bar{B} \to BT$

$$\begin{array}{l} A_1 \\ A_2 \\ \vdots \\ A_n \\ \hline B \end{array} \Bigg\} \text{ is valid}$$

Read this as "from $A_1, A_2, \ldots, A_n$, infer B".

Sometimes written "$A_1, A_2, \ldots, A_n \vdash B$".

RULES OF INFERENCE

↳ How can we "correctly" infer new info from existing info.

Some popular valid Argument forms ↴

popular ↓ frequently used

Argument forms: have special names

↳ we even use them in daily life

- **Modus Ponens**

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

- **Modus Tollens**

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

- **Conjuctive Simplification**

$$\begin{array}{l} p \\ q \\ \hline \therefore p \end{array}$$

- **Disjunctive Syllogism**

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

- **Hypothetical Syllogism** (Transsitivity of Implication)

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

- **Addition**

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

- Conjunctive Simplification (Alternate version)

$$\frac{p \wedge q}{\therefore p}$$

- Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

(ex) Identify the rules of inferences used in each of the following arguments.

(i) $\underbrace{\text{Aditya is a Maths major}}_{M}$. Therefore, $\overbrace{\text{Aditya is either a Math major}}^{M}$ OR $\underbrace{\text{a CS Major}}_{C}$

↳ $\dfrac{M}{\therefore M \vee C}$ } Addition ✓

(ii) If $\overbrace{\text{it snows today}}^{S}$, $\overbrace{\text{the college will close}}^{C}$. $\underbrace{\text{The college is not closed today}}_{\bar{C}}$. Therefore, $\underbrace{\text{it did not snow today}}_{\bar{S}}$

↳ $\begin{array}{l} S \to C \\ \bar{C} \\ \hline \therefore \bar{S} \end{array}$ } Modus Tollens ✓

(iii) If $\overbrace{\text{I go swimming}}^{S}$, then $\overbrace{\text{I will stay in the sun too long}}^{SS}$.
If $\underbrace{\text{I stay in sun too long}}_{SS}$, then $\underbrace{\text{I will sunburn}}_{SB}$.
Therefore, If $\underbrace{\text{I go swimming}}_{S}$, then $\underbrace{\text{I will get sunburn}}_{SB}$.

↳ $\begin{array}{l} S \to SS \\ SS \to SB \\ \hline \therefore S \to SB \end{array}$ } Hypothetical Syllogism ✓

(ex) Use rule of inference to show that the premises "Aditya works Hard.", "if Aditya works hard then he is a dull boy", and "if Aditya is a dull boy then he will not get the job" imply the conclusion "Aditya will not get the job".

→ let H = Aditya works hard

D = Aditya is a dull boy

J = Aditya will get the job

$$\begin{array}{l} H \\ H \to D \\ D \to \bar{J} \\ \hline \therefore \bar{J} \\ \hline \end{array}$$

$$\begin{array}{ll} H & \\ H \to D & \\ D & \text{(Modus Ponen)} \\ D \to \bar{J} & \\ \hline \therefore \bar{J} & \text{(Modus Ponen)} \end{array}$$

or we can prove it as

$$\begin{array}{ll} H \to D & \\ D \to \bar{J} & \\ H \to \bar{J} & \text{(Hypothetical syllogism)} \\ H & \\ \hline \therefore \bar{J} & \text{(Hypothetical syllogism)} \end{array}$$